Designing for the integration of dynamic software environments in the teaching of mathematics

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Karlstad, November 2015
Maria Fahlgren
List of papers


Co-authorship

Three of the papers were written together with my colleague and fellow PhD student, Mats Brunström. We have contributed equally to the project design, data collection, analysis and the writing of the papers.
1 Introduction

This chapter provides the background and aim of the thesis. It also describes how the papers included relate to each other and concludes with an outline of the thesis.

1.1 Background

During the last decade the availability of technologies in schools has increased in Sweden as in other countries, and many schools provide their students with a computer of their own (Valiente, 2010). This so-called one-to-one setting opens up new possibilities for the teaching and learning of school subjects, not least mathematics. Teachers no longer need to utilize shared computer rooms for the implementation of lessons where students use computers. However, despite increased access to hardware in classrooms and the wide agreement on the potentialities for teaching and learning mathematics provided by different types of mathematics software, the integration of computer use into mathematics classrooms is still sparse (Assude, Buteau, & Forgasz, 2010; Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Hoyles, Noss, Vahey, & Roschelle, 2013; Lagrange & Erdogan, 2009). In Swedish schools, while students often use their computer for searching for information and for writing essays or assignments, it is less common for students to use a computer during mathematics lessons (Swedish National Agency for Education, 2013).

At the same time, the view of what it means to master mathematics has changed over the past decades, which is reflected in mathematics curricula in many countries (e.g. National Council of Teachers of Mathematics, 2000; Swedish National Agency for Education, 2012). Alongside content knowledge, different competencies such as problem solving, reasoning and communication are highlighted. In 2009 the Swedish School Inspectorate made an evaluation of quality focusing particularly on mathematical abilities at upper secondary school. One of their main findings was that individual work with textbooks dominated most lessons and that reasoning and communication in mathematics was sparse (Swedish School Inspectorate, 2010). Thus, it remains an important challenge to create learning sit-
uations that foster reasoning and communication. Research shows that computers can be utilized to create such learning situations (e.g. Healy & Hoyles, 1999; Hennessy & Murphy, 1999). In computer environments students can work together and use computers to explore and discover mathematical properties and relations. When the results from these explorations are displayed on a common computer screen, they can serve as a basis for discussions (Goos, Galbraith, Renshaw, & Geiger, 2003; Granberg & Olsson, 2015; Hennessy & Murphy, 1999; Sinclair, 2003).

Even if teachers let their students use computers in mathematics classrooms, there is little known about what types of activity students are supposed to engage in. Previous surveys exploring mathematics teachers’ integration of technology tend not to differentiate between hardware and software use (Bretscher, 2014), as illustrated by a survey made by the Swedish National Agency for Education (2013). This means that there is a limited knowledge about “what types of software teachers choose to use in conjunction with particular types of hardware” (Bretscher, 2014, p. 45). Bottino and Kynigos (2009) recognize that software for drill and practice is commonplace in mathematics classrooms. Although this way of using technology might be of importance, it fails to exploit the potential of technology to provide opportunities for inquiry-oriented learning (Bottino & Kynigos, 2009; Fishman, Marx, Blumenfeld, Krajcik, & Soloway, 2004). Especially, researchers have demonstrated the opportunities for mathematical explorations provided by dynamic mathematics software environments (Baccaglini-Frank & Mariotti, 2010; e.g. Moreno-Armella, Hegedus, & Kaput, 2008). This thesis focuses on the integration of such software environments.

One reason for the low utilization of computers in mathematics classrooms is that the increased availability of technology adds to the already recognized complexity of mathematics teaching and learning (Lagrange & Erdogan, 2009). It is widely stated that the use of technology alone will have little or no impact on learning outcomes (Fishman et al., 2004; Hicks, 2010; Laborde, 2001; Lantz-Andersson, 2009; Ruthven & Hennessy, 2002). There is a need for new kinds of student task (Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012;
Hitt & Kieran, 2009; Laborde, 2001) and a revision of current teaching practice as well (Hicks, 2010; Joubert, 2013b; Lagrange & Monaghan, 2009; Perez, 2014). Thus, a major challenge for researchers and teachers developing inquiry-oriented practices is to create learning situations where students are given the opportunity to exploit the affordances provided in a computer software environment (Laborde, 2001; Monaghan, 2004). At the same time, research emphasizes the key role of the teacher in the integration of technology into mathematics classrooms (Drijvers et al., 2014; Kendal & Stacey, 2002; Lagrange & Erdogan, 2009; Tabach, 2011; Thomas & Palmer, 2014).

Instrumentation Theory has been widely used as a framework for researchers within the field of technology in mathematics education. Central within this theory is the process of instrumental genesis by which an artefact becomes an instrument for a user (Artigue, 2002; Drijvers & Gravemeijer, 2005; Trouche, 2004). To address the challenge of integrating technology into the mathematics classroom, Trouche (2005) introduces the notion of instrumental orchestration, which also takes account of the social dimension of instrumental genesis within a classroom. Although many researchers associate instrumental orchestration primarily with the organisation of classroom interaction (e.g. Drijvers et al., 2010), the original notion also includes the customization of an artefact to create a particular task environment (Ruthven, 2014).

In summary, the purpose of this thesis emerges from the recognized contradiction between the increasing availability of technology and the weak utilization of the potential of dynamic mathematics software environments. The focus of the thesis is on new types of task environments to foster students’ mathematical reasoning and instrumental genesis process. The next section describes the aim of the thesis and outlines the research questions.

1.2 Aim of the thesis
This is an applied education thesis focusing on designing for the integration of dynamic software environments in the teaching of mathematics. Specifically, it aims to investigate particular aspects of the de-
sign of classroom mathematical tasks which make use of this type of computer-based system, with a focus on improvement, both of particular tasks and of the design process through which they are developed and refined. The overarching research question of the thesis is:

*How can key aspects of the design and implementation of task sequences making use of dynamic software environments be better understood and improved accordingly?*

This question is divided into three sub-questions, based on the following three key aspects: (a) *task design model*, (b) *instrumental genesis and orchestration*, and (c) *follow-up lessons*. The first aspect concerns features of tasks and task environment relevant to developing a specific plan of action for a lesson. The second aspect deals with orchestration, in terms of creating a particular task environment to support the instrumental genesis of specific dynamic software tools. The third aspect examines how to follow up student work on computer-based tasks in a whole-class discussion.

To summarize, the three sub-questions which are addressed within the including papers are:

1) How well does a particular task design model guide the development of plans of actions which support mathematical reasoning, and how could it be improved? (Paper 1 and Paper 2)
2) What is the instrumental genesis for the manipulation of key tools (scales of the coordinate axes, movable points and slider bars), and how can the task (re)design achieve this instrumental genesis more effectively? (Paper 3 and Paper 4)
3) What do teacher-devised follow-up lessons to researcher-designed task sequences look like and how could they be improved (Paper 5 and Paper 6)

There is a common structure throughout the sub-questions in that an initial question concerning how a particular aspect works at the moment under the present conditions is followed by a question of how to improve this.
To be able to examine the first two questions, a design-based research approach seemed appropriate for several reasons. The specific educational context to be investigated, that is, one-to-one classroom settings using computer-based tasks designed to enhance student reasoning, is hard to find. This fits well with a design research approach, which, instead of studying what exists, focuses on what could be (Schwartz et al. 2008). Equally, we wanted to collaborate with teachers because, as Gravemeijer and Eerde (2009) argue, a close collaboration between researchers and teachers make “the research more practical and the teaching more scientific” (p. 511).

Nevertheless, the third question, and particularly the second part of it, is closer to action research in which practices of teaching are studied with a view to better understanding them and improving them (Calhoun, 2002). However, while the issues of designing tasks and task environment are central in the thesis, the focus is on the design research approach. The next section describes how each paper relates to the design research program reported in this thesis.

1.3 How the papers relate to the design research program

The thesis comprises two research projects: a small initial project (leading to the study reported in Paper 1) and a larger main project of which various aspects are examined in the studies reported in Papers 2 to 6. Both of these projects and all of these studies relate to aspects of design.

The initial project reported in Paper 1 comprises a distinct piece of work which provided useful guidance in preparing for the main study. This first study concerns the design of task sequences, something which plays a central role in the thesis. In this study, a local design theory – in terms of a task design model to develop a concrete plan of action – was tested and refined accordingly. Furthermore, the ideas of \textit{a priori} and \textit{a posteriori} analyses were used. In this way, the analysis process in this paper was along lines appropriate within a design research cycle. In addition, this study provided practical insights into the use of video recording as a mean to collect and analyse data when students work in pairs at one computer.
The main project involves three short teaching units each consisting of an opening researcher-designed lesson and a follow-up whole-class lesson devised by the class teacher. It is only the researcher-designed lessons that follow a design-based approach, and which is labeled design experiment in the work reported, following Cobb and Gravemeijer (Cobb, Confrey, Disessa, Lehrer, & Schauble, 2003; Gravemeijer & Cobb, 2006). Papers 2 to 4 focus on the design experiment while Papers 5 and 6 deal with the follow-up lessons.

With respect to the design, trial and analyses of sequences of tasks, the main study concerns only the first iteration in the design experiment (Paper 2 to 4). Although each task sequence was revised in the light of what happened in classrooms, they were not trialled out in a further iteration. However, there were some general aspects of the task design, not closely related to a specific topic, that indeed were taken into account in the design of a subsequent task sequence. Paper 2 directly describes a complete cycle including subsequent refinement in the design process of computer-based task sequences. While Paper 2 focuses on the researcher-designed lesson in the first teaching unit, Paper 3 and particularly Paper 4 concern the evolution of students’ instrumental knowledge over the three researcher-developed lessons. These papers are about a particular type of analysis which plays an important part in the design experiment which is to understand more about the process of instrumental genesis. Paper 4 goes further than just describing the instrumental genesis in that it analyses how it might relate to the tasks involved, and discusses the redesign of those. In this way, it contributes to a central topic of the thesis – the design of computer-based tasks.

Like Paper 3 and 4, Paper 5 relates to all three teaching units. However, this paper is not really design research as such but does relate to design research in that it is about how teachers continue their work with students following their participation in a design experiment. The paper investigates what happens when the teachers take over in terms of designing the follow-up lessons in each teaching unit and the researchers recede into the background. Finally, Paper 6 is about how to improve follow-up lessons by making them more student-centered.
1.4 Outline of the thesis

The thesis consists of two parts. The first part is a kind of extended summary, in Sweden termed “kappa”, which combines and discusses the central ideas of each paper. At the end of this volume, the second part comprises the entire six papers. The first part can be read without having read any of the papers. This section outlines the structure of this part.

The first part starts with Chapter 2, which presents the context for the main study, conducted in Swedish upper-secondary schools. The theories used in each of the papers are drawn from three areas: design research and particularly the design of tasks, instrumentation theory, and domain-specific theories of mathematical thinking and learning. These theories are introduced in Chapters 3, 4 and 5. Because almost all papers can be understood as contributing to a program of design research, Chapter 3 reviews relevant literature related to design research and particularly the literature concerning task design. Chapter 4 discusses instrumentation theory, which constitutes a consistent thread across all the papers. Chapter 5, then, introduces various theoretical sources denoted as domain-specific theories (Cobb & Gravemeijer, 2008; Edelson, 2002), that is, theories concerning the teaching and learning of mathematics. Depending on the particular mathematical focus, the different papers draw on various domain-specific instruction theories.

Chapter 6 describes and discusses the methods used within the thesis and Chapter 7 provides a summary of the findings reported in each paper by elaborating responses to the research questions introduced in Section 1.2. Finally, Chapter 8 discusses the academic and professional contribution of the thesis and provides suggestions for further research.
2 The context of the thesis

This chapter describes the context of the main study (reported in Papers 2 to 6), conducted in Swedish upper-secondary schools. It introduces the Swedish school context and the national curricula framing the particular mathematical topic for the study. The chapter ends with a description of the particular software environments employed in the design, which are the focus of the studies.

2.1 The Swedish context

This section introduces relevant (for the thesis) parts of the Swedish national curricula. It also describes some contextual issues concerning textbooks and the setting, here referred to as one-to-one computer setting.

2.1.1 The Swedish curricula

In Sweden new curricula were introduced in the autumn of 2011. In the Swedish syllabi, mathematical content and different mathematical abilities are introduced separately. In the syllabus for upper-secondary school mathematics five core content areas and seven general abilities are described. It is also worth noting that the official guidelines for assessing students are based on the abilities (Swedish National Agency for Education, 2012). Most relevant to this thesis is the description of the reasoning and communication abilities:

Teaching in mathematics should give students the opportunity to develop their ability to:

[...]

5) follow, apply and assess mathematical reasoning.

6) communicate mathematical thinking orally, in writing, and in action.
(Swedish National Agency for Education, 2012, pp. 1–2)

The mathematical content knowledge of relevance to this study is parts of the mathematical course Mathematics 1b. Below are some excerpts from syllabus related to the particular topic of functions and graphs:

13
Teaching in the course should cover the following core content:

[...]

The concept of linear inequality.

Algebraic and graphical methods for solving linear equations and inequalities and exponential equations.

[...]

The concept of a function, domain and range of a definition, and also properties of linear functions, and exponential functions.

Representations of functions, such as in the form of words, shapes, functional expressions, tables and graphs.

Differences between the concepts of equation, algebraic expressions and functions. (Swedish National Agency for Education, 2012, pp. 8–9)

2.1.2 Textbooks and their role

As in most countries (Gueudet, Pepin, & Trouche, 2012), textbooks are central resources for the teaching of mathematics in Swedish classrooms (Jablonka & Johansson, 2010). In Sweden, there is no governmental control of mathematics textbooks and the responsibility of choosing a textbook is a local issue (Johansson, 2006). The participating schools followed the same textbook, which facilitated the planning of the study.

In our view the outline of the textbook (Alfredsson, Bråting, Erixon, & Heikne, 2011) was not optimal since it separated out the algebraic and graphical aspects of functions. Actually, while the textbook introduces the algebraic aspects of functions at the beginning of the book, the related graphical aspects are introduced at the end. Considering that the course runs over a school year, this means that there is a fairly long period of time between the occasions for students to practice these two types of functional representations. For the participating students, the situation became different from the traditional one in that they were expected to work with graphical presentations of functions alongside working with corresponding algebraic representations.
Concerning the integration of technology in this particular textbook, it provides basic instruction in how to plot function graphs with a graphical calculator. For example, this includes instruction on how to achieve an appropriate viewing window.

2.1.3 One-to-one computer setting
The main study was conducted in two upper secondary schools that provide students with a computer of their own. This setting is often referred to as one-to-one computer setting. The trend in Sweden is that schools make investments to increase the availability of technologies in the schools. However, not every school in Sweden provides this kind of setting and there are variations within different municipalities and between different school levels (Fleischer, 2013; Perselli, 2014).

Moreover, a national report shows that there are variations between the uses of computers within different school subjects (Swedish National Agency for Education, 2013). Typically students in Swedish classrooms use the computer for searching for information on the Web, for word processing, and for making presentations on different kinds of schoolwork. Further, the report identifies mathematics as a subject in which the computer is underused (Skolverket, 2013). In line with several international reports (e.g. Bretsch, 2014; Cuban, Kirkpatrick, & Peck, 2001; Thomas, 2006), this report shows that an increased availability of technology in schools does not guarantee a changed teaching practice. In other words, presence of the technology itself it is not enough – there is a need for teachers to learn how and when to use computers for different purposes in the classroom (Hew & Brush, 2007).

2.2 Dynamic mathematics software
This section defines and introduces the technology, which was used in the designs developed in the two studies. The notion of Dynamic Mathematics Software (DMS) is used as an umbrella comprising software which provides opportunities for dynamic (physical) actions (Moreno-Armella et al., 2008). In contrast to the work with a static
medium, such as paper and pencil, a dynamic medium offer students possibilities to dynamically explore mathematical objects and relationships (Moreno-Armella et al., 2008). In the context of geometry, Dynamic Geometry Software (DGS), such as *Cabri Geometry* and *Geometer's Sketchpad*, have been used as educational tools for several decades (Ruthven, Hennessy, & Deaney, 2008). These types of software provide Euclidean tools for the construction of geometric objects, which “can be selected and dragged by mouse movements in which all user-defined mathematical relationships are preserved” (Moreno-Armella et al., 2008, p. 104).

There are also graphing software primarily developed for algebraic purposes that offers dynamic features, for example in the form of slider bar tools for dynamically changing function graphs. For example, a Computer Algebra System (CAS) provides these types of tool (Drijvers, 2003). The research reported in this study uses a desktop version of a computer package combining both these types of software (*GeoGebra*). According to Jones and Hohenwarter (2007), this type of package

... provides a closer connection between the symbolic manipulation and visualisation capabilities of CAS and the dynamic changeability of DGS. It does this by providing not only the functionality of DGS (in which the user can work with points, vectors, segments, lines, and conic sections) but also of CAS (in that equations and coordinates can be entered directly and functions can be defined algebraically and then changed dynamically). (p. 127)

The literature (e.g. Arzarello & Robutti, 2010; Falcade, Laborde, & Mariotti, 2007; Lagrange & Psycharis, 2014; Zehavi & Mann, 2011) emphasizes the affordance provided by DMS environments to dynamically link multiple representations of mathematical objects. The opportunity to simultaneously view dynamically linked representations allows students “to build strong connected schema” (Pierce & Stacey, 2013, p. 327). For instance, it is possible to link a graphical representation of a function to its corresponding symbolic representation. Section 5.4 provides a more detailed description about important features of the particular aspects of DMS environments which are the focus within this thesis: different dragging modalities, the scaling of coordinate axes, the manipulation of movable coordinate points and the use of slider bar tools to control function parameters.
3 Design traditions

*Design* is a recurring term throughout the thesis. Foremost, the term is used as a verb but sometimes it is referred to as a noun. This flexibility of the term is recognized as particularly useful in relation to education research (Hjalmarson & Lesh, 2008). In this way, “Designs are designed” (p. 99), and design research could be used as a means to document the development of a design by simultaneously investigating the process of design. That is, “research about a design is more than only the final product” (Hjalmarson & Lesh, 2008, p. 99).

Thus, the notion of design has a significant role throughout the thesis. The overarching research approach is design research, and the issue of task (re)design is central to the type of design research undertaken here. Hence, this chapter first provides some background to the research paradigm of design research, and introduces important aspects of it. Then the chapter examines ideas and findings from the literature on task design that are relevant to the thesis.

3.1 Design research

This section provides a brief background to design-based research approaches and the variety of labels and aims associated with them. Then the section articulates some characteristics of design research and shows how these fit with the research undertaken within this thesis. Finally, the development of theory within a design-based research approach, relevant to the thesis, is discussed.

3.1.1 Some historical background

By referring to the work by Thorndike (1910), O’Donnell (2004) gives a brief background on the research approach today known as *design-based research* or just *design research*. Thorndike was aware of the difficulty but also the importance of studying complex natural contexts such as classrooms. Thus, already a century ago, the importance of classroom-based research beyond laboratory-based research was recognized (O’Donnell, 2004). However, it was only about two decades ago, that a breakthrough was presented in two oft-cited pub-
lications by Brown and by Collins (Brown, 1992; Collins, 1992), and the term *design experiment* was introduced as a label for this popular new methodological approach in educational research. Although their reasons for this approach were the same, that is, they acknowledged limitations of laboratory studies to examine the complexity of real classrooms, their intentions differed somewhat (Lesh, Kelly, & Yoon, 2008). While Brown points out that theoretical aspects have always been a keystone of her work and that “this is intervention research designed to inform practice” (Brown, 1992, p. 143), Collins on the other hand, regards design experiment as a means to enable theorists “to benefit from the wisdom of practitioner” (Lesh et al., 2008, p. 131).

Schoenfeld (2006) argues that there has been work that could be considered as exemplifying the design experiment, conducted by a community of researchers decades before the introduction of this notion. Particularly in mathematics education, there is a long tradition of task design (Ruthven, 2015). Although the notion of design experiment, introduced by Brown and Collins has been adopted by several researchers (Cobb et al., 2003; Schoenfeld, 2006), there are several labels used for this type of research approach. Many authors use the notion design-based research (The Design-Based Research Collective, 2003; Wang & Hannafin, 2005) or just design research (Gravemeijer & van Eerde, 2009; Kelly, Lesh, & Baek, 2008; Wood & Berry, 2003), and a few other use *educational design research* (McKenney & Reeves, 2014; Plomp, 2009; Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) to emphasize educational concerns. Another common label used in the literature is *development* or *developmental research* (Goodchild, Fuglestad, & Jaworski, 2013; Gravemeijer, 1994; Van den Akker, 1999). This thesis uses the notion of design experiment when describing the work on designing, testing and redesigning the computer-based task sequences within each teaching unit.

There is wide agreement among educational researchers, teachers and administrators that there is a gap between on the one hand educational research and on the other hand educational practice (Burkhardt & Schoenfeld, 2003; Carlgren, 2011; The Design-Based Research Collective, 2003). The main reason behind this gap is partly that the re-
search is too detached from practice in that it often does not emerge
from practical needs and partly that research results have no or lim-
ited impact on practice, even if they provide useful information about
it. Design research is suggested as a promising research approach to
address this problem (Burkhardt & Schoenfeld, 2003; Carlgren, 2011;
Hjalmarson & Lesh, 2008; The Design-Based Research Collective,
2003).

3.1.2 Characteristics of design research

The literature referring to the methodological approach of design ex-
periment (even if using another label) agrees on some major charac-
teristics applying to the approach, which are interventionist, natural-
istic, iterative, and theory oriented. This section introduces these
characteristics and demonstrates how they fit the design research
program reported in this thesis.

**Interventionist**

According to Schoenfeld (2006), design is the act of creation. To be
able to research properties of an intervention that does not yet exist,
one has to create this intervention and the result is a design experi-
ment (Schoenfeld, 2006). In line with this, Cobb et al. (2003), suggest
that the design experiment represents a kind of interventionist
methodology; they mean that design experiments can function as
“test-beds” for innovations. The intervention could be of different
kinds, for example products, materials, activities, procedures, type of
assessment. The focus of the research is then to observe and examine
the intervention within classrooms (Fishman et al., 2004). This fea-
ture fits the main study since the computer-based task sequences that
were developed formed an intervention not previously trialled.

**Naturalistic**

Design experiments are practically oriented both since they involve
collaboration with practitioners and are situated in real educational
contexts (Anderson & Shattuck, 2012; Cobb et al., 2003). Therefore,
the context within which a design experiment is performed should be
naturalistic (Barab & Squire, 2004), although it is designed and systematically changed by the researcher. Further, the research team should involve participants with different expertise, knowledge and experience, for example developers, researchers, and teachers (McKenney & Reeves, 2014; Van den Akker, 1999). The design research project reported in this thesis is collaboration between two researchers and four upper-secondary school teachers. The object of the research is teachers’ and students’ activity in ordinary classroom settings.

**Iteratively**

A further characterizing feature of design research is that it incorporates multiple cycles of design, enactment, evaluation, and revision (Cobb et al., 2003; The Design-Based Research Collective, 2003). Often conjectures about an initial design are generated and, after implementation, reflected upon and revised. Then the new conjectures are made subjected to test (Hjalmarson & Lesh, 2008; Wood & Berry, 2003). This thesis, however, only describes the first cycle of a whole design experiment since the three designed task sequences have, to date, only been trialled and revised once.

**Theory oriented**

The interventionist nature of the approach means that design researchers often need to call upon multiple theories since the available research literature in many domains only provides limited guidance (Gravemeijer & Cobb, 2006). Gravemeijer and Cobb suggest drawing on articles about students’ learning in a particular domain together with descriptions of classroom settings, activities, representations and computer tools that have been shown to support that learning. In line with this, Hjalmarson and Lesh (2008) emphasize the need for a design researcher to draw on different knowledge bases, theories, but also experiences. They point out the large number of variables affecting a design, and refer to the fact that teachers and decision makers working in real-life contexts also have to incorporate multiple theories when designing instructional activities for their classroom. The Design-Based Research Collective (2003) points out that a piece of de-
sign research must lead to shareable theories that can be communicated with other educational designers and practitioners. In this thesis, these theories are denoted “local instruction theories” (Cobb & Gravemeijer, 2008; Gravemeijer & Cobb, 2006), and they are described in more detail in the next section.

### 3.1.3 Local instruction theory

Cobb et al. (2003) point out the importance of clarifying the theoretical intent with the design experiment, that is: “what is the point of the study?” (p. 11). In doing this, Gravemeijer and Cobb (2006), suggest identifying the end points or instructional goals, clarifying the intended learning goals and specifying the core ideas in a particular domain. This is a critical issue since the purpose of the instructional activities is to move towards the stated learning goals (Cobb & Gravemeijer, 2008; Gravemeijer & Cobb, 2006). Once the instructional goals are clarified, the next step is developing a so-called local instruction theory, which starts with development of a conjectured local instruction theory (Cobb & Gravemeijer, 2008; Gravemeijer & Cobb, 2006). Besides descriptions about the learning goals for students and the designed instructional materials, this theory consists of “conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process” (Gravemeijer & Cobb, 2006, p. 50). This closely relates to the construct of hypothetical learning trajectory (HLT), introduced by Simon (1995). “An HLT consists of the goal for the students’ learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the students’ learning” (Simon & Tzur, 2004, p. 93). Although Simon suggested a HLT as a rationale for teachers for choosing a specific instructional design, this construct has been employed by researchers as a framework for designing technology-based tasks (Drijvers, 2003; Sacristán et al., 2010; Sinclair, Mamolo, & Whiteley, 2011).

In conclusion, this thesis concerns the production of educational design in the form of task sequences and the local instruction theory that goes with that.
3.2 Task design

Task design is an important issue within mathematics education research, and Sierpinska suggests, “the design, analysis and empirical testing of mathematical tasks, whether for the purpose of research or teaching, as one of the most important responsibilities of mathematics education” (2004, p.10). However, this is a complex and subtle process that involves several issues for researchers and teachers to consider (Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013; Joubert, 2013b). In a commentary chapter, Ruthven (2015) suggests an organising scheme for frameworks and their principles concerning task design. He suggests four elements, of which at least one constitutes a part of any task design framework. These elements are: (a) a template for planning task activity, (b) criteria for devising a productive task, (c) organisation of the task environment and (d) management of crucial task variables. This section provides a theoretical background (relevant for the thesis) guided by these four elements.

3.2.1 A template for phasing task activity

This thesis draws on some models for task design providing a logical flow throughout the student activities. Leung (2011), for instance, proposes three aspects that could serve as guiding principles for task design in the context of technology-rich environments, in particular DGS environments: exploration, re-construction and explanation. Leung (2011) proposes a task design model composed of three epistemic modes that reflect the aspects mentioned above. These modes resemble the different phases recognized in the proving process, particularly in DGS environments (De Villiers, 2004; Marrades & Gutiérrez, 2000). Principally, these phases include activities such as exploration, conjecturing, construction of a formal proof, and exploration for further generalizations. For instance, Marrades and Gutiérrez (2000) utilized tasks structured in three phases when they investigated ways of using DGS environments to improve students’ proving skills. In the first phase students were asked to create a figure and explore it. In the next phase, students generated conjectures and in the third phase they were to justify their conjectures.
Laborde (2001) introduces the notion of prediction tasks in a DGS environment. In these tasks students are prompted to make predictions about a mathematical situation before investigating it in a DGS environment. The same idea, namely to use computer feedback to verify student predictions, has also been used in CAS environments. In a study reported by Kieran and Saldahna (2008), students were asked to anticipate the result of an algebraic product before doing any paper-and-pencil or CAS manipulation. Students were then supposed to use CAS to verify their conjectures. This aligns with the suggestion that predictions should be accompanied by reflections to increase the opportunities for students to resolve any conflicts that may arise between their predictions and the answers (Arcavi & Hadas, 2000; Kasmer & Kim, 2012). In cases where students’ predictions are inconsistent with the results achieved from their computer investigations, there are good opportunities for students to reflect and try to find an explanation (Laborde, 2001).

Related to the theory of Realistic Mathematics Education (RME), Gravemeijer (1999) suggests the idea of emergent modelling as a design heuristic informing the task design. This perspective proposes instructional activities at several levels to guide students in their process of progressive mathematization. This process starts with a context-specific situation in which students use informal solution strategies. Then students gradually meet more generic situations that require more abstract mathematical reasoning. One important aspect is that students can refer back to previous stages in this process of mathematization (Drijvers, 2003). Finally, the students reach a level that is context-independent and where mathematical reasoning requires conventional symbolizations, formal mathematical reasoning (Gravemeijer, 1999). This principle was taken into account to achieve coherence across the three task sequences designed in the main study.

### 3.2.2 Criteria for devising a productive task

The literature emphasizes that there is a need for novel types of task to utilize the opportunities provided by DMS environments (Doorman et al., 2012; Hitt & Kieran, 2009; Laborde, 2001). This section intro-
duces four design principles relevant to the work reported in this thesis: open problems, unexpected outcomes, two tool systems, and mathematical reality.

Open problems

The literature suggests that open problems create teaching and learning environments that allow students to explore and produce conjectures (Arcavi & Hadas, 2000; Baccaglini-Frank & Mariotti, 2010; Furinghetti & Paola, 2003; Mogetta, Olivero, & Jones, 1999). According to Mogetta et al. (1999) some characterizing properties of an open geometry problem are that it

… usually consists of a simple description of a configuration and a generic request for a statement about relationships between elements of the configuration or properties of the configuration. (…) The requests are different from traditional closed expressions such as “prove that…,” which present students with an already established result. (pp. 91–92)

Baccaglini-Frank and Mariotti (2010) refer to conjecturing open problems where the solution process consists of two phases. First, a conjecturing phase in which the students are engaged in the exploration of geometric constructions to find a conjecture. Then students are expected to attempt to prove their conjecture.

Unexpected outcomes

As mentioned earlier, several studies emphasize the advantage of letting students predict an outcome before investigating the situation further by using technology (Arcavi & Hadas, 2000; Kasmer & Kim, 2012; Laborde, 2001). If the computer feedback is not just inconsistent but also surprising, it could trigger and motivate students to scrutinise the mathematical situation further (Arcavi & Hadas, 2000; Kieran, Tanguay, & Solares, 2012). In relation to this, Joubert suggests epistemological obstacle as “key to the design of ‘good’ tasks” appropriate for computer software (Joubert, 2013a, p. 74). Epistemological obstacles arise when students, who encounter obstacles when tackling a mathematical problem, need to construct some new piece of knowledge to be able to solve the problem (Brousseau, 1997). However, it is challenging for teachers to find appropriate situations that
might produce this kind of unexpected outcomes (Arcavi & Hadas, 2000).

Two tool systems
One principle emerging from the instrumentation theory (Chapter 4) concerns the importance for students to encounter tasks that require solutions instrumented by both paper-and-pencil techniques and techniques instrumented by a particular DMS environment (Artigue, 2005; Bretscher, 2009; Drijvers, 2003; Guin & Trouche, 1998; Kieran & Saldanha, 2008). It is important to provide an opportunity for students to compare those two tool systems, since they have different epistemic value (Artigue, 2002) (see Section 4.3). A further reason is to enhance the connection between those two techniques, which is known to be a challenge for students (Bretscher, 2009).

Mathematical reality
In relation to the emergent modelling principle described in the previous section, Gravemeijer (1999) suggests creating mathematical situations which students experience as realistic. According to Gravemeijer, the mathematical situation does not need to refer to a real-life context. The idea is that the situation should be “experientially real for the students and can be used as starting points for progressive mathematization” (Gravemeijer, 1999, p. 158).

3.2.3 Organisation of the task environment
The designed tasks are not only intended to foster students’ instrumental genesis process (see Section 4.1). Another important aspect concerns the social task environment. Because we set out to design task sequences that would foster student mathematical reasoning and communication, the aspects of formulation in writing and working in pairs were essential.
Fostering instrumental genesis

One important design principle that Leung (2011) specifies is to include tasks that make students acquainted with the software and that give them possibilities to develop modalities for interaction with the DGS environment. Leung (2011) states “Constructing or manipulating virtual mathematical objects is a meaningful way to learn to turn virtual tools into pedagogical instruments” (p. 327). To let students make their own constructions is a way to make them aware of the construction process that lies behind the dynamic figures (Ruthven, 2009).

Formulations in writing

Several researchers emphasize, for various reasons, the value of asking students to express themselves in writing (Bartolini Bussi & Mariotti, 2008; Doerr, 2006; Kieran & Saldanha, 2008; Sinclair, 2003). In their work on designing task sequences in CAS environments, Kieran et al. (2012) asked students to “write about how they were interpreting their mathematical work and the answers produced by the CAS aimed at bringing mathematical notions to the surface, making them objects of explicit reflections and discourse in classroom” (pp. 9–10). Kieran and colleagues also point out the importance for students to make their predictions explicit. In this way students are encouraged “to be clearer of how they envision the situation they are working on” (Kieran et al., 2012, p. 26).

However, Sinclair (2003) directs attention to the tendency for students to give sparse responses when asked to explain their reasoning in writing. As a way to encourage students to explain, Doerr (2006) suggests a task design principle implying “that the response to the task will require students explicitly to reveal how they are thinking about the situation by documenting and representing their ideas” (Doerr, 2006, p. 8). In relation to this, Bartolini Bussi and Mariotti (2008) point out the importance for students to make “individual reports on their own experience and reflections” (p. 755) after collaborative activities with the artefact. This allows the teacher to take account of individual contributions in the subsequent whole-class discussion (Bartolini Bussi & Mariotti, 2008).
Working in pairs

Several researchers suggest that students can work together in computer environments since the computer screen can serve as a common referent to enhance joint reasoning (e.g. Arzarello & Robutti, 2010; Goos et al., 2003; Hennessy, 1999).

3.2.4 Management of crucial task variables

The thesis utilizes two important variables in the design and analysis of tasks: didactical variables and key elements of instrumented action scheme.

Didactical variables

Ruthven et al. (2009) show how design research has generated tools for “the design of learning environments and teaching sequences informed by close analysis of the specific topic of concern” (p. 329); these design tools provide a framework to identify and address specific aspects, both in the construction of an initial design and in subsequent revisions or refinements with respect to empirical findings. Ruthven et al. (2009) argue that didactical variables, that could be used to identify important choices to consider in the design process, which might affect students’ reasoning (Brousseau, 1997) is a design tool which can be used independently of the theoretical framework in use. According to Ruthven et al. (2009) didactical variables are features of the task environment which act as “key levers to precipitate and manage the unfolding of the expected trajectory of learning” because they “significantly affect students’ solving strategies” (p. 334). The identification of didactical variables “starts from analysis of the knowledge available to students. Observations of how situations play out with students in the classroom may then reveal further variables not identified through prior analysis” (p. 334).

Key elements of instrumented action scheme

Drijvers and Gravemeijer (2005) investigate the relationship between the use of CAS and algebraic thinking among students in ninth and tenth grade. They use the instrumental approach (see Chapter 4) to
provide a detailed description of key elements of the instrumented action scheme (see Section 4.1) that students develop during activities with some particular CAS tasks. In reporting this, they provide concrete examples for each of which they identify lists of key elements, which they regard as an emergence of successful instrumental genesis. These elements have primarily either a technical or a conceptual character. However, as Drijvers and Gravemeijer (2005) show, the elements are intertwined. That is, the identification of key elements of instrumented action scheme elucidates the intertwinement between technical and conceptual knowledge. For instance, they show how the identification of key elements throw light on how seemingly technical obstacles encountered by students might reflect the limitation of conceptual understanding. Besides using the instrumental approach and the notion of key elements of scheme, Drijvers and Gravemeijer (2005) suggest these as guidance in designing “tasks that enhance a productive instrumental genesis” (p. 189). The process of instrumental genesis is described in more detail in Section 4.1.

3.2.5 Task design models

Drawing on the discussion paper by Leung (2011), this thesis uses the terminology task design model. Using the terminology technopagogy mathematics task design model, Leung (2011) refers to a particular model of task design composed of three epistemic modes, which provide support for a logical flow. In this thesis, however, I use task design model to refer to any task design framework comprising the four elements suggested by Ruthven (2015), which are described above. That is, while Leung’s task design model only provides a template for planning task activity the task design models described in this thesis (Section 7.1) take all the four aspects suggested by Ruthven into account. In this thesis, I draw on principles from the literature to sketch out a prior task design model. The model is then tested out by using it to develop a concrete plan of action in terms of particular kinds of task. The experiences from the work on these tasks provide foundations for suggested changes of the initial design model.
3.3 Summary

This chapter has described relevant features of the design-based research approach, including a description of what constitutes a local instruction theory, and how those features were addressed in the planning of the main study. Further, the framework for task design models suggested by Ruthven (2015) was used to organize the literature which served as guidance in the design of the conjectured local instruction theories within the thesis. The theories chosen for this purpose were chosen from various theoretical frameworks in the field of mathematics education. One of these frameworks has played a particularly prominent role in the course of the entire study: Instrumentation theory. The next chapter provides a more detailed description of that framework.
4 Instrumentation theory

The instrumental approach plays a key role throughout the research work reported in this thesis. The notion of instrumental approach, however, is used in various ways in the literature and is comprehensive in that it involves several different aspects and constructs. For instance, researchers draw on two different lines of development of this approach. On the one hand what Verillon and Rabardel (1995) term the “cognitive ergonomic” approach drawing on cognitive psychological theories, and on the other hand what Chevallard (1992) terms an “anthropological” approach more in line with sociocultural perspectives. First, this chapter describes the instrumental genesis process. Then it clarifies the constructs of schemes and techniques and how they are used in the thesis. Finally, the construct of instrumental orchestration is discussed.

4.1 Instrumental genesis

Central in an instrumental approach is the process of instrumental genesis. The construct of instrumental genesis emerges from the theory of instrumented activity developed by Rabardel and Verillon (1995). One central notion in their model is instrument as a mediating component between an object and a subject. However, they emphasize that “no instrument exists in itself” (p. 84) but “becomes so when the subject has been able to appropriate it for himself (...) and, in this respect, has integrated it with his activity” (p. 84). When a subject engages with an artefact with a particular object in mind, s/he develops different so-called utilization schemes associated with the artefact (Verillon & Rabardel, 1995). It is during this interaction process, known as the instrumental genesis, a specific artefact becomes an instrument for a subject. In other words, when confronted with an artefact, a user has to develop cognitive schemes to be able to manipulate the artefact (Guin & Trouche, 1998).

The idea of instrumental genesis has been adopted by several researchers within the field of technology and its integration into mathematics classrooms (Artigue, 2002; Guin & Trouche, 1998; Lagrange, 1999; Leung, Chan, & Lopez-Real, 2006; Mariotti, 2002; Trouche,
Although a majority of the studies focusing on the process of instrumental genesis relates to the work by French researchers in the context of CAS, this construct has been used with other kinds of technologies. For instance, Haspekian (2005) recognized the importance of considering the instrumental genesis process of students’ interaction with spreadsheets and Leung, Chan and Lopez-Real (2006) used it to study students’ engagement in exploratory tasks within a DGS environment.

Concerning the terms *artefact* and *tool*, this thesis uses the terms interchangeably. Regarding what could be considered as an artefact/tool, it depends on the situation under consideration (Drijvers, Godino, Font, & Trouche, 2013; Trouche, 2004). For instance, a DMS environment could be considered as a collection of artefacts. According to Drijvers et al. (2013),

...it is a matter of granularity if one considers the software as one single artefact, or if one sees it as a collection of artefacts, such as the construction artefact, the measurement artefact, the dragging artefact, and so on (p. 26)

The artefacts used in the studies reported in the thesis are of these later types. In the initial project (reported in Paper 1), the focus is on the dragging artefact within a DGS environment whereas in the main project, the focus is on artefacts concerning scaling of axes, movable points and slider bars associated with functions and graphs.

In line with the Piagetian tradition, researchers drawing on the cognitive ergonomic line of theory (e.g. Trouche, Drijvers 2003), see the development of schemes as the core of the instrumental genesis (Drijvers, Kieran, & Mariotti, 2010), while researchers following the anthropological approach focus on the techniques (e.g. Artigue, 2002; Lagrange, 1999). In the next section the use of these two notions are described.

### 4.2 Schemes and techniques

As mentioned above, in the cognitive ergonomic approach developed by Verillon and Rabardel (1995), the construct of so-called utilization schemes, closely related to an artefact, are central in the process of
instrumental genesis. Researchers (e.g. Drijvers & Gravemeijer, 2005; Trouche, 2004) distinguish between two kinds of utilization schemes: usage schemes and instrumented action schemes. The usage schemes are basic and relate closely to the artefact while instrumented action schemes focus on actions upon objects, such as graphs or formulas. According to Drijvers and Gravemeijer (2005), “Instrumented action schemes are coherent and meaningful mental schemes, and they are built up from elementary usage schemes by means of instrumental genesis” (p. 167). However, since instrumented action schemes are a cognitive construct, not visible for observation, it is the observable part of instrumented action schemes that could be investigated (Drijvers, 2003; Guin & Trouche, 2002).

In Chevallards’ anthropological approach, technique is one of the four components of practices, or praxeologies (Artigue, 2002). Artigue and her colleagues adapted this approach, in which institutional conditions are important, in their work with CAS in mathematics education. She points out

... that the term “technique” has to be given a wider meaning than is usual in educational discourse. A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work. I would like to stress that techniques are most often perceived and evaluated in terms of pragmatic value, that is to say, by focusing on their productive potential (efficiency, cost, field of validity). But they have also an epistemic value, as they contribute to the understanding of the objects they involve, and thus techniques are a source of questions about mathematical knowledge. (Artigue, 2002, p. 248)

However, as Artigue (2007) argues, the distinction between epistemic and pragmatics values of techniques is not considered in the anthropological approach. Lagrange (1999), in his examination of the role of teaching by using CAS technology, found it important to consider the relationship between “the technical and conceptual part of mathematical activities” (p. 63). While he emphasizes “the role of schemes in the process of conceptualisation”, he also stresses “the need for techniques in the teaching of concepts” (p. 63) and he raises the question about the relationship between schemes and techniques. According to Lagrange (1999): “in an educational context, techniques can be seen
as official, rational objects for communicating whereas schemes are structures actually produced in students’ mind” (p. 63).

Although there are tensions between the two theoretical perspectives used within an instrumental approach, the central idea of both schemes and techniques is that conceptual and technical aspects are intertwined (Drijvers et al., 2010). Drijvers et al. (2010) argue that it is the relationship between those aspects “that makes instrumentation theory powerful” (p. 110) and that the difference between the words scheme and technique is of less importance.

In their empirical investigation of the relationship between computer algebra techniques and algebraic thinking, Drijvers and Gravemeijer (2005) provide an instructive framework for investigating both technical and conceptual aspects of instrumented action schemes. I found their terminology of key elements of instrumented action scheme (see Section 3.2.4) interesting and chose to follow their recommendation of using it in relation to other types of technology (than CAS).

In conclusion, this thesis uses the terms artefact and tool interchangeably and there is no distinction made between them. Moreover, the notion of instrumented action schemes is adopted and used in the two papers (Paper 3 and 4) that investigate the role of tasks in supporting students’ instrumental genesis process. The notion of technique is used while referring to the techniques instrumented by the two tool systems: paper and pencil and GeoGebra. Thus, the notion of instrument, as it is used throughout the thesis, could be summarized into the following ‘equation’:

\[ \text{Instrument} = \text{Artefact/Tool} + \text{Schemes/Techniques} \]

4.3 Instrumental orchestration

In spite of being mental structures for an individual user, the utilization schemes have a social dimension since they “concern the coordination of action, not only within the subject, but also inter-subjects in collective activities, (…) particularly important in an educational perspective” (Verillon & Rabardel, 1995, p. 87). Furthermore, “different
students may develop different schemes for the same type of task, or for using a similar command in the technological environment” (Drijvers & Gravemeijer, 2005, p. 168). To address the variability among students’ individual instrumental genesis processes present in the classroom, Guin and Trouche (2002) introduced the notion of instrumental orchestration. The theory of instrumental orchestration concerns the integration of technology within a class, taking individual processes of instrumental genesis into account (Trouche, 2005).

Drawing on Chevallard’s theory of didactic exploitation system, Trouche (2005) defines an instrumental orchestration as composed of didactical configurations and a set of exploitation modes. A didactical configuration concerns the arrangements of the artefacts available in the environment and associated exploitation modes are the way the teachers utilize the didactical configuration (Trouche, 2004). Later on, Drijvers et al. (2010) added a third element of an instrumental orchestration – a didactical performance – to address “the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode” (p. 215).

Ruthven (2014) argues that, “while Drijvers et al. take over Trouche’s constructs of ‘didactical configuration’ and ‘exploitation mode’, these become more closely tied to concerns with the organization of classroom activity around use of a tool” (p. 11). In this way the didactical intention behind each orchestration became less prominent. Ruthven (2014) suggests adding didactical intention to emphasize the objectives of a particular configuration. This aligns with the original definition of instrumental orchestration, which besides a type of configuration and related exploitation modes comprises “a set of objectives” (Trouche, 2003, p. 792). These objectives concern modifying the instrumented action schemes relating to “a type of tasks so as to encourage the construction of knowledge” (p. 793).

Although the main focus in this thesis is on aspects of instrumental orchestration concerning the customization of an artefact to create a particular task environment, the follow-up lesson within each teaching unit concerns the didactical configurations and associated exploi-
tation modes. This later aspect is the focus of Paper 5 (see Section 7.3).

4.4 Summary

This chapter has discussed one central theoretical approach underpinning the studies reported in the thesis – instrumentation theory. This theory, however, is not particularly developed for mathematical purposes and, thus, is a more generic theory. In the next chapter, theories related to mathematics education, which for this reason are described as domain-specific theories, are introduced.
5 Domain-specific theories

In this thesis, domain-specific theories are theories that are specifically tied to mathematics education. First, the chapter clarifies the view of mathematical reasoning used in the thesis. Next it examines the literature on whole-class discussions intended to follow up students’ mathematical work with (or indeed without) computers. Then it reviews research on the mathematical field of functions and graphs, the focus of the intervention designed in the main study. Finally, the chapter describes relevant (for the thesis) tools provided by DMS environments.

5.1 Mathematical reasoning

Although there is consensus concerning the importance of reasoning in mathematics, there is no agreement on its definition (Walton, 1990; Yackel & Hanna, 2003). According to Kilpatrick et al. (2001), it has been common to view mathematical reasoning as deductive reasoning, especially in the form of formal proofs. However, they introduce the expanded notion of adaptive reasoning and state that this notion “is much broader, including not only informal explanation and justification but also intuitive and inductive reasoning based on pattern, analogy, and metaphor” (p. 129). This rather wide conception of mathematical reasoning is in line with the one suggested by Boesen (2006), who argues that:

The competence also includes inductive reasoning, where general statements can be reached based on observations of patterns and regularities. This means that the competence could involve an element of investigative activity in finding patterns, formulating, improving and examine different hypotheses. (pp. 35–36)

The literature concerning mathematical reasoning in a school context frequently uses particular verbs, such as predict (or guess, propose), investigate (or explore), conjecture (or hypothesize), justify (or verify), generalize, explain, and prove (e.g. Arzarello, Olivero, Paola, & Robutti, 2002; Brunström, 2015; Hanna, 2000; Healy & Hoyles, 2002; Liljekvist, 2014). In a document providing commentary notes to the Swedish syllabus for upper-secondary school mathematics the different abilities are described in more detail:
Reasoning ability means being able to perform mathematical reasoning, involving mathematical concepts and methods that provide solutions to problems and modelling situations. Conducting reasoning also includes for example to test, propose, predict, guess, question, explain, find patterns, generalize, and argue, by oneself as well as together with others. It also includes being able to formulate and investigate hypotheses (Swedish National Agency for Education, 2012, author’s translation).

This thesis uses a wide conception of mathematical reasoning in line with the literature introduced above and with the one used in the Swedish syllabus (see Section 2.1).

Many researchers have elaborated on different aspects of reasoning in DMS environments (e.g. Arzarello et al., 2002; Brunström, 2015; Granberg & Olsson, 2015; Healy & Hoyles, 2002), and particularly in DGS environments. It is well substantiated that DGS environments provide opportunities for mathematical activities such as exploration, conjecturing, verification and explanation (e.g. Arzarello et al., 2002; Hanna, 2000; Healy & Hoyles, 2002). However, there is an identified risk that powerful visual images displayed on a computer screen impede students from developing reasoning based on conceptual understanding (Brunström, 2015; Healy & Hoyles, 1999).

5.2 Classroom mathematical discussions

There is a growing interest among researchers in studying ways of creating mathematical whole-class discussions in which students play a central role (Franke, Kazemi, & Battey, 2007). Often, the purpose is to follow up students’ previous work in pairs or in small groups and use it as a basis for a collective mathematical discussion in order to develop students’ mathematical understanding (Ponte, Branco, & Quaresma, 2014; Doorman et al., 2012; Kieran, Guzmán, Boileau, Tanguay, & Drijvers, 2008; Stein, Engle, Smith, & Hughes, 2008). The idea is to make students’ thinking public so it could be reflected on. However, engaging students in mathematical discussions in which each student is given an opportunity to participate actively while simultaneously making certain that the intended mathematical direction of the lesson is followed is not an easy undertaking for teachers (Franke et al., 2007; Ruthven, Hofmann, & Mercer, 2011; Stein et al., 2008). Often, whole-class discussions are characterized by the domi-
nating initiation-response-evaluation/follow-up structure, referred to as IRE or IRF (Franke et al., 2007; Ruthven et al., 2011). Communications that follow this structure are teacher-centred, that is, it is (most often) the teacher who initiates a question to which (most often) a student responds, and then the teacher evaluates or follows up on the response (Franke et al., 2007; Ruthven et al., 2011).

Based on a literature review concerning teachers’ role in communicative classroom discourses, Walshaw and Anthony (2008) suggest some implications for teachers to create productive classroom discourses. For instance, they emphasize the importance of making “students’ mathematical reasoning visible and open for reflection” (p. 539) so that it becomes a resource for their own learning as well for their classmates. Further, they pinpoint the significance of focusing on language to enhance the connection between students’ informal mathematical knowledge and mathematical conventions.

Stein et al. (2008) propose a model of five practices as a tool to support teachers in the orchestration of whole-class discussions while using students’ work as a departure:

1. anticipating likely student responses to cognitively demanding mathematical tasks,
2. monitoring students’ responses to the tasks during the explore phase,
3. selecting particular students to present their mathematical responses during the discuss-and-summarize phase,
4. purposefully sequencing the student responses that will be displayed, and
5. helping the class make mathematical connections between different students’ responses and between students’ responses and the key ideas. (p. 321)

In Sweden, this model has been given a prominent role as it has been translated into Swedish (Smith & Stein, 2014) and is recommended in the government CPD\(^1\) initiative for teachers of mathematics. This

\(^1\) Continuing Professional Development
model focuses on how to follow up students’ previous work on problem solving.

The issue of orchestrating whole-class discussions has also been highlighted in following up students’ work with computer activities (Bartolini Bussi & Mariotti, 2008; Doorman et al., 2012; Kieran et al., 2008). Kieran and her colleagues report research concerning teachers’ adaptation of research-designed resources aimed at supporting the learning of algebra with CAS technology (Kieran & Drijvers, 2006; Kieran & Saldanha, 2008; Kieran et al., 2008; Kieran et al., 2012). The resources involve designed task-sequences with accompanying teacher guides (Kieran et al., 2012). One guiding principle suggested by Kieran et al. is “focused substantive classroom discussion in which the teacher attempts to elicit student thinking rather than give them answers a little too quickly” (Kieran & Saldanha, 2008, p. 399). The students in their study were expected to formulate conjectures and explanations in writing, which could be used as a base in the follow-up classroom discussion (Kieran & Saldanha, 2008). Although the teacher guides offer suggestions concerning the collective discussion...

... the teacher guides did not specify how to conduct the collective discussion – how to encourage reflection, how to inquire into student thinking, how to have students share their thinking with their classmates during the collective sessions, how to use the blackboard to help students coordinate their CAS and paper-and-pencil techniques, or how to orchestrate discussions of a theoretical nature. (pp. 194–195)

By using classroom observations, Kieran et al. investigated how three teachers integrated the designed task-sequences into their teaching practice, for example whether they used any additional resources or in which way they adopted the resources. They found that, although the participating teachers used the same task-sequences, their use of the resources resulted in quite different teaching practices. For instance, they observed how one teacher did not draw on students’ solutions while demonstrating solutions to tasks, while one of the other teachers “inquired into students’ thinking and used this as a basis for discussing some of the different approaches” (Kieran et al., 2012, p. 17).

There are several studies that focus on the teacher’s role in orchestrating collective discussions with a focus on semiotic mediation (Bartolini Bussi & Mariotti, 2008; Maracci & Mariotti, 2009). In line with
the Vygotskian tradition, Bartolini Bussi and Mariotti (2008), regard artefacts as mediators of mathematical knowledge, primarily related to users’ accomplishment of a task. They suggest a didactical cycle model as a means to utilize the potential of an artefact. This cycle involves the following three phases: “activities with artifacts”, “individual production of signs”, and “collective production of signs” (pp. 754−755). It is in the last phase that a collective discussion is central. The teacher’s role in this discussion is to bring students’ individual development of artefact signs to the fore as a base for the evolution of mathematical signs. Maracci and Mariotti (2009) report on a study focusing on a teacher’s implementation of a collective follow-up discussion on students’ work with the tool Casyopée. In their study students worked in pairs or in small groups and were encouraged to produce common responses to particular tasks. Then students were expected to provide individual reports on the tasks, which formed the basis for part of the subsequent whole-class discussion. The aim of this discussion was “to arrive at a shared and de-contextualized formulation of the different mathematical notions at stake” (Maracci & Mariotti, 2009, p. 351). Their study confirms the crucial role of the teacher in promoting and guiding the class discussion.

5.3 Mathematical functions and graphs
The field of functions and graphs is a central topic in the thesis. For students to comprehend the concept of functions it is essential to be “able to move flexibly across representations and perspectives” (Moschkovich, Schoenfeld, & Arcavi, 1993, p.97). This section first describes aspects (relevant for the thesis) of different perspectives and representations of functions. Then it discusses issues relating to the various use of letters applied in the designed lesson materials.

5.3.1 Different perspectives on functions
The literature offers several dualities concerning perspectives on functions. First, functions can be conceived both structurally as objects and operationally as processes (Even, 1998; Lagrange & Psycharis, 2014; Moschkovich et al., 1993; Sfard, 1991). In the view of functions as processes, the computational character is the focus, for
example a function is considered as an input-output machine that links a $y$-value to each value of $x$. In an object view of functions, the focus is on structures and functions are seen as entities.

A second duality is the local/global duality (Even, 1998; Leinhardt, Zaslavsky, & Stein, 1990). The local view is characterized by a point by point attention where the focus is on specific values of a function. A global view, on the other hand, draws attention to global features of a function such as the shape of its graph or the structure of its closed form equation. The dualities mentioned so far are closely related in the way that a global approach supports the view of functions as objects.

A further duality is the distinction between a correspondence and covariation approach to functions (Confrey & Smith, 1994; Confrey & Smith, 1995; Davis, 2009; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Lagrange & Psycharis, 2014). A correspondence approach builds on a view of functions as rules that determine a unique $y$-value to each value of $x$, that is, the focus is on the correspondence between $x$ and $y$. In a covariation approach, on the other hand, the focus is on how the value of $y$ changes and how this is related to the change in $x$-value, that is, how $y$ covaries with $x$. Particularly, Confrey and Smith (1994; 1995) advise a covariation approach to support students’ understanding of exponential functions, by making the “rate of change” visible.

5.3.2 Different representations of functions

Another important dimension is the connection between different representations of functions (Ferrara, Pratt, & Robutti, 2006; Knuth, 2000; Leinhardt et al., 1990; Moschkovich et al., 1993). Leinhardt et al. (1990) discuss a specific kind of tasks where students are supposed to translate between different forms of representation, that is, algebraic, graphical, tabular and verbal representations. Examples of these kinds of task are when students are asked to translate from algebraic to graphical representation and vice versa. Leinhardt et al. suggest that
...movement from graphs to their equations would be the more difficult task because it involves pattern detection whereas graphing an equation involves, by comparison, a relatively straightforward series of steps, that is, generating ordered pairs, plotting them on a Cartesian grid, and connecting them with a line. (Leinhardt et al., 1990, p. 35)

As mentioned in Section 2.2, one important potential of DMS environments is the affordance of dynamically linking multiple representations of mathematical objects.

5.3.3 Unknowns, variables and parameters

Central in the teaching and learning of algebra is the appreciation of the various uses of letters. Among beginning algebra students the perception of letters as *unknowns* dominates (Ely & Adams, 2012; Trigueros & Jacobs, 2008). The use of letters as *variables* is more complex since it takes place in several different ways (Trigueros & Jacobs, 2008), for example as generalized numbers connected to pattern and structure or as related quantities in functional algebra (Ely & Adams, 2012; Usiskin, 1988). Moreover, besides not being a mathematically well-defined concept, a variable plays different roles in different settings, which can be confusing for students (Trigueros & Jacobs, 2008). Accordingly, much of the research within this domain concerns students’ difficulties in making sense of variables (e.g. Bardini, Radford, & Sabena, 2005; Bloedy-Vinner, 2001; Ely & Adams, 2012; Persson, 2010).

Besides the use of literal symbols as unknowns and variables, another usage of letters is as *parameters*. Researchers point out the importance of being able to distinguish parameters from unknowns and variables (Bloedy-Vinner, 2001; Drijvers, 2003; Ursini & Trigueros, 2004), which has proven difficult for students (Furinghetti & Paola, 1994). There are two main reasons for this difficulty (Bloedy-Vinner, 2001). First, the role of letters depends on the context (Bloedy-Vinner, 2001; Ely & Adams, 2012). For example, out of context an expression such as $y = kx$ could represent an equation with 3 unknowns or a function with 2 variables (Ely & Adams, 2012). The second difficulty relates to the notion of parameter and its contradictory epistemic nature (Bardini et al., 2005) causing students to experience a conflict. It is the apparent contradiction between a parameter as a
constant but one that varies that causes this conflict (Bardini et al., 2005; Bloedy-Vinner, 2001). The reason for this is that parameters can play roles as unknowns and variables, however, at a higher level (Bloedy-Vinner, 2001; Furinghetti & Paola, 1994; Ursini & Trigueros, 2004), or as Bloedy-Vinner explains it:

... on the one hand, the parameter is an argument of a (second order) function, and as it varies it determines corresponding equations or functions; on the other hand, within each equation or function which corresponds to a specific parameter value, the parameter is a constant while the other letters are unknowns or variables. (p. 180)

A further inherent difficulty concerning the role of a letter is that the terms in which it is framed often change in the course of a solution procedure (Drijvers, 2003; Ursini & Trigueros, 2004). To illustrate this, Usiskin (1988) uses the traditional problem of finding an equation for a line given the slope and a point that the line goes through. In this particular example, the role of $b$ changes from being a parameter in the equation $y = mx + b$ to being an unknown to be solved for as soon as the given data – the value of the slope $m$ and the coordinates of the point – are substituted into the equation.

The literature describes three main different approaches to algebra: through generalization, problem solving or a functional approach (Bardini et al., 2005; Drijvers, 2003; Usiskin, 1988). To emphasize the interpretation “of letters as variables rather than unknowns” (p. 132), Chazan and Yerushalmy (2003) suggest a function-based approach to algebra. In line with the research by Drijvers (2003), the main project reported in this thesis takes a functional approach as the point of departure.

5.4 Dynamic mathematical software environments

This section introduces the literature concerning particular elements of DMS environments relevant for the thesis. First, the dragging function, which is regarded as a defining feature of a DGS environment (Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Hölzl, 2001; Laborde, 2001; Leung, 2011), is introduced. Then research related to technology in the field of functions and graphs is intro-
duced, namely the literature concerning scaling of axes, movable points, and slider bars.

### 5.4.1 Different dragging modalities in DGS environments

Hölzl (2001) discusses two ways in which dragging can be used: drag to check, for example whether a construction has a desired property, and drag to search for new properties. The later is a way to use a DGS environment to explore new mathematical relations. The purpose of this kind of dragging is to search for regularities and invariances (Hölzl, 2001; Olivero & Robutti, 2007). Arzarello et al. (2002) introduce a hierarchy of different dragging modalities. They observed that students use different dragging modalities depending on purpose (Arzarello et al., 2002). For instance, to investigate and explore students used *wandering dragging*, *bound dragging* and *guided dragging*. By wandering dragging, a basic point is moved randomly on the screen “in order to discover interesting configurations or regularities in the drawings” (p. 67). In bound dragging, the point which is dragged is a *semi-draggable point*, that is, a point that belongs to an object. Guided dragging is used to give a drawing a particular shape, by moving one of its basic points (Arzarello et al., 2002).

Baccaglini-Frank and Mariotti (2010) advance the research on different dragging modalities. In their research they focus on students’ cognitive processes when they engage in explorations in a DGS environment (*Cabri*) and produce conjectures. One of the dragging modalities utilized in their study is *dragging with trace activated*. Baccaglini-Frank and Mariotti refer to this form of dragging as a combination of two *Cabri* tools: *dragging* plus *trace*, which together constitute a *new* global tool that can be used in the process of conjecture-generation” (pp. 230–231). This tool is also discussed by Santos-Trigo and Espinoza-Perez (2002). They point out that “a powerful tool to identify and explore mathematical relationships is to trace the locus of any specified object” (p. 47). They used three locus problems to show the power of DGS environments as a means to “identify and examine geometric properties” (p. 38). The task model in Paper 1 within this thesis is primarily designed to suit geometrical locus problem.
5.4.2 Scaling of axes

Already several decades ago, researchers pointed out the influence that new technology would have in the field of functions and graphs (e.g. Leinhardt et al., 1990). For instance, in comparison to the corresponding work with paper and pencil, they emphasized the ease, and thereby the speed of changing the scales of the axes to obtain several different views of a graph. However, some difficulties have also been identified relating to the issue of scales and scaling of axes (Hennessy, 1999; Mitchelmore & Cavanagh, 2000; Yerushalmy, 1991).

As an example of elements of instrumental genesis (see Section 4.1) observed in a CAS environment, Artigue (2002) discusses “framing schemes”:

When students use function graphs in a computer environment (or a graphic calculator), they are faced with the fact that a function graph is “window-dependent” and they have to develop specific “framing schemes” in order to cope efficiently with this phenomenon. (p. 250)

Mitchelmore and Cavanagh (2000) argue that one reason for students’ limited understanding of scaling might be their lack of experience in dealing with graphs where the axes are unequally scaled. In line with this, Goldenberg (1988) discusses students’ preference for symmetric scaling, that is, x- and y-axis are equally scaled. He refers to an example where students who had received an appropriate view of a graph, still changed the scales to obtain symmetric scaling. In this way, the students received a visual appearance obscuring important features of the graph. To receive a better visual appearance, they changed the scales of the axes by the same factor, that is, they used a zoom operation. Goldenberg (1988) argues that one reason for this might be that students’ intuition about scale changes is closely connected to real-world experiences: “our almost automatic approach is to change both scales by the same factor” (p. 36). However, since different units on the axes usually are required to see the graph in an appropriate way, Goldenberg (1988) stresses the importance for students to deal with unequally scaled axes. In relation to this, Yerushalmy (1991) emphasizes the importance of understanding the relation between the properties of a function and its picture in a
graphical view. The issue of scales and scaling in a DMS environment is addressed in Paper 3 in this thesis.

5.4.3 Manipulating movable points

The literature identifies some student difficulties in making connection between symbolic and graphical representations of points (Hennessy, 1999). For students to be able to make this connection, they need to understand the notion of ordered pairs of numbers represented by points in the coordinate systems (Goldenberg, 1988; Hennessy, 1999). According to Hennessy, students “fail to grasp the association between an ordered pair and corresponding x- and y-values” (Hennessy, 1999, p. 27). Furthermore, Goldenberg, Scher and Feurzeig (2008) report findings where students’ inaccessibility of controlling the variable $x$ in a graphical software environment made them confused since “It was called ‘the variable’, but they never varied it!” (p. 77). On the contrary, students could change the values of the parameters in a polynomial function such as $y = ax^2 + bx + c$, which in turn, might give the impression that the variables are $a$, $b$ and $c$, and not $x$ (Goldenberg, 1988).

One defining feature of DGS environments is the ability to drag points in geometric constructions and hence manipulate them dynamically (Goldenberg et al., 2008). According to Goldenberg et al. software designers and researchers have strived to develop analogue dynamical tools aimed at investigating variable coordinate-pair. Researchers working with DGS for investigating Euclidean geometry use the notion of draggable points (e.g. Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010) but this notion is seldom connected to an algebraic representation. In this thesis the notion “movable point” is used while referring to a draggable coordinate-pair. Here then, draggable correspond to the dragging modality denoted as semi-draggable where the object constraining the dragging is a graph (Arzarello et al., 2002).
5.4.4 Controlling slider bars

The affordance of technology to provide access to multiple representations of functions is well documented (Ferrara et al., 2006; Laborde, 2007; Pierce, Stacey, Wander, & Ball, 2011). The use of DMS environments provides opportunities to make direct manipulation of dynamically linked representations of functions (Falcade et al., 2007; Lagrange & Psycharis, 2014; Zehavi & Mann, 2011).

Drijvers (2003) and Zbiek and Heid (2001) have used a functional approach to investigate students’ use of technology to develop the understanding of parameters. The particular tool they used, and which they termed a slider bar\(^2\), enhances the manipulation of parameters. This, they argue, gives students the opportunity to examine the visual effect on a graph while changing the value of a parameter (Drijvers, 2003; Zbiek & Heid, 2001). However, in an overview, Zbiek et al. (2007), raises the following question: “Does the physical sense of moving a slider obscure rather than enhance the desired cognition of the connection between the parameter value and a salient visual consequence?” (p. 1177). According to them, besides the symbolic and graphical representation of the function itself, the slider bar could be regarded as a third representation for students to focus on while investigating the effect of different values of a parameter. A further risk with the use of slider bars, identified by Drijvers (2003) is that students tend to examine the effect of the ‘sliding parameter’ superficially, and thus not bother about reasons behind a particular behaviour.

\(^2\) “a number-line representation of the value of the parameter” (Zbiek, Heid, Blume, & Dick, 2007, p. 1177)
6 Methods

As mentioned in the Introduction chapter, the thesis comprises two research projects; a small initial project in the form of a case study (reported in Paper 1) and a larger main project of which various aspects are examined in the studies reported in Papers 2 to 6. In Chapter 3, the main characteristics of design research and particularly the issues of task design (relevant for the thesis) are introduced. This chapter provides an overview of the implementation of the studies. First, it describes issues relating to the research design. Then the chapter provides a discussion of the trustworthiness of the methods. A final section concerns the ethical considerations that have been taken into account. As regards the processing of data and the various interpretative frameworks that have been used in the analysis process, the reader is referred to each of the individual papers.

6.1 The research design

The first part of the research work is the initial project in the form of the case study reported in Paper 1. Alongside this study, we (the two researchers involved) took part in a year-long school development project with the aim of iteratively constructing, discussing and testing sequences of tasks designed for a DMS environment. Experiences from that project inspired us to then undertake a research project in collaboration with practicing teachers. The school context for the research project consisted of upper-secondary classes all following the same mathematical course, one which is not intended to prepare students for further studies in mathematics.

As noted earlier (Section 1.3), the design research project that we undertook involved the development of three teaching units, each consisting of an opening researcher-designed lesson and a follow-up whole-class lesson devised by the class teacher. The researcher-designed lessons involve the design of task sequences and it is the development of these lessons that followed a design-based approach, labeled as design experiment.
The choice of using this type of design-based approach has some implications. For instance, we only work with a small number of teachers and classes (see Section 6.1.1) due to the issue of time. Design-based research is recognized as being time-consuming because the adoption by practitioners often requires some reorganization of the mathematics curricula to fit into the time schedule planned for an intervention (Stylianides & Stylianides, 2013). Appendix A provides a timeline for the main project. This section first describes the selection of participants for the studies. Then the lesson materials used are presented. Finally, it provides an overview of the various data collection methods used.

### 6.1.1 The selection of participants

#### The initial project

One of the rationales behind the choice of using graduate students in the initial project was that their working strategies could be informative to improve the initial task design model. According to Alcock and Inglis (2008), it is widely accepted to study the behaviour of high achieving students for the design of mathematical tasks. Another rationale was that the specific topic under consideration (conic sections) is neither a part of the curriculum for Swedish upper-secondary school mathematics nor in most of the university courses. Therefore, graduate students were chosen because one important condition was that the participating students have studied the topic that they were asked to think about.

#### The main project

One year before the start of the main project, a workshop in GeoGebra was arranged to introduce teachers (from two schools) to a DMS environment and show some of its potentials. Once starting our search for participants in the main study, half a year later, we started by asking the teachers who had attended this workshop. An information meeting was advertised, which seven teachers attended. The teachers were informed about the background and aim of the study and an overview of the whole research project was presented. The lit-
erature points out the importance for the teachers involved in a design research project “to understand the goal of the design” (Hjalmarson & Lesh, 2008, p. 104). Hjalmarson and Lesh argue that teachers should see a need for the design so that the bridge between research and practice is addressed.

The roles of the researchers and the expected roles of the teachers were explained. At this meeting, there were four teachers who were interested in participating in the study. The teachers worked at two different schools providing one-to-one settings, which was an important condition for the study. A further desirable condition (which these teachers could fulfil) was that they would teach the same mathematics course in the forthcoming school year. The participating classes were in the tenth grade and the teachers considered it important that the teaching units should form part of the regular curriculum. At the first meeting, the topic for the first task sequence was decided.

6.1.2 The lesson materials

The designed instructional material, which form part of the instruction local theory (see Section 3.1.5), consists partly of the task design model developed in the initial project (see Section 7.1.1), partly of the three task sequences developed in the main project. The initial versions of the designed task sequences in the format of worksheets are presented in Appendix B, C and D. Although the tasks within a task sequence should be connected with each other, they need also to vary in type. There are basically four main types of task employed over the three task sequences:

(a) Prediction tasks where students are supposed to first predict an outcome before testing that prediction, either by using the computer or by making a routine calculation, and then reflecting on the outcome in relation to the prediction.

(b) Tasks aimed at eliciting numeric and/or algebraic solutions through using paper-and-pencil techniques.

(c) Tasks aimed at graphical solutions by using graphing techniques instrumented by GeoGebra.
(d) Description/explanation tasks where students are expected to describe/explain the outcome of particular investigations. This type of task was only used in the third task sequence where students were expected to use the slider tool to examine how different values of a parameter effect the shape and location of the corresponding graph.

To enhance the opportunity for students to make connections between paper-and-pencil techniques (see (b) above) and techniques instrumented by GeoGebra (see (c) above), some of the tasks involved both these technique systems. Moreover, to foster students’ technical skills, the task sequences (on the worksheets) intertwined mathematical development with instructions on using the computer.

6.1.3 Data collection

In the initial project, data was only collected on one occasion in terms of video recording of the pair work between the graduate students. The main project, on the other hand, involved the collection of data on several occasions. Data was collected both before and after the actual classroom implementations of the project following the three main phases of a design experiment: preparing for a design experiment, conducting a design experiment, and conducting retrospective analysis (Cobb et al., 2003; Cobb & Gravemeijer, 2008). Altogether, data was collected at the following six occasions: (a) during the preparation phase, (b) the research team meetings before each teaching unit, (c) the opening researcher-developed lessons within each teaching unit, (d) the follow-up teacher-planned lessons within each teaching unit, (e) the short debriefing meetings with each teacher after each teaching unit. Below, the different types of data collected are described in detail.

Preparing for the study

To get information about the students’ initial experience of and attitudes to using computers in mathematics classroom, a small questionnaire survey was performed with the participating students. Students were asked whether they had had any access to a computer of their own at school earlier. They were also asked if they had any pre-
vious experiences of using computers in mathematics lessons. If they had, they were asked what kind of activity they had done. The students were also asked about their opinion about using a computer in the teaching and learning of mathematics. To get the students acquainted with the software, an introductory workshop devised by the researchers was held.

Individual interviews with the participating teachers were conducted to get information about their teaching background in general and their previous experiences of using technology in the mathematics classroom. Of particular interest was whether the teachers had previous experiences of using computers. How did they use it? What kind of software did they use? What kind of tasks? and so on. They were also asked about their views on using technology in the teaching and learning of mathematics, both constraints and opportunities. Finally, they were posed questions about their interpretation of the curriculum text concerning mathematical reasoning competencies.

Research team meetings
The joint meetings (see Appendix A) between researchers and teachers during the project, which were audio recorded, all followed the same agenda. First, the researchers provided their overall impression of the previous teaching unit, primarily based on experiences from the researcher-developed lesson (only at the second and the third meeting). Then the teachers contributed their own experiences, issues such as observed student obstacles were shared, and these served as a basis for discussions concerning possible revisions of the tasks. Then an initial researcher-designed proposal of a task sequence for the next teaching unit (which had been sent out to the teachers in advance) was thoroughly reviewed. The researchers explained their intention behind each task and the teachers provided valuable information regarding the participating students’ capabilities and their current practices. During this phase, anticipations about possible student responses and behaviour were discussed.

While the organization of the researcher-designed lessons was quite straightforward to plan in advance, the follow-up lessons were more
challenging to envisage because they needed to take account of what had happened in the opening researcher-designed lesson. Accordingly, the follow-up lessons were only briefly discussed. All these meetings were audio recorded. After the completion of the third teaching unit, a reflective discussion with the participating teachers was held to strengthen the retrospective analysis of the entire design-based research project.

The researcher-developed lessons

Video and screen recordings from a chosen (by the teacher) pair of students were made in both the initial and the main study. The position of the camera was at an angle behind the students to capture the screen and student gestures, for example if and where they were pointing at the screen. To ensure the quality of the data in the form of screen pictures, the screen was recorded by using Camtasia software.

All teacher-student interactions throughout the lessons were audio recorded using a microphone attached to the teacher. In addition, field notes were taken to identify which students the teacher interacted with at specific moments. To be able to identify each student, a drawing of the physical classroom environment was made and each student position in the classroom was coded accordingly. These data were, for instance, employed in the process of analysis to establish whether students’ written responses were influenced by the teacher. The students written responses on the worksheets from all students were copied.

The follow-up lessons

The discussions in the follow-up whole-class lessons were audio recorded. The transcribed utterances were supplemented by time-coded field notes within square brackets to add information not captured by the audio recording. Furthermore, pictures of the white board and/or the projected computer screen were occasionally taken.
The short individual teacher meetings

The aim of these informal meetings was to capture the teacher's spontaneous impression after each teaching unit. During these short meetings field notes were made.

6.2 Trustworthiness

An important issue is the validation of data within a qualitative research approach. Denscombe (2007), drawing on the work of Lincoln and Guba (1985), discusses the difficulties of using conventional constructs such as validity, reliability, generalizability and objectivity. Instead, he suggests using the closely corresponding constructs of credibility, dependability, transferability and confirmability while discussing the trustworthiness of the research (Denscombe, 2007).

The credibility (or internal validity) of a study “asks the question: How congruent are one’s findings with reality?” (Merriam, 1995). To enhance the credibility in the main study, multiple data sources were used in the analysis of the empirical data. For example, through analysing students’ reasoning, transcripts from the video recording, students’ written responses and the audio recording of teacher-student interactions throughout the opening researcher-designed lessons were used. In the analysis of students’ instrumental genesis process, the video recording was triangulated against the audio recordings and the written records from the larger number of students to ensure that the observed behaviour was typical among the participating students.

Besides the triangulation of data, the joint meetings with the teachers to some degree allowed for “respondent validation” (Cohen, Manion, & Morrison, 2007). For instance, these meetings allowed discussions of particular incidents of student responses or behaviour, which offered opportunities for sharing interpretations of the data with the teachers. Further, the collaboration between two researchers allowed for an “investigator triangulation” (Denscombe, 2007). Typically, the coding process was done independently by each of the two researchers and then comparisons were made followed by discussions until full agreement was reached. Finally, according to Denscombe, the fact
that the data are grounded in ordinary classrooms also increases the credibility of the study.

The construct of reliability which is about the replicability of a study is not useful in qualitative research (Cohen et al., 2007; Merriam, 1995). Instead the notion of dependability is suggested. This notion addresses the issue of making the methodology of the study transparent for others. Drawing on Lincoln and Guba (1985), Denscombe (2007) suggests a so-called audit trail which “should be constructed and mapped out for the reader – allowing them to follow the path and key decisions taken by the researcher from conception of the research through to the findings and conclusions derived from the research” (p. 298). This aligns with design-based research where the goal is to make explicit elements of the design that typically remain implicit (Barab & Squire, 2004; Edelson, 2002).

The fact that we only conducted the initial iteration of a design-based research study created the opportunity for a rather detailed description of the task design process. Initially, the local instruction theory (see Section 3.1.5) involves anticipations about possible student responses while working on the task sequences. Papers 1–4 provide rather detailed descriptions of the hypothetical student learning trajectories. The empirical testing is reported in terms of the usability of the designed tasks and the revisions made based on the feedback from students. In reporting on the usability of a designed product, Hjalmarsone and Lesh (2008) suggest including “the potential users, the setting, prerequisites for use, and the purpose the product was tended to fulfil. Such documentation is important for the generation of theories because it provides evidence of the process of design” (p. 104). Besides, these types of documentation, the results sections of the papers include description of the usage of analytical tools to analyse student utterances and activities. However, this has been made explicit more or less rigorously within the different papers. Probably, Paper 2 provides the most detailed description of the analysis process.

Transferability (external validity) concerns whether the conclusions of a study are transferable to other contexts (Miles & Huberman, 1994). For a reader to be able to judge this requires a good enough
description of the research setting. The submission and publishing of papers in (international) journals and presentations at seminars and conferences have offered opportunities for review by people outside the local research context.

Finally, the confirmability (objectivity) of a study concerns the extent to which the research findings are uninfluenced by the researchers (Denscombe, 2007). As Denscombe points out, “no research is ever free from influence of those who conduct it.” (p. 300). Qualitative data consists of interpretations of text or images. One aspect of this has been taken into account in cases where the responses from all the students have been reported in order to provide an overall picture which is not involved in filtering of evidence by the researchers. Another aspect concerns the usage of the interpretative frameworks in the analysis of data. Suggested explanations by the author have been discussed with colleagues to check for rival explanations (Denscombe, 2007).

6.3 Ethical considerations

As a researcher, you are expected to follow good research practice, which includes following relevant legislation and ethical codes (Gustafsson, Hermerén, & Pettersson, 2011). An ethical code consists of a collection of ethical principles relevant to a specific research area. This thesis follows the ethical principles stipulated by the Swedish Research Council for research in humanities and social sciences (Swedish Research Council, 2002). There are four main areas of particular importance: information, consent, confidentiality and use. These guiding principles have been taken into account in preparing and implementing the research projects. To ensure good research practice, a proposal was sent to the Ethical Research Committee of Karlstad University, which approved the research project.

Before the projects started, all the participating students and teachers were informed, both orally and in writing, about the aim of the project and their roles within it. They were also informed that their participation in the project was voluntary and that they could choose to end participation at any time. All students except two were willing to participate and they signed an informed consent form, as did the four
teachers. Even though the students were over 15 years old, information letters about the project was sent to the parents.

Specific ethical attention was paid to the data gathered from the video recording of the pair of students. According to the Swedish Research Council, video recording should be used sparingly, only when the same result is impossible to obtain by using other kinds of data collection methods (Gustafsson et al., 2011). Therefore, we decided to only utilize video recording for observing one pair of students in each class and use only audio recordings in the collection of data from interviews, meetings and whole-class lessons. However, to reduce the intrusiveness of the video recording, the students were only filmed from behind, as facial expressions were not necessary for the analysis. The students, who agreed to be filmed, and the graduate students, as well, were provided with more detailed information about the way the material would be used.

To ensure that the requirement of confidentiality is fulfilled, all data have been coded. For instance, instead of using the name of the students, their position in the classroom during a specific lesson was used as a code.
7 Findings

This chapter provides a brief summary of the findings reported in each of the six individual papers, showing how these are related to the overarching research question of the thesis, introduced in Section 1.2:

*How can key aspects of the design and implementation of task sequences making use of dynamic software environments be better understood and improved accordingly?*

As described earlier, this question is divided into three sub-questions, each relating to the key aspects: task design models, instrumental genesis and orchestration, and follow-up lessons. These questions are addressed in the Sections 7.1, 7.2, and 7.3 respectively in this chapter.

A further purpose of the chapter is to illustrate the progression throughout the research work reported in the thesis. This is done by discussions on the relations between the different issues addressed within the including papers. The first section discusses key aspects of task design models by using the organising scheme suggested by Ruthven (2015) (see Section 3.2). The second section concerns the process of instrumental genesis, particularly in relation to task design. Finally, the last section is about how to follow up and take advantage of students’ work on computer-based tasks.

7.1 Task design models

Paper 1 and 2 deal with the design of tasks intended to foster students’ reasoning capabilities in terms of exploring, formulating, verifying and explaining conjectures. Both papers suggest a particular task design model which is used to develop a concrete plan of action that is then trialled and improved accordingly. The task design model concerns the characteristics of the tasks and the task environment. The overarching research question addressed in the two papers is: *How well does a task design model guide the development of plans of actions which support mathematical reasoning, and how could it be improved?* It is not just about how well the actual task sequences are working but how well the principles underpinning the task sequences
are functioning. This section first describes the models used within the two papers, and how they were refined. The models are referred to as P1-model and P2-model, respectively. Then the task design models are compared and contrasted to draw out similarities and differences between them.

7.1.1 The task design model used in Paper 1

The idea behind the design approach investigated in this paper is to construct a suitable task-situation by reformulating an existing problem, primarily a traditional proof task, into a more open problem. The term task-situation is adopted from Kieran and Saldanha (2008) and means “an extended set of questions related to a given central task or mathematical idea” (p. 408). The task design model in this paper was developed to suit geometrical locus problems to be undertaken with a DGS environment. The purpose of the model is to foster students’ capability to explore, conjecture, verify, explain and make generalizations with a particular focus on the last element. This section first introduces the guiding principles used in Paper 1. Then the initial version of the task design model is introduced followed by suggested refinements.

The guiding principles

The literature (see Section 3.2) provided a framework for the initial task design, which comprises ideas not only about productive types of tasks, but also about how to structure the tasks. It also involves ideas about how to organize the task environment, both the physical and the social task environment. The four guiding principles used are open problems, logical flow, fostering instrumental genesis, and formulations in writing.

Open problems. One way to create open problems that exploit the features provided by DGS environments to explore and produce conjectures could be to turn traditional closed tasks such as “prove that...” into open ones. We tackled this by not specifying what to prove. Instead the students are expected to make conjectures about a mathematical situation under given condition. Moreover, the mathematical
situations should provide possibilities for students to formulate conjectures that can be generalized.

Logical flow. The structuring of the tasks should provide a logical flow throughout the proving process. This means that the students first are expected to utilize the software to explore the mathematical situation under consideration to discern any pattern and formulate conjectures. Then the students are expected to test their conclusion and also to provide an explanation accounting for the validity of a conjecture. Finally, students are encouraged to investigate the mathematical situation further (by using the software) and try to make further generalisations.

Fostering instrumental genesis. The students are expected to do all necessary mathematical constructions in the software environment by themselves, that is, not using pre-designed applets. The intention is to make students acquainted with the software, for instance to enable them to add appropriate construction elements when needed during their search for further generalisations (Hölzl, 2001; Santos-Trigo & Espinosa-Perez, 2002).

Formulations in writing. The tasks should involve a mix of paper-and-pencil work and computer activities. For instance, students are encouraged to formulate their conjectures and explanations of them in writing. This might encourage them to make reflections to get ideas for further explorations to make generalisations.

The initial version
The initial version of the model (see Table 1) was tried in a case study with two graduate students, which provided guidance for improvement of the model.
Table 1
_The initial model for design of task-situations_

<table>
<thead>
<tr>
<th>Description of the mathematical situation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Make an appropriate construction in [the DGE, e.g. GeoGebra] and study the</td>
</tr>
<tr>
<td>position of [a dependent object, e.g. a point], for different positions of [an independ-</td>
</tr>
<tr>
<td>ent object]. Make a conjecture.</td>
</tr>
<tr>
<td>(b) Use [the DGE] to support or refute your conjecture.</td>
</tr>
<tr>
<td>(c) Explain in your own words why your conjecture is true.</td>
</tr>
<tr>
<td>(d) Construct a proof.</td>
</tr>
<tr>
<td>(e) Make new related investigations. Make conjectures, support or refute, explain</td>
</tr>
<tr>
<td>and prove.</td>
</tr>
</tbody>
</table>

**Suggested refinements**

The findings in Paper 1 resulted in the revised version of the model in Table 2. In the following, the suggested revisions are discussed. The principle that students should make their own conjectures is consistent with the notion _open problem_ (Baccaglini-Frank & Mariotti, 2010; Mogetta et al., 1999). However, to make the problem even more open, the more concrete and specific instruction “study the position of” could be replaced with the more abstract and generic “search for mathematical properties”. This formulation would have probably worked equally well for the students in the case study since they were high achieving students. Thus, we argue that the level of openness in the instruction should be adapted to the mathematical situation under consideration and the students’ capabilities. Moreover, an alternative to the formulation “Make an appropriate construction” would have been to give detailed descriptions of the different steps in the construction. However, the level of guidance should be adapted both to how difficult the construction is and to the students’ mathematical and instrumental knowledge.

In Task (a), the sentence “Make a conjecture” is replaced by “Formulate a conjecture”. The students in the case study gave a rather short answer to this request and they made little effort on the written formulation. The suggested revision might be a way to stress the importance of describing mathematical findings in a proper way (Edwards, 1997). Furthermore, the new formulation makes it more likely that the conjecture will be formulated as a conditional statement, that
is, with both premises and conclusions. This might in turn facilitate
the performance of Task (e), that is, a generalization of the conjecture.

Table 2
The revised model with the modifications highlighted in bold.

<table>
<thead>
<tr>
<th>Description of the mathematical situation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Make an appropriate construction in [the DGE, e.g. GeoGebra] and study the position of [a dependent object, e.g. a point], for different positions of [an independent object]. <strong>Formulate</strong> a conjecture.</td>
</tr>
<tr>
<td>(b) Are you convinced of the truth of your conjecture? If not, try to use [the DGE] to support your conjecture. When you are convinced, go to the next task.</td>
</tr>
<tr>
<td>(c) Explain in your own words why your conjecture is true.</td>
</tr>
<tr>
<td>(d) Construct a proof.</td>
</tr>
<tr>
<td>(e) <strong>Investigate if your conjecture can be generalized. Perform the tasks above with new premises, by using appropriate techniques, such as posing what if? or what if not? questions.</strong></td>
</tr>
</tbody>
</table>

The formulation of Task (b) is changed completely. Once the students in the case study had read the instruction for Task (b) they felt that they had already performed it. The reason for this was that they supported their conjecture by investigating a further example before they formulated the conjecture in writing. To avoid this kind of confusion, we chose to reformulate Task (b) so that students can proceed with the next task in case they are already convinced of the truth of their conjecture.

In Task (e), a new formulation is suggested (see Table 2). The revision was done to direct students to focus on their formulated conjecture. The case study showed how the students worked in a systematic way when they generalized their initial conjecture. They went back to the original description and looked for premises to vary. Further, the revision is made to focus more on the process of generalization. However, this is identified as a challenge for many students (e.g. Sinclair, 2003), and therefore we decided to provide a clue of how to search for further generalizations by suggesting to pose what-if or what-if-not questions. This strategy to help students make generalizations is suggested in the literature (e.g. Chazan, 1990; Lin, Yang, Lee, Tabach, & Stylianides, 2012; Mason, Burton, & Stacey, 2010; Yerushalmy, 1993).
7.1.2 The task design model used in Paper 2

Paper 2 provides a task design model of so-called prediction tasks intended to foster student reasoning concerning exponential functions in a DMS environment. The prediction tasks are embedded in a task sequence with the aim of developing students’ awareness of the difference between linear and exponential growth. The literature provided a framework for the initial task design, which comprises ideas of how to encourage students to perform different aspects of reasoning in relation to functions and graphs. This section follows the same structure as the previous one; first, the guiding principles are introduced, followed by a description of the initial version of the task design model and finally, the suggested refinements are discussed.

The guiding principles

In the design of prediction tasks, in total seven design principles guided by the literature (Section 3.2) were used. These are logical flow, unexpected outcomes, mathematical reality, fostering instrumental genesis, formulations in writing, and working in pairs.

Logical flow. The structuring of the prediction task should promote a logical flow by first requesting students to make a prediction about an outcome of a mathematical situation, and then to investigate the situation under consideration. Finally, the students are encouraged to reflect on and/or explain the outcome of the investigation in relation to their initial prediction.

Unexpected outcomes. Particularly suitable as prediction tasks are those capable of eliciting some misconception common among students or those that are considered challenging for many students, and hence might provide unexpected outcomes.

Mathematical reality. The tasks in the task sequence, in which the prediction tasks are embedded, should be related to a common theme, preferably connected to a context that students experience as realistic. The reason for this is to enhance the opportunity for students to make reflections and connections between different tasks in a task
sequence and to offer students a familiar situation and thereby increase their motivation.

**Fostering instrumental genesis.** The tasks were to be intertwined with computer instructions since students were expected to do the constructions by themselves, that is, they should not use prepared applets. The main reason for this choice is to enhance the scope for the mathematics software to become an instrument for the students.

**Formulations in writing.** Students were, when appropriate, expected to formulate their predictions and explanations in writing to make them reflect further on the situation they are working on. Moreover, in this way their reasoning becomes public to others to be used in a follow-up whole-class discussion.

**Working in pairs.** Students were encouraged to work in pairs at one computer. In this way, the results from explorations with the software are displayed on a common computer screen, which serves as a common referent to enhance joint reasoning. This was important because one main purpose with the prediction tasks was to foster student reasoning.

**Didactical variables.** The domain-specific literature related to functions and graphs in a dynamic graphical software environment (see Section 5.3), served as guidance in the *a priori* identification of four didactical variables. While tackling the tasks, students might (a) use different forms of representation of functions and make translations between them, (b) experience both a local and a global view of functions, (c) use both a correspondence and covariation approach to functions and (d) pay attention to the effects of scaling of coordinate axes.

**The initial version**
In total three different prediction tasks, embedded in a task sequence, were trialled with four tenth grade classes, involving a total of 85 students. To articulate the theoretical rationale for different design choices that might affect students’ reasoning and to analyse them
after empirical testing, the design tool of didactical variables was employed. In total, seven didactical variables based on the literature were identified \textit{a priori}, and during the analysis process four further didactical variables were identified. Besides the four domain-specific didactical variables mentioned above, one variable concerned which tools, paper and pencil or computer students were asked to use to solve a particular task. Still another variable concerned whether to ask students for explanations or not.

Moreover, the conjectured local instruction theory that goes with the designed tasks comprises a detailed description about expected student reasoning which might occur throughout the implementation of the activities in the classroom.

\textit{Suggested refinements}

Besides providing suggestions for revision of the particular prediction tasks, Paper 2 pinpoints key didactical variables that proved important to consider in designing prediction tasks. These didactical variables were grouped into the following three concerns: \textit{explanations, functions} and \textit{scaffolding issues}. Below follows a description of how some of the didactical variables provided guidance in the refinement of the prediction tasks to enhance the opportunity for students to develop reasoning based on conceptual understanding.

The didactical variables concerning explanations turned out to be crucial in the design of prediction tasks. The \textit{a priori} didactical variable identified \textit{Ask for an explanation or not} was discussed in all prediction tasks, and in the revised version we suggest adding some further requests for explanations. The choice whether to ask for an explanation or not was essential both in the prediction part and the comparison part of each prediction task. However, sometimes it might be appropriate to just ask for a description or a comment. Further, the results indicate that the wording is crucial in the formulation of questions where students are asked for explanations. This implied the identification of a new didactical variable regarding how to direct students’ focus on \textit{what} to explain. Actually, all requests for explanations in the study turned out to require some refinement concerning this
didactical variable. In the revised versions, we endeavour to make the requests more pointed.

The didactical variable concerning what form of representation to use is elaborated on in all prediction tasks. For instance, if the aim is to provoke a known misconception, it is important to reflect on whether this misconception interrelates to a specific form of representation. The didactical variable concerning the choice of covariation or correspondence approach turned out to be essential. In several cases revisions were made to emphasize a covariation approach. The main reason for these revisions was to encourage students to reflect on the mathematics behind exponential growth and thereby promote conceptual reasoning.

Scaffolding issues were elaborated on in all prediction tasks and all the new didactical variables identified during the design process are more or less related to these issues. For example, one didactical variable concerns the question whether to support students by specifying the scaling of axes or not. Another didactical variable was introduced in the revision of one of the tasks where students were given a numeric value determining what to count as a good enough guess. Thus, whether to give this scaffolding or not became a new didactical variable.

7.1.3 Comparison between the two task design models

The task design models used in the two studies are not the same. For instance, the use of didactical variables makes the model used in Paper 2 more sophisticated than the one used in Paper 1. Moreover, the intentions behind the two task design models differed; while the intention behind the P1-model is to foster students’ proof capability, and thus requires a rather high mathematical capability, the intention behind the P2-model is to foster students’ reasoning capability more generally. However, although there are differences between them, there are also similarities. To compare and contrast the two task design models, the “scheme for design” framework suggested by Ruthven (2015) is used as a framework.
A template for phasing task activity

Concerning the structuring of the tasks, both models are based on a sequence of phases that follow a logical flow. To some degree they are rather similar in that they both include the following three phases: (a) formulation of a prediction/conjecture, (b) verification or refutation of a prediction/conjecture, and (c) reflection on or explanation of a prediction/conjecture. The reason behind this similarity is that the purposes of the models are similar in that they both aim at fostering student reasoning, although in different contexts. However, the P1-model also includes a phase before (exploration) and a phase after (generalisation) these three phases. This difference between the models is due to the different levels of sophistication and confidence of the students for whom the tasks were designed.

Although both models emphasize student explanations, the requirements of those are at different levels. In the P1-model, students are encouraged to explain the truth of an explicitly formulated (by the students) conjecture, while in the P2-model students are asked to explain the reason behind a formulated prediction about a mathematical situation and/or the outcome of the investigation of the situation. It is worth noticing that there is quite a substantial difference between the notions of conjecture and prediction in the two studies. While a conjecture might be formulated as a conditional statement, that is, with both premises and conclusion, a prediction might be formulated as a guess of something in relation to a specific mathematical situation, for example the position of a point in a coordinate system or a closed form equation.

Finally, in Paper 2 a further kind of sequencing is used since the three prediction tasks that were trialled out were embedded in an overarching task sequence. That means that the prediction tasks together with some other types of task (and computer instructions as well) provided a logical flow throughout the whole task sequence.

Criteria for devising a productive task

In the P1-model, the suggestion of using conjecturing open problem is adopted. That is, problems in which students are expected to first
make explorations to find a conjecture, which they are then supposed to prove or at least verify and explain. The idea of the P1-model is to reformulate a traditional proof task into a more open task-situation. That is, this model requires a proof task as a base. With respect to this, the range of application of the P1-model is rather restricted in comparison to the task design model used in Paper 2, which is more generic.

Because one aim of the P1-model is to encourage students to search for further generalisations, the P1-model requires that the task-situation provides opportunities for making generalisations. The case study shows how task-situations that provide unexpected outcomes, that is, not anticipated by the students, might motivate students to investigate the situation further. However, this criterion was not explicated in Paper 1. In Paper 2, on the other hand, unexpected outcomes were desirable because they might elicit common misconceptions among students or situations that are known as challenging for students. These could then be brought to the fore to enhance the opportunity of clarification for the students.

Although the two models require a mix of paper-and-pencil work and computer activities, it is only the P2-model that involves tasks which provide students opportunities to make connections between the two tool systems (paper-and-pencil techniques and techniques instrumented by a particular DMS environment). Further, the idea of using a mathematical reality is only adopted in the P2-model.

**Organisation of Task environment**

Although the two task design models were designed for different pieces of DMS environment – dynamic geometry software environment in Paper 1 and dynamic graphing software environment in Paper 2 – the organisation of the physical and social task environments were similar. Both models assumed that students would make all the computer constructions by themselves. In contrast to the P1-model, the P2-model includes computer instructions. However, this issue was discussed in Paper 1 and it was highlighted that the need for computer instructions depends on students’ familiarity with the software.
Concerning the social task environment, the students were expected to *work in pairs* with one computer per pair. Although this was only explicated in terms of an *a priori* guiding principle in Paper 2, it was an implicit assumption in Paper 1. Moreover, both models encourage students to express their thinking *in writing*.

*Management of crucial task variables*

The task design model in Paper 2 is more sophisticated than the P1-model in that it introduces the idea of *didactical variables* as a design tool. However, although not using the language of didactical variables in Paper 1, some didactical variables were discussed. For example, the issue of scaffolding, which many of the didactical variables in the P2-model relate to, was discussed. Hence, in reflection, we believe that it might have been a useful tool in the work reported in Paper 1. Moreover, the didactical variable identified in Paper 2 about the importance of wording to direct students’ focus when formulating tasks is emphasised in Paper 1 as well.

*7.2 The instrumental genesis and orchestration*

Paper 3 and 4 concern two different aspects of instrumental genesis and the orchestration of that by creating a particular task environment. The overarching research question for the two papers is: *What is the instrumental genesis for the manipulation of key tools (scales of the coordinate axes, movable points and slider bars), and how can the task (re)design achieve this instrumental genesis more effectively?* While Paper 3 deals with aspects concerning scale and scaling of axes, Paper 4 is about aspects concerning movable points and slider bars. Moreover, Paper 4 is a more sophisticated paper than Paper 3 in that it links more to the design and redesign of task environment. Both papers adopt the idea, introduced by Drijvers and Gravemeijer (2005), of identifying key elements of instrumented action schemes to analyse students’ instrumental genesis process. In the following, the description within each paper of the instrumental genesis and its orchestration is presented. Then the relation between the findings in the two papers is discussed.
7.2.1 Instrumental genesis and orchestration in Paper 3

The literature highlights some issues concerning scales and scaling of axes with technology (see Section 5.4.2). For instance, it states that students’ capability to deal with scaling of axes is often taken for granted. While working with graphical technology, the scaling of axes is often left for students, which has proved to cause them some difficulties (Hennessy, 1999; Mitchelmore & Cavanagh, 2000).

The particular tools addressed in this paper are those associated with the coordinate system. The students were provided with computer instructions on how to scale the axes in GeoGebra. The findings concerning students’ use of these particular tools were to some degree surprising. During their work with the tasks, the participating students mainly encountered two situations which required rescaling of the axes: (a) to see an object in the coordinate system and (b) to obtain an appropriate visible appearance of the object(s). This resulted in the identification of the following key elements of instrumented action scheme:

1. Knowing how to change the scale of the axes by adjusting one axis at a time.
2. Realizing when it is necessary to adjust the $y$-axis to see an object, such as points or a graph.
3. Realizing when it is appropriate to change the scale of one axis at a time to obtain a better visible picture of the objects.

While the first item primarily requires technical capabilities the other two items primarily demand conceptual knowledge. In this study, the mathematical knowledge required to be able to rescale the axes in an appropriate way is about the range and domain of functions that represent real world situations.

Concerning the instrumented techniques of scaling the axes, the findings indicate that students are disposed to employ a zoom technique, although instructions were provided on the worksheet about how to adjust one axis at a time. This often resulted in an inappropriate visual appearance of the graph, which caused difficulties when the
students were to work further with the graph. For example, it turned out to be hard or even impossible for some students to solve problems graphically by attaching a point to the graph and using it to read off specific values of $x$ or $y$.

One reason for their preference for using the zoom technique might be that students are already familiar with it from the use of other screen-based technologies such as smartphones. In contrast, they did not adopt the technique of adjusting one axis at a time so easily. The reason for this might be lack of prior experiences of this kind of technique. Another reason might be the features of this kind of tool in the particular software under consideration. While the zooming tool, and the associated zoom operation technique, is readily available, the technique of scaling one axis at a time is more demanding, probably because the associated usage schemes involve knowledge of tools with limited accessibility in GeoGebra.

In comparison, we (the researchers and the teachers) who are accustomed to using a graphical calculator and familiar in particular with the issue of scaling the viewing window appreciate the ease of scaling one axis at a time in a dynamic software environment. The participating students, on the other hand, who had no previous experiences with graphical calculators, seemed not to appreciate this affordance. Thus, there is a risk that teachers might tend to overlook difficulties that students might encounter during their instrumental process.

However, the findings also show examples where technical obstacles were turned into so-called epistemological obstacles. For example, there were occasions where students could not see any points in the coordinate system even though they could see them in the algebra view. This kind of obstacle turned out to provide opportunities for valuable reflections among the students.

Altogether, the findings suggested ways in which the orchestration might be redesigned in order to better support students’ development of the desired instrumentation schemes. In particular, they highlight the potential importance of taking account of students’ previous expe-
riences both regarding the use of technology in mathematics and the use of every-day technology.

7.2.2 Instrumental genesis and orchestration in Paper 4

This paper examines the process of instrumental genesis amongst students concerning specific dynamic software tools to manipulate function variables and parameters. Research points out the inaccessibility for students of controlling the function variable \( x \) in graphical software environments (see Section 5.4.2). In this paper the notion of movable point is used when referring to a draggable coordinate-pair, as a tool for manipulating coordinates. To manipulate parameters, the well-known dynamic tool denoted as slider bar is used. This tool provides students with the opportunity to examine the visual effect on a graph of changing the value of a parameter (see Section 5.4.3).

This paper aims to analyse and reflect on student responses to three interrelated task sequences, designed to support a planned instrumental genesis. The literature served as guidance in the a priori identification of key elements of instrumented action schemes that I seek to develop. Table 3 provides an overview of them. In this paper, the conceptual and the technical elements of the key elements are separated out. In this way the conceptual aspects and their technical counterparts are clarified.

To provide suggestions for redesign to enhance the scope of students’ acquisition of the instrumented action schemes, this paper identifies some further key elements, marked with (*) in Table 3. For example, the findings concerning the use of a movable point in manipulating variable coordinates revealed a further conceptual aspect: Realize that a graph consists of (infinitely) many points.
Table 3
An overview of the key elements of instrumented action schemes

<table>
<thead>
<tr>
<th>Key elements – conceptual aspects</th>
<th>Key elements – technical aspects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand ordered pairs of numbers as represented by points</td>
<td>Enter a point into the (a) Input bar and (b) coordinate system</td>
</tr>
<tr>
<td>Interpret a variable coordinate as a general number in a functional relationship</td>
<td>Create a movable point on a graph</td>
</tr>
<tr>
<td>Realize that a graph consists of (infinitely) many points (*)</td>
<td>Solve equations graphically by using a movable point</td>
</tr>
<tr>
<td>Interpret a variable coordinate as an unknown in a functional relationship, for example by constructing and solving equations</td>
<td></td>
</tr>
<tr>
<td>Understand that, although open to variation, a function parameter can act as fixed</td>
<td>Create a slider bar</td>
</tr>
<tr>
<td>Interpret a parameter as an unknown to be found to solve a linear inequality</td>
<td>Use a slider bar to set different values of a parameter</td>
</tr>
<tr>
<td>Interpret a parameter as an unknown to be found to solve a problem given some specific conditions</td>
<td>Use a slider bar to solve an linear inequality by moving the slider until the conditions are fulfilled</td>
</tr>
<tr>
<td>Realize that different values of a functional parameter affect the location and/or shape of a corresponding graph</td>
<td>Use the slider bar to investigate parameters as changing quantities</td>
</tr>
<tr>
<td>Be able to define a suitable domain for a parameter, appropriate to a particular situation (*)</td>
<td>Be able to adjust the settings of a slider bar (*)</td>
</tr>
<tr>
<td>Realize that a slider bar corresponds to a parameter in a function formula (*)</td>
<td>Know how to place the formula expression close to the slider bar icon in the Graphic view (*)</td>
</tr>
</tbody>
</table>

Furthermore, there were some new key elements identified related to the use of slider bars to control parameters. Concerning the technical aspects of manipulation of parameters as unknowns, the findings indicate a need for instrumental knowledge about how to adjust the setting of a slider bar. The initial task design offered all necessary technical instructions over the course of the task sequences. However, the findings show occasions where it might be appropriate to address this as something for students to find out by themselves. That is, there is a need for instructions on how to adjust the setting of a slider.
Thus, I suggest the two interrelated key elements *Be able to define a suitable domain for a parameter, appropriate to a particular situation* and *Be able to adjust the settings of a slider bar to consider* in a revised version of the task design.

Finally, the findings show how students encountered difficulties in making connections between the slider bar and the corresponding parameter in a function expression. This confirms the concern raised by Zbiek et al. (2007) that students might regard the slider bar as a third representation of a function, besides the graph and the algebraic expression. The problem is that the parameter could not be manipulated directly. Instead, the slider bar provides a third representation not directly connected to neither the graph nor the formula. I found this problem as rather demanding to solve by revising the task design. I regard this issue as an important one worth paying attention to for software designers, as well as for researchers and teachers.

In the same way as Paper 1 provided guidance to Paper 2, Paper 3 served as guidance in Paper 4. The following section describes how Paper 3 relates to Paper 4.

### 7.2.3 The relation between Paper 3 and Paper 4

While Paper 3 only focuses on some tasks in the first and second task sequences, Paper 4 concerns tasks in all the task sequences. This allows a logical flow with aspect to the instrumental genesis process over the course of the three task sequences. Furthermore, the design heuristic of emergent modelling (see Section 3.2.1) is used in that students’ attention is intended to be directed from a context-specific situation in the first task sequence toward a context-free, and thus general, situation in the last task sequence.

In some way, Paper 3 worked as a methodological pre-study to Paper 4. The instrumental approach and the terminology suggested by Drijvers and Gravemeijer (2005) were tested on a limited part of the empirical data, that is, the data concerning scaling issues. Paper 4 is more sophisticated in that it clearly distinguishes between technical and conceptual character of the key elements. This distinction made it
possible to identify the reason behind different observed student obstacles. For example, the findings show how the computer feedback draws the students’ attention to a misunderstanding likely otherwise to have been overlooked. This might help teachers to become aware of students’ lack of conceptual knowledge.

Concerning the list of key elements identified in Table 3, the results from Paper 3, concerning the scaling of axes provides a further element, which needs to be taken into account in the instrumental orchestration of students’ process of instrumental genesis in DMS environments. The conceptual counterpart to this key element is that students need to be able to realize when an unequal change of the scales of the axes is necessary to obtain an appropriate visual appearance of the object.

When reflecting on the work reported so far in this chapter, there is a progression between on one hand Paper 1 and 2 and on the other hand Paper 3 and 4. Moreover, the work in Paper 1 and 2 led into looking more closely at the instrumental genesis and the trajectory of development of thinking in Paper 3 and 4. Paper 5 and 6, then, are different from the first four papers in that we as researchers took a step back. While the first papers have been focusing on the design work which, although we consulted the teachers, was done by us as researchers, the follow-up lessons were designed by the teachers. The next section reports findings from the observation of these follow-up lessons.

7.3 Follow-up lessons

While the previous papers focus on the part of the study referred to as the design experiment, that is, issues concerning the opening researcher-designed lesson, Paper 5 and 6 concern the follow-up lessons. Although the follow-up lessons were part of the study, it was the teachers that had the whole responsibility for these lessons. Our role as researchers was only observational throughout these lessons. However, at the joint meetings, there were some discussions about the planning of these lessons and there were instances where the teachers trialled some ideas emerging throughout the collective discussions. The overarching question for these papers is: What do follow-up les-
sons look like and how could they be improved? First, this section describes the findings concerning what the follow-up lessons look like, reported in Paper 5. Then, knowing more about those lessons, Paper 6 elaborates on how to improve them.

7.3.1 What do follow-up lessons look like?

The literature agrees on the importance of following up students’ computer work in whole-class discussions (Bartolini Bussi & Mariotti, 2008; Drijvers & Gravemeijer, 2005; Kieran & Saldanha, 2008). In this respect, the study would have been functionally incomplete if we had just focused our attention towards the opening researcher-designed lessons. To really understand how the designed task sequences would work out in the real world of classroom, we needed to understand something about the follow-up lessons. Consequently, the aim of Paper 5 is to investigate how the teachers in the main study plan and conduct such lessons.

As a theoretical framework, primarily the theory of instrumental orchestration (see Section 4.3) is used. The whole-class orchestration types identified by Drijvers et al. (2010) provided useful guidance in the examination of the teachers’ orchestration of the follow-up lessons. Particularly useful was the distinction between on the one hand the teacher-centred types of orchestration and on the other hand the student-centred types of orchestration. In the teacher-centred ways of orchestrating whole-class discussions, the teacher dominates the communication, which primarily follows a traditional IRE or IRF pattern (Drijvers et al., 2010) (see section 5.2). In contrast to these types of orchestration, the student-centred types of orchestration give students “the opportunity to react and have more input. Even though the teacher manages the orchestration, there is more interaction and students have more voice” (Drijvers et al., 2010, p. 220).

Besides the theoretical constructs of didactical configuration and exploitation modes, teachers’ didactical intention behind their orchestration was of interest. Thus, the research question addressed in Paper 5 is: What were the didactical intentions in follow-up lessons and what instrumental orchestrations were associated with them?
The result section of Paper 5 displays the findings from the classroom observations for each of the teachers in terms of a table comprising the teacher’s didactical intention and related instrumental orchestration (i.e. a combination of didactical configuration and exploitation mode). These tables are then, for each teacher, supplemented with information about the rationale behind the teacher’s different choices.

**A summary of the findings in Paper 5**

In accordance with the findings reported by Kieran et al. (2012), there were differences between individual teacher’s didactical intentions for the follow-up lessons even if they have adapted the same researcher-designed tasks. Mainly three lines of didactical intentions for the follow-up lessons were discerned among the participating teachers:

1. Utilize the tasks from the task sequences in the previous lesson, and occasionally take account of the strategies developed by students.
2. Make use of the dynamic software and the manner and focus of such use.
3. Encourage students to engage in classroom discussions.

While the first and the second item above were primarily observed in the follow-up lessons for the first two task sequences, the third item was only observed in teachers’ follow-up lesson for the third task sequence. One reason for this might be that the third task sequence included some new types of task, denoted as “description tasks” (see Section 6.1.2), in which students were asked to describe and explain in writing mathematical phenomena observed on the screen. These tasks provided a diversity of student responses in terms of different ways of expressing themselves mathematically, which teachers could utilize as a basis for the whole-class discussions. Another reason might be that the teachers had become more confident with these types of lesson and therefore dared to try out some new teaching strategies in which students are more involved.
Mostly the kinds of teacher orchestration observed were associated with teacher-centered types of orchestration where an IRE or IRF structure dominates. However, there were some episodes observed during the last follow-up lessons where teachers strove toward a more student-centered type of orchestration. During these episodes, students were encouraged to compare and discuss their answers in small groups or some students were asked to present their written responses so that they could be used as a base for a whole-class discussion. The findings from this study encouraged us as researchers to think about how to support teachers in their planning and implementation of productive follow-up lessons. This issue is addressed in the next section.

7.3.2 How could follow-up lessons be improved?

When taking a broader perspective on how student work on the task sequence during the opening researcher-designed lesson could be effectively taken up in the follow-up lesson, literature concerning whole-class discussions (with and without computers) became important. Consequently, Paper 6 relates to the literature introduced in Section 5.2.

Our study confirms the identified challenge for teachers to engage students in whole-class discussions taking their strategies into account (Franke et al., 2007; Stein et al., 2008). The aim of Paper 6 is to examine whether the model suggested by Stein et al. (2008) could be useful as guidance for teachers in orchestrating student-centered whole-class discussions based on students’ previous computer-based work. This aim was addressed by examining two types of task that were typical in the intervention: description/explanation tasks and prediction tasks (see Section 6.1.2).

For each type of task, one example is chosen as a base for examination of the five practices in the Stein et al. (2008) model. Then, for each example, empirical data, in the form of copies of students’ written responses from one of the four classes, is analysed. The audio recordings collected at the joint meetings with teachers provided some information related to the first of the five practices: “anticipating likely
student responses to cognitively demanding mathematical tasks” (Stein et al., 2008, p. 321). While examining each of the two examples in the light of the five practices, the identified categories of various student responses are discussed.

Reflecting on the finding (in Paper 5) that the opportunity for teachers to base the whole-class discussion on student responses is reduced if not all students have attended the earlier lesson, it would be worth considering whether to design a teaching unit that could be fitted into one single lesson. Indeed that is why, in Paper 6, we made the assumption that the practices 2 to 5 should be performed during one single lesson, as suggested by Stein et al. (2008). This might also save time, which is identified as an important influencing factor (Abboud-Blanchard, 2014; Assude, 2005; Ruthven, 2009).

The findings in Paper 6 indicate that there is a great potential in the Stein et al. model also in this type of setting. However, we have not trialled out the model by observing classrooms where teachers are following a lesson plan based on the five practices. Hence, more research is needed to examine the model further.
8 Discussion

Central to the development work reported in this thesis is the designing of tasks appropriate for DMS environments. The outcomes are not only the designed task sequences in the format of worksheets. There are also the local instruction theories developed around them. As a piece of applied educational research, the aim of this thesis is to supply knowledge about the practice of teaching mathematics with DMS environments while contributing to the development of theories that could be of use to other researchers in the same field and potentially to practitioners. This chapter starts by discussing the academic and professional contributions of the thesis. Then implications for further research are discussed.

8.1 Academic contributions

The primary theoretical contribution of the thesis is that it provides illustrative instances of the task design process and the reflexive use of the two design tools involved.

8.1.1 The process of design

By focusing on a stage that other studies discuss little, that is, the initial iteration of the design research (Bell, 2004; Cobb & Gravemeijer, 2008), this study provides insight into the detailed practice of such research. This means that the detailed description of the design process in terms of the development of local instruction theories within it can give other researchers insight into this kind of research. Although the study focuses on a particular mathematical domain, it might be used as an illustration of general features of design-based methodology. According to Kelly (2004), findings relating to domain-specific theories are of particular value for researchers who work in similar contexts because they “grow authentically and are grounded in the lived experience of teachers and students” (p. 122). Thus, even if the findings in the studies reported in the thesis only could be generalized to a limited extent, it might prospectively contribute to the development of more general theories of the process of design.
8.1.2 Design tools
The thesis uses two design tools that provided useful guidance in the design process: didactical variables and key elements of instrumented action schemes. A design tool is characterized as theoretical concepts used “to identify and address some specific aspect of task design to support both the initial formulation of a design and its subsequent refinement in the light of implementation” (Ruthven, 2015, p. 319). In a source paper, Ruthven et al. (2009) introduce and elaborate on examples of design tools, among which didactical variables (Brousseau, 1997) is suggested as one (see Section 3.2.4). Although the key elements have not been denoted as a design tool, I suggest that it could be designated as one because the way I used the tools in Paper 3 and 4 resembles how we used didactical variables in Paper 2. However, there is a distinct difference between the two tools. While didactical variables are suggested being used as a design tool independently of the theoretical framework in use (Ruthven et al., 2009), the key elements of this design tool are tightly connected to the instrumentation theory framework. Below, the findings concerning the two design tools are discussed.

Didactical variables
Paper 2 pinpoints and elaborates on didactical variables, which proved necessary to consider in designing prediction tasks to foster student mathematical reasoning. Didactical variables were used to articulate the theoretical rationale for different design choices that might affect students’ reasoning, in this case, concerning exponential functions in a DMS environment. Altogether, the conjectured local instruction theory developed in Paper 2 comprises not only the designed task sequence, and the tools that will be used, but also considerations on didactical variables and conjectures about student reasoning that might evolve when they engage with the tasks. Besides being useful in the design of the initial version of the task sequence, didactical variables provided useful guidance through the processes of analysis and refinement of the task sequence.

Several of the identified variables are domain-specific, for example they relate to the particular topic of functions and graphs. However,
some of the identified didactical variables, we argue, are more generic and could be used beyond the domain of functions and graphs. In this way, we suggest, many ideas that we have been able to show in the local instruction context of the study are of potential value and interest to other researchers working in other local contexts.

**Key elements of instrumented action scheme**

In reflection, the way that I used the idea of key elements of action schemes, adopted by Drijvers and Gravemeijer (2005) resembles how we used didactical variables. Based on the literature, these tools were used to identify specific conditions for the design of an initial version of the task design *a priori*. Then they served as guidance in the analysis process. Further, the empirical findings supplied some *a posteriori* identified didactical variables and key elements, which were used in the suggestions for redesign.

In conclusion, while the study presented in this thesis (Papers 3 and 4) has followed a similar approach to that pioneered in the study of the use of CAS to tackle algebraic equations reported by Drijvers and Gravemeijer (2005) it has introduced a variant approach. By giving attention to redesign of the task sequences, it has been able to provide not just an inventory of the conceptual and technical elements of instrumented action schemes but to bring related issues of instrumental orchestration to the fore and highlight the role of task sequences in this. This reinforces the suggestion by Drijvers and Gravemeijer (2005) that one advantage of reducing instrumented action schemes to a list of items is that it provides concise and concrete guidelines for designing tasks.

**8.2 Professional contributions**

This section is organized around three central issues of the thesis: task design, instrumental genesis and orchestration, and follow-up lessons.
8.2.1 Task design

Considering the need for new types of task adapted for the teaching and learning within DMS environments (Doorman et al., 2012; Hitt & Kieran, 2009; Laborde, 2001), one of the main contributions of the thesis is the researcher-designed computer-based task sequences. However, researchers agree that there is always a need for teachers to adapt designed tasks to their own practice (Hoyles et al., 2013; Joubert, 2013a). According to Hoyles et al. (2013), a major challenge concerning scaling the utilization of technology into mathematics classroom “is to encourage teachers to move from adoption to adaptation” (p. 1068).

In relation to this, Gravemeijer and Cobb (2006) discuss the value of an “ecological validity” of a design-based research study, which means that “the results should provide a basis for adaptation to other situations” (p. 77). They emphasize the importance of developing a local instruction theory “that can function as frame of reference for teachers who want to adapt the corresponding instructional sequence to their own classrooms, and their personal objectives” (p. 77). I argue that the outcome from the study in terms of an improved local instruction theory, offers a frame of reference to support teachers that may adopt or adapt the task sequence into their own practice (Gravemeijer & Cobb, 2006). Actually, the findings from the main study have already been adapted in an ongoing developmental project with first year engineering students at Karlstad University.

8.2.2 Instrumental genesis and orchestration

Although the findings related to the process of instrumental genesis reported here arise from the use of a particular piece of software on particular task sequences, I argue that the key elements elaborated are generic and, thus, worth considering when designing tasks for other DMS environments which provide similar tools. The findings about the conceptual and technical components of the instrumented action schemes which students need to develop provide a useful mapping of the major lines of instrumental genesis that the use of these tools calls for. The reciprocal relationship between key elements of technical and conceptual character clarifies how seemingly technical
difficulties among students might depend on lack of conceptual knowledge or vice versa. In this way, the findings show the importance for teachers to be aware of this relationship.

Equally, the considerations informing the design and redesign of task sequences point to important dimensions of the instrumental orchestration to support development of the requisite action schemes by students. The findings also highlight the potential importance of taking account of students’ previous experiences both regarding the use of technology in mathematics and the use of every-day technology, such as smartphones. The findings show how students draw on existing instrumentation schemes developed in relation to what they perceive as similar tools when they start to work with a new tool. This implies an instrumental orchestration that takes account both of the continuities and discontinuities between tools.

**8.2.3 Follow-up lessons**

As recognized in the Introduction section, new teaching strategies have to emerge to take advantage of the potential provided by technology in the teaching and learning of mathematics (Drijvers et al., 2010; Hicks, 2010; Joubert, 2013b; Lagrange & Monaghan, 2009; Pérez, 2014). There was full agreement among the researchers and the teachers in the main study that the researcher-developed lesson was to be followed-up in a subsequent whole-class lesson. Further, at the joint meetings, we discussed what such follow-up lessons might include and how to encourage student reasoning.

However, despite these attempts to agree the shape of follow-up lessons across the teaching team, this study confirms the finding reported by Kieran et al. (2012), that even where teachers are participating together in a collaborative project, and are using the same teaching resources to follow a designed task sequence, they can end up displaying quite different teaching strategies. Indeed, this study strengthens that finding because, in contrast to the study by Kieran and colleagues, the teachers involved here had been directly involved in the designed-based research that produced the resources. In line with Kieran et al. this thesis tried to identify what it is that underpins
teachers’ different adaptations, for example in terms of different ways of orchestrating a follow-up lesson.

The findings from the observational study of the teachers’ orchestration of follow-up lessons provided some ideas for how to improve these lessons in terms of making them more student-centred.

8.3 Further research
The methodological choice of using a design-based research approach resulted in our collecting a very comprehensive data set. Consequently, there remain some interesting areas that we have not yet examined. Particularly, we want to investigate the teachers’ role during the researcher-designed lessons where the teachers walked by and monitored students’ work. The literature recognizes various possible roles that emerge among teachers in technology-based mathematics classrooms (Zbiek & Hollebrands, 2008). For example, teachers might act as technical assistant, catalyst and facilitator, explainer, counsellor or evaluator. The data we have available from these lessons would allow us to follow the communication between the teacher and pairs of students.

Because we only have performed the first iteration of a design research study, it will be interesting to continue with a second iteration to investigate how our suggestions for improvement of tasks will work. Furthermore, it would be of interest to incorporate the follow-up lessons in the design research study, treating them as a more integral part of the intervention. The findings in Paper 5 and 6 suggest that it might be instructive to conduct a ‘teaching unit’ in one single lesson following the Stein et al. (2008) model. Accordingly, a suggestion for further research is to trial out the suggested versions of the task sequences but restrict the scope of them to fit into a lesson plan based on the five practices of the Stein et al. (2008) model.

Taking a broader perspective and going beyond the particular levels of education at which each of the two studies reported in this thesis was conducted, it would be interesting to examine the scope for the task design models which we developed to be used at other levels. For in-
stance, it would be interesting to perform a research study, using findings from the main study, with the focus on students’ work with computer-based task sequences at university level.

Finally, if one wants a widespread use of technologies, such as dynamic software environments in mathematics classroom, Fishman et al. (2004) suggest an expanded design-research agenda that goes beyond the level of classroom unit. They argue for the importance of acknowledging the larger school context by focusing on schools and school systems as the primary research unit. According to Fishman et al. (2004), by involving a school system in the design of a technology innovation, “their use will also spread to other teachers within or across schools” (p. 48). Therefore, I argue, it would be interesting to perform a research study involving not only teachers and their classes but school staff who are responsible for issues such as organizational structures and technology planning.
References


Appendix A: Timeline for the main project

Preparing for the study
- Student questionnaire
- Teacher interviews

Teaching unit 1
- Computer-based lesson 1
- Follow-up lesson 1
- Individual teacher meeting

Teaching unit 2
- Computer-based lesson 2
- Follow-up lesson 2
- Individual teacher meeting

Teaching unit 3
- Computer-based lesson 3
- Follow-up lesson 3
- Individual teacher meeting

Joint meeting 1
Planning Teaching Unit 1

Joint meeting 2
Planning Teaching Unit 2

Joint meeting 2
Planning Teaching Unit 3

Joint meeting 4
Reflection on the entire project
How does a sunflower grow?

The height of a sunflower is 50 cm when it is measured for the first time (June 1). After that the sunflower grows so that it becomes 30 % higher each week.

I. Calculate the height of the sunflower one week after the first measurement. Do the calculation in the box below.

The height of the sunflower is depending on the time. If we let \( x \) be the number of weeks that have passed since the first measurement and \( y \) be the height of the sunflower (in cm) it is possible to use a graph to study how the sunflower is growing. We will now use GeoGebra to do this.

1. Choose "Options" then "Labeling" and "No New Objects".
2. Right-click in the Graphics View and mark Grid.
3. Move the coordinate system so that the origin is located in the bottom left corner. This can be done with: ![Click somewhere in the graphics view and drag.]

When the first measurement is performed \( x = 0 \) (since 0 weeks have passed) and \( y = 50 \). The corresponding point in a coordinate system is \((0,50)\).

4. Insert this point by entering \((0,50)\) into the "Input Bar": ![](input.png)

**NOTE!** To be able to see the point you must adjust the scale on the \(y\)-axis. This can be done by "dragging" the \(y\)-axis. (first mark ![Drag the y-axis](drag.png))

5. Insert the point that shows the height of the sunflower after one week (due to your calculation in task 1) by entering its coordinates into the "Input Bar".

If appropriate, adjust the scale on the \(x\)-axis!
Appendix B: Task sequence 1 – the initial version

2. Guess (without any calculations) where the point showing the height of the sunflower after two weeks will be.

- Insert this point by using the tool:

  Change the color of the point! (Right-click on the point, choose “Object Properties” and then “Color”).

a) Explain why you chose to place the point at this specific position.

b) Note the point’s coordinates (in the Algebra View): ________________

c) Calculate the height of the sunflower after two weeks (round the answer to the nearest cm).

- Insert this point by entering its coordinates into the “Input Bar”

d) Compare the two points (the one you guessed and the one you calculated). If the points do not coincide, try to explain why.
Appendix B: Task sequence 1 – the initial version

3. What is the height of the sunflower after 3 weeks? After 4 weeks? After 5 weeks? Continue to fill in the table below (round the answers to the nearest centimeters):

<table>
<thead>
<tr>
<th>Number of weeks</th>
<th>The height of the sunflower (in cm)</th>
<th>The corresponding point</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
<td>(0,50)</td>
</tr>
<tr>
<td>1</td>
<td>50 \cdot 1.3 = 65</td>
<td>(1,65)</td>
</tr>
<tr>
<td>2</td>
<td>50 \cdot 1.3^2 = 84.5 \approx 85</td>
<td>(2,85)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Insert the rest of the points (from the table) in the coordinate system by entering its coordinates into the "Input Bar".

4. Guess (without any calculations) where the point showing the height of the sunflower after six weeks will be.

- Insert this point by using the tool: ![Change the color of the point!]

a) Note the point’s coordinates (in the Algebra View): ________________________

b) Calculate the height of the sunflower after six weeks: ________________________

- Insert this point by entering its coordinates into the "Input Bar".

c) Compare the two points (the one you guessed and the one you calculated).

Comments?
Appendix B: Task sequence 1 – the initial version

5. The height of the sunflower \( y \) (in cm) after \( x \) weeks follows a certain pattern that can be described by a formula. Study your calculations in the table (Task 3) and try to figure out what this formula could be like.

When you think you have found the formula for the sunflower: Try it by inserting your suggestion into the “Input Bar”.

a) The formula for the sunflower: ____________________________

b) Explain how you can know that you have found the right formula.

6. Use the graph to estimate the following:

a) How high is the sunflower after 3.5 weeks?

______________________________

b) When is the height of the sunflower 160 cm?

_____________________________

7. For how long time do you think that the sunflower can continue to grow 30\% in height each week? Motivate!

8. Another sunflower is 70 cm high (June 1) and grows so that it becomes 20 \% higher each week.

a) Find the formula that shows how this sunflower is growing. ____________
Appendix B: Task sequence 1 – the initial version

Enter the new formula in the same GeoGebra window as the old one. Change the color of the new graph!

b) The two graphs intersect in one point. Explain in your own words the meaning of this point.

9. Below you can see the graph for another sunflower (measured for the first time on June 1).

a) What can be told from the graph about the height of the sunflower? Describe with your own words.

b) Try to find the formula for this sunflower. Use GeoGebra to check your answer.

c) Explain how you found the formula.
Appendix C: Task sequence 2 – the initial version

Candy for Sale

To get money for a class trip, the Class 9b at Sugar School decided to rent a table and sell candy at the market place. It cost 100 SEK to rent a table. They purchase candy for 40 SEK per kg. Thus, the total cost is made up of a fixed cost of 100 SEK and a variable cost of 40 SEK per kg.

1. a) What will be the total cost if the class buys 10 kg candy? Do the calculation in the box below.

Let \( y \) be the total cost in SEK and let \( x \) be the weight in kg of the candy purchased.

b) Determine a formula that describes how the total cost depends on the weight in kg of the candy purchased.

This relation can also be displayed graphically. We will now use GeoGebra to do this.

- Right-click in the Graphics View and mark Grid.
- Move the coordinate system so that the origin is located in the bottom left corner.
- Insert the formula you determined in Task 1b) into the "Input Bar".

NOTE! To be able to see the graph (the line) there might be a need to adjust the axis.

- Change the color of the graph!
- Insert a point somewhere on the graph.
- Move the point along the graph and observe how the coordinates change in the Algebra View.
Appendix C: Task sequence 2 – the initial version

2. a) Describe how you can use the point to check your calculation in Task 1a).

b) Determine, by moving the point, how much candy the class can buy if they only have 700 SEK to invest.

c) Determine, by creating and solving an equation, how much candy the class can buy if they only have 700 SEK to invest. Then, compare with your answer in Task 2b).

The class decides to sell the candy for 50 SEK per kg. The revenue \( y \) depends on how much candy \( x \) in kg they sell.

3. Determine a formula that shows how the revenue depends on how much candy they sell.

\[ y = 50x \]

Insert this formula (in the same window), by entering it into the "Input Bar". Change the color of the line!

4. a) Use GeoGebra to graphically determine how much candy they need to buy and then sell to make a profit. Describe how this can be observed.
Appendix C: Task sequence 2 – the initial version

b) How would you formulate the result above (Task 4a) "mathematically"? Choose the right box below.

- $x < 10$
- $x = 10$
- $x > 10$

c) To make a profit out of the candy sale, revenues must exceed costs. This implies the following inequality: $50x > 100 + 40x$. Solve this inequality in the box below and compare with your answer above (Task 4b).

Use GeoGebra to solve the inequality by entering: $50x > 100 + 40x$

To remove the inequality, right-click on it (in the Algebra View) and chose "Delete".

The class is considering selling the candy at a higher price to make a profit without having to buy and sell as much candy.

Now we will use the “Slider tool” in GeoGebra to be able to change the price in an easy way.

Choose the “Slider tool” and click somewhere in the Graphics View. The following box will appear:

Change the default name to $p$ (the price per kg). Change “Min”, “Max”, and “Increment” as follows:

---

3
Appendix C: Task sequence 2 – the initial version

1. Insert the following formula: \( y = px \). Change the color of the line!

2. Investigate how the line changes when you drag the slider. Also, notice the changes of the formula!

3. Remove the graph showing the revenues when the price is 50 SEK per kg \( (y = 50x) \). Right-click on it and chose "Delete".

4. Set the slider so that the selling price is 60 SEK per kg.

5. a) How much candy do they have to buy and then sell to make a profit when the selling price is 60 SEK per kg? Use GeoGebra!

______________________________________________________________________

b) Calculate (without using GeoGebra) how much profit they make if they buy and sell 10 kg candy when the selling price is 60 SEK per kg?

______________________________________________________________________

c) Describe how you can solve the problem above (Task 5b) graphically using GeoGebra.

______________________________________________________________________

6. What price per kilo, do they need to have to make a profit if they buy and sell more than 4 kg? Use GeoGebra to solve the problem!
Exponential functions

We are now going to study different kinds of exponential functions. First, we will repeat one example of exponential growth that we have studied earlier: “How does a sunflower grow?” This example was about a sunflower that was 50 cm when it was measured for the first time (June 1) and grows so that it becomes 30% longer each week.

This means that the length (in cm) of the sunflower is:
- after 1 week: \( 50 \cdot 1.3 \) (the growth factor is 1.3)
- after 2 weeks: \( 50 \cdot 1.3^2 \)
- after \( x \) weeks: \( 50 \cdot 1.3^x \)

The formula describing the length of the sunflower (\( y \) cm) as a function of time (\( x \) weeks) is therefore: \( y = 50 \cdot 1.3^x \) (or \( f(x) = 50 \cdot 1.3^x \)).

Another sunflower with the starting length 70 cm (June 1) grows so that it becomes 20% longer each week. The formula describing the growth of this sunflower is \( y = 70 \cdot 1.2^x \) (or \( f(x) = 70 \cdot 1.2^x \)).

Now we will use the “Slider tool” in GeoGebra to make it easy to change the starting length (June 1) and the growth factor.

1. Right-click in the Graphics View and mark Grid.
2. Move the coordinate system so that the origin is located in the bottom left corner.
3. Choose the “Slider tool” and click somewhere in the Graphics View. The following box will appear:

   ![Slider tool](image)

4. Change “Min”, “Max”, and “Increment” as follows:

   ![Slider settings](image)

5. Create one more slider, BUT change the name to \( C \), and change “Min”, “Max”, and “Increment” as follows:

   ![Slider settings](image)
Appendix D: Task sequence 3 – the initial version

1. a) Which values should be used on the sliders C and a to get the graph of the sunflower that is 50 cm June 1 and that is growing in length with 30 % each week?

\[ f(x) = C \cdot a^x \]

\[ C = \quad a = \quad \text{And the formula is:} \]

\[ \text{____________________} \]

b) What is the y-value of the point? _______________________________

c) What does your answer in 1b) really say about the sunflower. Describe in your own words.

d) Use your formula from task 1a) to check your answer in 1b).

2. Determine the length of the sunflower 3 weeks after June 1 in two different ways

a) Use point A: ______________________________

b) Calculate:

Set the sliders C and a to get the graph of a sunflower that is 30 cm June 1 and that is growing in length with 40 % each week.
Appendix D: Task sequence 3 – the initial version

Now we will leave the example with the sunflowers and instead we will study the general exponential function, $f(x) = C \cdot a^x$, and how its graph is depending on the values of $C$ and $a$.

1. Set the slider $a$ at 2.

3. Drag the slider $C$ so that the value of $C$ varies. Describe in your own words how to see the value of $C$ in the graph.

2. Set the slider $C$ at 50.

4. Drag the slider $a$ so that the value of $a$ varies. Describe in your own words how the value of $a$ influences the graph.

3. Set the slider $a$ at 1 and drag the slider $C$ so that the value of $C$ varies.

5. Describe what the graph looks like and try to explain why it looks like this. (For example, what would it mean if $a=1$ in the sunflower example?)
6. a) Guess (without using GeoGebra) what the graph of the function \( f(x) = 80 \cdot 0,5^x \) will look like. Make a sketch in the coordinate system below.

b) Explain the thoughts behind your guess.

Right-click on the slider \( a \). First choose "Object Properties" then "Slider" and change the "Min" value to zero:

Set the sliders so that \( a = 0.5 \) and \( C = 80 \).

c) Compare your guess with the graph obtained in GeoGebra. Explain any differences that may occur!
Appendix D: Task sequence 3 – the initial version

d) Give a suggestion of an example from everyday life that could be described by this graph.

7. The value of a car drops from 100 000 SEK to 50 000 SEK in two years. What is the annual decrease in percentage if the value of the car is decreasing exponentially?

Use GeoGebra to solve this problem (by finding appropriate values of the sliders).

Tip! The increment of the slider $a$ could be changed to 0.01 to get a more accurate value.

a) The value of the car decreases by __________% each year.

b) Describe how you came up with the answer above.

c) Do an appropriate calculation to check your result above.
Designing for the integration of dynamic software environments in the teaching of mathematics

This thesis concerns the challenge of integrating dynamic software environments into the teaching of mathematics. It investigates particular aspects of the design of tasks which employ this type of computer-based system, with a focus on improvement, both of the tasks themselves and of the design process through which they are developed and refined.

The thesis reports two research projects: a small initial one preceding a larger main project. The initial case study, involving two graduate students in mathematics, develops a task design model for geometrical locus problems. The main study constitutes the first iteration of a design-based study, conducted in collaboration with four upper-secondary school teachers and their classes. It seeks to identify task design characteristics that foster students’ mathematical reasoning and proficient use of software tools, and examines teachers’ organisation of ‘follow-up’ lessons.

The findings concern three particular aspects: features of tasks and task environment relevant to developing a specific plan of action for a lesson; orchestration of a particular task environment to support the instrumental genesis of specific dynamic software tools; how to follow up students’ work on computer-based tasks in a whole-class discussion.