The Weighted Space Odyssey

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Faculty of Health, Science and Technology

Mathematics

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Abstract

The common topic of this thesis is boundedness of integral and supremal operators between function spaces with weights. The results of this work have the form of characterizations of validity of weighted operator inequalities for appropriate cones of functions. The outcome can be divided into three categories according to the particular type of studied operators and function spaces.

The first part involves a convolution operator acting on general weighted Lorentz spaces of types Λ, Γ and S defined in terms of the nonincreasing rearrangement, Hardy-Littlewood maximal function and the difference of these two, respectively. It is characterized when a convolution-type operator with a fixed kernel is bounded between the aforementioned function spaces. Furthermore, weighted Young-type convolution inequalities are obtained and a certain optimality property of involved rearrangement-invariant domain spaces is proved. The additional provided information includes a comparison of the results to the previously known ones and an overview of basic properties of some new function spaces appearing in the proven inequalities.

The second type of investigated objects are bilinear and multilinear operators defined as a product of linear Hardy-type operators or in a similar way. It is determined when a bilinear Hardy operator inequality holds either for all nonnegative or all nonnegative and nonincreasing functions on the real semiaxis. The proof technique is based on a reduction of the bilinear problems to linear ones to which known weighted inequalities are applicable. The use of this method to solve other questions concerning more general multilinear operators is described as well.

In the third part, the focus is laid on iterated supremal and integral Hardy operators, a basic Hardy operator with a kernel and applications of these to more complicated weighted problems and embeddings of generalized Lorentz spaces. Several open problems related to missing cases of parameters are solved, therefore completing the theory of the involved fundamental Hardy-type operators. The results have a standard explicit form of integral or supremal conditions which are compatible with those known previously. It allows for a straightforward application in various situations involving more complicated weighted inequalities.

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I would also like to thank my family and friends for their support and encouragement.

Last but not least, I am especially grateful to my mathematics teachers on all levels of the educational system.

Karlstad, April 2016

In my opinion, in academic publications like this one it is rather reasonable to restrict oneself to acknowledging people who actually had some effect on the work presented. The previous lines could be thus considered fair enough. Nevertheless, since staying alive enough seems to be a necessary condition for defending a thesis, the acknowledgement section may be slightly extended.

My deepest gratitude goes to the nurses and doctors at the Karlstad Central Hospital, especially at the Department № 7 but also the others, for the great effort they have put into my treatment and the enormous deal of help I have received from them. All these people perform an excellent work and I appreciate it greatly.

I am also indebted to my parents and all friends for their support and numerous visits during this summer.

Karlstad, November 2016 Martin Křepela

List of papers

[I] M. KŘEPELA, *Convolution inequalities in weighted Lorentz spaces*, Math. Inequal. Appl. **17** (2014), 1201–1223.

[II] M. KŘEPELA, *Convolution in rearrangement-invariant spaces defined in terms of oscillation and the maximal function*, Z. Anal. Anwend. **33** (2014), 369–383.

[III] M. KŘEPELA, *Convolution in weighted Lorentz spaces of type* Γ, Math. Scand. **119** (2016), 113–132.

[IV] M. KŘEPELA, *Bilinear weighted Hardy inequality for nonincreasing functions*, Publ. Mat. **61** (2017), 3–50.

[V] M. KŘEPELA, *Iterating bilinear Hardy inequalities*, to appear in Proc. Edinb. Math. Soc.

[VI] M. KŘEPELA, *Integral conditions for Hardy-type operators involving suprema*, Collect. Math. **68** (2017), 21–50.

[VII] A. GOGATISHVILI, M. KŘEPELA, L. PICK AND F. SOUDSKÝ, *Embeddings of classical Lorentz spaces involving weighted integral means*, preprint.

[VIII] M. KŘEPELA, *Boundedness of Hardy-type operators with a kernel: integral weighted conditions for the case* $0 < q < 1 \leq p < \infty$, preprint.

[IX] M. KŘEPELA, *Convolution inequalities in weighted Lorentz spaces, case* $0 < q < 1$, to appear in Math. Inequal. Appl.

Author's comment

My contribution to paper [VII] was proving the inequalities involving the iterated Hardy-type operators and devising some parts of the reduction method used in the paper.

Contents

1
Introduction

Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes.

JOHANN WOLFGANG GOETHE

This thesis is facing no less challenge than defeating the inequality, conquering the space, choosing the right operator and, of course, maintaining weight all the time. There is no doubt that such complicated issues need to be dealt with in a proper way. And the proper way, of course, means the mathematical way. The reader is perhaps beginning to suspect that those topics might not necessarily be just a collection of buzzword phrases from an average newspaper (including an advertisement part) but *something completely different*. And the reader may be right.

The main topic of the research presented in this thesis are operator inequalities related to weighted function spaces. In many ways, this topic is connected to harmonic analysis, interpolation and approximation theory and other branches of functional analysis. Results from these fields are of enormous interest in the theory of partial differential equations on all its levels, from the investigation of existence and regularity of solutions to the more practical outcome involving explicit constructions of approximative solutions. This finds applications in numerical modelling of "real-life" problems from physics as well as other sciences. Operator inequalities and theory of function spaces also find use in other parts of approximation theory than just those concerned with differential equations. Theoretical results which involve approximation of functions have a great practical impact in fields like signal and image processing, data compression, electrical engineering and other.

The particular problems studied in this thesis can be summarily described as follows:

(i) proving weighted Young-type inequalities, related to boundedness of a convolution operator between weighted Lorentz spaces, by reducing the problem to weighted Hardy inequalities;

(ii) characterizing boundedness of various bilinear operators, in particular of Hardy type, between weighted Lorentz and Lebesgue spaces by employing an iteration technique;

(iii) proving weighted inequalities involving some fundamental Hardy-type operators acting on weighted Lebesgue spaces, and an application of the results to embeddings of generalized Lorentz spaces.

The initial motivation of part (i) was to improve the classical Young-O'Neil inequalities which involved $L^{p,q}$ spaces, by proving their analogues in general weighted Lorentz spaces. These inequalities can be directly applied to get sufficient conditions of Lorentz-space boundedness of operators which have the form of a convolution with fixed kernel. The obtained results are optimal in the sense that the boundedness conditions implied by the Young-type inequality are both sufficient and necessary, provided that the kernel in the operator is positive and radially decreasing. Many of the classical operators (for example, the Riesz fractional integral) satisfy this geometrical requirement.

Part (ii) is connected to the previous one by the fact that bilinear Hardy operators play a significant role in the proof technique used in (i). In part (ii), that technique, based on a certain iteration process, is further developed for use in more situations. The iteration method is simpler than those used in older papers on similar topics, and the older results are simplified in most cases. Using the iteration method also demonstrates the importance of the theory of function spaces and inequalities with general weights since the knowledge of various general-weighted inequalities is an indispensable ingredient in the method.

Probably the most important achievement of part (iii) is closing of several gaps in the theory of Hardy operators (even on its fundamental level) and proving boundedness results for certain new iterated supremal Hardy-type operators. The particular choice of investigated operators was motivated by a subsequent application of the developed theory in solving a complicate problem concerning embeddings of generalized Lorentz spaces. Solving the open problems concerning the missing cases of weighted Hardy-type inequalities also allowed to complete the results involving weighted Young-type convolution inequalities. This makes a link between parts (iii) and (i).

The thesis itself consists of two main parts. The first one is this introductory summary while the second one consists of those nine papers listed in the preface. These papers are summarily referred to as “the main papers” in the introduction. The reference marks [I–IX] are used to specify a particular paper from the list.

The introductory text is divided into several chapters, the first of which is this “introduction to the introduction”. Chapter 2 contains an overview of the elementary theory of function spaces with focus on those spaces and classes which are relevant for the main papers. Above all, this means introducing weighted Lebesgue spaces, weighted Lorentz spaces Λ, Γ, S and some more related structures, listing their basic properties and summarizing various existing results. In a similar manner, Chapter 3 introduces the operators and inequalities which are the subject of investigation in the thesis. The text further continues with Chapter 4 where the contents of the main papers are briefly summarized. That chapter outlines the proof ideas and techniques of the papers, the relation of the obtained results to previous research and potential applications.

This doctoral thesis is linked to the author’s licentiate thesis [74] which contained the results of papers [I–III]. Some larger portions of the text in Sections 3.2 and 4.1 and certain minor parts of Chapter 2 already appeared in the introductory summary in [74].

2
Weighted function spaces

Empty spaces – what are we living for?
QUEEN, *THE SHOW MUST GO ON*

All the research questions and results of this thesis have a connection to function spaces. In this chapter, relevant function spaces and some of their basic properties are introduced.

Very vaguely said, a function space is a set – or a "family" – of functions sharing a certain property. Such property may be, for instance, integrability, differentiability, boundedness or other. Grouping functions to vector-space structures is practical in many ways and it formed a base for a great amount of important research in functional analysis.

Suppose, for example, that $\mathscr{M}$ denotes the cone of real-valued μ-measurable functions defined on certain measure space $(\Omega, \mathfrak{M}, \mu)$. A usual way to define a function space X, consisting of functions from $\mathscr{M}$, is by using a mapping $\|\cdot\|_X : \mathscr{M} \to [0, \infty]$. One defines the set X, which may be called the *function space generated by* $\|\cdot\|_X$, by

$$X := \{f \in \mathscr{M};\ \|f\|_X < \infty\}.$$

The functional $\|\cdot\|_X$ might be a norm, which then justifies calling X a function *space*. However, $\|\cdot\|_X$ may as well satisfy only conditions which are much weaker than those defining a norm. Nevertheless, X is often called a function *space* even in such relaxed cases although the term may be incorrect in a strict sense (for example, X does not have to be a linear space). The term *function space* may, in such potentially problematic cases, be also replaced by the "safer" form *function class*.

The norm property of a functional generating a function space may be important. So it is, for instance, in some classical theorems of functional analysis requiring the involved structures to be Banach spaces. On the other hand, there are many other problems for which the norm property is irrelevant. The problems solved in this thesis fall, in fact, mostly in the second category.

The functional $\|\cdot\|_X$ may control various properties of functions. They can have a global character (e.g. the value of the integral of $|f|$ over a set Ω) or a local one (modulus of continuity, properties of ∇f, etc.). In what follows, the spaces of interest are mostly those based on the global behavior of functions, namely on various "integral properties" of those. The simplest, though probably the most important, example of such a space is the Lebesgue L^p space. It is defined as follows.

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and let $p \in (0,\infty]$. For any real-valued function f measurable on $(\Omega, \mathfrak{M}, \mu)$ define

$$\begin{aligned} \|f\|_p &:= \left(\int_\Omega |f(x)|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} && \text{if } 0 < p < \infty, \\ \|f\|_\infty &:= \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| && \text{if } p = \infty. \end{aligned}$$

In here, one uses the notation

$$\operatorname*{ess\,sup}_{x\in\Omega} |f(x)| := \inf\{c > 0;\ |f| \le c \ \mu\text{-a.e. on } \Omega\}.$$

Then the space L^p is defined by

$$L^p(\Omega, \mathfrak{M}, \mu) := \left\{ f : (\Omega, \mathfrak{M}, \mu) \to \mathbb{R} \ \mu\text{-measurable};\ \|f\|_p < \infty \right\}.$$

Often in this text, the underlying measure space $(\Omega, \mathfrak{M}, \mu)$ is $\mathbb{R}^d$ with the d-dimensional Lebesgue measure (and the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^d$). It is written only L^p instead of $L^p(\Omega, \mathfrak{M}, \mu)$ if there is no risk of confusion about the underlying measure space.

For every $p \in (0,\infty]$, the L^p space is indeed a linear space. The mapping $\|\cdot\|_p$ is a norm if and only if $p \in [1,\infty]$. If $p \in (0,1)$, then $\|\cdot\|_p$ is merely a quasi-norm since the Minkowski inequality fails in this case.

The symbol $\mathscr{M}_\mu(\Omega)$ will be used to denote the cone of μ-measurable real-valued functions on $(\Omega, \mathfrak{M}, \mu)$, and $\mathscr{M}_\mu^+(\Omega)$ will stand for the cone of all $f \in \mathscr{M}_\mu(\Omega)$ such that $f \ge 0$ μ-a.e. If μ is the Lebesgue measure (and $\mathfrak{M}$ is the Lebesgue σ-algebra), one writes simply $\mathscr{M}(\Omega)$ and $\mathscr{M}^+(\Omega)$. The set Ω will be always specified, usually as $\mathbb{R}^d$ or an interval on the real axis.

A special case of an L^p space which is worth highlighting is the *weighted Lebesgue space* $L^p(v)$ over $(0,\infty)$. It consists of all functions $f \in \mathcal{M}(0,\infty)$ such that

$$\|f\|_{L^p(v)} := \left(\int_0^\infty |f(x)|^p v(x) \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty \qquad \text{if } 0 < p < \infty,$$
$$\|f\|_{L^\infty(v)} := \operatorname*{ess\,sup}_{x>0} |f(x)| v(x) < \infty \qquad \text{if } p = \infty,$$

where v is a given nonnegative measurable function on $(0,\infty)$. The essential supremum in the case $p=\infty$ is, naturally, taken with respect to the Lebesgue measure.

The L^p spaces have many useful properties. This motivated the introduction of the *Banach function spaces* by W. A. J. Luxemburg in [81]. Roughly speaking, the idea was to introduce a general type of spaces based on a set of properties inspired by the properties of L^p spaces. The proper definition reads as follows.

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and let $\varrho : \mathcal{M}_\mu^+(\Omega) \to [0,\infty]$ be a mapping. Then ϱ is called a *Banach function norm* if for all functions $f,\ g,\ f_n \in \mathcal{M}_\mu^+(\Omega)$ $(n \in \mathbb{N})$, for all constants $a \geq 0$ and all μ-measurable sets $E \subset \Omega$ the following conditions are satisfied:

(P1) $\varrho(f+g) \leq \varrho(f) + \varrho(g)$,

(P2) $\varrho(af) = a\varrho(f)$,

(P3) $\varrho(f) = 0 \iff f = 0$ μ-a.e.,

(P4) $0 \leq g \leq f$ μ-a.e. $\Rightarrow \varrho(g) \leq \varrho(f)$,

(P5) $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow \varrho(f_n) \uparrow \varrho(f)$,

(P6) $|E| < \infty \Rightarrow \varrho(\chi_E) < \infty$,

(P7) $|E| < \infty \Rightarrow \int_E f \, \mathrm{d}\mu \leq C_E \varrho(f)$ for some constant $C_E \in (0,\infty)$ depending on E and ϱ but independent of f.

If ϱ is a Banach function norm, the collection

$$X_\varrho := \left\{ f \in \mathcal{M}_\mu(\Omega),\ \varrho(|f|) < \infty \right\} \tag{1}$$

is called a *Banach function space*.

There is a particular subclass of Banach function spaces called the *rearrangement-invariant* spaces. They are based on the following definition.

Let $f \in \mathscr{M}_\mu(\Omega)$. The *nonincreasing rearrangement* of f, denoted by f^*, is defined by

$$f^*(t) := \inf\left\{s \geq 0;\ \mu(\{x \in \mathbb{R}^d,\ |f(x)| > s\}) \leq t\right\}, \quad t \in (0, \mu(\Omega)).$$

A Banach function norm ϱ is called a *rearrangement-invariant* (shortly *r.i.*) *norm* if, for all functions $f,\ g \in \mathscr{M}_\mu^+(\Omega)$, it satisfies

(P8) $f^* = g^*$ on $(0, \mu(\Omega)) \Rightarrow \varrho(f) = \varrho(g)$.

As it was suggested before, being a norm might be a rather unnecessarily strong property of a functional. One may therefore introduce some additional terms for function classes based on weaker conditions.

A mapping $\varrho : \mathscr{M}_\mu^+(\Omega) \to [0, \infty]$ is said to be a *rearrangement-invariant quasi-norm* if conditions (P2)–(P8) and

(P1*) $\varrho(f+g) \leq B(\varrho(f) + \varrho(g))$ with a constant $B \in (1, \infty)$ independent of $f,\ g$

are satisfied for all functions $f,\ g,\ f_n \in \mathscr{M}_\mu^+(\Omega)$ $(n \in \mathbb{N})$, all constants $a \geq 0$ and all μ-measurable sets $E \subset \Omega$. In this case, the collection X_ϱ defined by (1) will be called a *rearrangement-invariant quasi-space*.

Furthermore, the collection X_ϱ is called a *rearrangement-invariant lattice* if the mapping ϱ satisfies the conditions (P2), (P4), (P6) and (P8) for all $f,\ g \in \mathscr{M}_\mu^+(\Omega)$, all $a \geq 0$ and all μ-measurable $E \subset \Omega$.

If X_ϱ is, at least, an r.i. lattice generated by a mapping ϱ, the notation $\|f\|_X := \varrho(|f|)$ and $X := X_\varrho$ may and will be used. In this way, the notation corresponds to the one used in the beginning of this chapter where the symbol $\|\cdot\|_X$ denoted the functional generating a function space.

The simplest example of an r.i. space is the L^p space over $\mathbb{R}^d$ with the Lebesgue measure and with $p \in [1, \infty]$. If $p \in (0, 1)$, this structure becomes only an r.i. quasi-space. The weighted Lebesgue space $L^p(v)$ is not r.i. unless the weight function v is constant a.e. Other typical function spaces which are not r.i. are the Sobolev spaces. This is not surprising since the information about differentiability – which is by its nature a local property of a function – is lost when passing from a function to its nonincreasing rearrangement.

In what follows, we will always assume that $(\Omega, \mathfrak{M}, \mu)$ is a measure space such that $\mu(\Omega) = \infty$. It is, of course, possible to modify the definitions of the spaces introduced below so that they correspond to the case of functions defined on a measure space of finite measure.

By generalizing and refining the classical Lebesgue L^p spaces, it is possible to create a wider and finer scale of r.i. spaces (or lattices). The first step in such direction is made by defining the *Lorentz space* $L_{p,q}$. This structure is generated by the following functional:

$$||f||_{p,q} := \left(\int_0^\infty (f^*(t))^q t^{\frac{q}{p}-1} \, \mathrm{d}t \right)^{\frac{1}{q}}, \qquad 0 < p, q < \infty,$$
$$||f||_{p,\infty} := \sup_{t>0} f^*(t) t^{\frac{1}{p}}, \qquad 0 < p < q = \infty.$$

As usual, the mapping $||\cdot||_{p,q}$ is not necessarily a norm, but if $p \in (1,\infty)$ and $q \in (1,\infty]$, then $L_{p,q}$ is *normable*. This means that there exists a norm which is equivalent to the mapping $||\cdot||_{p,q}$. To show this, one introduces the following functionals:

$$||f||_{(p,q)} := \left(\int_0^\infty (f^{**}(t))^q t^{\frac{q}{p}-1} \, \mathrm{d}t \right)^{\frac{1}{q}}, \qquad 0 < p, q < \infty,$$
$$||f||_{(p,\infty)} := \sup_{t>0} f^{**}(t) t^{\frac{1}{p}}, \qquad 0 < p < q = \infty.$$

In here, the symbol f^{**} denotes the *Hardy-Littlewood maximal function* of f^* given by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, \mathrm{d}s, \quad t > 0.$$

It can be proved (see [11]) that $||\cdot||_{p,q}$ is equivalent to $||\cdot||_{(p,q)}$ when $p \in (1,\infty)$ and $q \in (1,\infty]$.

The $L_{p,q}$ and $L_{(p,q)}$ spaces play a significant role in interpolation theory [11, 12] since the $L_{(p,q)}$ spaces are real interpolation spaces between L^1 and L^∞. Besides that, these spaces also appear in various refinings of classical inequalities. An example of these are the generalizations of the Young inequality presented later in Section 3.2.

Introducing and studying new function spaces is motivated by various reasons. One of them may be that such new spaces are interpolation spaces between other previously known ones. Another reason might be, for instance, the fact that a new space is the dual or associated space to a known one, it is the optimal domain or range space for an operator, etc. Further on in this text, some of those aspects will be discussed in connection to various particular cases.

At this point, it is time to show the definition of an *associate space*. Let ϱ be a Banach function norm for functions on $\mathscr{M}_\mu^+(\Omega)$. Then its *associate norm* ϱ' is defined by

$$\varrho'(g) := \sup_{\substack{f \in \mathscr{M}_\mu^+(\Omega) \\ \varrho(f) \leq 1}} \int_\Omega f\, g \,\mathrm{d}\mu$$

for all $g \in \mathscr{M}_\mu^+(\Omega)$. The Banach function space $X' := X_{\varrho'}$ is called the *associate space* of X_ϱ.

It should be noted that the associate norm and associate space are indeed a Banach function norm and a Banach function space, respectively. For proofs of these claims as well as for more details see [11]. The definition may be also extended to cover even r.i. lattices.

From now on, the notion of a weight will be used in a somewhat restricted sense. Indeed, a *weight* will always mean a function $v \in \mathscr{M}^+(0,\infty)$ such that

$$0 < \int_0^t v(s)\,\mathrm{d}s < \infty \quad \text{for all } t > 0.$$

In [79, 80], G. G. Lorentz defined a more general class of function spaces – the Λ spaces. They may be understood as general weighted variants of the $L_{p,q}$ spaces, and are defined as follows.

Let $p \in (0,\infty]$ and let v be a weight. The *weighted Lorentz space* $\Lambda^p(v)$ is the set

$$\left\{ f \in \mathscr{M}_\mu(\Omega);\ \|f\|_{\Lambda^p(v)} < \infty \right\},$$

where

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} \qquad \text{if } 0 < p < \infty,$$

$$\|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f^*(t) v(t) \qquad \text{if } p = \infty.$$

The spaces $\Lambda^p(v)$ with $p < \infty$ are usually called *classical weighted Lorentz spaces* and, as was said, they appeared first in [79, 80]. The *weak-type weighted Lorentz spaces*, that means those with $p = \infty$, were introduced in [28] and further treated, for instance, in [24, 27, 29, 30].

The class of $\Lambda^p(v)$-spaces encompasses a variety of function spaces which are obtained as special cases of $\Lambda^p(v)$ by a particular choice of the weight v. These include the aforementioned $L_{p,q}$ spaces, Lorentz-Zygmund spaces [10] and their generalizations [92], Lorentz-Karamata spaces [86] and other.

A Λ space, in spite of the name, is not necessarily a linear set. The main cause of this problem is the fact that the rearrangement mapping $f \mapsto f^*$ is, in general, not sublinear. To formulate this precisely, if the measure space $(\Omega, \mathfrak{M}, \mu)$ contains at least two disjoint sets of positive μ-measure, then for each $n \in \mathbb{N}$ and $t \in (0, \mu(\Omega))$ there exist functions $f,\ g \in \mathscr{M}_\mu(\Omega)$ such that

$$(f+g)^*(t) \geq n(f^*(t)+g^*(t)).$$

Even though they do not have to be linear, let alone normed spaces, the name "weighted Lorentz spaces" is commonly used for the $\Lambda^p(v)$ spaces. The questions of linearity of $\Lambda^p(v)$ and of their (quasi-)normability were studied in [33].

E. Sawyer [104] first described the associate space to $\Lambda^p(v)$. This type of a function space is called the Γ space and is defined in the following way.

If $p \in (0, \infty]$ and v is a weight, the weighted Lorentz space $\Gamma^p(v)$ is the set

$$\left\{ f \in \mathscr{M}_\mu(\Omega);\ \|f\|_{\Gamma^p(v)} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{\Gamma^p(v)} &:= \left(\int_0^\infty (f^{**}(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}}, & \text{if } 0 < p < \infty, \\ \|f\|_{\Gamma^\infty(v)} &:= \operatorname*{ess\,sup}_{t>0} f^{**}(t) v(t), & \text{if } p = \infty. \end{aligned}$$

The classical $\Gamma^p(v)$ space with $p < \infty$ is the one introduced in [104], although a space with a norm involving f^{**} was explicitly presented already in A.-P. Calderón's paper [21]. The weak-type spaces $\Gamma^\infty(v)$ appeared in [28] and were, as well as their weak-Λ counterparts, further studied for example in [24, 27, 29, 30, 48].

The relation between Λ and Γ spaces is rather strong. The aforementioned associatedness property has the following form (cf. [104]): if $p \in (1, \infty)$, $p' := \frac{p}{p-1}$ and v is a weight, then the r.i. space (lattice) X generated by the functional

$$\|f\|_X := \left(\int_0^\infty (f^{**}(t))^{p'} t^{p'} \left(\int_0^t v(s)\,\mathrm{d}s \right)^{-p'} v(t)\,\mathrm{d}t \right)^{\frac{1}{p'}} + \|f\|_1 \|v\|_1^{-\frac{1}{p}}$$

is the associate space to $\Lambda^p(v)$. If $v \notin L^1$, then the second term is not present. For other cases of p, see [27, 104].

Another relation between Λ and Γ concerns the normability issue. Indeed, it holds (see [6, 104]) that with $p \in (1,\infty)$ the functional $\|\cdot\|_{\Lambda^p(v)}$ is equivalent to a norm if and only if $v \in B_p$, where

$$B_p := \left\{ v \in \mathscr{M}^+(0,\infty);\ \exists C \in (0,\infty)\ \forall t > 0:\ t^p \int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s \le C \int_0^t v(s)\,\mathrm{d}s \right\}.$$

Notice that if $v \in B_p$, then $\Lambda^p(v) = \Gamma^p(v)$ with equivalent norms. (For more details see [29–31, 42, 110].)

The question of linearity and normability of $\Gamma^p(v)$ is considerably simpler than in the case of $\Lambda^p(v)$. The reason is that the Hardy-Littlewood maximal function does satisfy

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t) \tag{2}$$

for all $t > 0$ and all locally integrable functions f, g (see [11]). Thanks to (2), the functional $\|\cdot\|_{\Gamma^p(v)}$ is always an r.i. quasi-norm at least, for $p \ge 1$ it is an r.i. norm by the Minkowski inequality. Functional properties of Γ were studied in more detail in [64], for example.

There are further generalizations of Γ spaces [38, 39, 47, 51], based on generalized versions of the Hardy-Littlewood maximal operator. These are represented, for instance, by the $\Gamma^p_u(v)$ space (see [47]) generated by

$$\|f\|_{\Gamma^p_u(v)} := \left(\int_0^\infty \left(\int_0^t u(s)\,\mathrm{d}s \right)^{-p} \left(\int_0^t f^*(s)u(s)\,\mathrm{d}s \right)^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}},$$

and the *generalized* Γ *space* $\mathrm{G}\Gamma^{m,p}(u,v)$ (see [VII]) generated by

$$\|f\|_{\mathrm{G}\Gamma^{m,p}(u,v)} := \left(\int_0^\infty \left(\int_0^t (f^*(s))^m u(s)\,\mathrm{d}s \right)^{\frac{p}{m}} v(t)\,\mathrm{d}t \right)^{\frac{1}{p}}. \tag{3}$$

In both cases, u and v are weights and m, $p \in (0,\infty)$, with further extensions to the weak-type cases $m = \infty$, $p = \infty$ possible in the standard way. These spaces are the subject of investigation in paper [VII] and are discussed further below in this introductory summary.

The last Lorentz-type "space" considered here is the class S, introduced in [25]. If $p \in (0,\infty]$ and v is a weight, the class $S^p(v)$ is defined as

$$\left\{ f \in \mathscr{M}_\mu(\Omega);\ \lim_{s\to\infty} f^*(s) = 0,\ \|f\|_{S^p(v)} < \infty \right\},$$

where

$$\|f\|_{S^p(v)} := \left(\int_0^\infty (f^{**}(t) - f^*(t))^p v(t)\,dt \right)^{\frac{1}{p}} \qquad \text{if } 0 < p < \infty,$$
$$\|f\|_{S^\infty(v)} := \operatorname*{ess\,sup}_{t>0} (f^{**}(t) - f^*(t))\, v(t) \qquad \text{if } p = \infty.$$

Unlike the Λ and Γ spaces, the class $S^p(v)$ is not even an r.i. lattice (for a detailed study of this and related issues, see [25]). The functional $f^{**} - f^*$ is important in various parts of analysis [7–9, 11, 18, 25, 50, 66, 68, 69, 71, 73, 83, 84, 100] and represents a natural tool to measure oscillation of f, see [9, 11].

It might be reasonable to compare the class $S^p(v)$ to the $L_{\infty,q}$ spaces which consist of functions $f \in \mathscr{M}_\mu(\Omega)$ such that

$$\|f\|_{L_{\infty,q}} := \|f^*\|_1 + \left(\int_0^\infty (f^{**}(t) - f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

(In here, the case $q < \infty$ is considered.) This shows that the $S^p(v)$ spaces in a sense generalize the $L_{\infty,q}$ spaces. Details and applications of the $L_{\infty,q}$ spaces can be found in [9, 11].

3
Operators and inequalities

Well, here's another nice mess you've gotten me into.
Oliver Hardy

With the framework of function spaces established, it is time to take a closer look to the function-space-related problems treated in the thesis. In general, all of these problems involve finding conditions under which a certain operator is bounded between given function spaces. By definition, this means to determine when a certain functional inequality is valid. Expressing the problem in terms of inequalities is also practical for applications.

This chapter introduces relevant operators, their properties of interest as well as inequalities which are produced when those operators are studied. After a general summary relevant for all of the involved operators, the text is divided to two sections corresponding to the two important classes of operators investigated in the main papers.

Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and $\mathscr{X} \subset \mathscr{M}_\mu(\Omega)$. Then an *operator* T is any mapping $T : \mathscr{X} \to \mathscr{M}_\mu(\Omega)$. Such a "toothless" general definition may be further specified in various ways.

Suppose that addition and scalar multiplication are defined on $\mathscr{M}_\mu(\Omega)$ by means of performing these operations pointwise, and that $\mathscr{X}$ is a linear set. Then, an operator $T : \mathscr{X} \to \mathscr{M}_\mu(\Omega)$ is said to be *homogeneous* if for all $f \in \mathscr{X}$ and $a \in \mathbb{R}$ it satisfies

$$T(af) = aTf.$$

Next, T is said to be *linear* if it is homogeneous and for all $f, g \in \mathscr{X}$ it satisfies

$$T(f+g) = Tf + Tg.$$

An operator $T : \mathscr{X}_1 \times \mathscr{X}_2 \to \mathscr{M}_\mu(\Omega)$ is called *bilinear* if the operators $T(\cdot, g)$ and $T(f, \cdot)$ are linear for any fixed $g \in \mathscr{X}_2$ and $f \in \mathscr{X}_1$, respectively. This definition may be clearly extended to *multilinear* operators.

Furthermore, $T : \mathscr{X} \to \mathscr{M}_\mu(\Omega)$ is said to be *positive* if it maps nonnegative functions to nonnegative functions, i.e., if $T(\mathscr{X} \cap \mathscr{M}_\mu^+(\Omega)) \subset \mathscr{M}_\mu^+(\Omega)$. A positive operator T is called *quasi-linear* if it is homogeneous and there exist constants $C_1, C_2 \in [0, \infty)$ such that for all $f, g \in \mathscr{X} \cap \mathscr{M}_\mu^+(\Omega)$ it holds that

$$C_1(Tf + Tg) \leq T(f + g) \leq C_2(Tf + Tg)$$

pointwise μ-a.e. on Ω. A positive homogeneous operator T is called *sublinear* if it satisfies

$$T(f + g) \leq Tf + Tg$$

pointwise μ-a.e. on Ω for all $f, g \in \mathscr{X} \cap \mathscr{M}_\mu^+(\Omega)$.

A standard question concerning operators and function spaces is whether an operator T maps a function space X into a function space Y, i.e., whether $T(X) \subset Y$. In fact, the simple set inclusion is preferably replaced by *boundedness* of $T : X \to Y$ in the sense of the following definition.

Let X, Y be two function spaces (classes) of functions from the cone $\mathscr{M}_\mu(\Omega)$. An operator $T : X \to \mathscr{M}_\mu(\Omega)$ is said to be *bounded between X and Y* if there exists a constant $C \in (0, \infty)$ such that for all $f \in X$ (or all $f \in \mathscr{M}_\mu(\Omega)$) the inequality

$$\|Tf\|_Y \leq C\|f\|_X \tag{4}$$

is satisfied. An important particular case is attained by the choice $T = I$, where I is the identity operator. Inequality (4) then has the form

$$\|f\|_Y \leq C\|f\|_X. \tag{5}$$

If there exits a constant $C \in (0, \infty)$ such that (5) holds for all $f \in X$, then it is said that X is *embedded in* Y, and one writes $X \hookrightarrow Y$.

Often, there is an interest in finding the *optimal constant* C in (4) or (5), i.e., the least C the respective inequality holds with for all $f \in X$. The optimal C in (4) can be expressed by

$$C = \sup_{f \in X} \frac{\|Tf\|_Y}{\|f\|_X}, \tag{6}$$

if the convention $\frac{0}{0} := 0$, $\frac{a}{0} := \infty$ for $a \in (0, \infty]$ is considered. Obviously, if the optimal constant is infinite, then T is not bounded between X and Y.

The definitions of boundedness, an embedding or the optimal constant do not require any special properties of X, Y and $\|\cdot\|_X$, $\|\cdot\|_Y$. The functionals $\|\cdot\|_X$ and $\|\cdot\|_Y$

generating X and Y, respectively, might be rather arbitrary and the definitions will still make sense. If T is linear and the functional $\|\cdot\|_X$ satisfies at least $\|af\|_X = |a|\|f\|_X$ for all $f \in X$, $a \in \mathbb{R}$ and $\|\cdot\|_Y$ has an analogous property, then (6) can be rewritten as

$$C = \sup_{\substack{f\in X \\ \|f\|_X=1}} \|Tf\|_Y.$$

Finally, it might be worth noticing that if X and Y are Banach function spaces (in the sense of Luxemburg's definition), then the assertions $X \subset Y$ and $X \hookrightarrow Y$ coincide (see [11, p. 7]).

3.1 Hardy operators

Variants of the Hardy operator appear almost everywhere in this publication. A simple form of an operator from this family is the *Hardy average operator* A defined by

$$Af(t) := \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s, \qquad t > 0,$$

for $f \in \mathcal{M}^+(0,\infty)$ for which the integral makes sense (even as being infinite). The Hardy-Littlewood maximal function g^{**}, as presented in the previous chapter, is therefore the image of g^* under the mapping A, i.e., $g^{**} = A(g^*)$.

The research on Hardy operators and Hardy inequalities has a long and complex history and making a thorough exposition of it is definitely not an ambition of this introduction. Many publications about this topic exist (see [75, 76, 91]) and an interested reader may therefore consult them. In what follows, the focus is laid on those parts of the existing theory of Hardy operators which are relevant to the research presented in the main papers.

The Hardy average operator can be viewed as an one-dimensional relative of the *maximal operator*. The latter is defined as follows. For $f \in \mathcal{M}(\mathbb{R}^d)$ put

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(x)|\,\mathrm{d}x, \qquad x \in \mathbb{R}^d.$$

In here, $B(x,r)$ denotes the ball of radius r centered at the point x, and $|B(x,r)|$ the (Lebesgue) measure of the ball. This is not the only type of maximal operator in existence and use. Other frequently used variants involve non-centered balls, cubes, weighted forms, etc., see [11, 57, 58, 70]. Such types of maximal operators also have different properties and each of them might be useful in different situations.

The maximal operator is indispensable in various areas of analysis. In particular, it is crucial in the theory of differentiability of functions, e.g., in proving the Lebesgue and Rademacher differentiability theorems [36, 114]. Besides that, it finds applications in interpolation theory, Fourier analysis, singular integral theory and other fields, especially those in which smoothness of functions plays a significant role (see [11, 36, 57, 58, 113]).

One of the reasons for the interest in the Hardy operator is its close relation to the maximal operator in the framework of r.i. spaces. Namely (see [11, p. 122]), there exist positive real constants C_1, C_2 such that for all $f \in \mathscr{M}(\mathbb{R}^d)$ and all $t > 0$ one has

$$C_1(Mf)^*(t) \leq f^{**}(t) \leq C_2(Mf)^*(t). \tag{7}$$

These two relations are sometimes called the *Herz estimates*. They greatly simplify problems concerning the maximal operator. In particular, this affects the investigation of boundedness of M between r.i. spaces since this problem reduces to the question of boundedness of the one-dimensional Hardy operator restricted to nonincreasing functions.

The research presented here features various forms of weighted Hardy inequalities, i.e., it focuses on boundedness of Hardy-type operators in weighted (Lebesgue and Lorentz) spaces. Whenever general weights are in play, it is more practical to work with an even simpler form of the basic Hardy operator. It is defined by

$$Hf(t) := \int_0^t f(s)\,\mathrm{d}s, \qquad t > 0,$$

for any $f \in \mathscr{M}(0, \infty)$ for which the integral makes sense. Omitting the factor $\frac{1}{t}$ (in comparison with the classical average operator A) makes no difference in the weighted settings since this factor may be always incorporated in the weight.

Similarly, one defines the *Copson operator* $\widetilde{H}$ by

$$\widetilde{H}f(t) := \int_t^\infty f(s)\,\mathrm{d}s, \qquad t > 0,$$

for any $f \in \mathscr{M}(0, \infty)$ for which the integral makes sense. This operator is sometimes nicknamed *dual Hardy* or *Hardy adjoint*, since it is adjoint to the operator H in the sense of the identity

$$\int_0^\infty Hf(t)g(t)\,\mathrm{d}t = \int_0^\infty f(t)\widetilde{H}g(t)\,\mathrm{d}t$$

being satisfied for all $f, g \in \mathscr{M}(0, \infty)$ for which both sides of the equation make sense. Both H and $\widetilde{H}$ are linear operators.

The Hardy operator H is bounded between $L^p(v)$ and $L^q(w)$, with $p, q \in (0,\infty)$, if and only if there exists a constant $C \in [0,\infty)$ such that the *weighted Hardy inequality*

$$\left(\int_0^\infty \left(\int_0^t f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{8}$$

holds for all $f \in \mathscr{M}^+(0,\infty)$. Explicit conditions characterizing when this occurs are known. Namely, if $1 < p \le q < \infty$, then (8) holds for all $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{A}_1 := \sup_{t>0}\left(\int_t^\infty w(s)\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_0^t v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} < \infty.$$

If $0 < q < p < \infty$ and $q \ne 1 < p$, then (8) holds for all $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{A}_2 := \left(\int_0^\infty \left(\int_t^\infty w(s)\,\mathrm{d}s\right)^{\frac{r}{q}}\left(\int_0^t v^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{r}{q'}} v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty,$$

where $p' := \frac{p}{p-1}$ and $r := \frac{pq}{p-q}$. Moreover, these expressions even provide estimates of the optimal constant C in (8). Indeed, if $1 < p \le q < \infty$, then the optimal constant C in (8), i.e., the least $C \in [0,\infty]$ such that (8) holds for all $f \in \mathscr{M}^+(0,\infty)$ satisfies

$$C \approx \mathscr{A}_1.$$

The equivalence symbol "$\approx$" means that there exist positive real numbers $D_1 = D_1(p,q)$, $D_2 = D_2(p,q)$ dependent on p, q and such that

$$D_1 C \le \mathscr{A}_1 \le D_2 C.$$

Analogously, in the case $0 < q < p < \infty$, $q \ne 1 < p$ one has $C \approx \mathscr{A}_2$ for the optimal C. The characterizations involving expressions $\mathscr{A}_1$, $\mathscr{A}_2$ were proved by B. Muckenhoupt [85], V. G. Mazja [82] and G. Sinnamon [106]. Their variants for the limit cases $p = 1$, $q = 1$ ore the weak cases $p = \infty$, $q = \infty$ are also known, see [91].

Notice that if $0 < p < 1$, then inequality (8) with nontrivial weights v, w can never hold for all $f \in \mathscr{M}^+(0,\infty)$. It is caused by the fact that the $L^p(v)$ space with a parameter $0 < p < 1$ admits locally nonintegrable functions with a singularity possible at any point $t \in (0,\infty)$. For more details, see [78], [VIII].

By the change of variables $t \to \frac{1}{t}$, the weighted Hardy inequality (8) involving the operator H is transformed to a corresponding inequality involving the adjoint operator

$\widetilde{H}$. Hence, boundedness of H between weighted Lebesgue spaces is equivalent to boundedness of $\widetilde{H}$ between another two weighted Lebesgue spaces.

There is a generalization of the Hardy operator which is especially important in the theory of weighted inequalities and weighted function spaces. It has the following form.

Let $U : [0,\infty)^2 \to [0,\infty)$ be a measurable function satisfying:

(i) $U(x,y)$ is nonincreasing in x and nondecreasing in y;

(ii) there exists a constant $\vartheta \in (0,\infty)$ such that

$$U(x,z) \le \vartheta\,(U(x,y) + U(y,z))$$

for all $0 \le x < y < z < \infty$;

(iii) $U(0,y) > 0$ for all $y > 0$.

Then define

$$H_U f(t) := \int_0^t f(s)U(s,t)\,\mathrm{d}s, \quad \widetilde{H}_U f(t) := \int_t^\infty f(s)U(t,s)\,\mathrm{d}s, \quad t > 0, \tag{9}$$

for any $f \in \mathscr{M}(0,\infty)$ such that the involved integral makes sense. The *kernel* U satisfying the conditions (i)–(iii) is sometimes called the *Oinarov kernel*. In paper [VIII], the name *ϑ-regular kernel* is used instead to emphasize the exact value of the constant ϑ (and to hint that R. Oinarov was not the first to use such a kernel).

Boundedness of the operator H_U between the weighted Lebesgue spaces $L^p(v)$ and $L^q(w)$ was, with respect to various settings of parameters p, q, studied and characterized by S. Bloom and R. Kerman [14], R. Oinarov [88], V. D. Stepanov [115, 117], Q. Lai [77], D. V. Prokhorov [99] and by the author in paper [VIII]. The characterizing conditions obtained in these papers are not listed in this summary. Instead, the reader may find them in [VIII] and the references therein.

The operator H_U includes the ordinary Hardy operator H as a special case ($U \equiv 1$). Another typical example of a ϑ-regular kernel is the function $U(s,t) := \int_s^t u(x)\,\mathrm{d}x$, where u is a nonnegative locally integrable function (of one variable). Many complicated problems related to weighted inequalities, in particular those involving various kinds of iterated Hardy-type operators, may be approached by methods which in their final phase reduce the problem to dealing with a H_U operator (including the case $U \equiv 1$). In this sense, operators with ϑ-regular kernels can be viewed as a cornerstone of the theory of weighted Hardy-type inequalities and related function spaces.

As the example (7) showed, there is an interest in studying restricted Hardy-type inequalities stemming from their immediate application to problems concerning boundedness of maximal operators. By the term *restricted* it is meant here that a certain inequality holds for all functions from a given subset of $\mathscr{M}^+(0,\infty)$. Examples of such subsets may be the cones of all nonincreasing or nondecreasing functions from $\mathscr{M}^+(0,\infty)$, of all convex functions from there, etc. On the other hand, the term *nonrestricted* refers to an inequality being satisfied for all $f \in \mathscr{M}^+(0,\infty)$.

A simple example of a restricted inequality problem is characterizing when the weighted Hardy inequality (8) holds for all *nonincreasing* $f \in \mathscr{M}^+(0,\infty)$. The problem may be obviously rephrased as a question of finding conditions under which $\Lambda^p(v)$ is embedded to $\Gamma^q(\widetilde{w})$ with $\widetilde{w}(t) := w(t)t^{-q}$ for all $t > 0$. Clearly, the conditions $\mathscr{A}_1$, $\mathscr{A}_2$ are sufficient, in the respective settings of parameters p, q, for (8) to hold for all nonincreasing $f \in \mathscr{M}^+(0,\infty)$. However, they are not necessary in this case. The validity of this restricted Hardy inequality was studied by many authors in numerous papers such as [26–30, 40, 55, 104, 109, 110, 116]. The corresponding characterizations are now known for the full range $p, q \in (0,\infty]$. To observe the difference between a nonrestricted and restricted inequality, the reader may compare the conditions $\mathscr{A}_1$, $\mathscr{A}_2$ with their counterparts related to the restricted problem (cf. [27]) which are shown below.

If $0 < p \le q < \infty$, then (8) holds for all nonincreasing $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{B}_1 := \sup_{t>0} \left(\int_0^t w(s)s^q \,\mathrm{d}s \right)^{\frac{1}{q}} \left(\int_0^t v(s)\,\mathrm{d}s \right)^{-\frac{1}{p}} < \infty$$

and

$$\mathscr{B}_2 := \sup_{t>0} \left(\int_t^\infty w(s)\,\mathrm{d}s \right)^{\frac{1}{q}} \left(\int_0^t V^{-p'}(s)v(s)s^{p'} \,\mathrm{d}s \right)^{\frac{1}{p'}} < \infty.$$

If $0 < q < p < \infty$, then (8) holds for all nonincreasing $f \in \mathscr{M}^+(0,\infty)$ if and only if

$$\mathscr{B}_3 := \left(\int_0^\infty \left(\int_0^t w(s)s^q \,\mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t v(s)\,\mathrm{d}s \right)^{-\frac{r}{p}} w(t)t^q \,\mathrm{d}t \right)^{\frac{1}{r}} < \infty$$

and

$$\mathscr{B}_4 := \left(\int_0^\infty \left(\int_t^\infty w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t V^{-p'}(s)v(s)s^{p'} \,\mathrm{d}s \right)^{\frac{r}{p'}} w(t)\,\mathrm{d}t \right)^{\frac{1}{r}} < \infty,$$

where $p' := \frac{p}{p-1}$ and $r := \frac{pq}{p-q}$. In the respective cases, the optimal constants C in the inequality (8) (restricted to nonnegative nonincreasing functions) satisfy $C \approx \mathscr{B}_1 + \mathscr{B}_2$ and $C \approx \mathscr{B}_3 + \mathscr{B}_4$.

The reader may also notice that the inequality (8) may hold in this restricted sense even for $0 < p < 1$ with nontrivial weights v, w. It contrasts with the nonrestricted case where this was impossible.

Another example of restricted inequalities are those restricted to the cone of quasiconcave functions. Naturally, this type represents embeddings and operator inequalities involving the Γ spaces. These were studied, for instance, in [41, 54, 107, 108].

In general, it can be said that working with restricted inequalities is more difficult than doing so with the nonrestricted ones. This observation led to the development of the so-called *reduction methods*. These have gained certain popularity since the 2000's [41, 43, 46, 49, 52–54, 107]. The idea behind these methods is to reduce a restricted weighted operator inequality to an equivalent nonrestricted one. The new inequality generally involves some new weights and probably a more complicated operator. However, in most cases the new problem becomes easier by the mere fact that the inequality is nonrestricted.

The slightly ambivalent term "more complicated operator" has been already used in this text several times. In the context of this thesis, it mostly means various variants of *iterated Hardy-type operators*. These are operators constructed by iterating or mixing the integral operators H, $\widetilde{H}$ and their *supremal* counterparts S, $\widetilde{S}$ defined by

$$Sf(t) := \operatorname*{ess\,sup}_{s \in (0,t)} f(s), \qquad t > 0,$$

and

$$\widetilde{S}f(t) := \operatorname*{ess\,sup}_{s \in (t,\infty)} f(s), \qquad t > 0,$$

for $f \in \mathscr{M}^+(0,\infty)$. Certain applications also require adding some "inner weights" to such iterated operators. Accordingly, some of the problems solved in the main papers involve iterated operators defined in the following way.

Let u be a weight and $m \in (0,\infty)$. Then for $f \in \mathscr{M}^+(0,\infty)$ define

$$\begin{aligned} G_I f(t) &:= H^{\frac{1}{m}}(u\widetilde{H}^m f)(t), \qquad & G_S s(t) &:= S(u\widetilde{H} f)(t), \\ A_I(t) &:= H^{\frac{1}{m}}(u H^m f)(t), \qquad & A_S(t) &:= S(uHf)(t) \end{aligned} \tag{10}$$

at each point $t > 0$. Similarly, the "adjoint" variants of these operators are defined by replacing each operator H and S by its respective "adjoint" version $\widetilde{H}$ and $\widetilde{S}$ and vice versa in the above definitions. For example,

$$\widetilde{G}_I f(t) := \widetilde{H}^{\frac{1}{m}}(u H^m f)(t).$$

The operators G_I, G_S, $\widetilde{G}_I$ and $\widetilde{G}_S$, each of which is composed of one "ordinary" operator H or I and one "adjoint" operator $\widetilde{H}$ or $\widetilde{S}$, will be summarily called *gop operators*. This name refers to the initials of the authors of the paper [44] where boundedness of G_S between weighted Lebesgue spaces was studied. Not surprisingly, operators A_I, A_S, $\widetilde{A}_I$ and $\widetilde{A}_S$ then bear the name *antigop operators*. The letters I and S in the lower indices stand for "integral" and "supremal".

Gop and antigop operators play a prominent role in interpolation theory [10–12, 17, 22]. Besides that, they frequently appear as an outcome of applying a reduction method to problems involving restricted weighted inequalities for (iterated) Hardy operators. This is the case in paper [VII], where a certain weighted double-operator inequality is studied and reduced into problems of boundedness of gop and antigop operators between $L^p(v)$ and $L^q(w)$.

3.2 Convolution

Investigation of various properties of the convolution operator was the main task in the original thesis topic proposal. In the final outcome, convolution is the topic of papers [I–III] and [IX]. A general background is provided by this section.

Given two functions $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, their *convolution* $f * g$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x-y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^d, \tag{11}$$

if the integral makes sense. The space $\mathbb{R}^d$ (with the d-dimensional Lebesgue measure) is considered to be the integration domain, unless specified else. Definitions with different underlying measure spaces are possible and some of them will be mentioned later.

The concept of convolution has a very broad use both in the theory and practical applications. On the theoretical level, it is, above all, prominent in Fourier analysis and approximation theory (see e.g. [34, 57, 58, 120]). As it was said in the beginning of this introductory summary, results from these fields of mathematics have direct practical applications. In the particular case of convolution, the fields in which the concept is applied are, for example, signal and image processing, electrical engineering, probability, statistics, etc.

The convolution, understood as an operator acting on $L^1_{\mathrm{loc}}(\mathbb{R}^d) \times L^1_{\mathrm{loc}}(\mathbb{R}^d)$, is a bilinear operator. One may also fix a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and consider the linear operator

$$T_g f(x) := (f * g)(x), \qquad f \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{12}$$

The function g will be called a kernel, similarly to the previous use of the word. Numerous important operators can be expressed as T_g with a particularly chosen kernel g. Examples include the Riesz potential (fractional integral) operator or Riemann-Liouville integral [11, 58], Bessel [58] and Newton [35] potential operators, Hilbert and Riesz transforms [11,57], Stieltjes transform [118], mollifiers [1,35], etc. In Fourier analysis, convolution with Dirichlet, Fejér and Jackson kernels appears frequently in the theory (see [34,57,120]).

As it can be expected by a reader who has gone through the previous section, boundedness of convolution operators between function spaces is one of the main questions in this thesis. In case of the operator T_g, the problem may be stated as follows. Given the domain function space X and the range space Z, find conditions on the kernel g, under which it holds that

$$\|f * g\|_Z \le C(g)\|f\|_X, \qquad f \in X. \tag{13}$$

This notation means that (for each fixed g) there exists a constant $C(g) \in (0,\infty)$ such that the inequality holds for all functions $f \in X$. Analogous notation is used from now on. The validity of (13) therefore defines boundedness of T_g between X and Z. The sought conditions should, as usual, characterize the boundedness, i.e., they should be both necessary and sufficient.

In the main papers which deal with convolution, the main focus is laid on inequalities of the form (13) in which the term $C(g)$ is equal or equivalent to a norm of the kernel g in a certain function space Y. In this case, the concerned convolution inequality gets the visually pleasant shape

$$\|f * g\|_Z \le C\|f\|_X\|g\|_Y, \qquad f \in X,\ g \in Y. \tag{14}$$

Naturally, the constant C is meant to be independent of both f and g.

The most famous and fundamental result of the above type is the classical *Young inequality*. It reads as follows. If $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$, then

$$\|f * g\|_q \le \|f\|_p\|g\|_r, \qquad f \in L^p,\ g \in L^r.$$

The assumption $p \le q$ may not be dropped, see [62] for a proof that T_g, unless it is trivial, is not bounded between L^p and L^q when $q < p$.

The Young inequality is essential whenever convolution is used in connection with function spaces. Its classical applications are found in pure analysis and theory of partial differential equations (see [1, 11, 12, 57, 120]). A more peculiar example is given by the use of the Young inequality within the kinetic theory of gases (see [3, 4, 60] and the references therein).

The original Young inequality might be considered a model result for many further developments. Not surprisingly, convolution inequalities having the form (14) are often called *Young-type (convolution) inequalities*.

There has been an extensive research into more general Young-type inequalities (14) with spaces X, Y, Z other than L^p. In his fundamental paper [89], R. O'Neil proved a theorem which has since become known as the *Young-O'Neil inequality*. This theorem states that, if $1 < p, q, r < \infty$ and $1 \leq a, b, c \leq \infty$ are such that $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, then

$$\|f * g\|_{q,a} \leq C\|f\|_{p,b}\|g\|_{r,c}, \qquad f \in L_{p,b},\ g \in L_{r,c}. \tag{15}$$

An essential contribution of the paper [89] is the proof of a particularly important pointwise inequality, which will be referred to as the *O'Neil inequality*. It states that, for any $f, g \in L^1_{\mathrm{loc}}$ and any $t > 0$, one has

$$(f * g)^{**}(t) \leq t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s. \tag{16}$$

The proof of this inequality, as presented in [89], works correctly for the ordinary convolution (as given by (11)) but it contains some flaws if used with O'Neil's more general definition of a convolution operator. This was observed and corrected by L. Y. H. Yap in [119] by adding certain assumptions into the definition of a general convolution operator.

The O'Neil inequality is sharp in the following sense. There exists a constant D depending on the dimension of $\mathbb{R}^d$ and such that for all radially decreasing functions $f, g \in \mathscr{M}^+(\mathbb{R}^d)$ and all $t > 0$ one has

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \leq D (f * g)^{**}(t). \tag{17}$$

A function $f \in \mathscr{M}^+(\mathbb{R}^d)$ is called *radially decreasing* if there exists a nonincreasing function $\varphi \in \mathscr{M}^+(0,\infty)$ such that $f(x) = \varphi(|x|)$ for all $x \in \mathbb{R}^d$. The reverse inequality (17) for $d = 1$ was mentioned in [89] without proof. In paper [I], an elementary proof for that case is shown (cf. also [105]). The proof for a general dimension d may be found in [72].

In [15, 63, 119], inequality (15) was shown to hold even for an extended range of parameters $0 < a, b, c \leq \infty$ (while the other conditions on a, b, c, p, q, r remain the same as above). Furthermore, a limiting case of (15) with $1 < p < \infty$, $1 \leq b, c \leq \infty$,

$1=\frac{1}{p}+\frac{1}{r}$ and $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}<1$ was studied by H. Brézis and S. Wainger in [19], leading to the following result:

$$\|f*g\|_{BW_a}\le C\|f\|_{p,b}\left(\|g\|_{r,c}+\|g\|_1\right),\qquad f\in L_{p,b},\ g\in L_{r,c}\cap L_1,$$

where

$$\|f\|_{BW_a}:=\left(\int_0^1\left(\frac{f^*(t)}{1+|\log t|}\right)^a\frac{\mathrm{d}t}{t}\right)^{\frac{1}{a}}.$$

A. P. Blozinski [16] considered another limiting case of the parameters, namely such that $p=q>1$, $r=1$ and $0<a,b\le\infty$. He showed that, with these parameters, if $g\ge 0$ and $T_g:L_{p,b}\to L_{p,a}$, then necessarily $g=0$ a.e. It is important to notice that in this case the Lebesgue spaces are defined over the whole $\mathbb{R}^d$, i.e., $L_{p,a}=L_{p,a}(\mathbb{R}^d)$ and $L_{p,b}=L_{p,b}(\mathbb{R}^d)$. If $\mathbb{R}^d$ is replaced by an underlying measure space with a finite measure, then the aforementioned result of [16] does not need to be true, as it is shown below.

E. Nursultanov and S. Tikhonov [87] investigated boundedness of convolution of 1-periodic functions in $L_{p,q}$ spaces. Such functions may be equivalently represented by functions on a torus. Naturally, the involved $L_{p,q}$ spaces are also defined so that the underlying measure space is the interval $(0,1)$ (or the torus) equipped with the Lebesgue measure. The authors of [87] showed that, in the setting $1\le p<\infty$, $1\le a,b,c\le\infty$ and $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$, one has

$$\|f*g\|_{L_{(p,a)}(0,1)}\le C\|f\|_{L_{(p,b)}(0,1)}\|g\|_{L_{(1,c)}(0,1)},\qquad f\in L_{(p,b)}(0,1),\ g\in L_{(1,c)}(0,1).$$

This contrasts with the previous negative result of Blozinski (recall that $L_{p,q}=L_{(p,q)}$ if $1<p<\infty$ and $1\le q\le\infty$). Among other results of [87] is the Young-O'Neil inequality for spaces $L_{\infty,p}(0,1)$, which states that if $0<\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$, then

$$\|f*g\|_{L_{\infty,a}(0,1)}\le 4\|f\|_{L_{\infty,b}(0,1)}\|g\|_{L_{(1,c)}(0,1)},\qquad f\in L_{\infty,b}(0,1),\ g\in L_{(1,c)}(0,1).$$

Young-type inequalities and boundedness of convolution operators were further studied in the framework of weighted Lebesgue spaces with power weights [20, 65], L^p spaces with general Borel measures [5] and Wiener amalgam spaces [63]. In [13, 37, 67], the authors investigated under which conditions the $L^p(w)$ space is a convolution algebra, i.e., when the inequality

$$\|f*g\|_{L^p(w)}\le\|f\|_{L^p(w)}\|g\|_{L^p(w)},\qquad f,g\in L^p(w),$$

is satisfied. The convolution algebra property of r.i. spaces and various general properties of the convolution operator acting on r.i. spaces were also investigated by E. A. Pavlov in [93–98].

Analogues of the Young inequality in the Lebesgue spaces with variable exponent $L^{p(x)}$ were obtained by S. Samko in [101, 102] (see also [23, 103] and the references given therein).

Moreover, in [90] R. O'Neil investigated the behavior of a convolution operator in Orlicz spaces, providing a corresponding Young-type convolution inequality for these spaces.

4
Content summary of the main papers

Dope on the damn table.
CEDRIC DANIELS

In this chapter, the reader may find an overview of the content of the main papers. It includes the research questions, provided answers and other contributions of the papers, an outline of used methods, relations of the obtained results to the previously existing ones, applications, etc. All of these issues are, of course, treated in much better detail in the main papers themselves. This chapter should, however, provide the reader with a shorter summary.

The common denominator of all the work presented here is boundedness of integral or supremal operators in weighted function spaces, represented by weighted operator inequalities. The topic then gets more specific according to the particular kind of an operator that comes to question. The operators of interest can be divided into three subgroups: convolution, bilinear/multilinear Hardy operators, and iterated Hardy-type operators.

4.1 Forever Young

The goal of papers [I–III, IX] is to provide conditions of boundedness of the convolution operator in the weighted Lorentz-type spaces/classes Γ, Λ and S, and related Young-type convolution inequalities. The three first papers, i.e., [I–III], were contained in the author's licentiate thesis [74] bearing the appropriate name "Forever Young". The remaining paper [IX] was not a part of [74] but was finished later, complementing the results of [I].

The problems treated in the aforementioned papers were not investigated by other authors before, except for some special cases of weights such as those establishing the $L_{p,q}$ spaces. (Those older results were listed in the survey in Section 3.2.) The setting of papers [I–III, IX] offers a considerably greater generality, making little or no assumptions on the weights. Moreover, the technique implemented in these papers is different from those used in the previously existing works.

4.1.1 Papers [I] and [IX]

Let two weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ be given. The research questions of papers [I, IX] are stated as follows.

(i) Characterize the conditions on the kernel g, the weights and exponents under which the convolution operator T_g (see (12)) is bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$.

(ii) Find the optimal r.i. lattice Y such that the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \leq C\|f\|_{\Lambda^p(v)}\|g\|_Y, \qquad f \in \Lambda^p(v),\ g \in Y, \tag{18}$$

holds (with C independent of f, g).

Optimality of Y has the following meaning: if there exists another r.i. lattice $\widetilde{Y}$ such that (18) is satisfied with $\widetilde{Y}$ in place of Y, then necessarily $\widetilde{Y} \hookrightarrow Y$. In this sense, the optimal r.i. lattice Y is the largest r.i. lattice for which (18) holds.

In order to provide answers to the questions, the following method is implemented. At first, the O'Neil inequality (16) is used, giving

$$\|f * g\|_{\Gamma^q(w)} \leq \left\| t \mapsto t f^{**}(t) g^{**}(t) + \int_t^\infty f^* g^* \right\|_{L^q(w)}. \tag{19}$$

If the right-hand side can be estimated by the term $\|f\|_{\Lambda^p(v)}$, then T_g is bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$. Such estimates correspond to the inequalities

$$\|t \mapsto t f^{**}(t) g^{**}(t)\|_{L^q(w)} \leq C_1\|f\|_{\Lambda^p(v)}, \qquad f \in \Lambda^p(v),$$

and

$$\left\| t \mapsto \int_t^\infty f^* g^* \right\|_{L^q(w)} \leq C_2\|f\|_{\Lambda^p(v)}, \qquad f \in \Lambda^p(v). \tag{20}$$

Both of these are weighted Hardy-type inequalities restricted to nonnegative nonincreasing functions. They have been systematically studied (cf. Section 3.1) and the optimal constants $C_i = C_i(q, v, w, p, q)$, $i \in \{1, 2\}$ the inequalities hold with are known, see [25, 27, 54], [VIII]. (This knowledge had a certain gap leading to the split of articles [I] and [IX], see below.) One thus gets the condition

$$C_i < \infty, \qquad i \in \{1, 2\}, \tag{21}$$

which is obviously sufficient for boundedness of T_g between $\Lambda^p(v)$ and $\Gamma^q(w)$. To prove that it is also necessary, one uses the reverse O'Neil inequality (17). It yields that if $g \in \mathscr{M}^+(\mathbb{R}^d)$ is radially decreasing, then (21) is a necessary condition for boundedness of T_g from $\Lambda^p(v)$ to $\Gamma^q(w)$.

In the next step, it is observed that the sum $C_1 + C_2$ is equivalent to an r.i. norm (or quasi-norm) of g, denoted by $\|g\|_Y$. It gives the Young-type inequality (18). Optimality of Y is granted thanks to the necessity part (valid for nonnegative radially decreasing functions g) and thanks to the space Y being rearrangement-invariant.

The range of exponents covered by [I] is $0 < p \le q \le \infty$, $1 \le q < p < \infty$ and $0 < q < p = \infty$. The range restriction (compared to the whole quadrant $p, q \in (0, \infty]$) is caused by the fact that, at the time of [I], the validity of (20) was not satisfactorily characterized in the notorious case $0 < q < 1$ & $q < p < \infty$. The latter problem is equivalent (see [54, Theorem 4.1]) to characterizing boundedness of a particular type of the operator $\widetilde{H}_U$ (see (9)) between weighted Lebesgue spaces. For the setting of parameters required by this particular situation, the solution was known only in form of a discrete condition due to Q. Lai [78]. It was, however, inappropriate for the intended application. This whole problem was later eliminated by paper [VIII]. Based on that improvement, paper [IX] could be written, completing the results of [I]. Hence, [IX] deals with the same problem as [I] in the originally missing case $0 < q < 1$ & $q < p < \infty$. The complete range of parameters provided by both [I] and [IX] becomes $p, q \in (0, \infty]$.

The article [I] is written in such way that the results cover convolution on both $\mathbb{R}^d$ and on a compact interval for periodic functions. It is shown that the "classical" results [63, 87, 89, 119] follow as special cases of the presented theorems. In particular, it is pointed out that both the result of [16], stating that T_g with a nonnegative nontrivial g is not bounded between $L_{p,b}(\mathbb{R}^d)$ and $L_{p,a}(\mathbb{R}^d)$, $p > 1$, and the result of [87], stating that the same boundedness is possible in case of 1-periodic functions on $[0, 1]$, are consequences of a single theorem of [I]. The proven optimality of the domain space Y is a key point for drawing such conclusions.

The optimality aspect together with the general-weight setting are the main advantages of the results in [I, IX]. Thanks to the proven necessity of the provided conditions in case of a nonnegative radially decreasing kernel g, the general results of these papers may be directly applied to particular operators which have a form of convolution with a symmetrical kernel. The Riesz fractional integral operator (convolution with the

Riesz potential) is a typical example, other similar and plausible operators were named in Section 3.2. In this way, one obtains characterizations of boundedness of such operators between the concerned Lorentz spaces.

Furthermore, the last part of [I] deals with r.i. spaces which appear as the optimal domain Y. As a rule, the space Y may be expressed as an intersection of certain Γ spaces and another type of an r.i. space with a norm based on an iterated Hardy operator. The latter type of a function space is denoted by "K" in [I]. It is generated by the functional

$$\|f\|_{K^{p,q}(u,v)} := \left(\int_0^\infty \left(\int_x^\infty (f^{**}(t))^p u(t)\,\mathrm{d}t \right)^{\frac{q}{p}} v(x)\,\mathrm{d}x \right)^{\frac{1}{q}}, \tag{22}$$

or the "weak" variants of it created by the standard replacement of one of the integrals by an essential supremum over the same domain. (Are both the integrals replaced in such manner, the space becomes a weak Γ space.)

A K-type space with a special choice of weights appeared, for example, in [32] in connection with Sobolev embeddings of Morrey spaces. Recently, this type of a space was identified in [51] as the associate space to a generalized Γ space. In [IV, V], embeddings of such spaces are used to handle bilinear Hardy operator inequalities. Above all, most relevant for [I] is the role of these spaces as the optimal domain Y in the investigated Young-type inequalities. The K spaces (with other weight and exponent settings) play the same role in [II, III, IX] as well. The final section of [I] contains a summary of their elementary properties.

Both papers [II] and [III] deal with questions which are analogous to those of [I] in other Lorentz-space settings. The same method is implemented, using the O'Neil inequality and reduction of the problem to weighted Hardy inequalities. The results also feature similar optimality properties. Corresponding details are therefore omitted in the content descriptions of [II] and [III] below.

4.1.2 Paper [II]

The investigated problem reads as follows. Given $p, q \in (0, \infty]$ and weights v, w, characterize when T_g is bounded between $S^p(v)(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\Gamma^q(w)(\mathbb{R})$. Again, the main goal is to give a result in the shape of a Young-type inequality, in this case of the form

$$\|f * g\|_{\Gamma^q(w)} \leq C\|f\|_{S^p(v)}\|g\|_Y, \qquad f \in S^p(v),\ g \in Y \cap L^1.$$

The O'Neil-inequality-based method from [I] is used to solve this problem. Another ingredient is observing that the right-hand side of O'Neil's inequality (16) is equal to

$$\lim_{s\to\infty} s f^{**}(s)g^{**}(s) + \int_t^\infty (f^{**}-f^*)(g^{**}-g^*)$$

for any $t > 0$. If $f \in S^p(v)$ and $g \in L^1$, the first term is equal to zero, thus the whole problem is equivalent to characterizing the validity of the inequality

$$\left\| t \mapsto \int_t^\infty (f^{**}-f^*)(g^{**}-g^*) \right\|_{L^q(w)} \le C\|f\|_{S^p(v)}, \qquad f \in S^p(v). \tag{23}$$

For any $f \in \mathscr{M}(\mathbb{R})$, the function

$$t \mapsto t(f^{**}(t) - f^*(t)) \tag{24}$$

is nonnegative and nondecreasing on $(0,\infty)$, and any nonnegative nondecreasing functions may be approximated by functions having the form (24) for some $f \in \mathscr{M}(\mathbb{R})$ (cf. [108, Lemma 1.2]). Hence, (23) is equivalent to

$$\left\| t \mapsto \int_t^\infty \varphi(s)\frac{g^{**}(s)-g^*(s)}{s}\,\mathrm{d}s \right\|_{L^q(w)} \le C\|\varphi\|_{L^p(\varrho)}, \qquad \varphi \in \mathscr{M}^+(0,\infty),\ \text{nonincreasing},$$

where $\varrho(t) := v(t)t^{-p}$. Then, known characterizations of the validity of the last inequality are used to complete the work.

The results have similar properties (e.g., optimality) as their counterparts in [I]. The range of parameters covered by [II] is $p, q \in (0,\infty]$.

4.1.3 Paper [III]

This paper focuses on the problem of boundedness of T_g between spaces $\Gamma^p(v)$ and $\Gamma^q(w)$. Once again, the final result is the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \qquad f \in \Gamma^q(w),\ g \in Y,$$

with the r.i. space Y being optimal for the given pair of spaces $\Gamma^p(v)$ and $\Gamma^q(w)$ in the same sense as in the previous papers.

The method from [I] is applicable again. Similarly to [II], rewriting the right-hand side of the O'Neil inequality (16) in different terms proves to be advantageous. In particular, the following observation is made.

Consider $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$. If there exists a function $\gamma \in \mathscr{M}^+(0,\infty)$ with compact support and such that

$$g^*(t) = \int_t^\infty \frac{\gamma(s)}{s}\,\mathrm{d}s, \qquad t > 0, \tag{25}$$

then for all $t > 0$ it holds that

$$tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s = f^{**}(t)\int_0^t \gamma(s)\,\mathrm{d}s + \int_t^\infty \gamma(s)f^{**}(s)\,\mathrm{d}s =: T(f^{**})(t).$$

Hence, it is needed to characterize the validity of the inequality

$$\|T(f^{**})\|_{L^q(w)} \le C\|f^{**}\|_{L^p(v)}, \qquad f \in \mathscr{M}(\mathbb{R}^d).$$

To this end, a reduction theorem from [41] for linear operators acting on the cone of quasi-concave functions is applied. The standard observation confirming that the nonincreasing rearrangement of any function g^* can be approximated by functions of the form (25) is also used to extend the results to a general function g.

The described approach based on [41] is adopted in the case $p, q \in (1,\infty)$. In the other cases presented in the paper, different methods are used, involving other known results about Hardy-type inequalities.

The range of exponents covered by [III] is $1 \le p, q \le \infty$, $0 < p < 1$ & $q \in \{1,\infty\}$ and $p = \infty$ & $0 < q < 1$.

4.2 Bilinear Hardy operators

Numerous bilinear or multilinear operators can be produced by combining classical linear operators (such as Hardy or Copson) in form of a product or in various other ways.

An application of such bilinear mappings is illustrated by the central role of the operators

$$R_1(f,g)(t) := \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \int_0^t g(s)\,\mathrm{d}s, \quad R_2(f,g)(t) := \int_t^\infty f(s)g(s)\,\mathrm{d}s, \quad t > 0, \tag{26}$$

in the papers on convolution presented in the previous section (cf. the O'Neil inequality (16) and its subsequent use). The idea of characterizing boundedness of bilinear Hardy operators by further developing some techniques from [I] was suggested to the author by J. Soria and led to the creation of papers [IV] and [V].

4.2.1 Paper [IV]

A simple bilinear Hardy-type operator is defined by

$$T(f,g)(t) := \int_0^t f(s)\,\mathrm{d}s \int_0^t g(s)\,\mathrm{d}s, \qquad t>0, \tag{27}$$

for any f, $g \in L^1_{\mathrm{loc}}(0,\infty)$. With exception of the factor $\frac{1}{t}$, this operator is identical to the operator R_1 mentioned in the previous paragraph. The purpose of paper [IV] is to give characterizations of boundedness of T, restricted to nonnegative nonincreasing functions, between $\Lambda^{p_1}(v_1) \times \Lambda^{p_2}(v_2)$ and $L^q(w)$ (or, equivalently, between $L^{p_1}(v_1) \times L^{p_2}(v_2)$ and $L^q(w)$). In other words, the desired result is a characterization of the validity of the weighted bilinear Hardy inequality

$$\left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s \int_0^t g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C \left(\int_0^\infty (f^*(t))^{p_1} v_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}} \left(\int_0^\infty (g^*(t))^{p_2} v_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}$$

for all functions f, $g \in \mathscr{M}(\mathbb{R})$, with obvious modifications for the weak spaces with p_1, p_2 or q equal to ∞. Notice that the real line $\mathbb{R}$ as the domain of the functions f, $g \in \mathscr{M}(\mathbb{R})$ can be replaced by any reasonable measure space $(\Omega, \mathfrak{M}, \mu)$, if needed.

The result is proved by a so-called *iteration method*. The idea of it is somewhat similar to the one used in the articles about convolution operators. In the first step, the function g is fixed and considered a part of the weight

$$\psi(t) := \left(\int_0^t g^*(s)\,\mathrm{d}s\right)^q w(t), \qquad t>0.$$

The problem is then approached as a standard weighted Hardy inequality for nonincreasing functions,

$$\left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s\right)^q \psi(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le D \left(\int_0^\infty (f^*(t))^{p_1} v_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}, \qquad f \in \mathscr{M}(\mathbb{R}),$$

allowing the use of the known descriptions of the optimal constant D which can be written as

$$D = \sup_{\substack{f \in \mathscr{M}(\mathbb{R}) \\ \|f\|_{\Lambda^{p_1}(v_1)} \neq 0}} \frac{\left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s\right)^q \psi(t)\,\mathrm{d}t\right)^{\frac{1}{q}}}{\left(\int_0^\infty (f^*(t))^{p_1} v_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}}.$$

This quantity depends on the function g (contained in the weight ψ) and can be in all cases expressed as $\|g\|_X$, where X is an r.i. lattice that can be described as an intersection of certain "K spaces" (see (22)) and "J spaces". The latter type is an analogue to the K space and it is obtained by replacing the integral $\int_x^\infty$ by $\int_0^x$ in the (quasi-)norm (22). Details are, naturally, to be found in [IV].

The next step of the iteration method is to characterize, in terms of p_1, p_2, q, v_1, v_2 and w, when the inequality

$$\|g\|_X \le C \left(\int_0^\infty (g^*(t))^{p_2} v_2(t)\,\mathrm{d}t \right)^{\frac{1}{p_2}}, \qquad g \in \mathscr{M}(\mathbb{R}),$$

is satisfied. Thanks to the construction, the optimal C in here is also the requested optimal C in the original bilinear Hardy inequality. Due to the nature of $\|\cdot\|_X$, the problem reduces to characterizing certain embeddings $\Lambda \hookrightarrow J$ and $\Lambda \hookrightarrow K$. Providing such characterizations makes a substantial part of the work in [IV]. In that paper however, those characterizations play a rather auxiliary role and are used as means of solving the main problem concerning the bilinear Hardy inequality. Nevertheless, the description of the involved embeddings is of independent interest exceeding the particular application in [IV].

One of the ambitions of [IV] was to provide a complete list of conditions for all possible cases of exponents $p_1, p_2, q \in (0, \infty]$. This was achieved indeed, with certain logical consequences for the final length of the paper (there are 23 different cases).

4.2.2 Paper [V]

The "point of departure" of the author's research on bilinear Hardy operator inequalities carried out in papers [IV, V] was the article by M. I. Aguilar et al. [2]. It contains a characterization of boundedness of the bilinear Hardy operator T from (27) between $L^{p_1}(v_1) \times L^{p_2}(v_2)$ and $L^q(w)$. This result motivated the question whether an analogy could be proved in the restricted case – that was the problem solved in [IV].

In the first part of [V], the original problem from [2] was revisited. Namely, it was shown that the results of [2] can be obtained in a significantly simpler way by the iteration method. Moreover, more equivalent forms of the characterizing conditions were found, in most cases reducing the number of terms required in the expressions. Existence of equivalent conditions is a common feature in problems concerning weighted inequalities (cf. [27, 42, 45]) but it was not observed in [2]. Knowledge of the equivalent expressions is rather practical, especially when it is needed to combine or compare various weighted conditions. Frequently, this was the case in papers [I–IV].

Paper [V] continues with another part, the purpose of which is to demonstrate the application of the iteration method to other problems related to bilinear and multilinear operators. Several variants of Hardy and similar bilinear operator inequalities are chosen as examples. The point was not to give full characterizations as in the first part of [V] or in [IV] but rather to show a universal way how to find these. Whenever there is interest in doing so, the reader should be able to apply the techniques described in [V] to get explicit solutions to the problems presented in the paper.

4.3 Iterated operators

Iterated operators, in particular those associated with the name Hardy, were introduced in Section 3.1. It might be useful to emphasize the difference between *iterated operators* and the *iteration method* of treating bilinear operators, since both notions appear frequently here. An *iterated operator* T is constructed by composition of two or more "known" operators T_i, i.e., $T = T_1 \circ T_2$. Above all, the name is used in here for iterated Hardy operators, for instance such as the gop and antigop operators from (10). In contrast, the *iteration method* is simply the technique used in papers [I–V] to treat bilinear operators.

Studying nonrestricted inequalities representing boundedness of Hardy operators, simple or iterated, between weighted Lebesgue spaces is a fundamental problem. It can be illustrated by the following observations.

The point of the reduction methods (see Section 3.1) is to represent a restricted weighted inequality by one or more nonrestricted weighted inequalities. The price to pay in the latter case is usually the presence of a more complicated (e.g., iterated) operator in the nonrestricted inequality. Reverting the process, i.e., representing a nonrestricted inequality by a restricted one, is possible for some weights but not in general.

Next, the reduction only says that one problem is equivalent to another, a direct solution of one of them thus still needs to be found. If one has to choose whether to aim for a direct proof of a nonrestricted problem or of a restricted one, the first option is usually preferable, even if it involves dealing with a more complicated operator.

Since the restriction is given in terms of monotonicity of functions, Hardy operators naturally appear when reduction methods are used – this stems from representing a nonincreasing nonnegative function f by $\int_{\cdot}^{\infty} h(s)\,ds$, where $h \in \mathscr{M}^+(0,\infty)$. Nondecreasing functions are represented in an analogous way. All these aspects make nonrestricted inequalities with (iterated) Hardy operators a "root case".

Hardy operators which are fundamental, in the sense of the previous description, were researched in the papers presented below, in particular in [VI, VIII]. Paper [VII] deals with a more complicated problem by its systematic reduction to more of the "root-case" operator inequalities, therefore it represents an application of these fundamental results.

4.3.1 Paper [VI]

The first research problem of this paper is to characterize under which conditions the inequality

$$\left(\int_0^\infty \left(\sup_{s\in[t,\infty)} u(s)\int_s^\infty f(x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \qquad f\in\mathscr{M}^+(0,\infty),$$

is satisfied, with $p\in[1,\infty)$ and $q\in(0,\infty)$ and with u, v, w being weights. In other words, one is asking for a characterization of boundedness of the supremal antigop operator

$$\widetilde{A}_S : f \mapsto \widetilde{S}(u\widetilde{H}f), \qquad f\in\mathscr{M}^+(0,\infty)$$

(see (10)) with the inner weight u, between $L^p(v)$ and $L^q(w)$.

The second question answered in [VI] is under which conditions the inequality

$$\left(\int_0^\infty \left(\sup_{s\in[t,\infty)} u(s)g(s)\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C\left(\int_0^\infty g^p(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}$$

is satisfied for all nonincreasing functions $g\in\mathscr{M}^+(0,\infty)$. By a simple reduction argument (shown in Theorem 8 of [VI]), the answer to the second question follows from the answer to the first one.

The second problem was investigated already in [44] and an explicit characterization was given there for exponents satisfying $0<p\le q<\infty$. However, in the remaining case $0<q<p<\infty$, the authors of [44] produced only a discrete condition that involves a supremum over all possible partitions of the interval $(0,\infty)$. Such conditions are unfortunately almost nonverifiable and this effectively prevents them from being used in any applications.

In paper [VI], both the first and the second problem were solved by providing explicit integral conditions for all cases of positive p and q. Finding the correct form of the explicit condition related to the $q<p$ case is the main achievement of [VI]. This had not been done before and it opened the door to completing the theory of related weighted inequalities by providing reasonable conditions for all plausible cases of exponents.

The proofs in [VI] are based on the method of *(dyadic) discretization*, also called the *blocking technique*. This method is an excellent means of dealing with Hardy-type inequalities in weighted settings. A classical introduction to the technique may be found in the book [59].

The core of the discretization method is a simple but extremely useful proposition which reads as follows.

Let $\alpha \in (0,\infty)$. Then there exists a positive constant $C = C(\alpha)$ such that for all $k_{\min}, k_{\max} \in \mathbb{Z} \cup \{\pm\infty\}$ such that $k_{\min} < k_{\max}$ and all nonnegative sequences $\{a_k\}_{k=k_{\min}}^{k_{\max}}$ one has

$$\sum_{k=k_{\min}}^{k_{\max}} 2^{-k} \left(\sum_{j=k_{\min}}^{k} a_j \right)^{\alpha} \leq C \sum_{k=k_{\min}}^{k_{\max}} 2^{-k} a_k^{\alpha}, \qquad \sum_{k=k_{\min}}^{k_{\max}} 2^{k} \left(\sum_{j=k}^{k_{\max}} a_j \right)^{\alpha} \leq C \sum_{k=k_{\min}}^{k_{\max}} 2^{k} a_k^{\alpha}.$$

These inequalities have more variants (see [54, 56] and [VI]). Namely, suprema can be used in place of sums, and the sequence $\{2^k\}$ may be replaced by any sequence of real numbers b_k such that $\beta := \inf_{k_{\min} \leq k < k_{\max}} \frac{b_{k+1}}{b_k} > 1$. In the latter case, the constant C also depends on the parameter β (C increases with decreasing β). In either case, by means of these inequalities one can eliminate the discrete Hardy operator (represented by the "inner sum") in the expression.

The discrete inequalities from above are applied to the Hardy operators acting on functions in the following way. Consider, for example, the expression $\|\widetilde{H} f\|_{L^q(w)}$, i.e.,

$$\left(\int_0^{\infty} \left(\int_t^{\infty} f(s) \, \mathrm{d}s \right)^{q} w(t) \, \mathrm{d}t \right)^{\frac{1}{q}}. \tag{28}$$

For simplicity, suppose that $\int_0^{\infty} w(s)\,\mathrm{d}s = \infty$ and $0 < \int_0^{t} w(s)\,\mathrm{d}s < \infty$ for all $t \in (0,\infty)$. Then there exists a sequence $\{t_k\}_{k\in\mathbb{Z}}$ of points from $(0,\infty)$ such that $\int_0^{t_k} w(s)\,\mathrm{d}s = 2^k$ for each $k \in \mathbb{Z}$. Then one gets

$$\begin{aligned}
\left(\int_0^{\infty} \left(\int_t^{\infty} f(s) \, \mathrm{d}s \right)^{q} w(t) \, \mathrm{d}t \right)^{\frac{1}{q}} &= \left(\sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{\infty} f(s) \, \mathrm{d}s \right)^{q} w(t) \, \mathrm{d}t \right)^{\frac{1}{q}} \\
&\approx \left(\sum_{k\in\mathbb{Z}} 2^k \left(\int_{t_k}^{\infty} f(s) \, \mathrm{d}s \right)^{q} \right)^{\frac{1}{q}} \\
&= \left(\sum_{k\in\mathbb{Z}} 2^k \left(\sum_{j=k}^{\infty} \int_{t_j}^{t_{j+1}} f(s) \, \mathrm{d}s \right)^{q} \right)^{\frac{1}{q}} \\
&\approx \left(\sum_{k\in\mathbb{Z}} 2^k \left(\int_{t_k}^{t_{k+1}} f(s) \, \mathrm{d}s \right)^{q} \right)^{\frac{1}{q}}.
\end{aligned}$$

The symbol "$\approx$" has the usual meaning, in this particular case the equivalence constants may depend only on q. In the last expression there is no longer a Hardy operator present. If, for example, the goal is to compare this expression with the $L^p(v)$ norm of f, one can now proceed just by using the Hölder inequality (in both its variants for functions and sequences). The term *discretization* refers to the pass from the integral expression at the beginning to the discrete sum at the end.

The discretization in its original form is able to eliminate one Hardy operator, either integral or supremal. However, the operator of interest in [VI] is the supremal antigop operator, so instead of (28), the initial expression is

$$\left(\int_0^\infty \left(\sup_{z\in[t,\infty)} u(z)\int_t^\infty f(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.$$

Therefore, one needs to treat the inner operator as well.

The basic discretization method was improved in paper [VI] in order to meet this goal. The idea was to use a two-stage discretization constructed in the following way. The first stage is the same as in the simple discretization method, using a point sequence $\{t_k\}$ satisfying

$$\int_{t_k}^{t_{k+1}} w(s)\,\mathrm{d}s = 2\int_{t_{k-1}}^{t_k} w(s)\,\mathrm{d}s$$

for all relevant indices k. By this means, the outer supremal operator is eliminated in the way shown above in the example treating (28). In the second stage, a subsequence $\{t_{k_n}\}$ is constructed in such way that

$$\sum_{k=k_n}^{k_{n+1}} 2^k \sup_{t\in[t_k,t_{k+1}]} u^q(t) \geq 2\sum_{k=k_{n-1}}^{k_n} 2^k \sup_{t\in[t_k,t_{k+1}]} u^q(t)$$

for all relevant n. Clustering the "k-terms" into the "n-blocks" and using an appropriate elementary discrete proposition again, one can then eliminate the inner integral operator as well.

Naturally, what was shown in here is only a simplified description of the main idea of the technique. The reader may consult the complete version with all details in the text of paper [IV]. The discretization method of this kind works with no restrictions on the parameters $p,q\in(0,\infty)$ or on the weights. It can be applied to any kind of an "once-iterated" Hardy operator such as the gop and antigop operators.

4.3.2 Paper [VII]

The generalized Γ space $\mathrm{G}\Gamma^{m,p}(u,w)$ generated by the functional defined in (3) is the central object of interest in paper [VII]. The particular aspect which is studied in there is the existence of the embedding $\mathrm{G}\Gamma^{m_1,p_1}(u_1,w_1) \hookrightarrow \mathrm{G}\Gamma^{m_2,p_2}(u_2,w_2)$ between two different generalized Γ spaces. In other words, one wants to characterize, in terms of the exponents p_1, p_2, m_1, m_2 and the weights u_1, u_2, w_1, w_2, when the inequality

$$\left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s\right)^{\frac{p_2}{m_2}} w_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}} \leq C \left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_1} u_1(s)\,\mathrm{d}s\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}} \quad (29)$$

holds for all $f \in \mathscr{M}(\mathbb{R}^d)$.

The inequality above is an example of a two-operator inequality with a different operator on each side. This type of an inequality is usually rather hard to deal with. A particular case of the presented problem with $u_1 = u_2$ and $m_1 = m_2$ was solved in [47]. However, adding the inner exponents and, especially, the different inner weights u_1, u_2 makes the problem significantly more difficult.

The motivation for investigating inequality (29) comes from certain problems in partial differential equations theory, (29) can be also used to provide a comparison between different weighted maximal operators in a Λ space setting via the Herz estimates (7). Besides that, inequalities of the form (29) also play an essential role in determining normability of the generalized Γ spaces by using the technique of [111].

The subject of [VII] is closely related to iterated Hardy-type operators. Indeed, if one substitutes f^* for $\int_\cdot^\infty h$, each side of (29) expresses a weighted-Lebesgue-space norm of a certain integral gop operator evaluated at h. More importantly, the proofs of the results in [VII] providing characterizations of the validity of (29) also rely strongly on reducing the problem into iterated Hardy operator inequalities. Paper [VII] makes use of a great amount of results concerning weighted Hardy-type inequalities. For instance, all the gop and antigop operators from (10) find their applications in [VII]. This justifies placing the paper in the section about iterated operators.

The proofs in [VII] are based on using the fact that, for $p \in (1,\infty)$, the space $L^{p'}(v^{1-p'})$ (with $p' = \frac{p}{p-1}$) is the associate space to $L^p(v)$. By definition, this means that for every $f \in \mathscr{M}(0,\infty)$, every weight v and exponent $p \in (1,\infty)$ one has

$$\|f\|_{L^p(v)} = \sup_{g \in \mathscr{M}^+(0,\infty)} \frac{\|fg\|_1}{\|g\|_{L^{p'}(v^{1-p'})}},$$

with the convention "$\frac{0}{0} = 0$", "$\frac{a}{0} = \infty$", "$\frac{a}{\infty} = 0$" ($a > 0$) applied. The argument is used to eliminate the inner integral in the expression on the left-hand side of (29). Namely, if

$p_2 > m_2$, the left-hand side of (29) is equal to

$$\sup_{g\in\mathscr{M}^+} \frac{\left(\int_0^\infty g(t)\int_0^t (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2m_2}}} = \sup_{g\in\mathscr{M}^+} \frac{\left(\int_0^\infty (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2m_2}}},$$

where $\mathscr{M}^+$ stands for $\mathscr{M}^+(0,\infty)$. The optimal constant C in (29) may be then written as follows.

$$\begin{aligned}
C &= \sup_{f\in\mathscr{M}(\mathbb{R}^d)} \frac{\left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s\right)^{\frac{p_2}{m_2}} w_2(t)\,\mathrm{d}t\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_1} u_1(s)\,\mathrm{d}s\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}} \\
&= \sup_{f\in\mathscr{M}(\mathbb{R}^d)} \sup_{g\in\mathscr{M}^+} \frac{\left(\int_0^\infty (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2m_2}} \left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_1} u_1(s)\,\mathrm{d}s\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}} \\
&= \sup_{g\in\mathscr{M}^+} \frac{1}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2m_2}}} \sup_{f\in\mathscr{M}(\mathbb{R}^d)} \frac{\left(\int_0^\infty (f^*(s))^{m_2} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t (f^*(s))^{m_1} u_1(s)\,\mathrm{d}s\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{p_1}}} \\
&= \sup_{g\in\mathscr{M}^+} \frac{1}{\left(\int_0^\infty g^{\frac{p_2}{p_2-m_2}}(s) w_2^{\frac{m_2}{m_2-p_2}}(s)\,\mathrm{d}s\right)^{\frac{p_2-m_2}{p_2m_2}}} \left[\sup_{f\in\mathscr{M}(\mathbb{R}^d)} \frac{\left(\int_0^\infty (f^*(s))^{\frac{m_2}{m_1}} u_2(s)\,\mathrm{d}s \int_s^\infty g(t)\,\mathrm{d}t\right)^{\frac{m_1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s) u_1(s)\,\mathrm{d}s\right)^{\frac{p_1}{m_1}} w_1(t)\,\mathrm{d}t\right)^{\frac{m_1}{p_1}}} \right]^{\frac{1}{m_1}}.
\end{aligned}$$

The expression in the square bracket on the last line equals the optimal constant (cf. (5), (6)) of the embedding $\Gamma_{u_1}^{\frac{p_1}{m_1}}(\varphi) \hookrightarrow \Lambda^{\frac{m_2}{m_1}}(\psi)$, where $\varphi := w_1\left(\int_0^{\cdot} u_1\right)^{\frac{p_1}{m_1}}$ and $\psi := u_2 \int_{\cdot}^\infty g$. This constant can be expressed by the known characterizations from [47] as a term depending on the weights, exponents and the function g (but independent of f). It may have various forms depending on the exponents. Nevertheless, in all cases this form corresponds to a weighted-Lebesgue-space norm of the image of g under a certain gop or antigop operator, or to a sum of more such norms. Therefore, the whole problem reduces to a greater number of simpler problems concerning gop and antigop operators. These are handled by using appropriate known results, among them also those of paper [VI].

The used *duality method* relying on the expression of $L^p(v)$ as the associate space to $L^{p'}(v^{1-p'})$ can be, of course, applied to other questions, for example those concerning integral gop and antigop operators. However, its relative simplicity comes at the cost of the parameter restriction $p > 1$. In [VII], this condition is reflected by the restriction $p_2 > m_2$ which is present throughout the whole paper and cannot be lifted as long as the duality method is used. It is possible that the case $p_2 < m_2$ could be treated by a technique created by further improving the discretization method used in [47]. However, this is beyond the scope of paper [VII].

4.3.3 Paper [VIII]

The success in finding the missing explicit condition in paper [VI] clearly suggested how to solve another open problem. This problem has an even more fundamental character since it concerns the Hardy operator H_U with a ϑ-regular kernel U (see (9)). It was the last remaining case for which boundedness of H_U between $L^p(v)$ and $L^q(w)$ had not been characterized by an explicit integral condition. Namely, it involved the troublesome parameter setting $0 < q < 1 \leq p < \infty$.

In the other cases, i.e., for $p, q \in [1, \infty]$, simple integral conditions have been known since the time the problem had gained interest in the 1990's. These results may be found in the articles [14, 88, 117]. In the setting $0 < q < 1 \leq p < \infty$ however, only a discrete condition was known [77]. Recently, it was complemented by an integral condition [99] which though still contained a complicated implicit expression involving one of the weights. In [117] there was also shown that some conditions related to the case $1 < q < p < \infty$ were sufficient or necessary in the case $0 < q < 1 \leq p < \infty$ but no combination of them gave the desired characterization (i.e., both necessity and sufficiency). Finding the correct integral conditions remained an open problem.

This problem was successfully solved in paper [VIII], therefore filling the gap and completing the theory concerning Hardy operators with ϑ-regular kernels. It should be noted that even though the parameter combination $0 < q < 1 \leq p < \infty$ may seem rather obscure ($L^q(w)$ is not a Banach space then), the H_U-operator inequality with this setting is far from being useless. For example, using reduction methods to problems involving more complicated mappings (operators) between $L^q(w)$ to $L^p(v)$ with the setting $1 \leq q < p < \infty$ often results in getting inequalities with the (left-hand-side) exponent q between 0 and 1, and the (right-hand-side) exponent p equal to 1. See, for instance, the reduction in [VI, Theorem 8] which is exactly the case when a certain characterization for $0 < q < 1 = p$ is necessary for solving the problem studied in there.

The proof technique employed in [VIII] is essentially the same as in [VI], thus it relies on a two-stage discretization method. A minor difference is taking the constant ϑ (related to the ϑ-regular kernel U) into account when constructing the sequence $\{t_k\}$. Obtained results are then applied to solve another open problem involving the Copson operator restricted to nonincreasing functions. This was later used in paper [IX] to complete the results concerning Young-O'Neil inequalities, as it was described in the section devoted to papers [I] and [IX].

5
Remarks

I'm sciencing as fast as I can!
HUBERT J. FARNSWORTH

It is always pleasant when promises are kept and advertisements do not lie. This final section will therefore return to the challenges mentioned at the very beginning of the introductory summary in order to check if and how these were handled.

Hardly it can be denied that many *inequalities were defeated* during the course of the project. Indeed, what else can that mean than resolving whether the inequality holds or not. Defeating inequalities is, naturally, a long-time process or, better to say, a never-ending one. The interest in inequalities within mathematical analysis is enormous as the proof of nearly every theorem in the field includes an inequality of some kind. The control of operator inequalities made possible by the results of this thesis will hopefully contribute to further progress in related branches of analysis. Especially, since a part of the results has a rather fundamental character, at least from the perspective of function space theory, there is definitely a good potential for a further application.

There are certainly many ways to *conquer a space*. The list of options includes deciding whether the space is normable, finding its dual or associate space with possible consequences for determining reflexivity, characterizing embeddings between the space and other ones, and so on. Embeddings, for example, are the main topic of one of the main papers. Results from the other papers can be used elsewhere for similar purposes. From the perspective of their embeddings, the generalized Γ spaces were to a large extent conquered (with the exception of the resistance pocket in the region $p_2 < m_2$).

Questions concerning duality can be usually formulated by means of inequalities (especially the "reverse" ones with an operator on the right-hand side). Normability can be often approached in a similar way. Although normability and duality were not particularly addressed in the main papers, the content of these papers may be also applied to deal with problems of such kind.

Choosing the right operator is a tricky thing to do. There is usually some motivation to study just a given problem. If that problem involves an operator, then the latter is hardly a matter of "choice". On the other hand, results such as the reduction theorems show that, under favorable circumstances, an operator inequality may be equivalent to another one. In a vague language one could therefore say that "one operator is just as good as the next one". Such a revelation would be probably a nightmare in the world of advertising. In mathematics however, it is a very useful feature. Most of the solutions appearing in the thesis were obtained by a systematic reduction of operator inequalities to less complex ones.

Finally, *maintaining the weight* is indeed absolutely crucial. However, quite often one may get a question about any benefits of considering function spaces or inequalities with general weights. This is a legitimate question since most of the important applications of weighted spaces and inequalities involve specific weights such as the polynomial, logarithmic or monotone ones, and introducing general weights might thus seem to be only an unnecessary burden.

Nevertheless, there are good reasons for such a generalization. The simplest one is that the generalization in fact makes the work easier, not harder as it could seem. Using methods even as technical as the discretization is most likely still easier than keeping track of many parameters involved in polynomial or logarithmic weights, integrability of the weights, complicated constructions related to necessity issues and more. Even more importantly, particular results for special weights can be always derived from a general result as its special cases. Doing so needs much less effort than devising a complete proof for each particular case, not to mention that the knowledge of "general rules" offers a much better insight into the investigated problems and their potential broader context.

Another aspect that is particularly visible in this thesis is the use of general weighted inequalities to handle multilinear operators. As it was shown, certain multilinear operator inequalities are derived rather easily from corresponding linear operator ones by using the iteration method. To apply this method, it is necessary that the used linear-operator inequalities are valid for general weights, not only for specific ones. The reason is that during the iteration process, one or more of the functions by which the inequality is "tested" (the function g played such a role in the bilinear problems presented here) makes a part of a weight. Since the "test function" normally does not satisfy any more specific conditions than positivity or monotonicity, the weight the function is a part of can be indeed very general. Even if the goal was, for example, to get multilinear operator inequalities in spaces with power weights only (or even plain L^p spaces with no weights at all), corresponding linear-operator results valid only for power weights would not suffice for the use in the iteration method.

Besides the multilinear operator inequalities, there are even more situations where a weight is formed by an unknown function and therefore may be rather arbitrary. The existence of all such problems is a good motivation for the research of inequalities and function spaces with general weights.

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The Weighted Space Odyssey

Operators acting on function spaces are classical subjects of study in functional analysis. This thesis contributes to the research on this topic, focusing particularly on integral and supremal operators on weighted function spaces.

Proving boundedness conditions of a convolution-type operator between weighted Lorentz spaces is the first type of a problem investigated here. The results have a form of weighted Young-type convolution inequalities, addressing also optimality properties of involved domain spaces. In addition to that, the outcome includes an overview of basic properties of some new function spaces appearing in the proven inequalities.

Product-based bilinear and multilinear Hardy-type operators are another matter of focus. The main result in this part is a characterization of the validity of a bilinear Hardy operator inequality either for all nonnegative or all nonnegative and nonincreasing functions on the real semiaxis. The proof technique is based on a reduction of the bilinear problems to linear ones to which known weighted inequalities are applicable.

The last part of the presented work concerns iterated supremal and integral Hardy operators, a basic Hardy operator with a kernel, and applications of these to more complicated weighted problems and embeddings of generalized Lorentz spaces. Several open problems related to missing cases of parameters are solved, completing the theory of the involved fundamental Hardy-type operators.

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