Nikolai Kotik

Solution to boundary-contact problems of elasticity in mathematical models of the printing-plate contact system for flexographic printing
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**DISSERTATION**

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Abstract

Boundary-contact problems (BCPs) are studied within the frames of classical mathematical theory of elasticity and plasticity elaborated by Landau, Kupradze, Timoshenko, Goodier, Fichera and many others on the basis of analysis of two- and three-dimensional boundary value problems (BVPs) for linear partial differential equations. A great attention is traditionally paid both to theoretical investigations using variational methods and boundary singular integral equations (Muskhelishvili) and construction of solutions in the form that admit efficient numerical evaluation (Kupradze). A special family of BCPs considered by Shtaerman, Vorovich, Alblas, Nowell, and others arises within the frames of the models of squeezing thin multilayer elastic sheets. We show that mathematical models based on the analysis of BCPs can be also applied to modeling of the cliché-surface printing contacts and paper surface compressibility in flexographic printing.

The main result of this work is formulation and complete investigation of BCPs in layered structures, which includes both the theoretical (statement of the problems, solvability and uniqueness) and applied parts (approximate and numerical solutions, codes, simulation). An objective is creating mathematical and computational methods aimed at solution of a specific family of BVPs for elliptic systems in thin layers with the boundary functions defined on a set of “small” domains, which complicates applications of the methods based on discretization. To overcome this difficulty we elaborate analytical techniques, obtaining the solutions in the form of asymptotic series with the help of boundary integral equations, and the method of successive iterations (approximate decomposition) taking into account the structure and dimensions of the domains where BCPs are solved.

We elaborate a mathematical model of squeezing a thin elastic sheet placed on a stiff base without friction by weak loads through several openings on one of its boundary surfaces. We formulate and consider the corresponding BCPs in two- and three-dimensional bands, prove the existence and uniqueness of solutions, and investigate their smoothness including the behavior at infinity and in the vicinity of critical points. The BCP in a two-dimensional band is reduced to a Fredholm integral equation (IE) with a logarithmic singularity of the kernel. The theory of logarithmic IEs developed in the study includes the analysis of solvability and development of solution techniques when the set of integration consists of several intervals. The IE associated with the BCP is solved by three methods based on the use of Fourier-Chebyshev series, matrix-algebraic determination of the entries in the resulting infinite system matrix, and semi-inversion. An asymptotic theory for the BCP is developed and the solutions are obtained as asymptotic series in powers of the characteristic small
We propose and justify a technique for the solution of BCPs and boundary value problems with boundary conditions of mixed type called the approximate decomposition method (ADM). The main idea of ADM is simplifying general BCPs and reducing them to a chain of auxiliary problems for ‘shifted’ Laplacian in long rectangles or parallelepipeds and then to a sequence of iterative problems such that each of them can be solved (explicitly) by the Fourier method. The solution to the initial BCP is then obtained as a limit using a contraction operator, which constitutes in particular an independent proof of the BCP unique solvability.

We elaborate a numerical method and algorithms based on the approximate decomposition and the computer codes and perform comprehensive numerical analysis of the BCPs including the simulation for problems of practical interest. A variety of computational results are presented and discussed which form the basis for further applications for the modeling and simulation of printing-plate contact systems and other structures of flexographic printing. A comparison with finite-element solution is performed.
Introduction

Flexography is an important printing process that is gaining wide acceptance [6]. This is a method of direct rotary printing using resilient raised image printing plates consisting of screen dots. The printing plate, called cliché, is made of a flexible photopolymer. Ink is applied to the raised image on the cliché by a roll or a doctor blade wiped engraved cylinder, and is transferred to the substrate in the printing nip. Paper substrates are usually printed, using water-based inks while plastics and foils are sometimes printed with fast drying solvent-based inks. This makes flexography the predominant method used for printing flexible bags, wrappers, and similar forms of packaging. The soft rubber plates are also well-suited to printing on thick, compressible surfaces such as cardboard packaging.

The heart of the flexographic printing process is its simple inking system (see Fig. 1). The ink-fountain pan supplies ink to a rubber ink-fountain roll, which supplies ink to the ink-metering (anilox) roll and may come equipped with a reverse-angle doctor blade. The anilox roll transfers a precise amount of ink onto the printing plate, which is mounted on the printing cylinder. The printing plate on the printing-plate cylinder and the impression cylinder form a nip where the ink is transferred onto the substrate.

The appearance of the print can be described by its print density, i.e. the logarithm of the reflectance of the unprinted substrate divided by the reflectance of the printed surface. The ink transfer on the substrate depends on the surface
chemistry of the substrate and the ink, the porosity of the substrate, the time available in the printing nip and the applied pressure in the printing nip. Since paper is a non-uniform material, ink transfer can vary, leading to variations in the print density. The result is a non-uniform print with a cloudy appearance, i.e. print mottle. Print mottle is affected by compressibility of the substrate and the compliance of the cliché; a more compressible substrate or a more compliant cliché will give a lower print mottle [14].

Since print mottle is influenced by the compliance of both the substrate and the cliché, it seems reasonable to relate print density variations to contact mechanics conditions in flexographic presses. Hofstrand [13] showed that variations in print density of post-printed corrugated board depend to a great extend on the variations in contact pressure. It must in this case also be mentioned that variations in structure of corrugated board are periodic and well defined while variations of paper and board exhibit a more complicated nature. However, even if paper and board show a more complicated nature, it seems reasonable to assume that variations in contact pressure are of utmost importance for variations in print density of paper and board.

Contact problems were first treated by Hertz [11] who considered homogeneous elastic bodies in contact. However, if the contacting solids have a layered structure with surface irregularity (both the cylinder and the sheet), the Hertzian solution does not work. We are particulary interested in the squeezing of an elastic sheet of layered structure by two cylinders, one of which is rigid and the other has an elastic layered structure. Such problems were investigated in [10], [27], [26] and [9]. Deshpande [7] determined the nip width and the nip pressure when an incompressible elastic layer was squeezed between the cylinders and the layer thickness was large enough to reduce the analysis to the Hertzian contact problem. Hills and Sackfield [12] solved the problem of sliding contact between dissimilar elastic cylinders. Alexandrov and Vorovich [2] as well as Alblas and Kuipers [18] obtained asymptotical solutions for a thin layer on a rigid base for a plane and cylindrical punch respectively. Nowell and Hills [29] studied numerically the elastic contact of a substrate between two symmetric cylinders. One can also mention the work of Komvopoulos et al. [16], who analyzed the normal and shear plane strain at the contact between a layered strip and a rigid cylinder using finite elements. Recently the contact problems in question have also been studied by many authors.
INTRODUCTION

The study of the contact between the screen dots on the plate cylinder and a thin layer of a compressible material, such as board or plastic in the flexographic printing is a field where the mathematical modelling can successfully be applied. Mathematically, these contacts can be described by a specific class of boundary-value problems (BVPs), the so called boundary-contact problems (BCPs).

The stress state of an elastic body under compression may be described by Lamé equations and corresponding boundary conditions of mixed type. Such non-classical BCPs formulated and investigated in [41] may serve as useful model settings to form the cliché-contact system in flexographic printing. However, the solution techniques and results, presented in [41] and by other researchers, which are related to the analysis of BCPs are mostly of theoretical importance. In fact, it is hardly possible to apply these results in order to efficiently compute all stress-strain components and forces and to analyze and visualize complicated processes that take place in compressed elastic sheets.

With the advent of modern computers in the 1950s, it became possible to analyze more complex structural systems and design routine structures more efficiently. The finite element method is a good example of an analysis methodology that tends naturally to an implementation as a computer software system. The finite element method has commonly been used to solve a wide range of contact problems. The advantage of this method is its universal character and availability of commercial solvers which facilitates the modelling. However, as for all the numerical methods, the application of the finite element method has its own restrictions. As a method based on a discretization of the volume it creates difficulties if attempts are made to take into account irregularities with sizes much smaller that these of the whole body (e.g. the holes). Irregularities therefore complicate the mesh generation considerably, leading to a drastic increase of computation time and memory usage. Nevertheless, these very irregularities are of particular interest for contact problems in the flexographic printing as they describe the screen dots and the sheet geometry. Therefore, the elaboration of new alternative methods which capture these aspects and are both fast and efficient is still an urgent task. This work is devoted to the development of such a method.

The BCPs are studied within the frames of the classical mathematical theory of elasticity and plasticity elaborated by Landau [22], Kupradze [19], Timoshenko and Goodier [38], Fichera [8] and several others on the basis of analysis of two- and three-dimensional BVPs for linear partial differential equations. Traditionally, great attention has been paid both to theoretical investigations using variational methods and boundary singular integral equations (see e.g. Muskheishvili, [28]) and the construction of solutions in a form that admits efficient numerical evaluation (see e.g. Kupradze, [20]). A special family of BCPs considered by Vorovich [41], Alblas [18], Nowell [29] and others, emerges within the frames of the modelling of squeezing thin multilayered elastic sheets. In this work we show that mathematical models based on the analysis of BCPs can also be applied to the modelling of the cliché surface printing contacts and paper surface compressibility in flexographic printing. We develop the theory of solvability for a family of BCPs and the method of boundary singular integral equations together with the techniques aimed at the construction of solution using explicit functional methods.

This work focuses on mathematical modelling and computer simulation of geometrical and elastic properties of the sheet at the contact with the screen dots
INTRODUCTION

during the flexographic printing. We state and examine the BCPs describing the contact between the screen dots and the sheet. The present study includes both theoretical (statement of problem, solvability and uniqueness) and applied aspects (approximate and numerical solutions).

Certain assumptions about the system geometry and boundary loads were made. We assume that the radii of the cylinder \( r \approx 10^{-1} \text{m} \) is much greater than geometrical sizes of the screen dots \( \approx 10^{-3} \text{m} \) and surface irregularity of the sheet \( \approx 10^{-4} \text{m} \). Therefore curvature of the cylinder can be neglected and we consider this problem as a planar one in the domain occupied by the cylinder and the sheet. We assume also that the boundary loads are sufficiently weak. This is a characteristic feature of flexographic printing which makes it possible to use the assumption that the substrate and the cliché can be modelled within the frames of classical theory of elasticity (see Chapter 1), leading finally to the formulation of BCPs (see Chapter 2) considered in the linear settings (see Fig. 2).

In Chapter 1, we introduce basic facts and notations of the classical theory of elasticity, classify statements of the problems, write some fundamental equations and relations, and present fundamental solutions.

In Chapter 2, we state the two-dimensional BCP for the contact between screen dots and sheet lying on a stiff base without friction. We introduce the functional spaces we need to prove the existence and uniqueness of the solution to BCP. We give explicit representation of the solution for BCP and describe the behavior of the solution at infinity and at critical points. We state also the three-dimensional BCP and formulate the conditions of existence and uniqueness of the solution to BCP.

In Chapter 3, we reduce the BCP to a Fredholm integral equation with a logarithmic singularity in the kernel. We introduce general properties of the integral operators and operator-valued functions with the logarithmic singularity in the kernel (including the case of several intervals of integration). To solve the integral equation we describe three methods based on the use of the Fourier-Chebyshev series, matrix-algebraic determination of the entries in the resulting infinite-system matrix, and asymptotic semi-inversion. We obtain the solution to BCPs in a band explicitly in the form of asymptotic series.

In Chapter 4, we present the method of approximate decomposition in two- and three-dimensional cases. The main idea of this method is to simplify the general BCP and to reduce it to a chain of auxiliary problems for a ‘shifted’ Laplacian in a long rectangle or a parallelepiped and then to a sequence of iterative problems such that each of them can be solved (explicitly) by the Fourier method. The solution to the initial BCP is then obtained as a limit using a contraction operator and fixed-point iterations.

In the last chapters we present the results of computations and description of the algorithms and software.

The main results of the dissertation are:

I. Formulation of well-posed BCPs (2.5)–(2.7) and (2.23)–(2.25) (that are uniquely solvable and have the solution depending continuously on the boundary data, Theorems 2.1 and 2.2) which can be used in mathematical models of squeezing a thin elastic sheet by weak loads relevant for simulating the contact mechanics of flexographic printing on board.

II. Development of the boundary integral equation method based on semi-
inversion of logarithmic integral operators on several intervals of integration (Theorems 3.12 and 3.13).

III. Explicit solution of BCPs in the form of asymptotic series (Theorem 3.14) using asymptotic solution to boundary integral equation, and justification of the boundary integral equation method (Theorem 3.3).

IV. Development of the approximate decomposition method (ADM) for the solution to BCPs using successive iterations (Theorems 4.1, 4.2, and 4.3) in which intermediate problems are solved explicitly. ADM can be applied for the solution to a family of BVPs for the Laplace, Helmholtz, and Poisson equations (Theorems 5.1–5.7) and used as an alternative proof of the existence and uniqueness of solution to the BCPs.

The structure of the dissertation is shown in Fig. 3.

![Figure 3. Scheme of dissertation.](image-url)
Chapter 1

Introduction to general elasticity

In this chapter we introduce basic facts and notations of the classical theory of elasticity, classify statements of the problems, write some fundamental equations and relations, and give fundamental solutions. At the end we describe some particular statements we are interested in.

Almost all engineering materials, including paper and cardboard, possess the property of elasticity to a certain extent. If the external forces producing deformation do not exceed a certain limit, the deformation disappears with the removal of the forces. Throughout this work it will be assumed that the bodies under consideration are perfectly elastic, i.e. that they resume their initial form completely after removal of the forces.

It will be assumed that the bodies under consideration are homogeneous and continuous over their volume so that the smallest element cut from the body possesses the same specific properties as the body. To simplify the theoretical part of the work it will be assumed that the bodies under consideration are isotropic, i.e., that the elastic properties are the same in all directions.

1.1 Basic facts and notations

1.1.1 Stresses and displacements

If no external force is applied to the body under consideration and the body is not deformed, then all its parts are in mechanical equilibrium. If the body is disturbed by some force, i.e., if it is deformed, then the initial equilibrium of molecules is changed; the part of the body is no longer in mechanical equilibrium.

In the deformed body internal forces are activated and strive to return the body to its original state.

The internal forces are of molecular nature and their radii of action are very small in comparison with the distance considered in the theory of elasticity. It is generally assumed that the internal forces acting on some part of the body from the side of the remainder of the body act only through the boundary of this part.
The forces acting from the environment are called external forces and are divided into mass and surface forces.

If the body under consideration comes into contact with some external medium, then "short-range forces" arise on the surface of the contact. Such forces are called surface forces and are of the same nature as those described above.

Gravity forces, magnetic forces and others cannot be represented by surface forces. Therefore, in the theory of elasticity, mass forces are introduced in addition to surface forces. It is assumed that the action of such forces on an elementary particle of the body is statically equivalent to a force applied to the center of the particle mass. These forces are assumed to be proportional to the masses of the particles on which they act. They are called mass forces.

Consider a particle with the mass $\Delta m$ and let a point $x$ be the center of the particle mass. The result vector of the force acting on $x$ is denoted by $\mathbf{F}(\Delta)$. Assume that there exists the limit

$$\mathbf{F}(x, t) = \lim_{\Delta m \to 0} \frac{\mathbf{F}(\Delta m)}{\Delta m},$$

which depends only on the point $x$ and, in dynamics, on $t$ and is called the mass force.

To describe the internal forces consider a point $x$ and speculatively draw through it a small surface $\Delta S$. The internal forces produced by the interaction of the parts on the opposite sides of $\Delta S$ can be represented as forces applied to the points of $\Delta S$. Draw the normal $\mathbf{n}$ at the point $x$ to the surface $\Delta S$, give it a definite positive direction and consider the force which the part lying on the positive side of the normal $\mathbf{n}$ exerts on the part lying on the opposite side. Denote the force vector by $\mathbf{T}$ and consider the limit

$$\sigma^{(n)}(x, t) = \lim_{\Delta S \to 0} \frac{\mathbf{T}}{\Delta S},$$

which exists and depend on the direction of $\mathbf{n}$, on the point $x$ and, in dynamics, on $t$; this quantity is called the stress. The sign of the limit is changed if the opposite direction is taken as positive.

Let us take a Cartesian coordinate system $X_1X_2X_3$ and denote the stresses $\sigma^{(n)}$, when $\mathbf{n}$ coincides with the $X_i$-axis, by $\sigma^{(i)}$ and the coordinates of these vectors in the system $X_1X_2X_3$ by $\sigma_1, \sigma_2, \sigma_3$. The vector $\mathbf{\sigma}^{(n)}$ is called the force-stress vector. Consider the matrix $[\sigma^{ij}]_{3 \times 3}$, whose components are called the force-stress components. These nine scalars form a second rank tensor which is called the force-stress tensor.

Draw through an arbitrary point $x$ three planes, parallel to the coordinate planes, and consider a small tetrahedron formed by them and by another plane, normal to an arbitrary unit vector whose direction neither coincides with nor is opposite to the coordinate axes. Determine the force-stresses acting on the tetrahedron mass and take into account that the sum of all vectors which represents the resulting vector of the external forces is equal to zero and we obtain the vector relation

$$\mathbf{\sigma}^{(n)}(x, t) - \sum_{i=1}^{3} \sigma^{(i)}(x, t) n_i = 0$$
or in terms of the components

\[
\sigma_j^{(n)}(x, t) = \sum_{i=1}^{3} \sigma_{ij}(x, t)n_i, \quad j = 1, 2, 3, \tag{1.1}
\]

which give the required force-stresses at the point in any direction in terms of
the components of the force-stress tensor at the same point.

Consider some point \( x = (x_1, x_2, x_3) \) of the body at rest. During the
deformation the point \( x \) changes its position, i.e., it is displaced. Its position
at the time \( t \) is denoted by \( x = (x'_1, x'_2, x'_3) \). The difference

\[
u(x, t) = (x'_1 - x_1, x'_2 - x_2, x'_3 - x_3)
\]
is called the value of the displacement vector at the point \( x \) at the time \( t \) and
its components are \( \mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \).

1.1.2 Basic equations in terms of stress components

Detach some part from the body and denote its domain by \( \Omega \) and the boundary
by \( \Gamma \). To write an equilibrium condition for this part sum up the external forces
acting on it and equate the sum to the internal forces with the sign reversed.
The sum of external forces is expressed by the integral

\[
\int_{\Gamma} \sigma^{(n)}(y, t) \, d\Gamma,
\]
where \( n \) is the outward normal to \( \Gamma \) at the point \( y \) and \( d\Gamma \) is an element of
the area of \( \Gamma \). At the time \( t \) each point \( x \) of the domain \( \Omega \) the mass force
\( \mathbf{F}(x, t) \) is acting. The sum of these forces is expressed by the integral

\[
\int_{\Omega} \rho \mathbf{F}(y, t) \, dx,
\]
where \( dx \) is an element of the volume and \( \rho \) is the body density. If \( \mathbf{u}(x, t) \) is
the displacement of the point \( x \), then the sum of the internal forces is given by
integral

\[
-\int_{\Omega} \rho \frac{\partial^2 \mathbf{u}(x, t)}{\partial t^2} \, dx.
\]

It follows from equilibrium condition that

\[
\int_{\Gamma} \sigma^{(n)}(y, t) \, d\Gamma + \int_{\Omega} \rho \mathbf{F}(y, t) \, dx = \int_{\Omega} \rho \frac{\partial^2 \mathbf{u}(x, t)}{\partial t^2} \, dx,
\]
or componentwise

\[
\int_{\Gamma} \sum_{i=1}^{3} \sigma_{ij}(y, t) \, n_i \, d\Gamma + \int_{\Omega} \rho F_j(y, t) \, dx = \int_{\Omega} \rho \frac{\partial^2 u_j(x, t)}{\partial t^2} \, dx, \quad j = 1, 2, 3.
\]
Similar calculations of moments and the equilibrium condition (the vanishing sum of the resultant moment of active forces and the resultant moment of internal forces) give the formula

$$\int_y \sigma^{(n)}(y, t) \, d\Gamma + \int_\Omega \rho x \cdot \mathbf{F}(y, t) \, dx = \int_\Omega \rho x \cdot \frac{\partial^2 u(x, t)}{\partial t^2} \, dx. \quad (1.2)$$

Apply the Gauss-Ostrogradski formula and obtain the equations

$$\sum_{i=1}^3 \sigma_{ij}(x, t) \frac{\partial}{\partial x_i} n_i + \rho \mathbf{F}_j(x, t) = \rho \frac{\partial^2 u_j(x, t)}{\partial t^2}, \quad j = 1, 2, 3, \quad (1.3)$$

which are called the basic equations of motion in dynamics of classical elasticity in terms of stress components.

Taking (1.1) into account, the same manipulation with (1.2) results in the equality

$$\sigma_{ij} = \sigma_{ji},$$

which shows that the components of stress form a symmetrical matrix and consequently only six scalars $\sigma_{ij}$ remain independent.

Assume now that the external forces do not depend on time. Then the displacements and stresses are also independent and relations (1.3) assume the form

$$\sum_{i=1}^3 \sigma_{ij}(x) \frac{\partial}{\partial x_i} n_i + \rho \mathbf{F}_j(x) = 0 \quad j = 1, 2, 3,$$

which are called the equilibrium equations.

### 1.1.3 Components of the strain tensor and Hooke’s law

The theory of elasticity is concerned with elastic media. The elastic properties of the medium are described by a specific dependence (which is called Hooke’s law) existing between stresses and strains or, more precisely, between quantities characterizing the stresses and deformed states of the medium.

Choose an arbitrary point $x = (x_1, x_2, x_3)$ in undeformed medium (at the time $t_0$) and consider a point $x + \xi$ in a small neighborhood of the chosen one. Calculate a change of the small vector $\xi = (\xi_1, \xi_2, \xi_3)$ causes by strain. Here $x$ coincides with the origin of the vector and $x + \xi$ with its end. At the time $t$ the point $x$ will occupy the position $x + u(x, t)$ and the point $x + \xi$ the position $x + \xi + u(x + \xi, t)$. Therefore, the change of the vector $\xi$, which is denoted by $\Delta \xi(x, t)$, or simply by $\Delta \xi$, may be calculated by the formula

$$\Delta \xi(x, t) = u(x + \xi, t) - u(x, t).$$

Applying Taylor’s expansion and neglecting, due to the smallness of the vector $\xi$, the terms of the order higher than $|\xi|$, we obtain

$$\Delta \xi_i(x, t) = \sum_j \frac{\partial u_i(x, t)}{\partial x_j} \xi_j,$$
where $\Delta \xi = (\Delta \xi_1, \Delta \xi_2, \Delta \xi_3)$. The quantities

$$\frac{1}{2} \left( \frac{\partial u_i(x,t)}{\partial x_j} + \frac{\partial u_j(x,t)}{\partial x_i} \right) \equiv \varepsilon_{i,j}(x,t)$$

form a symmetrical matrix. They are called the components of the strain tensor or the components of strain (at the point $x$ at the time $t$).

The deformed state is characterized by changes in the distances between the points. The deformed state at any point $x$ may be therefore characterized by changes in length of every vector of the type $\xi$ which, with its derivatives, are assumed so small with respect to the Cartesian coordinates that their product may be neglected

$$(\xi + \Delta \xi)^2 - \xi^2 = \sum_i \left( \xi_i + \sum_j \frac{\partial u_i(x,t)}{\partial x_j} \xi_j \right)^2 - \sum_i \xi_i^2 = 2 \sum_{i,j} \frac{\partial u_i(x,t)}{\partial x_j} \xi_i \xi_j.$$

This formula shows that changes in the distance between the points and hence the deformed state are characterized exclusively by the strain components.

It is clear that there must exist some dependence between the stress components and corresponding strain components. Hooke’s law suggests the following simplest linear dependence between them:

$$\sigma_{ij}(x,t) = \sum_{l,k} c_{ijkl}(x,t) \varepsilon_{lk}(x,t), \quad (1.4)$$

where $c_{ijkl}(x,t)$ are certain quantities called elastic constants are not dependent on the strain and stress components. The symmetry of the matrices $\|\sigma_{ij}\|$ and $\|\varepsilon_{ij}\|$ implies $c_{ijkl} = c_{ikjl} = c_{jikl}$.

Hooke’s law also assumes that the strain components are expressed (linearly and uniquely) through the stress components

$$\varepsilon_{ij} = \sum_{l,k} s_{ijkl}(x,t) \sigma_{lk}(x,t). \quad (1.5)$$

Formulas (1.4) and (1.5) imply elasticity of the medium. By elasticity is meant an ability of the medium to restore its original shape after the forces that have caused deformation are removed. More precisely, elasticity is such a state of the continuous medium which is characterized by the one-to-one relation between stresses and strains; zero stresses correspond to zero strains.

The medium is said to be isotropic, if its elastic constants $c_{ijkl}$ do not depend on the orientation of the coordinate axes. In this case the number of different elastic constants is reduces to two and following relations are valid

$$c_{ijkl} = \lambda \delta_{ij} \delta_{lk} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}), \quad (1.6)$$

where $\lambda$ and $\mu$ are called the Lamé constants and $\delta_{ij}$ is Kronecker symbol. Substitution of (1.6) in (1.4) gives Hooke’s law in the form

$$\sigma_{ij}(x,t) = \lambda \delta_{ij} \sum_k \varepsilon_{kk}(x,t) + 2\mu \varepsilon_{ij}(x,t). \quad (1.7)$$
The constants $\lambda$ and $\mu$ do not generally depend on the position of a point in the medium and satisfy the conditions

$$\mu \neq 0, \quad 3\lambda + 2\mu \neq 0$$

and we obtain from (1.7)

$$\varepsilon_{ij}(x, t) = \frac{1}{2\mu} \sigma_{ij}(x, t) - \frac{\delta_{ij}\lambda}{2\mu(3\lambda + 2\mu)} \sum_k \sigma_{kk}(x, t). \quad (1.8)$$

### 1.2 Problem settings in classical elasticity

#### 1.2.1 Main definitions

We introduce some classes of functions which will be frequently used [19]. Let $\varphi$ be the function determined in the domain $\Omega$ of the $m$-dimensional Euclidian space $E_m$.

**Definition 1.1** $\varphi$ belongs to Class $C^0(\Omega)$ if $\varphi$ is continuous in $\Omega$; $\varphi$ belongs to Class $C^k(\Omega)$, where $k$ is an integer, if at every point $x$ of the domain $\Omega$ there exist all partial derivatives of $\varphi$ with respect to the Cartesian coordinates of $x$ up to the order $k$ and they are continuous in $\Omega$.

**Definition 1.2** $\varphi$ is continuously extendible at the point $y \in \Sigma$, where $\Sigma$ is the boundary of the set $\Omega$, if there exists a limit $\lim_{x \to y} \varphi(x)$, where $x \in \Omega$.

**Definition 1.3** $\varphi \in C^k(\Omega \cup \Sigma) = C^k(\overline{\Omega})$ if $\varphi \in C^k(\Omega)$ and if, in addition, $\varphi$ and all its derivatives with respect to the Cartesian coordinates up to the order $k$ are continuously extendible at each point of the set $\Sigma$.

**Definition 1.4** If $\varphi$ is a vector or a matrix determined in $\Omega$, then $\varphi \in C^k(\Omega)$ means that every its component belongs to $C^k(\Omega)$.

**Proposition 1.1** [20] If $\varphi \in C^0(\overline{\Omega})$, then the function $\psi$ determined by the equalities $\psi(y) = \varphi(y)$ if $y \in \Omega$ and $\psi(y) = \lim_{x \to y} \varphi(x)$ if $y \in \Sigma$ is continuous on $\Omega$.

**Definition 1.5** The function $\varphi$ defined in the domain $\Omega$ will be called regular if $\varphi \in C^2(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$.

An elastic medium is a domain $D$ of the three-dimensional Euclidian space and a set of quantities $\mu, \lambda$ (Lamé constants), and $\rho$ (the density of the medium) satisfying the conditions

$$\rho > 0, \quad \mu > 0, \quad 3\lambda + 2\mu > 0.$$
1.2. PROBLEM SETTINGS IN CLASSICAL ELASTICITY

1.2.2 Dynamic, static, and oscillation states

An elasto-dynamic state of the medium \( D(\rho, \lambda, \mu) \) in the time interval \( T = [t_0, t_1] \), corresponding to the mass force \( \mathcal{F} \), is the pair \([u, \sigma]\), satisfying the conditions

\[
\sum_{j=1}^{3} \sigma_{ij} \frac{\partial}{\partial x_j} + \rho \mathcal{F}_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1, 2, 3, \tag{1.9}
\]

\[
\sigma \in C^1(D \times T), \quad u \in C^2(D \times T) \cap C^1(\overline{D} \times T),
\]

\[
\sigma_{ij} = \lambda \delta_{ij} \text{div} u + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3, \tag{1.10}
\]

where \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \), \( \mathcal{F}_i : \overline{D} \times T \to \mathbb{R} \) and \( u = (u_1, u_2, u_3) \), \( u_i : \overline{D} \times T \to \mathbb{R} \) are real vectors, \( \sigma = \|\sigma_{ij}\|_{3 \times 3} \), \( \sigma_{ij} : \overline{D} \times T \to \mathbb{R} \) is a real matrix. Relation (1.9) is called the equation of the elasto-dynamic state [19] and relation (1.10) Hooke’s law.

Substituting (1.10) in (1.9), we obtain

\[
\mu \Delta u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} \text{div} u + \rho \mathcal{F}_i = \rho \frac{\partial^2 u_i}{\partial t^2},
\]

which is called the dynamic equation (of classical elasticity) or, more precisely, the equation of the elasto-dynamic state of the medium \( D(\rho, \lambda, \mu) \), corresponding to the mass force \( \mathcal{F} \), in terms of the displacement components. Its vector form is

\[
\mu \Delta u + (\lambda + \mu) \text{grad} \text{div} u + \rho \mathcal{F} = \rho \frac{\partial^2 u}{\partial t^2},
\]

where

\[
\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3), \quad \Delta u_i = \sum_j \frac{\partial^2 u_i}{\partial x_j^2},
\]

\[
\text{grad} \varphi = \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \frac{\partial \varphi}{\partial x_3} \right), \quad \text{div} u = \sum_i \frac{\partial u_i}{\partial x_i}.
\]

The elasto-static state of the medium \( D(\rho, \lambda, \mu) \), corresponding to the mass force \( \mathcal{F} \), is the pair \([u, \sigma]\), satisfying the conditions

\[
\sum_{j=1}^{3} \sigma_{ij} \frac{\partial}{\partial x_j} + \rho \mathcal{F}_i = 0, \quad i = 1, 2, 3, \tag{1.11}
\]

\[
\sigma \in C^1(D), \quad u \in C^2(D) \cap C^1(\overline{D}),
\]

\[
\sigma_{ij} = \lambda \delta_{ij} \text{div} u + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3, \tag{1.12}
\]

where \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \), \( \mathcal{F}_i : \overline{D} \to \mathbb{R} \) and \( u = (u_1, u_2, u_3) \), \( u_i : \overline{D} \to \mathbb{R} \) are real vectors, \( \sigma = \|\sigma_{ij}\|_{3 \times 3} \), \( \sigma_{ij} : \overline{D} \to \mathbb{R} \) is a real matrix. Relation (1.11) is called the equation of the elasto-static state [19] and relation (1.12) Hooke’s law.

Substituting (1.12) in (1.11), we obtain the static equation (of classical elasticity) or, more precisely, the equation of the elasto-static state of the medium
CHAPTER 1. INTRODUCTION TO GENERAL ELASTICITY

\( D(\rho, \lambda, \mu) \), corresponding to the mass force \( \mathbf{F} \), in terms of the displacement components

\[
\mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad div} \mathbf{u} + \rho \mathbf{F} = 0. \tag{1.13}
\]

The elasto-oscillation state of the medium \( D(\rho, \lambda, \mu) \), corresponding to the mass force \( \mathbf{F} \), is the pair \([\mathbf{u}, \sigma]\), satisfying the conditions

\[
\sum_{j=1}^{3} \frac{\sigma_{ij}}{\partial x_j} + \rho \omega^2 u_i + \rho F_i = 0, \quad i = 1, 2, 3, \tag{1.14}
\]

\[
\sigma \in C^1(D), \quad \mathbf{u} \in C^2(D) \cap C^1(\bar{D}),
\]

\[
\sigma_{ij} = \lambda \delta_{ij} \text{div} \mathbf{u} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3, \tag{1.15}
\]

where \( \mathbf{F} = (F_1, F_2, F_3) \), \( F_j : \bar{D} \mapsto \mathbb{Z} \) and \( \mathbf{u} = (u_1, u_2, u_3) \), \( u_i : \bar{D} \mapsto \mathbb{Z} \) are complex vectors, \( \sigma = \|\sigma_{ij}\|_{3 \times 3} \), \( \sigma_{ij} : \bar{D} \mapsto \mathbb{Z} \) is a complex matrix and \( \omega \) is an arbitrary real number called the oscillation frequency. Relation (1.14) is called the equation of the elasto-oscillation state [19] and relation (1.15) Hooke’s law.

Substituting (1.15) in (1.14), we obtain the oscillation equation (of classical elasticity) or, more precisely, the equation of the elasto-oscillation state of the medium \( D(\rho, \lambda, \mu) \), corresponding to the mass force \( \mathbf{F} \) and to the oscillation frequency \( \omega \) in terms of the displacement components

\[
\mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad div} \mathbf{u} + \rho \omega^2 \mathbf{u} + \rho \mathbf{F} = 0. \tag{1.16}
\]

1.2.3 Matrix representations

Introduce the matrix differential operator \( A(\partial_x) = \|A_{ij}(\partial_x)\|_{3 \times 3} \), where

\[
A_{ij}(\partial_x) = \delta_{ij} \mu \Delta + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j}
\]

and the matrix differential operator \( A(\partial_x, \omega) = \|A_{ij}(\partial_x, \omega)\|_{3 \times 3} \), where

\[
A_{ij}(\partial_x, \omega) = A_{ij}(\partial_x) + \delta_{ij} \rho \omega^2 = \delta_{ij} (\mu \Delta + \rho \omega^2) + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j}.
\]

Then the equation of the elasto-dynamic state of the medium \( D(\rho, \lambda, \mu) \), corresponding to the mass force \( \mathbf{F} \), may be written as

\[
A(\partial_x) \mathbf{u} + \rho \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2},
\]

the elasto-static equation as

\[
A(\partial_x) \mathbf{u} + \rho \mathbf{F} = 0
\]

and, corresponding to the oscillation frequency \( \omega \), the oscillation equation takes the form

\[
A(\partial_x, \omega) \mathbf{u} + \rho \mathbf{F} = 0.
\]
1.2. PROBLEM SETTINGS IN CLASSICAL ELASTICITY

Let \( x = (x_1, x_2, x_3) \) be a point of medium and \( n(x) = (n_1(x), n_2(x), x_3(x)) \) be an arbitrary unit vector. Substituting in (1.1) the stress components expressed in terms of the displacement components, we obtain

\[
\sigma^{(n)} = \lambda n_i \operatorname{div} u(x) + 2\mu \frac{\partial u_i(x, t)}{\partial n_i(x)} + \mu \sum_j \left( \frac{\partial u_j(x, t)}{\partial x_i} - \frac{\partial u_i(x, t)}{\partial x_j} \right) n_j(x),
\]

or

\[
\sigma^{(n)} = \lambda n_i \operatorname{div} u(x) + 2\mu \frac{\partial u_i(x, t)}{\partial n_i(x)} + \mu (n(x) \times \text{rot} u(x, t)).
\]

The matrix differential operator \( T(\partial_x, n(x)) = \| T_{ij}(\partial_x, n(x)) \|_{3 \times 3} \), where

\[
T_{ij}(\partial_x, n(x)) = \lambda n_i(x) \frac{\partial}{\partial x_j} + \mu n_j(x) \frac{\partial}{\partial x_i} + \mu n_i(x) \delta_{ij} \frac{\partial}{\partial n_j(x)}
\]

is called the stress operator and \( \sigma^{(n)} = T(\partial_x, n(x)) u \).

1.2.4 Formulation of basic problems

Classical elasticity is concerned with three main types of problems for determining the elastic state. They are dynamic, static and harmonic oscillation problems. Each type includes six basic problems [19].

We consider the dynamic internal case. In the same way we may consider the external cases.

First internal basic problem of dynamics.

Find an elasto-dynamic state \([u, \sigma]\) of the medium \( D(\rho, \lambda, \mu) \) with the boundary \( S \), corresponding to the mass force \( F \), in the time interval \( T = [t_0, t_1] \) by the boundary condition

\[
u(x, t) = f(x, t), \quad \forall (x, t) \in S \times T
\]

and the initial conditions

\[u(x, t_0) = \varphi(x), \quad \frac{\partial u(x, t_0)}{\partial t} = \psi(x), \quad \forall x \in D,\]

where \( f : S \times T \to \mathbb{R}; \quad \mathcal{F}, \varphi, \psi : D \times T \to \mathbb{R} \) are given vectors. This problem is denoted by (I) and called the first internal problem of dynamics. Similarly we can define the corresponding external problem.

The second internal basic problem of dynamics (denoted by (II)) is formulated in the same way as the first one, with the only difference that the displacements from the boundary condition (1.17) are replaced by the surface stresses

\[
u(x, t) = f(x, t), \quad \forall (x, t) \in S \times T.
\]

In the third internal basic problem of dynamics the normal component of displacement and the tangential components of stresses are given and the boundary conditions take the form

\[
u(x, t) = g(x, t), \quad \forall (x, t) \in S \times T,
\]

\[T(\partial_x, n)u(x, t) - n[n T(\partial_x, n)u(x, t)] = h(x, t),\]

and the initial conditions

\[u(x, t_0) = \varphi(x), \quad \frac{\partial u(x, t_0)}{\partial t} = \psi(x), \quad \forall x \in D,\]

where \( h : S \times T \to \mathbb{R}; \quad \mathcal{F}, \varphi, \psi : D \times T \to \mathbb{R} \) are given vectors. This problem is denoted by (III) and called the second internal problem of dynamics. Similarly we can define the corresponding external problem.

The fourth internal basic problem of dynamics (denoted by (IV)) is formulated in the same way as the first one, with the only difference that the stresses from the boundary condition (1.17) are replaced by the surface forces.

Second internal basic problem of dynamics.

Find an elasto-static state \([u, \sigma]\) of the medium \( D(\rho, \lambda, \mu) \) with the boundary \( S \), corresponding to the mass force \( F \), in the time interval \( T = [t_0, t_1] \) by the boundary condition

\[
u(x, t) = f(x, t), \quad \forall (x, t) \in S \times T
\]

and the initial conditions

\[u(x, t_0) = \varphi(x), \quad \frac{\partial u(x, t_0)}{\partial t} = \psi(x), \quad \forall x \in D,\]

where \( f : S \times T \to \mathbb{R}; \quad \mathcal{F}, \varphi, \psi : D \times T \to \mathbb{R} \) are given vectors. This problem is denoted by (II) and called the second internal problem of dynamics. Similarly we can define the corresponding external problem.

The fourth internal basic problem of dynamics (denoted by (IV)) is formulated in the same way as the first one, with the only difference that the stresses from the boundary condition (1.17) are replaced by the surface forces.

Fourth internal basic problem of dynamics.

Find an elasto-static state \([u, \sigma]\) of the medium \( D(\rho, \lambda, \mu) \) with the boundary \( S \), corresponding to the mass force \( F \), in the time interval \( T = [t_0, t_1] \) by the boundary condition

\[
u(x, t) = f(x, t), \quad \forall (x, t) \in S \times T
\]

and the initial conditions

\[u(x, t_0) = \varphi(x), \quad \frac{\partial u(x, t_0)}{\partial t} = \psi(x), \quad \forall x \in D,\]

where \( f : S \times T \to \mathbb{R}; \quad \mathcal{F}, \varphi, \psi : D \times T \to \mathbb{R} \) are given vectors. This problem is denoted by (IV) and called the fourth internal problem of dynamics. Similarly we can define the corresponding external problem.
where \( g : S \times T \to \mathbb{R} \) is a scalar function and \( h \) is a vector function and its components are \( h_i : S \times T \to \mathbb{R} \).

In the fourth internal basic problem of dynamics the normal component of stress and the tangential components of displacement are given and the boundary conditions take the form

\[
\mathbf{n} \cdot \mathbf{T}(\partial_x, \mathbf{n})\mathbf{u}(x, t) = g(x, t), \quad \forall (x, t) \in S \times T,
\]

\[
\mathbf{u}(x, t) - \mathbf{n}[\mathbf{u}(x, t) \mathbf{n}] = \mathbf{h}(x, t).
\]

In the fifth internal basic problem of dynamics the boundary of the medium is divided into four portions; the displacements are given on one of the portions, the stresses on the second, the normal components of displacement and the tangential components of stress on the third, the normal components of stress and the tangential components of displacement on the four.

Finally, in the sixth internal basic problem of dynamics specific combinations of displacements and stresses are given on the boundary

\[
T(\partial_x, \mathbf{n})\mathbf{u}(x, t) + \mathbf{u}(x, t) = f(x, t), \quad \forall (x, t) \in S \times T.
\]

The basic problems of statics are formulated similarly to the basic problems of dynamics but without the initial conditions and when the time dependence is ignored.

1.3 Fundamental relations of classical elasticity

Consider the elasto-oscillation state of the medium \( D(\rho, \lambda, \mu) \), corresponding to the body force \( \mathbf{F} \) in the form

\[
\mathbf{F}(x, t) = \Phi^{(1)}(x) \cos \omega t + \Phi^{(2)}(x) \sin \omega t
\]

in terms of the displacement components in the form

\[
\mathbf{v}(x, t) = \mathbf{v}^{(1)}(x) \cos \omega t + \mathbf{v}^{(2)}(x) \sin \omega t,
\]

with the oscillation frequency \( \omega \). Introducing the complex quantities

\[
\Phi = \Phi^{(1)} + \Phi^{(2)}, \quad \mathbf{u} = \mathbf{v}^{(1)} + \mathbf{v}^{(2)},
\]

\( \mathbf{u} \) being a vector with the components \( u_1, u_2, u_3 \), we have

\[
\mathbf{F}(x, t) = \text{Re}\{\Phi(x)e^{-i\omega t}\}, \quad \mathbf{v}(x, t) = \text{Re}\{\mathbf{u}(x)e^{-i\omega t}\}
\]

and \( \mathbf{u} \) satisfies

\[
\mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad} \text{ div} \mathbf{u} + \omega^2 \mathbf{u} + \Phi = 0.
\]

Denote

\[
\Delta^* \equiv \mu \Delta + (\lambda + \mu) \text{grad} \text{ div} = (\lambda + 2\mu) \text{grad} \text{ div} - \mu \text{ rot} \text{ rot}
\]

in terms of which equation (1.16) will be written

\[
\Delta^* \mathbf{u}(x) + \omega^2 \mathbf{u}(x) = -\Phi(x), \quad (1.21)
\]
or in the elasto-static state equation (1.13)
\[ \Delta^* u(x) = -\Phi(x). \] (1.22)

Derive now the Betti formulas \[20\]. Let \( D \) be a finite region of three-dimensional space occupied by an elastic medium; \( S \) its boundary surface, to which \( n \) is an outward-drawn unit normal vector; \( dS \) a surface element; and \( \alpha \) and \( \beta \) any two real numbers which satisfy \( \alpha + \beta = \lambda + \mu \).

Applying the identity
\[
(\alpha + \beta) \int_D \frac{\partial^2 u_k}{\partial x_l \partial x_m} \, dv = \alpha \int_S \frac{\partial u_k}{\partial x_l} \cos(n, x_m) \, dS + \beta \int_S \frac{\partial u_k}{\partial x_m} \cos(n, x_l) \, dS,
\]
where \( l, m = 1, 2, 3 \) and \( x_m \) is a unit vector along the \( x_m \)-axis, we obtain
\[
\int_D \Delta^* udv = \int_S P^{(n)} u dS,
\]
where the vector \( P^{(n)} u \) is called the generalized stress vector at the surface element with the normal \( n \) and
\[
P^{(n)} u = P^{(1)} \cos(n, x_1) + P^{(2)} \cos(n, x_2) + P^{(3)} \cos(n, x_3),
\]
where
\[
P^{(k)} = (p_{1k}, p_{2k}, p_{3k}), \quad k = 1, 2, 3,
\]
\[
p_{jk} = \begin{cases} \frac{\partial u_k}{\partial x_j} + \mu \frac{\partial u_j}{\partial x_k}, & j \neq k, \\ \beta \operatorname{div} u + (\alpha + \mu) \frac{\partial u_k}{\partial x_k}, & j = k. \end{cases}
\]

Let \( u \) and \( v \) be two vector fields which satisfy the conditions of the Gauss-Ostrogradskii theorem in \( D \) and on \( S \). Represent the scalar product \( u \cdot P^{(n)} v \) in the form
\[
u \cdot P^{(n)} v = Q_1 \cos(n, x_1) + Q_2 \cos(n, x_2) + Q_3 \cos(n, x_3),
\]
\[
Q_k = \sum_{j=1}^3 p_{jk}(v) u_j, \quad k = 1, 2, 3.
\]
The divergence of the vector \( Q = (Q_1, Q_2, Q_3) \) is
\[
\operatorname{div} Q = \sum_j \sum_k \frac{\partial p_{jk}}{\partial x_k} u_j + \mathcal{E}(u, v),
\]
where
\[
\mathcal{E}(u, v) = (\lambda + 2\mu) \sum_k \frac{\partial u_k}{\partial x_k} \frac{\partial v_k}{\partial x_k} + \mu \left( \frac{\partial u_1}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial v_2}{\partial x_1} \right) + \beta \left( \frac{\partial u_1}{\partial x_2} \frac{\partial v_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1} \frac{\partial v_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \frac{\partial v_3}{\partial x_1} \right) + \alpha \left( \frac{\partial u_1}{\partial x_2} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_1} + \frac{\partial u_3}{\partial x_2} \frac{\partial v_3}{\partial x_1} \right) + \alpha \left( \frac{\partial u_1}{\partial x_2} \frac{\partial v_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_3}{\partial x_2} \frac{\partial v_3}{\partial x_1} \right) + \alpha \left( \frac{\partial u_1}{\partial x_2} \frac{\partial v_3}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_3}{\partial x_2} + \frac{\partial u_3}{\partial x_2} \frac{\partial v_3}{\partial x_2} \right).
is a bilinear form in the derivatives $\partial u_j/\partial x_k$ and $\partial v_l/\partial x_m$, symmetric with respect to the vectors $u$ and $v$.

On the other hand, inserting the $p_{jk}(v)$ from (1.24)

$$\sum_{k=1}^{3} \frac{\partial p_{jk}}{\partial x_k} = \Delta^* v, \quad j = 1, 2, 3,$$

where

$$\Delta_j^* v \equiv \mu \Delta v_j + (\lambda + \mu) \frac{\partial}{\partial x_j} \text{div } v,$$

we obtain

$$\text{div } Q = u \cdot \Delta^* v + \mathcal{E}(u, v), \quad (1.25)$$

where $\Delta^* v = (\Delta_1^* v, \Delta_2^* v, \Delta_3^* v)$. Applying the Gauss-Ostrogradskii formula to (1.25), we obtain the first generalized Betti formula

$$\int_D u \cdot \Delta^* v \, dv = \int_S u \cdot P(n) v \, dS - \int_D \mathcal{E}(u, v) \, dv. \quad (1.26)$$

Setting here $u = v$ we obtain the second generalized Betti formula

$$\int_D u \cdot \Delta^* u \, dv = \int_S u \cdot P(n) u \, dS - \int_D \mathcal{E}(u, u) \, dv. \quad (1.27)$$

Interchanging $u$ and $v$ in (1.26) and subtracting the two equations, we obtain by virtue of $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ the third generalized Betti formula

$$\int_D (u \cdot \Delta^* v - v \cdot \Delta^* u) \, dv = \int_S (u \cdot P(n) v - v \cdot P(n) u) \, dS. \quad (1.28)$$

Assigning various values to the constants $\alpha$ and $\beta$, we obtain various familiar formulas of the theory of elasticity. In the case $\alpha = \mu, \beta = \lambda$ from (1.24) we have $p_{jk} = \tau_{jk}$, $j, k = 1, 2, 3$ and the generalized stress vector $P(n) u$ from (1.23) reduces to the stress vector

$$\sigma(n) u = \sigma_1^{(n)} \cos(n, x_1) + \sigma_2^{(n)} \cos(n, x_2) + \sigma_3^{(n)} \cos(n, x_3) \quad (1.29)$$

and relations (1.26), (1.27), (1.28) respectively become the first Betti formula

$$\int_D u \cdot \Delta^* v \, dv = \int_S u \cdot \sigma(n) v \, dS - \int_D \mathcal{W}(u, v) \, dv,$$

the second Betti formula

$$\int_D u \cdot \Delta^* u \, dv = \int_S u \cdot \sigma(n) u \, dS - \int_D \mathcal{W}(u, u) \, dv,$$

and the third Betti formula

$$\int_D (u \cdot \Delta^* v - v \cdot \Delta^* u) \, dv = \int_S (u \cdot \sigma(n) v - v \cdot \sigma(n) u) \, dS,$$
where \( W(\mathbf{u}, \mathbf{u}) = \mathcal{E}(\mathbf{u}, \mathbf{u}) \) for \( \mathbf{u} = \mathbf{v}, \alpha = \mu \) and \( \beta = \lambda \), that is
\[
W(\mathbf{u}, \mathbf{u}) = 2\mu \left( 2(\varepsilon_{1,2}^2 + \varepsilon_{2,3}^2 + \varepsilon_{3,1}^2) + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k}^2 \right) + \lambda (\text{div } \mathbf{u})^2.
\]

1.4 Fundamental solutions

The force \( \Phi(x) \) from (1.21) and (1.22) generally differs from zero only in a finite spatial region \( D \). Consider the scalar and vector potentials
\[
\begin{align*}
\Phi(x) &= -\frac{1}{4\pi} \int_D \Phi(y) \cdot \text{grad} \frac{1}{r(x,y)} \, dv_y, \\
\Psi(x) &= \frac{1}{4\pi} \int_D \Phi(y) \times \text{grad} \frac{1}{r(x,y)} \, dv_y,
\end{align*}
\]
where \( r(x,y) \) is the distance between the points \( x(x_1, x_2, x_3) \) and \( y(y_1, y_2, y_3) \), and \( dv_y \) is volume element at \( y \).

For \( x \in D \) we have by (1.30)
\[
\Phi(x) = \text{grad } \Phi(x) + \text{rot } \Psi(x).
\]

Splitting similarly the displacement vector \( \mathbf{u} \) into two parts
\[
\mathbf{u}(x) = \text{grad } \phi(x) + \text{rot } \psi(x)
\]
and substituting the expressions for \( \Phi(x) \) and \( \mathbf{u}(x) \) into (1.21), we may verify that (1.21) will be fulfilled if the scalar potential \( \phi(x) \) and the vector potential \( \psi(x) \) are solutions of the equations
\[
\begin{align*}
\Delta \phi(x) + k_1^2 \phi(x) &= -\frac{1}{a^2} \phi(x), \\
\Delta \psi(x) + k_2^2 \psi(x) &= -\frac{1}{b^2} \psi(x),
\end{align*}
\]
where \( a^2 = \lambda + 2\mu, \ b^2 = \mu, \ k_1^2 = \omega^2/a^2, \) and \( k_2^2 = \omega^2/b^2. \)

The case when the force distribution \( \Phi(x) \) reduces to a constant force concentrated at a point is of special interest. Let such a concentrated force of magnitude \( 4\pi \) and directed along the \( x_1 \)-axis be applied at \( y(y_1, y_2, y_3) \). We will represent it as the limit approached by the resultant of some system of forces \( \Phi^*(x) \) continuously distributed over the sphere \( \sigma(y, \varepsilon) \) of radius \( \varepsilon \) and centered at \( y \), as tends to zero. We stipulate that the \( x_2 \)- and \( x_3 \)-components of \( \Phi^*(x) \) are to remain bounded, while the \( x_1 \)-component, \( \Phi_1^*(x) \), increases so that
\[
\lim_{\varepsilon \to 0} \int_{\sigma(y, \varepsilon)} \Phi_1^*(x) \, d\sigma_x = 4\pi.
\]

For the force \( \Phi(x) \) thus defined (1.30) gives
\[
\begin{align*}
\Phi'(x) &= -\frac{\partial}{\partial x_1} \frac{1}{r(x,y)}, \quad \Psi_1(x) = 0, \\
\Psi_2(x) &= \frac{\partial}{\partial x_3} \frac{1}{r(x,y)}, \quad \Psi_3(x) = -\frac{\partial}{\partial x_2} \frac{1}{r(x,y)}.
\end{align*}
\]
Substituting these values in (1.32), we obtain

\[
\Delta \phi(x) + k_1^2 \phi(x) = \frac{1}{\omega^2} \frac{\partial}{\partial x_1} \left( \frac{e^{ik_1r}}{r} - \frac{1}{r} \right),
\]

\[
\Delta \psi_1(x) + k_2^2 \psi_1(x) = 0,
\]

\[
\Delta \psi_2(x) + k_2^2 \psi_2(x) = -\frac{1}{b^2} \frac{\partial}{\partial x_3} \left( \frac{e^{ik_2r}}{r} - \frac{1}{r} \right),
\]

\[
\Delta \psi_3(x) + k_3^2 \psi_3(x) = \frac{1}{b^2} \frac{\partial}{\partial x_3} \left( \frac{e^{ik_3r}}{r} - \frac{1}{r} \right),
\]

It is easy to verify that the functions

\[
\phi(x, y) = -\frac{1}{\omega^2} \frac{\partial}{\partial x_1} \left( \frac{e^{ik_1r}}{r} - \frac{1}{r} \right),
\]

\[
\psi_1(x, y) = 0,
\]

\[
\psi_2(x, y) = \frac{1}{\omega^2} \frac{\partial}{\partial x_3} \left( \frac{e^{ik_2r}}{r} - \frac{1}{r} \right),
\]

\[
\psi_3(x, y) = -\frac{1}{\omega^2} \frac{\partial}{\partial x_3} \left( \frac{e^{ik_3r}}{r} - \frac{1}{r} \right),
\]

respectively, solve equations (1.33) for fixed \( y \) and \( x \neq y \).

Let us introduce the vectors

\[
u_{p} = \text{grad } \phi(x, y), \quad u_s = \text{rot } \psi(x, y).
\]

The impressed force concentrated at \( y \) has been assumed above to lie parallel to the \( x_1 \)-axis and therefore such specification will be indicated by a superscript. From (1.33), (1.34) and (1.35) we obtain [24] the following expressions for the components of the vectors \( u_p \) and \( u_s \)

\[
u_{p,j}^{(1)} = -\frac{1}{\omega^2} \frac{\partial^2}{\partial x_j \partial x_1} \left( \frac{e^{ik_1r}}{r} - \frac{1}{r} \right),
\]

\[
u_{p,j}^{(2)} = \frac{1}{\omega^2} \frac{\partial^2}{\partial x_j \partial x_1} \left( \frac{e^{ik_2r}}{r} - \frac{1}{r} \right) + \frac{1}{b^2} \delta_{1j} e^{ik_1r},
\]

where \( j = 1, 2, 3 \), \( \delta_{1j} \) is Kronecker symbol and the required solution of (1.21) is, by (1.31),

\[
u_{p,j}^{(1)} = \frac{1}{b^2} \delta_{1j} e^{ik_2r} \frac{e^{ik_1r}}{r} - \frac{1}{\omega^2} \frac{\partial^2}{\partial x_j \partial x_1} \left( \frac{e^{ik_1r}}{r} - \frac{e^{ik_2r}}{r} \right).
\]

The solution for a force applied at \( y \) along the \( x_2 \)- and \( x_3 \)-axes is analogous

\[
u_{j}^{(1)} = \frac{1}{b^2} \delta_{1j} e^{ik_2r} \frac{e^{ik_1r}}{r} - \frac{1}{\omega^2} \frac{\partial^2}{\partial x_j \partial x_k} \left( \frac{e^{ik_1r}}{r} - \frac{e^{ik_2r}}{r} \right),
\]

where \( k, j = 1, 2, 3 \). Solution (1.36) can be written in the form

\[
u^{(k)}(x, y) = \text{rot rot } \delta^{(k)} \frac{e^{ik_2r}}{r} - \text{grad div } \delta^{(k)} \frac{e^{ik_1r}}{r},
\]
1.4. FUNDAMENTAL SOLUTIONS

where \( \delta^{(k)} = (\delta_{11}, \delta_{12}, \delta_{k3}) \).

The vectors \( u^{(1)}(x, y), u^{(2)}(x, y), u^{(3)}(x, y) \) constitute the fundamental solutions of equation (1.21). They represent in complex form the amplitudes of the even and odd components of the harmonically oscillating displacement vector \( v(x, t) \). Separating their real and imaginary parts, we can construct solutions of equation (1.16) that will represent the oscillatory displacements in at infinite, isotropic and homogeneous medium by a concentrated periodic force applied at \( y(y_1, y_2, y_3) \) and acting parallel to one of the coordinate axes [20]. The matrix

\[
\Gamma(x, y) = \begin{bmatrix}
u_1^{(1)} & u_1^{(2)} & u_1^{(3)} \\
u_2^{(1)} & u_2^{(2)} & u_2^{(3)} \\
u_3^{(1)} & u_3^{(2)} & u_3^{(3)}
\end{bmatrix}
\]

is called the matrix of fundamental solutions of equation (1.21).

The matrix of fundamental solutions of the static equation (1.22) may be obtained from the preceding analysis as a particular case by setting \( \omega = 0 \) in (1.34) because the function \( e^{ib\omega r'/r - 1/\omega^2} \) tends to a constant \( b \) when \( \omega \) tends to 0 and evaluating the ensuing indeterminate expressions by passing to the limit. Doing this for \( \varphi(x, y) \) and \( \psi^{(k)}, k = 1, 2, 3 \) we obtain [20] the following static counterpart of (1.36)

\[
\delta^{(k)} \frac{\partial}{\partial x} \left( \alpha^2 - b^2 \right) \frac{\partial}{\partial y} \left( \alpha^2 + b^2 \right) \delta_{jk} \frac{1}{r(x, y)}
\]

or

\[
u^{(k)}(x, y) = \text{grad div} \left( \delta^{(k)} \frac{r}{2\alpha^2} \right) - \text{rot rot} \left( \delta^{(k)} \frac{r}{2\alpha^2} \right).
\]

The matrix

\[
\delta (x, y) = \begin{bmatrix}
u_1^{(1)} & \nu_1^{(2)} & \nu_1^{(3)} \\
v_2^{(1)} & \nu_2^{(2)} & \nu_2^{(3)} \\
v_3^{(1)} & \nu_3^{(2)} & \nu_3^{(3)}
\end{bmatrix}
\]

is called the matrix of fundamental solutions in the static case of equation (1.22).

1.4.1 Generalized fundamental solutions

Consider the generalized stress vector \( P u^{(k)}(x, y), k = 1, 2, 3 \) corresponding to the displacements of an infinite continuum under the action of concentrated forces. Since \( u^{(k)}(x, y) \) are two-point vector functions it is necessary to indicate the point to which the operation of \( P \) refers

\[
P_{\nu}(x, y) u^{(k)}(x, y), \quad P_{\psi}(y) u^{(k)}(x, y).
\]

The corresponding components will be

\[
P_{\nu}^{(s)} u^{(k)}(x, y), \quad P_{\psi}^{(s)} u^{(k)}(x, y), \quad s = 1, 2, 3.
\]

Let \( n_x \) and \( n_y \) be the unit normals at \( x \) and \( y \) to surface elements through these points. According to (1.23)-(1.24)

\[
P(x) u^{(k)}(x, y) = (\alpha + \mu) \frac{\partial u^{(k)}(x, y)}{\partial n_x} + \beta n_x \text{div}_x u^{(k)}(x, y) + \alpha (n_x \times \text{rot}_x u^{(k)}(x, y))
\]
of the matrix satisfies as a vector function of vector 

Consider the case when the generalized stress vector reduces to the stress

1.4.2 Fundamental solutions of the first, second, and third

Proposition 1.2 [19] The column vectors of the associated matrix considered as functions of \( x = (x_1, x_2, x_3) \) satisfy corresponding homogeneous equation (1.21).

1.4.2 Fundamental solutions of the first, second, and third kind

Consider the case when the generalized stress vector reduces to the stress vector \((\alpha = \mu, \beta = \lambda)\). According to (1.29) and by Proposition 1.2 each column of the matrix

satisfies as a vector function of \( x = (x_1, x_2, x_3) \) corresponding homogeneous equation (1.21), and each column of the matrix

\[
\begin{vmatrix}
0 & 0 & 0 \\
1 & u_1 & u_1 \\
0 & u_2 & u_2 \\
0 & u_3 & u_3
\end{vmatrix}
\]
1.4. FUNDAMENTAL SOLUTIONS

satisfies as a vector function of \( x = (x_1, x_2, x_3) \) corresponding homogeneous equation (1.22). These solutions are called the fundamental solutions of the first kind of equations (1.21) and (1.22) respectively. Here

\[
\sigma_j^{(e)}(x, y) = 2\mu \frac{\partial u_j^{(e)}(x, y)}{\partial n_y} + \frac{\lambda}{\lambda + 2\mu} \cos(n_y, x_j) \frac{\partial}{\partial y_j} \varepsilon^{ijr} e^{ikr} - \delta_{jk} \frac{\partial}{\partial n_y} e^{ikr}
\]

\[
\sigma_j^{(o)}(x, y) = 2\mu \frac{\partial u_j^{(o)}(x, y)}{\partial n_y} + \frac{\lambda}{\lambda + 2\mu} \cos(n_y, x_j) \frac{1}{\partial y_j} + \frac{1}{\partial r(x, y)} - \delta_{jk} \frac{1}{\partial n_y} r(x, y)
\]

Consider the case than

\[
a = \frac{\mu(\lambda + \mu)}{\lambda + 3\mu}; \quad \beta = \frac{(\lambda + \mu)(\lambda + 2\mu)}{\lambda + 3\mu}.
\]

By Proposition 1.2 each column of the matrix

\[
\Pi_2(x, y) = \begin{bmatrix}
N_1^{(y)} u^{(1)} & N_2^{(y)} u^{(1)} & N_3^{(y)} u^{(1)} \\
N_1^{(y)} u^{(2)} & N_2^{(y)} u^{(2)} & N_3^{(y)} u^{(2)} \\
N_1^{(y)} u^{(3)} & N_2^{(y)} u^{(3)} & N_3^{(y)} u^{(3)}
\end{bmatrix}
\]

satisfies as a vector function of \( x = (x_1, x_2, x_3) \) corresponding homogeneous equation (1.21), and each column of the matrix

\[
\Pi_2 (x, y) = \begin{bmatrix}
N_1^{(y)} u^{(1)} & N_2^{(y)} u^{(1)} & N_3^{(y)} u^{(1)} \\
N_1^{(y)} u^{(2)} & N_2^{(y)} u^{(2)} & N_3^{(y)} u^{(2)} \\
N_1^{(y)} u^{(3)} & N_2^{(y)} u^{(3)} & N_3^{(y)} u^{(3)}
\end{bmatrix}
\]

satisfies as a vector function of \( x = (x_1, x_2, x_3) \) corresponding homogeneous equation (1.22). These solutions are called the fundamental solutions of the second kind of equations (1.21) and (1.22) respectively. Here

\[
N_j^{(e)} u^{(e)}(x, y) = \frac{1}{\lambda + 3\mu} \left( 2\mu \delta_{jkl} + 3(\lambda + \mu) \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial x_l} \right) \frac{1}{\partial n_x} r(x, y)
\]

and

\[
N_j^{(e)} u^{(o)}(x, y) = \frac{2\mu(\lambda + 2\mu)}{\lambda + 3\mu} \frac{\partial u_j(x, y)}{\partial n_x} + \frac{(\lambda + \mu)(\lambda + 2\mu)}{\lambda + 3\mu} \cos(n_x, x_j) \varepsilon^{ikr} e^{ikr}
\]

\[
N_j^{(e)} u^{(o)}(x, y) = \frac{2\mu(\lambda + 2\mu)}{\lambda + 3\mu} \frac{\partial u_j(x, y)}{\partial n_x} + \frac{(\lambda + \mu)(\lambda + 2\mu)}{\lambda + 3\mu} \cos(n_x, x_j) \varepsilon^{ikr} e^{ikr}
\]

Let \( S \) be a closed convex surface such that none of its outward-drawn normals, endlessly prolonged, ever meets it again. For the point \( y = (y_1, y_2, y_3) \) on \( S \), and an arbitrary \( x = (x_1, x_2, x_3) \) consider the function

\[
v(x, y) = r \cos(r_0, n_y) [r + r \cos(r_0, n_y)] - r,
\]
where \( \mathbf{n}_y \) is the inward normal at \( y \) and \( \mathbf{r}_0 \) a unit vector along the segment \( r(x, y) \) directed from \( y \) to \( x \).

**Proposition 1.3** \([20]\) \( v(x, y) \) is a harmonic function of the variable \( x \).

Since \( v(x, y) \) does not depend on the origin and orientation of the coordinate system, we may place the origin at \( y \) and direct the \( x_1 \)-axes along the inward normal \( \mathbf{n}_y \) then

\[
v = x_1 \ln(r - x_1) - r.
\]

By virtue of the assumed property of the surface \( S \) and its outward normals, \( x_1 + r \neq 0 \) for all its interior points.

Construct the matrix

\[
\mathbf{Z} = \begin{vmatrix}
\frac{\partial^2 v}{\partial x_1^2} & \frac{\partial^2 v}{\partial x_1 \partial x_2} & \frac{\partial^2 v}{\partial x_1 \partial x_3} \\
\frac{\partial^2 v}{\partial x_2 \partial x_1} & \frac{\partial^2 v}{\partial x_2^2} & \frac{\partial^2 v}{\partial x_2 \partial x_3} \\
\frac{\partial^2 v}{\partial x_3 \partial x_1} & \frac{\partial^2 v}{\partial x_3 \partial x_2} & \frac{\partial^2 v}{\partial x_3^2}
\end{vmatrix}
\]

Each column of this matrix is a vector function of \( x(x_1, x_2, x_3) \) satisfying

\[
\text{div} \mathbf{Z}^{(k)} = 0, \quad \text{rot rot} \mathbf{Z}^{(k)} = 0, \quad k = 1, 2, 3.
\]

The matrix \( \mathbf{Z}(x, y) \) is consequently a matrix of solutions of corresponding homogeneous equation (1.22). Introduce another matrix

\[
\mathbf{M}(x, y) = \frac{1}{3(\lambda + \mu)} \left( \mathbf{Z}(x, y) - (\lambda + 2\mu) \mathbf{u}(x, y) \right).
\]

This matrix, which evidently satisfies corresponding homogeneous equation (1.22), are called the fundamental solution of the third kind with respect to \( x(x_1, x_2, x_3) \). This solution is defined only for points \( x \) contained in the finite domain bounded by the surface \( S \).

### 1.5 Boundary-contact problems

Consider some special statements of the problems that are related to the analysis of a printing-plate contact system. Such problems arise when the contact between screen dots and a sheet in the flexographic printing is simulated.

Let an elastic layer \( \mathcal{V} = \{ -\infty < x_1 < +\infty, -\infty < x_2 < +\infty, 0 < x_3 < h \} \) with Poisson’s ratio \( \nu \) be situated on a stiff base \( x_3 = 0 \) without friction. The plane \( x_3 = h \) is denoted by \( \mathcal{X}_1 \) and the plane \( x_3 = 0 \) by \( \mathcal{X}_2 \) (Fig. 1.1).
Recall that we consider weak boundary loads and apply an assumption that the cliché-substrate contact system is governed by the laws of classical elasticity. We also take into account that paper and cardboard possess to a certain extent the property of elasticity.

Suppose that we have a set of disjoint domains $\Omega = \bigcup_{m=1}^{N} \Omega_m$ bounded by closed piecewise smooth curves $\Gamma_m$, and situated on the plane $\mathcal{K}_1$. Denote $\Omega^* = \mathcal{K}_1 \setminus \Omega$. Distribution of shear strains $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ on plane $\mathcal{K}_1$, displacements $f_3(x_1, x_2)$ on $\Omega$, and elongations $f_4(x_1, x_2)$ on $\Omega^*$ are given. We denote by $u_j$ and $\mathcal{T}_j$, ($j = 1, 2, 3$) the displacements and respectively projections of the body forces in directions $x_j$. In this case we have the boundary conditions as in, respectively, the third internal basic problem of statics (1.20) on $\mathcal{K}_2$, the second internal basic problem of statics (1.19) on $\mathcal{K}_1 \cap \Omega^*$, and the third internal basic problem of statics (1.20) on $\mathcal{K}_1 \cap \Omega$. Note that the basic problems of statics are formulated [19] as the basic problems of dynamics without the initial conditions where the time dependence is ignored.

Domain $V$, set $\Omega$, and planes $\mathcal{K}_1$ and $\mathcal{K}_2$ simulate, respectively, the substrate (paper or cardboard), the cliché with screen dots on it, the cliché base, and the base of the printing plate cylinder.

The determination of $u_j$ (1.13) reduces to a mixed BVP for the Lamé equations in $V$

$$\Delta u_j + k_0 \frac{\partial}{\partial x_j} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = \mathcal{T}_j, \quad k_0 = \frac{1}{1 - 2\nu}, \quad j = 1, 2, 3,$$  \hspace{1cm} (1.37)

with the boundary conditions

$$u_3 = 0, \quad \frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} = 0 \quad \text{on } \mathcal{K}_2,$$

$$\frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} = f_j(x_1, x_2) \quad \text{on } \mathcal{K}_1,$$

$$u_3 = f_3(x_1, x_2) \quad \text{on } \Omega,$$

$$(k_0 - 1) \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + (k_0 + 1) \frac{\partial u_3}{\partial x_3} = f_4(x_1, x_2) \quad \text{on } \Omega^*, \hspace{1cm} (1.38)$$

where $j = 1, 2$, and the conditions at infinity

$$\Phi_V(u_1, u_2, u_3) = \int_V \Pi_V \, dV < \infty,$$

$$\Pi_V = (k_0 - 1) \left( \sum_{j=1}^{3} \left( \frac{\partial u_j}{\partial x_j} \right)^2 \right) + \sum_{j=1}^{3} \left( \frac{\partial u_j}{\partial x_j} \right)^2 + \sum_{j \neq k=1}^{3} \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right)^2.$$  \hspace{1cm} (1.39)
The physical meaning of boundary conditions (1.38) on \( \mathcal{K}_2 \) is that the layer lies on the undistorted base \( \mathcal{K}_2 \) without friction. Other boundary conditions (1.38) specify, respectively, the shear strain on \( \mathcal{K}_1 \), the normal displacements on \( \Omega \), and the normal strain on \( \Omega^* \). Condition at infinity (1.39) expresses boundedness of the potential energy of deformations in the layer.

Problem (1.37)\---(1.39) constitutes formulation of a BCP in a three-dimensional band which will be considered in this work.

In the case when the elastic layer \( V \) is fixed to the plane \( \mathcal{K}_2 \) we have the boundary conditions as in, respectively, the first internal basic problem of statics (1.18) on \( \mathcal{K}_2 \), the second internal basic problem of statics (1.19) on \( \mathcal{K}_1 \cap \Omega^* \), and the third internal basic problem of statics (1.20) on \( \mathcal{K}_1 \cap \Omega \). The determination of \( u_i \) reduces to a mixed BVP for the Lamé equations in \( V \) with the boundary conditions

\[
\begin{align*}
    u_j &= 0, & j &= 1, 2, 3 & \text{on } \mathcal{K}_2, \\
    \frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} &= f_j(x_1, x_2), & j &= 1, 2 & \text{on } \mathcal{K}_1, \\
    u_3 &= f_3(x_1, x_2) & \text{on } \Omega, \\
    (k_0 - 1) \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + (k_0 + 1) \frac{\partial u_3}{\partial x_3} &= f_4(x_1, x_2) & \text{on } \Omega^* \\
\end{align*}
\]

and the same conditions at infinity.

The strain and stress components can be expressed in terms of the displacement components \( u_j, j = 1, 2, 3 \). We shall use the letters \( \epsilon, \gamma, \sigma, \) and \( \tau \) for, respectively, unit elongation, shear strain, normal stress, and shear stress. In this case the components of the strain tensor are

\[
\begin{align*}
    \epsilon_1 &= \frac{\partial u_1}{\partial x_1}, & \gamma_{12} &= \frac{\partial u_2}{\partial x_1}, & \gamma_{13} &= \frac{\partial u_3}{\partial x_1}, \\
    \gamma_{23} &= \frac{\partial u_3}{\partial x_2} & \gamma_{23} &= \frac{\partial u_3}{\partial x_2} & \gamma_{23} &= \frac{\partial u_3}{\partial x_2} \\
\end{align*}
\]

Linear relations between the components of stress and components of strain (Hooke’s law) (1.7) and (1.8) can be written as

\[
\begin{align*}
    \sigma_{x_1} &= 2\mu\epsilon_1 + \lambda(\epsilon_1 + \epsilon_2 + \epsilon_3), & \tau_{12} &= \mu\gamma_{12}, \\
    \sigma_{x_2} &= 2\mu\epsilon_2 + \lambda(\epsilon_1 + \epsilon_2 + \epsilon_3), & \tau_{13} &= \mu\gamma_{13}, \\
    \sigma_{x_3} &= 2\mu\epsilon_3 + \lambda(\epsilon_1 + \epsilon_2 + \epsilon_3), & \tau_{23} &= \mu\gamma_{23}, \\
\end{align*}
\]

and

\[
\begin{align*}
    \epsilon_{x_1} &= \frac{1}{E} \left( \sigma_{x_1} - \nu(\sigma_{x_2} + \sigma_{x_3}) \right), & \gamma_{12} &= \frac{\tau_{12}}{\mu}, \\
    \epsilon_{x_2} &= \frac{1}{E} \left( \sigma_{x_2} - \nu(\sigma_{x_3} + \sigma_{x_1}) \right), & \gamma_{13} &= \frac{\tau_{13}}{\mu}, \\
    \epsilon_{x_3} &= \frac{1}{E} \left( \sigma_{x_3} - \nu(\sigma_{x_2} + \sigma_{x_1}) \right), & \gamma_{23} &= \frac{\tau_{23}}{\mu}, \\
\end{align*}
\]

where \( E \) is the modulus of elasticity in tension, \( \nu \) is Poisson’s ratio, and

\[
\begin{align*}
    \mu &= \frac{E}{2(1 + \nu)}, & \lambda &= \frac{E\nu}{(1 - 2\nu)(1 + \nu)}
\end{align*}
\]
are Lamé constants.

In the next chapter we develop the solvability theory for the BCPs.
CHAPTER 1. INTRODUCTION TO GENERAL ELASTICITY
Chapter 2

Solvability and uniqueness

In this chapter we consider two- and three-dimensional BCPs based on statement (1.37)–(1.39) arising when the contact between screen dots and a sheet lying on a stiff base without friction is simulated. We introduce functional spaces required in the problem statement and prove the existence and uniqueness of the solution to BCPs. We give explicit representation and describe the behavior of the solution at infinity and at critical points.

2.1 Function spaces

Let us introduce some functional spaces we need below. Let \( E \) be a domain in \( n \)-dimensional Euclidean space. Below by \( u^{(j)} \) we denote partial derivatives of order \( j \).

**Definition 2.1** \( L_2(E) \) is a Hilbert space with the inner product and the norm

\[
(u_1, u_2)_{L_2(E)} = \int_E u_1 u_2 \, dx, \quad \|u\| = \left( \int_E u^2 \, dx \right)^{\frac{1}{2}}.
\]

A vector \( u = (u_1, u_2, \ldots, u_n) \in L_2(E) \) if each \( u_i \in L_2(E) \) and

\[
(u_1, u_2)_{L_2(E)} = \int_E \sum_{t=1}^n u_{1t} u_{2t} \, dx, \quad \|u\| = \left( \int_E \sum_{t=1}^n u_{t}^2 \, dx \right)^{\frac{1}{2}}.
\]

**Definition 2.2** \( L_p(E) \) is a space of functions defined on \( E \) with the norm

\[
\|u\|_{L_p(E)} = \left( \int_E |u|^p \, dx \right)^{\frac{1}{p}}, \quad p > 1.
\]

A vector \( u = (u_1, u_2, \ldots, u_n) \in L_p(E) \) if each \( u_i \in L_p(E) \) and

\[
\|u\|_{L_p(E)} = \left( \sum_{t=1}^n \|u_{t}\|_{L_p(E)}^p \right)^{\frac{1}{p}}.
\]
Chapter 2. Solvability and Uniqueness

Definition 2.3 \( u \in C(E) \) if \( u \) is continuous in \( E \) with the norm
\[
\|u\|_{C(E)} = \max_{E} |u|.
\]
A vector \( u = (u_1, u_2, \ldots, u_n) \in C(E) \) if each \( u_i \in C(E) \) and
\[
\|u\|_{C(E)} = \sum_{i=1}^{n} \|u_i\|_{C(E)}.
\]

Definition 2.4 \( u \in C^k(E) \) if:
1) \( \exists u^{(j)}, j = 1, \ldots, k; \)
2) \( u^{(j)} \in C(E), j = 1, \ldots, k; \)
3) the norm
\[
\|u\|_{C^k(E)} = \sum_{m=0}^{k} \sum_{a_1 + \ldots + a_n = m} \max_{P \in E} \left| \frac{\partial^a u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}(P) \right|
\]
where
\[
\frac{\partial^a u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}} = \frac{\partial^{a_1} \ldots \partial^{a_n}}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}.
\]
A vector \( u = (u_1, u_2, \ldots, u_n) \in C^k(E) \) if each \( u_i \in C^k(E) \) and
\[
\|u\|_{C^k(E)} = \sum_{i=1}^{n} \|u_i\|_{C^k(E)}.
\]

Definition 2.5 \( u \in C^{k,\mu}(E) \) if:
1) \( \exists u^{(j)}, j = 1, \ldots, k; \)
2) \( u^{(j)} \) satisfies the Hölder condition \( |u^{(j)}(Q_1) - u^{(j)}(Q_2)| \leq C r_{Q_1,Q_2}^\mu \) in the domain \( E \) with index \( \mu, C = \text{const}; \)
3) the norm
\[
\|u\|_{C^{k,\mu}(E)} = \sum_{m=0}^{k} \sum_{a_1 + \ldots + a_n = m} \max_{P \in E} \left| \frac{\partial^a u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}(P) \right| + \sum_{a_1 + \ldots + a_n = k} \max_{Q_1,Q_2 \in E} r_{Q_1,Q_2}^{-\mu} \sum_{a_1 + \ldots + a_n = k} \left| \frac{\partial^a u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}(Q_1) - \frac{\partial^a u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}(Q_2) \right|
\]
where \( r_{Q_1,Q_2} \) is the distance between \( Q_1 \) and \( Q_2 \).
A vector \( u = (u_1, u_2, \ldots, u_n) \in C^{k,\mu}(E) \) if each \( u_i \in C^{k,\mu}(E) \) and
\[
\|u\|_{C^{k,\mu}(E)} = \sum_{i=1}^{n} \|u_i\|_{C^{k,\mu}(E)}.
\]

Definition 2.6 \( W_p^k(E) \) is the closure of \( C^k(E) \) with respect to the norm
\[
\|u\|_{W_p^k(E)} = \left( \int_{E} \left( \sum_{a_1 + \ldots + a_n = m} \left( \frac{\partial^a u}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}} \right)^2 \right)^{p/2} \, dE \right)^{1/p}.
\]
A vector \( u = (u_1, u_2, \ldots, u_n) \in W_p^k(E) \) if each \( u_i \in W_p^k(E) \) and
\[
\|u\|_{W_p^k(E)} = \left( \sum_{i=1}^{n} \|u_i\|_{W_p^k(E)}^{p/2} \right)^{2/p}.
\]
Consider the band \( S = \{-\infty < x_1 < +\infty, \ 0 < x_2 < h\} \). Denote by \( K_1 \) the line \( x_2 = h \) and by \( K_2 \) the line \( x_2 = 0 \). Suppose that we have a system of disjoint segments \( \omega = \bigcup_{k=1}^{\nu} \omega_k \) on \( K_1 \).

Let \( u = (u_1, u_2) \in C^1(S) \). Introduce the inner product in \( C^1(S) \)
\[
(u^{(1)}, u^{(2)}) = \int_S \left( (k_0 - 1) \sum_{i=1}^{2} \frac{\partial u_1^{(1)}}{\partial x_i} \frac{\partial u_2^{(2)}}{\partial x_i} + 2 \sum_{i=1}^{2} \frac{\partial u_1^{(1)}}{\partial x_i} \frac{\partial u_2^{(2)}}{\partial x_i} + \left( \frac{\partial u_1^{(1)}}{\partial x_2} + \frac{\partial u_2^{(1)}}{\partial x_1} \right) \cdot \left( \frac{\partial u_1^{(2)}}{\partial x_2} + \frac{\partial u_2^{(2)}}{\partial x_1} \right) \right) dS, \quad (2.1)
\]
where \( u^{(1)} = (u_1^{(1)}, u_2^{(1)}) \) and \( u^{(2)} = (u_1^{(2)}, u_2^{(2)}) \) are any two vectors from \( C^1(S) \). The corresponding semi-norm is
\[
\|u\|_1 = \left( \int_S \left( (k_0 - 1) \sum_{i=1}^{2} \frac{\partial u_1^{(1)}}{\partial x_i} \frac{\partial u_1^{(1)}}{\partial x_i} + 2 \sum_{i=1}^{2} \frac{\partial u_1^{(1)}}{\partial x_i} \frac{\partial u_1^{(1)}}{\partial x_i} + \left( \frac{\partial u_1^{(1)}}{\partial x_2} + \frac{\partial u_1^{(2)}}{\partial x_1} \right) \cdot \left( \frac{\partial u_1^{(2)}}{\partial x_2} + \frac{\partial u_1^{(2)}}{\partial x_1} \right) \right) dS \right)^{\frac{1}{2}}. \quad (2.2)
\]

**Definition 2.7** \( C^1_1(S) \) is a subspace of \( C^1(S) \) and \( u \in C^1_1(S) \) if
\[
u_2|x_2 = 0, \quad \int_{S_\delta(P_0)} u_1 \ dS = 0,
\]
where \( S_\delta(P_0) \) is a disc with the center at the point \( P_0(0, h/2) \) and the radius \( \delta < h/2 \).

**Definition 2.8** \( C^1_1(S) \) is the closure of \( C^1_1(S) \) with respect to the semi-norm \( (2.2) \).

**Definition 2.9** \( C^1_1(S) \) is a subspace of \( C^1(S) \) and \( u \in C^1_1(S) \) if \( u_2|_\omega = 0 \).

**Definition 2.10** \( C^1_1(S) \) is the closure of \( C^1_1(S) \) with respect to the semi-norm \( (2.2) \).

**Proposition 2.1** \( [37] \) Let \( u \in C^1_1(S) \). Then
1) \( (u_1 - c_0) b^\alpha \in L_p(S) \), where \( p' < -1 - 0.5p \) if \( 1 < p \leq 2 \) and \( p' < -p \) if \( p > 2 \);
2) \( (u_1 - c_0) b^\alpha \in L_p(K), \) where \( p' < -1 - 0.5p \) if \( 1 < p \leq 2 \) and \( p' < -p \) if \( p > 2 \);
3) \( u_2 b^\alpha \in L_p(S), \) where \( p' < -1 + 0.5p \) if \( 1 < p < 2 \) and \( p' \leq 0 \) if \( p \geq 2 \);
4) \( u_2 b^\alpha \in L_p(K), \) where \( p' < -1 + 0.5p \) if \( 1 < p < 2 \) and \( p' \leq 0 \) if \( p \geq 2 \);
where \( s \) is a line \( x_2 = h_1, 0 \leq h_1 \leq h, \ c_0 \) is a constant, and \( b = \sqrt{x_1^2 + h^2} \);
5) the following inequalities hold
\[
norm{(u_1 - c_0) b^\alpha}_{L_p(S)}, \ \norm{u_2 b^\alpha}_{L_p(K)} \leq m \norm{u}_{C^1_1(S)}.
\]

**Proposition 2.2** \( [41] \) Let \( u \in C^1_1(S) \). Then \( u \in L_p(S'), \ L_p(K_0) \) where \( p \geq 1, \ S' \) is any bounded closed subset of \( S \) (such that \( S_\delta(P) \subset S' \) for a certain \( \delta > 0 \), where \( P \in S' \) and \( S_\delta(P) \) is a disk centered at \( P \) with the radius \( \delta \)), \( K_0 \) is any finite interval on \( K_2 \), and the following inequalities hold:
\[
\norm{u}_{L_p(S')} \leq m \norm{u}_{C^1_1(S)}.
\]
Consider the layer $V = \{-\infty < x_1 < +\infty, \ -\infty < x_2 < +\infty, \ 0 < x_3 < h\}$. Denote by $\mathcal{X}_1$ the plane $x_3 = h$ and by $\mathcal{X}_2$ the plane $x_3 = 0$. Suppose that we have a set of disjoint domains $\Omega = \bigcup_{n=1}^{m} \Omega_m$ bounded by closed piecewise smooth curves $\Gamma_m$ and situated on the plane $\mathcal{X}_1$.

Let $u = (u_1, u_2, u_3) \in C^1(V)$. Introduce the inner product in $C^1(V)$

$$(u^{(1)}, u^{(2)}) = \int_V \left( (k_0 - 1) \sum_{t=1}^{3} \frac{\partial u_1^{(1)}}{\partial x_t} \cdot \sum_{t=1}^{3} \frac{\partial u_2^{(2)}}{\partial x_t} + 2 \sum_{t=1}^{3} \frac{\partial u_1^{(1)}}{\partial x_t} \cdot \frac{\partial u_2^{(2)}}{\partial x_t} + \sum_{t \neq j = 1}^{3} \left( \frac{\partial u_1^{(1)}}{\partial x_j} \cdot \frac{\partial u_2^{(2)}}{\partial x_j} + \frac{\partial u_1^{(1)}}{\partial x_j} \cdot \frac{\partial u_2^{(2)}}{\partial x_j} \right) \right) dV,$$

(2.3)

where $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})$ and $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})$ are any two vectors from $C^1(V)$. The corresponding semi-norm is

$$\|u\|_2 = \left( \int_V \left( (k_0 - 1) \sum_{t=1}^{3} \left( \frac{\partial u_1^{(1)}}{\partial x_t} \right)^2 + 2 \sum_{t=1}^{3} \left( \frac{\partial u_2^{(2)}}{\partial x_t} \right)^2 + \sum_{t \neq j = 1}^{3} \left( \frac{\partial u_1^{(1)}}{\partial x_j} \cdot \frac{\partial u_2^{(2)}}{\partial x_j} + \frac{\partial u_1^{(1)}}{\partial x_j} \cdot \frac{\partial u_2^{(2)}}{\partial x_j} \right) \right)^2 dV \right)^{\frac{1}{2}}. \tag{2.4}$$

Definition 2.11 $C^1_1(V)$ is a subspace of $C^1(V)$ and $u \in C^1_1(S)$ if

$$u|_{\mathcal{X}_2} = 0, \quad \int_{B_h(P_0)} u_t dV = 0, \quad t = 1, 2, \quad \int_{B_h(P_0)} (u_1 x_2 - u_2 x_1) dV = 0,$$

where $B_h(P_0)$ is a ball with the center at the point $P_0(0, 0, h/2)$ and the radius $\delta < h/2$.

Definition 2.12 $\bar{C}_1^1(V)$ is the closure of $C_1^1(V)$ with respect to the semi-norm (2.4).

Definition 2.13 $C_2^1(V)$ is a subspace of $C^1(V)$ and $u \in C_2^1(V)$ if $u_3|_{\partial \Omega} = 0$.

Definition 2.14 $\bar{C}_2^1(V)$ is the closure of $C_2^1(V)$ with respect to the semi-norm (2.4).

Proposition 2.3 [37] Let $u \in \bar{C}_2^1(V)$. In this case:

1) $(u_1 - c_1)b^{\xi_1^p} + (u_2 - c_2)b^{\xi_2^p} \in L_p(V)$, where $p^* < -2 + 2p$ if $p < 0$ and $p^* < -2p$ if $-2 < p < 0$;

2) $(u_1 - c_1)b^{\xi_1^p} + (u_2 - c_2)b^{\xi_2^p} \in L_p(\mathcal{X})$, where $p^* < -2 + 2p$ if $p < 0$ and $p^* < -2p$ if $-2 < p < 0$;

3) $u_3 b^{\xi_3^p} \in L_p(V)$, where $p^* < -2 + 2p$ if $1 < p < 2$ and $p^* < 0$ if $2 < p < 6$;

4) $u_3 b^{\xi_3^p} \in L_p(\mathcal{X})$, where $p^* < -1 + 5p$ if $1 < p < 2$ and $p^* < 0$ if $2 < p < 4$; here $\mathcal{X}$ is a plane $x_3 = h_1, 0 \leq h_1 \leq h, c_1, c_2$ are constants, and $b = \sqrt{x_1^2 + x_2^2 + h^2}$;

5) the following inequalities hold

$$\|u_1 - c_1\|_{L_p(V)} \leq m \|u\|_{\bar{C}_2^1(V)},$$

$$\|u_2 - c_2\|_{L_p(V)} \leq m \|u\|_{\bar{C}_2^1(V)},$$

$$\|u_3\|_{L_p(\mathcal{X})} \leq m \|u\|_{\bar{C}_2^1(V)}.$$
2.2 Two-dimensional case

In this chapter, we summarize, following [41], the fundamental results concerning solvability and uniqueness of BCPs in two-dimensional bands.

2.2.1 Statement of the problem in the two-dimensional case

A BCP (1.37)–(1.39) in a three-dimensional band was formulated in Section 1.5. Here we consider the corresponding formulation in a two-dimensional parallel-plane band.

Consider an elastic band $S = \{ -\infty < x_1 < +\infty, 0 < x_2 < h \}$ with Poisson’s ratio $\nu$ situated on a stiff base (steel) $x_2 = 0$ (see Figure 2.1). The boundary (cliché) $x_2 = h$ is denoted by $K_1$ and the boundary $x_2 = 0$ by $K_2$.

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{figure2.1.png}
\caption{Statement of the problem.}
\end{figure}

Suppose that we have a system of disjoint segments $\omega = \bigcup_{k=1}^{N} \omega_k$, where $a_k, b_k$ are the edges of $\omega_k$. Let $\omega^* = K_1 \setminus \omega$.

We assume that the band $S$ lies on the undistorted base $K_2$ without friction and that distribution of shear strains on line $K_1$, displacements on $\omega$, and elongations on $\omega^*$ are given.

We denote by $u_k$ and $f_k, k = 1, 2$, the displacements and, respectively, projections of the body forces in directions $x_k$. The determination of displacements $u_k$ reduces to a mixed BVP for the Lamé equations in the band

$$
\Delta u_k + k_0 \left( \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial u_1}{\partial x_1} \right) = f_k, \quad k_0 = \frac{1}{1 - 2\nu}, \quad k = 1, 2, \quad (2.5)
$$

with the boundary conditions

$$
\begin{align*}
\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} &= 0 \quad \text{on } K_2, \\
\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} &= f_1(x_1) \quad \text{on } K_1, \\
\frac{\partial u_1}{\partial x_2} + (k_0 - 1) \frac{\partial u_1}{\partial x_2} &= f_2(x_1) \quad \text{on } \omega^*, \\
\frac{\partial u_1}{\partial x_2} + (k_0 + 1) \frac{\partial u_1}{\partial x_2} &= f_2(x_1) \quad \text{on } \omega,
\end{align*}
$$

(2.6)
and the conditions at infinity

$$\Phi_\sigma(u_1, u_2) = \int_S H_\sigma \, ds < \infty,$$

$$H_\sigma = (k_0 - 1) \left( \sum_{j=1}^2 \left( \frac{\partial u_j}{\partial x_j} \right)^2 + \sum_{k=1}^2 \left( \frac{\partial u_k}{\partial x_k} \right)^2 + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2. \tag{2.7}$$

The physical meaning of boundary conditions (2.6) on $K_2$ is that the band lies on the undistorted base $K_2$ without friction. Other boundary conditions (2.6) specify, respectively, the shear strain on $K_1$, the normal displacements on $\omega$, and the normal strain on $\omega^\ast$. Condition at infinity (2.7) expresses boundedness of the potential energy of deformations in the layer.

### 2.2.2 Existence and uniqueness

**Definition 2.15** A vector-function $u = (u_1, u_2) \in \bar{C}^1(I)(S)$ is the generalized solution of BCP (2.5)–(2.7) if

$$u_2(x_1, h) = f_2(x_1) \text{ on } \omega$$

and

$$(u, \varphi)_{\bar{C}^1(S)} = \int_{K_1} f_1(\varphi_1 - c) \, dx_1 + \int_{\omega^\ast} f_3 \varphi_2 \, dx_1 + \int_S (F_1(\varphi_1 - c) + F_2 \varphi_2) \, dS \tag{2.8}$$

for each vector-function $\varphi = (\varphi_1, \varphi_2) \in \bar{C}^1(I)(S)$ and a constant $c$.

**Theorem 2.1** Let:

1) body forces satisfy the conditions

$$F_1 \cdot b \cdot \frac{\nu_1}{\nu_1} \in L_{p_{11}}(S), \quad F_2 \cdot b \cdot \frac{\nu_2}{\nu_2} \in L_{p_{22}}(S), \quad p_{11}, p_{22} > 1,$$

where $p_{11}$ is defined by condition 1) from Proposition 2.1 with $p = \frac{\nu_1}{\nu_1 - 1}$ and $p_{22}$ is defined by condition 3) from Proposition 2.1 with $p = \frac{\nu_2}{\nu_2 - 1}$;

2) $f_1$ and $f_3$ satisfy the conditions

$$f_1 \cdot b \cdot \frac{\nu_1}{\nu_1} \in L_{p_{21}}(K_1), \quad f_3 \cdot b \cdot \frac{\nu_2}{\nu_2} \in L_{p_{22}}(\omega^\ast), \quad p_{21}, p_{22} > 1,$$

where $p_{21}$ is defined by condition 2) from Proposition 2.1 with $p = \frac{\nu_2}{\nu_2 - 1}$, $p_{22}$ is defined by condition 4) from Proposition 2.1 with $p = \frac{\nu_2}{\nu_2 - 1}$, and $b = \sqrt{x_1^2 + h^2}$;

3) $f_2(x_1) \in W_1^1(\omega)$.

Then problem (2.5)–(2.7) is uniquely solvable if and only if

$$\int_{K_1} f_1 \, dx_1 + \int_S F_1 \, dS = 0. \tag{2.9}$$

**Proof.**

**Necessity.** Assume that there is an $u \in \bar{C}^1(I)(S)$ satisfying (2.8) for every $\varphi \in \bar{C}^1(I)(S)$. Substitute $\varphi = 0$ into (2.8) and obtain (2.9).

**Sufficiency.** Set

$$u = v + f, \tag{2.10}$$
where \( u = (u_1, u_2) \), \( v = (v_1, v_2) \), \( f = (0, \bar{f}_2) \), and \( \bar{f}_2 \) is an extension of \( f_2 \) according to Lemma 5.12 (see Appendix). Substituting (2.10) into (2.8) we obtain an integral identity

\[
(w, \varphi)_{C^1_j(S)} = \int_{X_1} f_1 \varphi_1 \, dx_1 + \int_{\omega^*} f_3 \varphi_2 \, dx_1 + \int_S (\mathcal{F}_1 \varphi_1 + \mathcal{F}_2 \varphi_2) \, dS - (f, \varphi)_{C^1_j(S)} = T(\varphi). \tag{2.11}
\]

Prove that \( T(\varphi) \) is a linear bounded functional in \( C^1_j(S) \). Linearity is obvious. To prove boundedness of \( T(\varphi) \) note that according to (2.9) \( T(\varphi) \) can be represented in the form

\[
T(\varphi) = \int_{X_1} f_1 (\varphi_1 - c_0) \, dx_1 + \int_{\omega^*} f_3 \varphi_2 \, dx_1 + \int_S (\mathcal{F}_1 (\varphi_1 - c_0) + \mathcal{F}_2 \varphi_2) \, dS - (f, \varphi)_{C^1_j(S)},
\]

where \( c_0 \) is a constant. By the Hölder inequality

\[
\left| \int_{X_1} f_1 (\varphi_1 - c_0) \, dx_1 \right| \leq \left\| f_1 b^{-\frac{\alpha}{\gamma_1}} \right\|_{L_{1/(\alpha_1)}} \cdot \left\| (\varphi_1 - c_0) b^{\frac{\alpha}{}\gamma_1} \right\|_{L_{\alpha_1}(\omega_1)} \tag{2.12}
\]

and according to Proposition 2.1

\[
\left| \int_{X_1} f_1 (\varphi_1 - c_0) \, dx_1 \right| \leq m \left\| f_1 b^{-\frac{\alpha}{\gamma_1}} \right\|_{L_{1/(\alpha_1)}} \cdot \left\| \varphi \right\|_{C^1_j(S)}. \tag{2.13}
\]

In the same manner we use the Hölder inequality and Proposition 2.1 to obtain the following estimates

\[
\left| \int_S \mathcal{F}_1 (\varphi_1 - c_0) \, dS \right| \leq m \left\| \mathcal{F}_1 b^{-\frac{\alpha}{\gamma_1}} \right\|_{L_{1/(\alpha_1)}} \cdot \left\| \varphi \right\|_{C^1_j(S)}, \tag{2.14}
\]

\[
\left| \int_{\omega^*} f_3 (\varphi_2) \, dx_1 \right| \leq m \left\| f_3 b^{-\frac{\alpha}{\gamma_2}} \right\|_{L_{1/(\alpha_2)}} \cdot \left\| \varphi \right\|_{C^1_j(S)}, \tag{2.15}
\]

\[
\left| \int_S \mathcal{F}_2 (\varphi_2) \, dS \right| \leq m \left\| \mathcal{F}_2 b^{-\frac{\alpha}{\gamma_2}} \right\|_{L_{1/(\alpha_2)}} \cdot \left\| \varphi \right\|_{C^1_j(S)}. \tag{2.16}
\]

By the Cauchy-Schwarz inequality

\[
\left\| (f, \varphi)_{C^1_j(S)} \right\| \leq \left\| f \right\|_{C^1_j(S)} \cdot \left\| \varphi \right\|_{C^1_j(S)}. \tag{2.18}
\]

Take into account (2.12)–(2.18) and obtain the inequality

\[
\left| T(\varphi) \right| \leq m \left\| \varphi \right\|_{C^1_j(S)} \left( \left\| f_1 b^{-\frac{\alpha}{\gamma_1}} \right\|_{L_{1/(\alpha_1)}} + \left\| f_3 b^{-\frac{\alpha}{\gamma_2}} \right\|_{L_{1/(\alpha_2)}} + \left\| \mathcal{F}_1 b^{-\frac{\alpha}{\gamma_1}} \right\|_{L_{1/(\alpha_1)}} + \left\| \mathcal{F}_2 b^{-\frac{\alpha}{\gamma_2}} \right\|_{L_{1/(\alpha_2)}} + \left\| f \right\|_{C^1_j(S)} \right), \tag{2.19}
\]
which shows boundedness of $T(\varphi)$ in $C^1_1(S)$ and therefore in $C^1_1(\omega)$.

In order to prove the unique solvability in terms of variational statement (2.8), (2.11), we need the following Riesz’s theorem (concerning functionals on Hilbert spaces).

**Proposition 2.4 [23]** Let $H$ be a Hilbert space with an inner product. For each bounded linear functional $f$ on $H$ there is a unique $y \in H$ such that $f(x) = (x, y)$, where $x \in H$. Moreover, $\|f\|_{H^*} = \|y\|_H$, where $H^*$ is the dual space of $H$.

Apply Proposition 2.4 to $T(\varphi)$ to prove the existence and uniqueness of $v$. According to (2.10) and (2.19)

$$(v, \varphi)_{C^1_1(S)} = |T(\varphi)| \leq m \|\varphi\|_{C^1_1(S)} \left(\|f_1 b^{-\frac{r_1}{p_1}}\|_{L_{r_1}^1(\omega^s)} + \|f_3 b^{-\frac{r_3}{p_3}}\|_{L_{r_3}^2(\omega^s)} + \|f_2\|_{W_2^1(\omega^s)}\right). \quad (2.20)$$

By (2.20) and definition of the norm of a functional

$$\|v\|_{C^1_1(S)} \leq m \left(\|f_1 b^{-\frac{r_1}{p_1}}\|_{L_{r_1}^1(\omega^s)} + \|f_3 b^{-\frac{r_3}{p_3}}\|_{L_{r_3}^2(\omega^s)} + \|f_2\|_{W_2^1(\omega^s)}\right). \quad (2.21)$$

Taking into account the representation of $u$ in the form (2.10) and (2.21) we obtain

$$\|u\|_{C^1_1(S)} \leq m \left(\|f_1 b^{-\frac{r_1}{p_1}}\|_{L_{r_1}^1(\omega^s)} + \|f_3 b^{-\frac{r_3}{p_3}}\|_{L_{r_3}^2(\omega^s)} + \|f_2\|_{W_2^1(\omega^s)}\right). \quad (2.22)$$

(2.22) and (2.10) yield now uniqueness of solution $u$. $\square$

The next statement concerns the smoothness of solution.

**Proposition 2.5 [41]** Assume all the conditions from Theorem 2.1 are valid. Then:

1) if $F = (F_1, F_2) \in C^{k, \mu}(S^s)$, where $S^s$ is a subset of $S$, then $u \in C^{k, +2, \mu}(S^s)$, where $S^s$ is a closed subset of $S^s$ and the boundary of $S^s$ can have common points with $X_3$ (see Fig. 2.2 (a));

2) if $F = (F_1, F_2) \in C^{k, \mu}(S^s)$, where $S^s$ is a subset of $S$ and $f_1 \in C^{k, +2, \mu}(X_1)$, $f_3 \in C^{k, +2, \mu}(\omega^s)$ then $u \in C^{k, +2, \mu}(S^s)$, where $S^s$ is a closed subset of $S^s$ and the boundary of $S^s$ can have common points with $X_1$ and inner points of $\omega^s$, $k \geq 0$ (see Fig. 2.2 (b));

3) if $F = (F_1, F_2) \in C^{k, \mu}(S^s)$, where $S^s$ is a subset of $S$ and $f_1 \in C^{k, +2, \mu}(X_1)$, $f_3 \in C^{k, +2, \mu}(\omega)$ then $u \in C^{k, +2, \mu}(S^s)$, where $S^s$ is a closed subset of $S^s$ and the boundary of $S^s$ can have common points with $X_1$ and inner points of $\omega^s$, $k \geq 0$ (see Fig. 2.2 (c));

4) if $F = (F_1, F_2) \in C^{k, \mu}(S^s)$, where $S^s$ is a subset of $S$ and $f_1 \in C^{k, +1, \mu}(X_1)$, $f_2 \in C^{k, +2, \mu}(\omega)$, $f_3 \in C^{k, +1, \mu}(\omega^s)$ then $u \in C^{k, +2, \mu}(S^s)$, where
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$S''$ is a closed subset of $S'$ and the boundary of $S''$ can have common points with $K_1$ and inner points of $\omega, \omega^*$, $k \geq 0$ (see Fig. 2.2 (d));

Here $\mu' = \mu$ if $\mu < 1$ and $\mu' \to 1$ if $\mu = 1$ and $k \geq 0$.

In addition,

$$
\|u\|_{C^{k+2, \mu'}(S')} \leq m(\|F\|_{C^k(S')} + \|f_1\|_{C^{k+1, \mu}(K_1)} + \|f_2\|_{C^{k+2, \mu} (\omega)} + \|f_3\|_{C^{k+1, \mu} (\omega^*)}).
$$

Corollary 2.1 1) if $F \in C^{0, \mu}(S)$ then $u$ is twice continuously differentiable in $S$ and satisfies equation (2.5) and boundary condition (2.6);

2) if $f_1 \in C^{1, \mu}(K_1)$, $f_2 \in C^{2, \mu}(\omega)$ and $f_3 \in C^{1, \mu}(\omega^*)$ then $u$ satisfies boundary conditions (2.6) at all inner points of $\omega^*, \omega$ and $K_1$.

Therefore the generalized solution of problem (2.5)-(2.7) $u$ is a classical solution of this problem.

\[ \text{Figure 2.2. Subdomains of } S \text{ used in the proof of smoothness of the solution (Proposition 2.5).} \]

2.2.3 Representations of solutions

Let us obtain local representations of the solution $u_t(x_1, x_2)$, $t = 1, 2$ in the vicinities of the points belonging to different parts of $S$ and its boundary. The representations of $u_t(x_1, x_2)$ and the proofs of Propositions 2.6–2.9 are given in [41] and [4].

Let $P_0$ be an internal point of $S$ and $S_D(P_0)$ a disc with a radius $D$ and center at $P_0$ such that $S_D(P_0) \in S$ (see Fig. 2.3).

Proposition 2.6 Assume the all conditions from Theorem 2.1 are valid. Then in the disc $S_D(P_0) \in S$, every solution of problem (2.5)-(2.7) can be represented
in the form

\[ u_t(x_1, x_2) = \int_{S_D(P_0)} \sum_{j=1}^{2} G_{ij}(x_1 - \xi_1, x_2 - \xi_2) \mathcal{F}_j \, d\xi_1 \, d\xi_2, \quad t = 1, 2, \]

where

\[ G_{ij}(x_1 - \xi_1, x_2 - \xi_2) = \begin{cases} \frac{1}{4\pi k_0 + 1} \left( \ln R - k_0(x_i - \xi_i)(\ln R)_{x_i} \right), & t = j, \\ \frac{1}{4\pi(k_0 + 1)} (x_1 - \xi_1)k_0 (\ln R)_{x_2} = \frac{(x_2 - \xi_2)k_0}{4\pi(k_0 + 1)} (\ln R)_{x_1}, & t \neq j, \end{cases} \]

and

\[ \mathcal{F}_j = f_j^* + \chi \mathcal{F}_j, \]

\[ f_j^* = u_j \Delta \chi + 2(u_j x_1 \chi_{x_1} + u_j x_2 \chi_{x_2}) + k_0 \left( (u_{1x_1} \chi_{x_1} + u_{2x_2} \chi_{x_2}) + u_{1x_1} \chi_{x_1} + u_{2x_2} \chi_{x_2} + (\theta \chi)_{x_1} \right), \quad t \neq j = 1, 2, \]

here \( \chi \) is from (A.64).

Let \( P_0 \) be on \( \mathcal{K}_2 \). Consider a semicircle \( S_{D(P_0)}^K \cap S \) (see Fig. 2.3).

**Proposition 2.7** Assume all conditions from Theorem 2.1 are valid. Then in the semicircle \( S_{D(P_0)}^K \cap S \) every solution of problem (2.5)–(2.7) can be represented in the form

\[ u_1(x_1, x_2) = J_1^1 + J_1^2, \quad u_2(x_1, x_2) = J_2^1 + J_2^2, \]

where

\[ J_1^1 = \int_{S_{D(P_0)}^K} \sum_{j=1}^{2} \tilde{G}_{ij}(x_1 - \xi_1, x_2 - \xi_2) \mathcal{F}_j \, d\xi_1 \, d\xi_2, \]

\[ J_2^1 = \int_{S_{K_2}} \tilde{G}_{ij}(x_1 - \xi_1, x_2, h) u_1 \chi_{\xi_1} \, d\xi_1, \quad t = 1, 2, \]
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and

\[ \tilde{G}_{ij}(x_1 - \xi_1, x_2, \xi_2) = G_{ij}(x_1 - \xi_1, x_2, \xi_2) + G_{ij}(x_1 - \xi_1, x_2 + \xi_2), \]

\[ \tilde{G}_{2j}(x_1 - \xi_1, x_2, \xi_2) = G_{2j}(x_1 - \xi_1, x_2 - \xi_2) - G_{2j}(x_1 - \xi_1, x_2 + \xi_2), \]

where \( j = 1, 2 \), and \( \mathcal{F}_j \) are from Proposition 2.6.

Let \( P_0 \in \omega^* \). Consider a semicircle \( S^\omega_\mathcal{D} = S_\mathcal{D}(P_0) \cap S \) and \( S^\omega_\mathcal{D} \cap \omega = \emptyset \) (see Fig. 2.3).

**Proposition 2.8** Assume the all conditions from Theorem 2.1 are valid. Then in the semicircle \( S^\omega_\mathcal{D} \in S \) every solution of problem (2.5)–(2.7) can be represented in the form

\[ u_t(x_1, x_2) = J^1_t - J^2_t, \]

where

\[ J^1_t = \int_{S^\omega_\mathcal{D}(P_0)} \sum_{j=1}^{2} \tilde{G}_{ij}(x_1 - \xi_1, x_2, \xi_2) \mathcal{F}_j d\xi_1 d\xi_2, \]

\[ J^2_t = \int_{x_1 \cap S^\omega_\mathcal{D}(P_0)} \left( (f_1 x + u_1 x_1 + u_2 x_2, 1) \tilde{G}_{11}(x_1 - \xi_1, x_2, h) + (f_2 x + (k_0 - 1)u_1 x_1 + (k_0 + 1)u_2 x_2, 1) \tilde{G}_{11}(x_1 - \xi_1, x_2, h) u_1 x_1 \right) d\xi_1, \]

\[ t = 1, 2, \quad \tilde{G}_{ij} \text{ are from Proposition 2.7, and } \mathcal{F}_j \text{ are from Proposition 2.6}. \]

Let \( P_0 \in \omega \). Consider a semicircle \( S^\omega_\mathcal{D} = S_\mathcal{D}(P_0) \cap S \) and \( S^\omega_\mathcal{D} \cap \omega^* = \emptyset \) (see Fig. 2.3).

**Proposition 2.9** Assume the all conditions from Theorem 2.1 are valid. Then in the semicircle \( S^\omega_\mathcal{D} \in S \) every solution of problem (2.5)–(2.7) can be represented in the form

\[ u_t(x_1, x_2) = J^1_t - J^2_t + J^3_t, \]

where

\[ J^1_t = \int_{S^\omega_\mathcal{D}(P_0)} \sum_{j=1}^{2} \tilde{G}_{ij}(x_1 - \xi_1, x_2, \xi_2) \mathcal{F}_j d\xi_1 d\xi_2, \]

\[ J^2_t = \int_{x_1 \cap S^\omega_\mathcal{D}(P_0)} \tilde{G}_{11}(x_1 - \xi_1, x_2, \xi_2)(f_1 x + u_1 x_1 + u_2 x_2) d\xi_1, \]

\[ J^3_t = \int_{x_1 \cap S^\omega_\mathcal{D}(P_0)} \chi \tilde{f}_2 ((k_0 + 1)(\tilde{G}_{12})_{\xi_2} + (k_0 - 1)(\tilde{G}_{11})_{\xi_1}) d\xi_1, \]

\[ t = 1, 2, \quad \tilde{G}_{ij} \text{ are from Proposition 2.7, and } \mathcal{F}_j \text{ are from Proposition 2.6}. \]
2.2.4 Behavior of the solutions at infinity and at critical points

Proposition 2.10 [41] For any generalized solution \( u \) we have:

1) \[
|u_1 - c_0| \leq mb^{1+\varepsilon}(|x_1| + h), \quad \varepsilon > 0,
|u_2| \leq mb^{1+\varepsilon}(|x_1| + h), \quad \varepsilon > 0;
\]

2) if \( F = 0 \) then
\[
\partial_{x_1 x_2} (u_1 - c_0) \leq mb^{1+\varepsilon}(|x_1| + h), \quad \varepsilon > 0, \quad k_1 + k_2 \geq 1.
\]
\[
\partial_{x_1 x_2} u_2 \leq mb^{1+\varepsilon}(|x_1| + h), \quad \varepsilon > 0, \quad k_1 + k_2 \geq 1
\]
uniformly with respect to \( x_2 \) in \( 0 \leq x_2 \leq h_1 < h; \)

3) if in addition \( f_1 = f_3 = 0 \) on \( \omega^* \) then
\[
\partial_{x_1 x_2} (u_1 - c_0) \leq mb^{1+\varepsilon}(|x_1| + h), \quad \varepsilon > 0, \quad k_1 + k_2 \geq 1.
\]
\[
\partial_{x_1 x_2} u_2 \leq mb^{1+\varepsilon}(|x_1| + h), \quad \varepsilon > 0, \quad k_1 + k_2 \geq 1
\]
uniformly with respect to \( x_2 \) in \( 0 \leq x_2 \leq h. \)

Proposition 2.11 [41] Let \( f_2 \in C^{k,\mu}(\omega) \), \( k \geq 3, \mu > 0 \). Then for any generalized solution \( u \) in \( 0 < r \leq D - \delta \) the following representations are valid
\[
\partial_{x_1 x_2} \left( u_1 - 2\pi i \sum_{j=0}^{N_2} N_{j+1}(\theta) r^{-\lambda_1} - 2\pi i \sum_{p=0}^{N_1} Q_{p,\mu}(\theta) r^{-\lambda_2} \right) = \psi_k r^k (r, \theta),
\]
where \( u_1 = u_1(r) \) are radial and \( u_2 = u_2(\theta) \) tangential components of \( u \) in the coordinate system with the origin at \( a_k \); \( N_{j+1}(\theta) \) and \( Q_{p,\mu}(\theta) \) are some functions with respect to \( \theta \); \( \lambda_p = -p \), \( N_1 = k - 1 \), \( \lambda_j = -j - 0.5 \), \( N_2 = k - 1 \) if \( \mu < 0.5 \), \( N_2 = k \) if \( \mu > 0.5 \), \( k_1 + k_2 = k - 3 \); and
\[
|\psi_k|^2 \leq m(D, \delta, k, k_1, k_2) \| f_2 \|_{C^{k,\mu}(\omega)} \frac{1}{r^{k+\mu-k_1-\varepsilon}}
\]
for any \( \varepsilon \).

Note that in the case \( k_1 + k_2 = 0 \) we have \( u_2 \sim r^{\frac{1}{2}} + o(r^{\frac{1}{2}}), \ j = 1, 2. \)

2.3 Three-dimensional case

In the three-dimensional case we have a mixed BVP (1.37)–(1.39)
\[
\Delta u_j + k_0 \frac{\partial}{\partial x_j} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 3_j,
\]
\[
k_0 = \frac{1}{1 - 2\nu}, \quad j = 1, 2, 3.
\]
with the boundary conditions

\[
\begin{align*}
    u_3 &= 0, \quad \frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} = 0 \quad \text{on } \mathcal{K}_2, \\
    \frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3} &= f_j(x_1, x_2) \quad \text{on } \mathcal{K}_1, \\
    u_3 &= f_3(x_1, x_2) \quad \text{on } \Omega, \\
    (k_0 - 1)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + (k_0 + 1)\frac{\partial u_3}{\partial x_3} &= f_4(x_1, x_2) \quad \text{on } \Omega^*,
\end{align*}
\]

where \( j = 1, 2 \), and the conditions at infinity

\[
\Phi_V(u_1, u_2, u_3) = \int V \Phi dV < \infty,
\]

\[
H_V = (k_0 - 1)\left(\sum_{j=1}^{2} \frac{\partial u_3}{\partial x_j}\right)^2 + \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial x_j}\right)^2 + \sum_{j \neq k=1}^{3} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)^2.
\]

**Definition 2.16** A vector-function \( u = (u_1, u_2, u_3) \in C^1_0(V) \) is the generalized solution of BCP (2.23)-(2.25) if \( u_3(x_1, x_2, h) = f_3(x_1, x_2) \) on \( \Omega \) and

\[
\begin{align*}
(u, \varphi)_{C^1_0(S)} &= \int_{\mathcal{K}_1} \left(f_1(\varphi_1 - c_1 + c_3x_2) + f_2(\varphi_2 - c_2 - c_3x_1)\right) dx_1 dx_2 + \\
&\quad \int_{\Omega^*} \left(f_4\varphi_3 dx_1 dx_2 + \int_{V} \left(\mathcal{F}_1(\varphi_1 - c_1 + c_3x_2) - \mathcal{F}_2(\varphi_2 - c_2 - c_3x_1) - \mathcal{F}_3\varphi_3\right) dV \right) \\
&\quad = \int_{V} \left(\sum_{j=1}^{2} \frac{\partial u_3}{\partial x_j}\right)^2 + \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial x_j}\right)^2 + \sum_{j \neq k=1}^{3} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)^2.
\end{align*}
\]

for every vector-function \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \in C^1_0(S) \) and any constants \( c_1, c_2, c_3 \).

Existence and uniqueness for problem (2.23)-(2.25) are proved within the frames of the following two theorems:

**Theorem 2.2** Assume that the following conditions hold:

1) body forces satisfy the conditions

\[
\mathcal{F}_j \cdot b \cdot \frac{r_j^{p_j}}{p_j} \in L_{p_j}(V), \quad j = 1, 2, \quad \mathcal{F}_3 \cdot b \cdot \frac{r_3^{p_3}}{p_3} \in L_{p_3}(V),
\]

where \( p_j > 6/5 \), \( j = 1, 2, 3 \), and \( p_j^{*} \), \( j = 1, 2, 3 \), are defined by condition 1) from Proposition 2.3 with \( p = \frac{p_j}{p_j^{*}} \) and \( p_3^{*} \) is defined by condition 3) from Proposition 2.3 with \( p = \frac{p_3}{p_3^{*}} \),

2) \( f_1, f_2, \) and \( f_4 \) satisfy the conditions

\[
\begin{align*}
    f_j \cdot b \cdot \frac{r_j^{p_j}}{p_j} \in L_{p_j}(\mathcal{K}_1), \quad j = 1, 2, \quad f_4 \cdot b \cdot \frac{r_4^{p_4}}{p_4} \in L_{p_4}(\Omega^*),
\end{align*}
\]

where \( p_j > 4/3 \), \( j = 1, 2, 3 \), and \( p_j^{*} \), \( j = 1, 2, 3 \), are defined by condition 2) from Proposition 2.3 with \( p = \frac{p_j}{p_j^{*}} \), \( p_3^{*} \) is defined by condition 4) from Proposition 2.3 with \( p = \frac{p_3}{p_3^{*}} \), and \( b = \sqrt{x_1^2 + x_2^2 + h^2} \).
3) \( f_3 \in W_2^2(\Omega) \);
4) curves \( \Gamma_m \) have differentiable curvature at every point.

Then problem (2.23)–(2.25) is uniquely solvable if and only if

\[
\int_{\xi_1} f_j \, dx_1 \, dx_2 + \int_V F_j \, dV = 0, \quad j = 1, 2,
\]

\[
\int_{\xi_1} (f_1 x_2 - f_2 x_1) \, dx_1 \, dx_2 + \int_V (F_1 x_2 - F_2 x_1) \, dV = 0. \tag{2.27}
\]

Proof.

Necessity. Assume that there is an \( u \in C_1^1(\Gamma) \) satisfying (2.26) for every \( \varphi \in C_1^1(\Gamma) \). Substitute \( \varphi = 0 \) into (2.26) and obtain (2.27) (\( c_1, c_2, c_3 \) are arbitrary constants).

Sufficiency. Set

\[
u = v + f, \tag{2.28}
\]

where \( u = (u_1, u_2, u_3) \), \( v = (v_1, v_2, v_3) \), \( f = (0, 0, f_3) \), and \( \tilde{f}_3 \) is an extension of \( f_3 \) according to Lemma 5.13. Substituting (2.28) into (2.26) we obtain an integral identity

\[
(v, \varphi)_{C_1^1(V)} = \int_{\xi_1} (f_1 \varphi_1 + f_2 \varphi_2) \, dx_1 \, dx_2 + \int_V f_3 \varphi_3 \, dx_1 \, dx_2 +
\]

\[
+ \int_V (F_1 \varphi_1 + F_2 \varphi_2 + F_3 \varphi_3) \, dv - (f, \varphi)_{C_1^1(V)} \equiv T(\varphi).
\]

\( T(\varphi) \) is a linear bounded functional in \( C_1^1(V) \). Linearity is obvious. To prove boundedness of \( T(\varphi) \) note that according to (2.27) \( T(\varphi) \) can be represented in the form

\[
T(\varphi) = \int_{\xi_1} (f_1 (\varphi_1 - c_1) + f_2 (\varphi_2 - c_2)) \, dx_1 \, dx_2 + \int_V f_3 \varphi_3 \, dx_1 \, dx_2 +
\]

\[
+ \int_V (F_1 (\varphi_1 - c_1) + F_2 (\varphi_2 - c_2) + F_3 \varphi_3) \, dv - (f, \varphi)_{C_1^1(V)},
\]

where \( c_1, c_2 \) are some constant. By the Hölder inequality

\[
|T(\varphi)| \leq \sum_{i=1}^2 \| f_i b^{-\frac{p_i}{2n}} \|_{L_{p_i}^2(\xi_1)} \cdot \| (\varphi_1 - c_1) b^{\frac{1}{2n}} \|_{L_p^2(\xi_1)} +
\]

\[
+ \| f_i b^{-\frac{p_i}{2n}} \|_{L_{p_i}^2(\Gamma')} \cdot \| (\varphi_2 - c_2) b^{\frac{1}{2n}} \|_{L_{p_i}^2(\Gamma')} +
\]

\[
+ \sum_{i=1}^2 \| F_i b^{\frac{1}{2n}} \|_{L_{p_i}^2(\Gamma')} \cdot \| (\varphi_1 - c_1) b^{\frac{1}{2n}} \|_{L_{p_i}^2(\Gamma')} +
\]

\[
+ \| F_i b^{\frac{1}{2n}} \|_{L_{p_i}^2(\Gamma')} \cdot \| (\varphi_2 - c_2) b^{\frac{1}{2n}} \|_{L_{p_i}^2(\Gamma')} + \| f \|_{C_1^1(V)} \cdot \| \varphi \|_{C_1^1(V)}.\]
2.3. THREE-DIMENSIONAL CASE

Taking into account the conditions of the Theorem for $p_k$ and $p'_k$, we apply Proposition 2.3 and obtain, according to Lemma 5.13, the boundedness of $T(\varphi)$ in $C^2_{t,1}(V)$

$$
|T(\varphi)| \leq m \|\varphi\|_{C^1_{t}(V)} \left( \sum_{j=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega^*)} + \right. \\
+ \left. \sum_{t=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \right. \\
\left. \left. \|f_{3}\| \|u\|_{W_{2}^{2}(\Omega)} \right) \right). \tag{2.29}
$$

Applying Proposition 2.4 to $T(\varphi)$ to prove the existence and uniqueness of $v$. According to (2.28) and (2.29)

$$(w, \varphi)_{C^2_{j}(S)} = |T(\varphi)| \leq$$

$$
\leq m \|\varphi\|_{C^1_{t}(V)} \left( \sum_{j=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega^*)} + \right. \\
+ \left. \sum_{t=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \right. \\
\left. \left. \|f_{3}\| \|u\|_{W_{2}^{2}(\Omega)} \right) \right). \tag{2.30}
$$

By (2.30) and definition of the norm of a functional

$$
\|w\|_{C^2_{j}(S)} \leq m \|\varphi\|_{C^1_{t}(V)} \left( \sum_{j=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega^*)} + \right. \\
+ \left. \sum_{t=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \right. \\
\left. \left. \|f_{3}\| \|u\|_{W_{2}^{2}(\Omega)} \right) \right). \tag{2.31}
$$

Taking into account the representation of $u$ in the form (2.28) and (2.31) we obtain

$$
\|u\|_{C^2_{j}(S)} \leq m \|\varphi\|_{C^1_{t}(V)} \left( \sum_{j=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(\Omega^*)} + \right. \\
+ \left. \sum_{t=1}^{2} \left| f_{j} \right| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \|f_{4} \| b^{-\frac{r_{1}}{2r_{2}}} \|u\|_{L_{r_{1}}^{1}(V)} + \right. \\
\left. \left. \|f_{3}\| \|u\|_{W_{2}^{2}(\Omega)} \right) \right). \tag{2.32}
$$

(2.28) and (2.32) yield now uniqueness of solution $u$. □

The next proposition states the existence of the classical solution.

**Proposition 2.12** [41] Assume the all conditions from Theorem 2.2 are valid. Then:

1) if $f_j \in C^{0,\nu}(V)$ ($j = 1, 2, 3$) then $u_j$ ($j = 1, 2, 3$) are twice continuously differentiable at all inner points of $V$ and satisfy equation (2.23) in $V$ and boundary condition (2.24) on $\Omega_1$;

2) if $f_j \in C^{1,\nu}(X_1)$ ($j = 1, 2$), $f_3 \in C^{0,\nu}(\Omega)$, and $f_4 \in C^{1,\nu}(\Omega^*)$ then $u_j$ ($j = 1, 2, 3$) satisfy boundary conditions (2.24) at all inner points of $\Omega^*$; $\Omega$ and $\Omega_2$.

Therefore the solution $(u_1, u_2, u_3)$ of problem (2.23)–(2.25) is a classical solution.
The solvability theory for the BCPs is constructed in this chapter on the basis of variational approach. It may be difficult to apply the theory for obtaining the solutions explicitly and working out analytical, semi-analytical, and numerical methods. In the next chapter we develop the method of boundary integral equations for the solution to BCPs which complements the general technique based on variational statements.
Chapter 3

Method of integral equations

In this chapter we reduce BCPs to a Fredholm integral equation with a logarithmic singularity of the kernel. We introduce general properties of the integral operators with a logarithmic singularity of the kernel including the case of several intervals of integration. To solve the integral equation we describe two methods based on the use of the Fourier-Chebyshev series and matrix-algebraic determination of the entries in the resulting infinite-system matrix. We obtain the solution in the form of series and also asymptotic series in powers of the characteristic small parameter of the problem.

3.1 Reduction to a Fredholm integral equation

We will reduce problem (2.5)–(2.7) to an integral equation under the following conditions

1. Body forces \( \mathbf{F} = 0 \) in the band \( S = \{ x = (x_1,x_2) : (-\infty < x_1 < +\infty, 0 < x_2 < h) \} \);
2. Shear stresses \( f_1 = 0 \) on the line \( K_2 \);
3. Normal stresses \( f_3 = 0 \) on the set \( \omega^* \);
4. \( f_2(x_1) = f(x_1) \in C^{k,\mu} (\omega), \ k \geq 2, \ \mu > 0. \)

Here \( \omega = \bigcup \omega_k \subset \{ x_2 = h, 0 < x_1 < a \} \) denotes a set of disjoint intervals.

Under these conditions problem (2.5)–(2.7) takes the form

\[
\Delta u_k + k_0 \frac{\partial}{\partial x_k} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = 0, \quad x \in S, \ k = 1, 2,
\]

\[
u_2 = 0 \quad \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \quad \text{on } K_2,
\]

\[
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1} = 0 \quad \text{on } K_1,
\]

\[
\frac{\partial u_2}{\partial x_2} = f(x_1) \quad \text{on } \omega,
\]

\[
(k_0 - 1) \frac{\partial u_1}{\partial x_1} + (k_0 + 1) \frac{\partial u_2}{\partial x_2} = 0 \quad \text{on } \omega^*.
\]
Consider the following BVP
\[ \Delta u_k + k_0 \frac{\partial}{\partial x_k} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = 0, \quad x \in S, \quad k = 1, 2, \]
\[ u_2 = 0 \quad \text{on} \quad K_2, \]
\[ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \quad \text{on} \quad K_1, \]
\[ u_2 = f(x) \quad \text{on} \quad \omega, \]
\[ (k_0 - 1) \frac{\partial u_1}{\partial x_1} + (k_0 + 1) \frac{\partial u_2}{\partial x_2} = \begin{cases} 3(k_0 - 1) E_0 q(x_1), & x_1 \in \omega, \\ 0, & x_1 \in \omega^*. \end{cases} \]  
\[
(3.2)
\]

In this case integral identity (2.8) takes the form
\[ (u, \varphi)_{\bar{C}_1} = -\frac{3k_0 - 1}{E k_0} \int_\omega q(x_1) \varphi_2(x_1) dx_1, \]
\[
(3.3)
\]
where \( q(x_1) \in L_p(\pi) \) for \( p > 1 \) and \( \varphi = (\varphi_1, \varphi_2) \) is any function from \( \bar{C}_1^{1}(S) \).

**Theorem 3.1** The generalized solution of problem (3.2) exists for any \( q(x_1) \in L_p(\omega) \), \( p > 1 \).

**Proof.** By the H"older inequality and Proposition 2.2
\[ \left| \int_\omega q(x_1) \varphi_2(x_1) dx_1 \right| \leq \|q\|_{L_p(\omega)} \cdot \|\varphi_2\|_{L_p(\omega)} \leq m \cdot \|q\|_{L_p(\omega)} \cdot \|\varphi\|_{\bar{C}_1}. \]
It follows that the right-hand side of (3.3) is a linear bounded functional on \( \bar{C}_1^{1} \) so that by Riesz’s Theorem (Proposition 2.4) the solution to (3.2) exists. \( \square \)

Introduce the function
\[ \varphi_i(x_1, x_2) = e^{i\lambda x_1} \chi(x_1) c_i(x_2), \quad \tilde{\chi}(x_1) = \chi(x_1) \chi(-x_1), \]
where \( \chi \) is from (A.64), \( c_i(x_2) \) are continuously differentiable on \([0, h]\), \( c_i(0) = 0 \), and \(-\infty < \lambda < \infty\) is a parameter. Substitute \( \varphi \) into (3.3) and let \( D \to \infty \) to obtain
\[ (u, \varphi)_{\bar{C}_1} = \int_S e^{i\lambda x_1} \left( i\lambda c_1 (k_0 + 1) u_{1x_1} + (k_0 - 1) u_{2x_1} \right) + \lambda c_2 (u_{1x_2} + u_{2x_1}) +
+ c_2' (k_0 - 1) u_{1x_2} + (k_0 + 1) u_{2x_2} + c_1' (u_{1x_2} + u_{2x_1}) \right) dx_1 dx_2 =
-\frac{3k_0 - 1}{E k_0} \int_\omega q(x_1) e^{i\lambda x_1} dx_1 \cdot c_2(h). \]
\[
(3.4)
\]
3.1. REDUCTION TO A FREDHOLM INTEGRAL EQUATION

**Theorem 3.2** Integral identity (3.4) is equivalent to the BVP

\[
\frac{\partial^2 u^*_1}{\partial x_2^2} - k_0^2 \lambda \frac{\partial u^*_1}{\partial x_2} - (k_0 + 1)\lambda^2 u^*_1 = 0, \quad 0 < x_2 < h,
\]

\[
(k_0 + 1) \frac{\partial^2 u^*_2}{\partial x_2^2} - k_0^2 \lambda \frac{\partial u^*_2}{\partial x_2} - \lambda^2 u^*_2 = 0, \quad 0 < x_2 < h,
\]

\[
(k_0 + 1) \frac{\partial u^*_1}{\partial x_2} - (k_0 - 1)i\lambda u^*_1 + \frac{3k_0 - 1}{\sqrt{2\pi k_0} E} \int_\omega q(x_1)e^{i\lambda x_1} dx_1 \bigg|_{x_2 = h} = 0,
\]

\[
\frac{\partial u^*_1}{\partial x_2} - i\lambda u^*_2 \bigg|_{x_2 = h} = 0, \quad \frac{\partial u^*_1}{\partial x_2} - i\lambda u^*_2 \bigg|_{x_2 = 0} = 0, \quad u^*_2 \bigg|_{x_2 = 0} = 0;
\]

here

\[
u^*_k(\lambda, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_k(x_1, x_2)e^{i\lambda x_1} dx_1
\]

are the Fourier transforms of \(u_k\), \(k = 1, 2\).

Proof of Theorem 3.2 consists of several steps based on the theory of integral equations with a logarithmic singularity of the kernel developed in Chapter 3 and includes the proof of the existence of the solution to (3.4) obtained using the inverse Fourier transform of \(u^*_k\), \(k = 1, 2\).

The general solution to (3.5) is

\[
u^*_1(\lambda, x_2) = a_1(\lambda) \sinh(\lambda x_2) + a_2(\lambda) \cosh(\lambda x_2) +
\]

\[+ a_3(\lambda) \lambda x_2 \sinh(\lambda x_2) + a_4(\lambda) \lambda x_2 \cosh(\lambda x_2), \quad (3.7)
\]

\[
u^*_2(\lambda, x_2) = a_1(\lambda) i \cosh(\lambda x_2) + a_2(\lambda) i \sinh(\lambda x_2) +
\]

\[+ a_3(\lambda) i (\lambda x_2 \cosh(\lambda x_2) - \frac{2 + k_0}{k_0} \sinh(\lambda x_2)) +
\]

\[+ a_4(\lambda) i (\lambda x_2 \sinh(\lambda x_2) - \frac{2 + k_0}{k_0} \cosh(\lambda x_2)), \quad (3.8)
\]

Taking into account the boundary conditions in (3.5) we obtain

\[a_1(\lambda) = a_4(\lambda) = 0,
\]

\[a_2(\lambda) = -A_0 q^*(\lambda) \frac{1}{\lambda} \frac{\sinh(\lambda h) - \lambda h \cosh(\lambda h)}{\lambda(2\lambda h + \sinh(2\lambda h))}, \quad (3.9)
\]

\[a_3(\lambda) = -A_0 q^*(\lambda) \frac{\sinh(\lambda h)}{\lambda(2\lambda h + \sinh(2\lambda h))},
\]

where \(A_0 = (3k_0 - 1)/(k_0 E i)\),

\[q^*(\lambda) = \frac{1}{\sqrt{2\pi}} \int_\omega q(x_1)e^{i\lambda x_1} dx_1, \quad (3.10)
\]

and \(q(x_1)\) is the solution to an integral equation of the first kind

\[K q \equiv \int_\omega K \left(\frac{x_1 - \xi}{h}\right) q(\xi) d\xi = \pi E_0 f(x_1), \quad (3.11)
\]
where
\[ \mathcal{K}\left(\frac{x_1 - \xi}{h}\right) = \int_{0}^{\infty} \frac{2\sinh^2 \eta}{2\eta^2 + \eta \sinh 2\eta} \cos\left(\eta \frac{x_1 - \xi}{h}\right) d\eta. \quad (3.12) \]

\[ E_0 = \frac{2k_0^2 E}{(3k_0 - 1)(k_0 + 1)}, \quad \text{and } E \text{ is the modulus of elasticity.} \]

Therefore solutions (3.7) and (3.8) take the form
\[ u^*_k(\lambda, x_2) = A_0 q^*(\lambda)U^*_k(\lambda, x_2), \quad k = 1, 2, \quad (3.13) \]

where
\[ U_1^*(\lambda, x_2) = b_1(\lambda) \cosh(\lambda x_2) + b_2(\lambda) \lambda x_2 \sinh(\lambda x_2), \]
\[ U_2^*(\lambda, x_2) = b_1(\lambda)i \sinh(\lambda x_2) + b_2(\lambda)i(\lambda x_2 \cosh(\lambda x_2) - \frac{2 + k_0}{k_0} \sinh(\lambda x_2)), \quad (3.14) \]
\[ b_1 = \frac{\sinh(\lambda h) - \lambda h \cosh(\lambda h)}{\lambda(2h + \sinh(2\lambda h))}, \]
\[ b_2 = \frac{\sinh(\lambda h)}{\lambda(2\lambda h + \sinh(2\lambda h))}. \quad (3.15) \]

A detailed investigation of the properties of the kernel of (3.11) is performed in Section 3.3. The unique solvability of (3.11) is proved in [41] (see Proposition 3.3 in Section 3.3).

Solving integral equation (3.11) with respect to \( q(x_1) \) we obtain \( q^*(\lambda) \) from (3.10). After that we can obtain \( u^*_k(\lambda, x_2) \) from (3.13) and finally \( u_k(x_1, x_2) \) from (3.6) using the inverse Fourier transform. This procedure is justified below.

First we prove several estimates leading to the existence of the inverse Fourier transform that gives the solution to BCP (3.2).

**Lemma 3.1** The following estimates hold
\[ |b_j(\lambda)| \leq B_j e^{-|\lambda|/h}, \quad j = 1, 2, \quad (3.16) \]
for \( |\lambda| \geq \lambda_0 > 0 \), where \( B_j, j = 1, 2 \) do not depend on \( \lambda \).

**Proof.** Estimates can be obtained using the explicit form (3.15). \( \square \)

**Lemma 3.2** The following estimates are valid for \( |\lambda| \geq \lambda_0 > 1 \) and \( 0 < x_2 \leq k_0 h < h \)
\[ |u^*_k(\lambda, x_2)| \leq A_k \max_{\lambda} |q^*(\lambda)| e^{-|\lambda|/(h-x_2)}, \quad k = 1, 2, \quad (3.17) \]
where \( A_k \) do not depend on \( \lambda \).

**Proof.** We obtain (3.17) using the inequalities \( |\sinh(\lambda x_2)| \leq e^{\lambda x_2}, |\cosh(\lambda x_2)| \leq e^{\lambda x_2} \). Lemma (3.1), and the explicit expressions (3.13) and (3.14). \( \square \)

Estimates (3.17) yield the existence of the inverse Fourier transforms
\[ u_k(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^*_k(\lambda, x_2) e^{-i\lambda x_1} d\lambda, \quad k = 1, 2, \quad (3.18) \]
in the strip \( 0 < x_2 < h \) because, according to (3.10) and the properties of \( q(x_1) \) established in Proposition 3.3, \( |q^*(\lambda)| < M, \lambda \in R, \) where \( M = const. \) In [41] it is shown that the functions \( u_k(x_1, x_2), \ k = 1, 2, \) obtained according to (3.13)–(3.12) and (3.18) satisfy (3.4). Theorem 3.2 is proved. \( \square \)
3.2. Solution to the BCP in the form of series

In this section we will show that the solution to BCP (3.2) can be evaluated using the solution of integral equation (3.11) with a logarithmic singularity of the kernel obtained in the form of Fourier-Chebyshev series.

3.2.1 Evaluation of the series solution

Consider the case when the set \( \omega \) in problem (3.2) consists of one interval, \( \omega = (a, b) \) and denote by \( d = (a + b)/2 \) and \( w = (b - a)/2 \) its center and half-length. Set \( x_1 = wt + d \) and denote \( \tilde{q}(t) = q(wt + d), \quad t \in (-1, 1). \) (3.19)

Lemma 3.3 Assume that

\[ \tilde{q}(t) = \rho(t) \sum_{n=0}^{\infty} q_n T_n(t), \] (3.20)

where \( \rho(t) = (1 - t^2)^{-1/2}, T_n(t) = \cos(n \arccos t) \) are the Chebyshev polynomials of the first kind, and \( \tilde{q} = \{q_n\}_{n=0}^{\infty} \in l^2 = \{\xi = (\xi_1, \xi_2, \ldots) : \sum_{n=0}^{\infty} |\xi_n|^2 < \infty\} \).

Then

\[ \tilde{q}^*(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \tilde{q}(t)e^{i\lambda t} dt = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} q_n i^n J_n(\lambda), \] (3.21)

where \( J_n(\lambda) \) are the Bessel functions of order \( n \), \( |\tilde{q}^*(\lambda)| < Q^*, \lambda \in R \), and series (3.21) converges uniformly on compact subsets of \( R \) and admits termwise differentiation arbitrary number of times.

Proof. Using the integral representations [1] for the Bessel functions

\[ J_n(z) = \frac{1}{\pi} \int_{0}^{\pi} e^{iz \cos \tau} \cos(nt) d\tau, \]

we determine the integrals

\[ \int_{-1}^{1} T_n(t) \rho(t)e^{i\lambda t} dt = \int_{0}^{\pi} \cos(\pi \tau)e^{i\lambda \cos \tau} d\tau = \pi i^n J_n(\lambda), \]

where \( \tau = \arccos t \) and obtain then formula (3.21) multiplying both sides in (3.20) by \( e^{i\lambda t} \) and integrating over \((-1, 1)\). \( J_n(\lambda) \) are analytic functions, and from the series representation of the Bessel functions [1] it follows that

\[ J_n(\lambda) = \frac{\lambda^n}{2^n n!} \left( 1 + O\left( \frac{1}{n} \right) \right), \quad n \to \infty \] (3.22)
uniformly on compact subsets of $R$, which yields the statement of Lemma 3.3.

According to (3.19) and Lemma 3.3 we have

$$q^*(\lambda) = w e^{i\lambda d} \tilde{q}^*(\lambda w) = \sqrt{\frac{\pi}{2}} w e^{i\lambda d} \sum_{n=0}^{\infty} q_n i^n J_n(\lambda w), \quad \lambda \in R, \quad (3.23)$$

where $q^*(\lambda)$ is from (3.10). 

Consider the case when $\omega$ in (3.2) consists of several intervals. Let

$$\omega = \bigcup_{j=1}^{m} \omega_j, \quad \omega_j = (d_j - w_j, d_j + w_j), \quad \omega_j \cap \omega_k = \emptyset, \quad j \neq k, \quad j, k = 1, 2, \ldots, m. \quad (3.24)$$

Denote by $q_j(x_1) = q(x_1)$, $x_1 \in \omega_j$, the restriction of $q(x_1)$ on interval $\omega_j$ and set $x_1 = w_j t + d_j$, $-1 < t < 1$, and $\tilde{q}_j(t) = q_j(w_j t + d_j)$, $j = 1, 2, \ldots, m$. Assuming that

$$\tilde{q}_j(t) = \rho(t) \sum_{n=0}^{\infty} q_n^{(j)} T_n(t), \quad -1 < t < 1, \quad q_n^{(j)} = (q_n^{(j)})_{n=0}^{\infty} \in l^2,$$

we have

$$q_j(x_1) = \rho_j(x_1) \sum_{n=0}^{\infty} q_n^{(j)} T_n(\frac{x_1 - d_j}{w_j}). \quad (3.25)$$

where

$$\rho_j(x_1) = w_j \left( (d_j + w_j) - x_1 \right) \left( x_1 - (d_j - w_j) \right)^{-\frac{1}{2}}, \quad j = 1, 2, \ldots, m. \quad (3.26)$$

According to Lemma 3.3 and (3.23) we have

$$q_j^*(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\omega_j} q_j(x_1) e^{i\lambda x_1} dx_1 = w_j e^{i\lambda d} \tilde{q}_j^*(\lambda w_j),$$

where

$$\tilde{q}^*(z) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \tilde{q}_j(t) e^{itz} dt = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} q_n^{(j)} i^n J_n(z), \quad j = 1, 2, \ldots, m,$$

so that

$$q^*(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\omega} q(x_1) e^{i\lambda x_1} dx_1 = \sum_{j=1}^{m} q_j^*(\lambda) = \sum_{j=1}^{m} w_j e^{i\lambda d} \tilde{q}_j^*(\lambda w_j) = \sqrt{\frac{\pi}{2}} \sum_{j=1}^{m} w_j e^{i\lambda d} \left( \sum_{n=0}^{\infty} q_n^{(j)} i^n J_n(\lambda w_j) \right). \quad (3.27)$$

From (3.13) and (3.27) it follows that

$$u_k^*(\lambda, x_2) = \frac{\pi}{2} A_0 U_k^*(\lambda, x_2) \sum_{j=1}^{m} w_j e^{i\lambda d_j} \sum_{n=0}^{\infty} q_n^{(j)} i^n J_n(\lambda w_j), \quad k = 1, 2. \quad (3.28)$$
3.2. SOLUTION TO THE BCP IN THE FORM OF SERIES

Substituting (3.28) into (3.18) we obtain

$$u_k(x_1, x_2) = \frac{A_0}{2} \sum_{j=1}^{m} w_j \left( \sum_{n=0}^{\infty} \frac{q_j(n, j)}{n^j} u_{n,k}(x_1, x_2) \right),$$

(3.29)

where

$$u_{n,k}(x_1, x_2) = \int_{-\infty}^{\infty} \tilde{u}_{n,k}^{(j)}(\lambda; x_1, x_2) d\lambda,$$

(3.30)

and

$$\tilde{u}_{n,k}^{(j)}(\lambda; x_1, x_2) = U_{n}^{(j)}(\lambda, x_2) J_n(\lambda w_j) e^{-i\lambda(x_1 - d_j)},$$

$$n = 0, 1, 2, \ldots, j = 1, 2, \ldots, k = 1, 2.$$

(3.31)

Theorem 3.3 Assume that (i) the solution $q(x_1)$ of integral equation (3.11) has the form (3.25), (3.26); (ii) the set $\omega$ is given by (3.24) and $\omega = \max_i w_i \in I_0 = (0, w_0)$; (iii) the conditions of Lemma 3.3 are satisfied; and (iv) $0 < h_0 < h$.

Then the (classical) solution $u = (u_1, u_2)$ of problem (3.2) is given by (3.29)–(3.31) for $(x_1, x_2) \in S_{h_0} = R \times [0, h_0]$, where $u_k \in C^2(S_{h_0})$, $k = 1, 2$.

Proof. In what follows we omit the upper index $(j)$ of quantities $u_{n,k}^{(j)}$ and $\tilde{u}_{n,k}^{(j)}$ estimating every $j$th term of the finite sum (3.29).

According to Lemmas 3.1–3.3 and properties of the Bessel functions [1], there is a $\lambda_0 > 1$ such that

$$|\tilde{u}_{n,k}(\lambda; x_1, x_2)| = |e^{-i\lambda(x_1 - d_j)} U_{n}^{(j)}(\lambda, x_2) J_n(\lambda w_j)| \leq A_0^j e^{-(\lambda - x_2)},$$

$$k = 1, 2, j = 1, 2, \ldots, m,$$

(3.32)

which holds for $|\lambda| \geq \lambda_0$, $x_1 \in R$, $\omega \in I_0 = (0, w_0)$, and $0 \leq x_2 \leq h_0$, where $A_0^j = \text{const.}$ $j = 1, 2, \ldots, m$, and $0 < h_0 < h$. It is easy to check that estimates similar to (3.32) are valid for the derivatives of the integrand in (3.30) with respect to $\lambda, x_1,$ and $x_2$. Thus, improper integrals in (3.30) converge absolutely and uniformly with respect to $x \in R$, $\omega \in I_0 = (0, w_0)$, and $x_2 \in [0, h_0]$ and $u_{n,k}(x_1, x_2)$ in (3.30) are differentiable functions of $x_1$ and $x_2$ on $[0, h_0] \times R$.

In order to verify the rate of decay of $u_{n,k}$ with respect to $n$, write

$$u_{n,k}(x_1, x_2) = I_{0,n}^{(k)} + I_{1,n}^{(k)} = \int_{|\lambda|<\lambda_0} + \int_{|\lambda|>\lambda_0} \tilde{u}_{n,k}(\lambda; x_1, x_2) d\lambda,$$

(3.33)

Estimate $I_{0,n}^{(k)}$, $n \geq 0$. Using (3.22), properties of the Bessel functions, and the estimate $|I_{0,n}^{(k)}(\lambda, x_2)| \leq M_0^k$, $|\lambda| \leq \lambda_0$, where $M_0^k = M_0^k(\lambda_0) = \text{const}$, which holds, according to Lemma 3.3, because $U_{n}^{(j)}(\lambda, x_2) \in C^2[-\lambda_0, \lambda_0]$, we have

$$|I_{0,n}^{(k)}(\lambda; x_1, x_2)| \leq \frac{C_0^n}{2\pi n} |\lambda_0|^n,$$

where $|C_0^n| < \tilde{C}$, $n \geq 1$, uniformly with respect to $x_1 \in R$, $\omega \in I_0 = (0, w_0)$, and $x_2 \in [0, h_0]$. 


In order to obtain asymptotic series for the integral using the formula (3.22) we obtain
\[
|I_{r,n}^{(k)}(\lambda; x_1, x_2)| \leq \frac{A_n^{*k}}{2^n n!} |\lambda|^{n} e^{-|\lambda| h(x_1 - x_2)}, \quad |\lambda| \geq \lambda_0 > 1, \tag{3.34}
\]
where \(A_n^{*k} = \text{const.}\), which holds uniformly with respect to \(x_1 \in R, \ w \in I_0 = (0, w_0), \) and \(x_2 \in [0, h_0].\) Integrating inequality (3.34) and estimating the integral using the formula \(n! n^{-n+1} = \int_0^{\infty} t^n e^{-tp} dt,\) we have
\[
|I_r^{(k)}(\lambda; x_1, x_2)| \leq \frac{A_n^{*k}}{(2(h - x_2))^{n+1}}.
\]
Thus,
\[
|u_n,k(x_1, x_2)| \leq \frac{A_n^{*k}}{(2(h - x_2))^{n+1}} \tag{3.35}
\]
for any \(\lambda_0 > 1\) and sufficiently large \(n.\) Using (3.32) and explicit representations (3.14) for \(U_k^* (\lambda, x_2)\) one can show that similar estimates hold for the derivatives of \(u_n,k(x_1, x_2).\)

Finally, taking into account the assumptions of Lemma 3.3 concerning the convergence of series (3.20) and using (3.35) we conclude that series entering every \(j\)th term of (3.29) converges absolutely and uniformly on \(S_{h_0} = R \times [0, h_0]\) and define the functions \(u_k(x_1, x_2) \in C^2(S_{h_0}),\) \(k = 1, 2.\) \(\boxdot\)

If \(q(t) = \rho(t) \sum_{n=0}^{N} q_n T_n(t)\) is a finite sum of Chebyshev polynomials then the Fourier transform (3.21) of \(q(t)\) (Lemma 3.3) is also a finite sum, as well as the representation (3.29) of the solution to BCP (3.2); the latter statement follows directly from Lemma 3.3.

### 3.2.2 Asymptotic series

Consider the case when \(\omega = (d - w, d + w)\) in (3.2) consists of one interval. In order to obtain asymptotic series for \(u_k\) and \(u_{n,k}\) in powers of the small parameter \(w,\) write (3.29)–(3.31) for \(m = 1:\)
\[
u_k(x_1, x_2) = \frac{A_0}{2} w \sum_{n=0}^{\infty} q_n \nu_n u_{n,k}(x_1, x_2), \tag{3.36}
\]
where
\[
u_{n,k}(x_1, x_2) = \int_{-\infty}^{\infty} \tilde{u}_{n,k}(\lambda; x_1, x_2) d\lambda,
\]
\[	ilde{u}_{n,k}(\lambda; x_1, x_2) = J_n(\lambda w) v_k(\lambda; x_1, x_2), \quad v_k(\lambda; x_1, x_2) = U_k^* (\lambda, x_2) e^{-i\lambda(x_1 - d)},
\]
\(n = 0, 1, 2, \ldots, \quad k = 1, 2,\)
and \(U_k^* (\lambda, x_2)\) \((k = 1, 2)\) are given by (3.14).

Taking into account the order of asymptotic representations for the logarithmic inverse operators evaluated in Section 3.6, we restrict the analysis to the first three terms.
3.2. SOLUTION TO THE BCP IN THE FORM OF SERIES

Using formula (3.33) and asymptotic series for the Bessel functions [1]

\[ J_n(z) = \frac{z^n}{n!} \left( 1 - \frac{z^2}{2(n+1)} + \frac{z^4}{32(n+1)(n+2)} + \ldots \right), \]

where \(|J_n(z)| < A\), \(A = \text{const}\), \(n = 0, 1, 2, \ldots\), obtain asymptotic representations for the first three terms \(J_{0,n}^{(k)}\) and \(J_{1,n}^{(k)}\), \(n = 0, 1, 2\). We have

\[ u_{0,k}(\lambda; x_1, x_2) = J_0(\lambda w) v_k(\lambda; x_1, x_2) = \left( 1 - w^2 \frac{\lambda^2}{2} + O(w^4) \right) v_k(\lambda; x_1, x_2), \]

\[ u_{1,k}(\lambda; x_1, x_2) = J_1(\lambda w) v_k(\lambda; x_1, x_2) = \left( w \lambda - w^3 \frac{\lambda^3}{4} + O(w^5) \right) v_k(\lambda; x_1, x_2), \]

\[ u_{2,k}(\lambda; x_1, x_2) = J_2(\lambda w) v_k(\lambda; x_1, x_2) = w^2 \frac{\lambda^2}{2} \left( 1 + O(w^2) \right) v_k(\lambda; x_1, x_2), \]

\[ u_{n,k}(\lambda; x_1, x_2) = J_n(\lambda w) v_k(\lambda; x_1, x_2) = O(w^n), \quad n \geq 3, \quad k = 1, 2, \]

(3.37)

which hold uniformly for \(|\lambda| \leq \lambda_0\) according to (3.17).

If \(|\lambda| > \lambda_0\) we can also use (3.37) because, in line with (3.17), \(|\lambda^n v_k(\lambda; x_1, x_2)| \leq B_k e^{-s_k|\lambda|}\), \(m \geq 1\), for sufficiently large \(\lambda_0\), where \(B_k\) and \(s_k\) are positive constants \((k = 1, 2)\).

Substituting (3.37) into (3.33) and taking into account the estimates for the Bessel functions and \(v_k(\lambda; x_1, x_2)\), we obtain

\[ u_{0,k}(x_1, x_2) = \mu_{0,k}^{(0)} + w^2 \mu_{0,k}^{(2)} + O(w^4), \]

\[ u_{1,k}(x_1, x_2) = w \mu_{1,k}^{(1)} + w^3 \mu_{1,k}^{(3)} + O(w^5), \]

\[ u_{2,k}(x_1, x_2) = w^2 \mu_{2,k}^{(2)} + O(w^4), \]

\[ u_{n,k}(x_1, x_2) = O(w^n), \quad n \geq 3, \quad k = 1, 2, \]

here

\[ \mu_{0,k}^{(0)} = \int_{-\infty}^{\infty} v_k(\lambda; x_1, x_2) d\lambda, \quad \mu_{0,k}^{(2)} = -\frac{1}{2} \int_{-\infty}^{\infty} \lambda^2 v_k(\lambda; x_1, x_2) d\lambda, \]

\[ \mu_{1,k}^{(1)} = \int_{-\infty}^{\infty} \lambda v_k(\lambda; x_1, x_2) d\lambda, \quad \mu_{1,k}^{(3)} = -\frac{1}{4} \int_{-\infty}^{\infty} \lambda^3 v_k(\lambda; x_1, x_2) d\lambda, \]

where \(\mu_{n,k}^{(j)} = \mu_{n,k}^{(j)}(x_1, x_2), \ j = 0, 1, 2, \) and \(k = 1, 2\). Finally write the sought-for asymptotic series for \(u_k(x_1, x_2)\) using (3.36)

\[ u_k(x_1, x_2) = \frac{A_k}{2} \left( w q_0 \mu_{0,k}^{(0)} + i w^2 q_1 \mu_{1,k}^{(1)} + w^3 \mu_{2,k}^{(2)} (q_0 - q_2) \right) + O(w^4), \quad k = 1, 2, \]

(3.38)

This series can be used for the efficient evaluation of the solution to BCP in a band when the size of \(\omega\) (simulating a screen dot) is sufficiently small.

Asymptotic series in the case of \(\omega\) consisting of several intervals are obtained in Section 3.7.
3.3 Investigation of the integral equation

In this section we will examine integral equation (3.11) with a logarithmic singularity. First we investigate its kernel.

Proposition 3.2 [41] For all \(0 \leq |t| < \infty\) the following representation is valid

\[ \mathcal{K}(t) = -\ln|t| - \mathcal{F}(t), \]

where the function

\[ \mathcal{F}(t) = \int_0^\infty \frac{(1-k(x))\cos xt - e^{-x}}{x} dx \]

is a regular function in the region \(|t| < \infty, |\tau| < 2\) of the complex plane \(x = t + i\tau\). Moreover the function \(\mathcal{F}\) can be represented in the band \(|t| < 2\) as an absolutely convergent series

\[ \mathcal{F}(t) = \sum_{i=0}^{\infty} d_it^{2i}, \]  \hspace{1cm} (3.39)

where

\[ d_0 = \int_0^\infty \frac{1-k(x)-e^{-x}}{x} dx, \quad d_i = \frac{(-1)^i}{(2i)!} \int_0^\infty (1-k(x))x^{2i-1} dx, \]

\[ k(x) = \frac{\cosh 2x - 1}{\sinh 2x + 2x}. \]

Denote \(k_i(x) = x^{2i}e^{-2x}\). It is easy to check that the following relationships hold:

\[ 1 - k(x) > 0, \]

\[ \frac{1}{2} k_i(x) < (1-k(x))x^{2i-1} < 6k_i(x), \]

\[ \int_0^\infty k_i(x) dx = \frac{(2i)!}{2^{2i+1}i!}, \]  \hspace{1cm} (3.40)

\[ \left| \int_0^1 (1-k(x))x^{2i-1} dx \right| \leq \frac{m_0}{2i}, \quad m_0 = \max_{0 \leq x \leq 1} |1-k(x)|, \]

\[ \left| \int_1^\infty (1-k(x))x^{2i-1} dx \right| \leq \frac{(2i)!}{2^{2i+1}i!}, \]

where \(i = 1, 2, \ldots\).

Using these properties of function \(k(x)\) and taking into account the explicit form of the coefficients \(d_i\) in Proposition 3.2 we obtain the rate of convergence of series (3.39)
3.4. LOGARITHMIC INTEGRAL EQUATIONS

Lemma 3.4 Let $d_i$ be given by Proposition 3.2. Then the following estimate holds

$$|d_i| \leq \frac{m_0}{2n(2i)!} + \frac{1}{2^{2i-1}}, \quad i = 1, 2, \ldots.$$ 

From Lemma 3.4, it follows in particular the statement concerning the radius of convergence for power series $\sum d_i t^{2i}$ from Proposition 3.2.

Proposition 3.3 \[41\] Assume that $f(x_1) \in C^{k,\mu}(\omega)$. Then integral equation (3.11) is uniquely solvable and:

1) $q \in C^{k,\mu}(\pi)$, where $\pi$ is any interior of $\omega$, $k \geq 1$;

2) $q \in L_p(\omega)$, where $1 < p < 2$.

3.4 Logarithmic integral equations

In this section, we present fundamentals of the methods applied to the study of the integral operators with a logarithmic singularity of the kernel.

3.4.1 General properties

Consider the simplest integral equation with a logarithmic singularity of the kernel

$$L\varphi \equiv \frac{1}{\pi} \int_a^b \ln \frac{1}{|t_0 - t|} \varphi(t) dt = f(t_0), \quad t_0 \in (a, b),$$

with respect to unknown function $\varphi$ for a given right-hand side $f$.

It is natural to consider the integral equation with a Cauchy kernel in the class of functions satisfying the Hölder condition [28].

Definition 3.1 $\varphi \in H_\mu(\gamma)$ if:

1) $\varphi$ is a complex function on $\gamma = (a, b)$;

2) $\varphi$ satisfies the Hölder condition with an index $\mu$, $0 < \mu \leq 1$;

3) the norm is

$$\|\varphi\|_{H_\mu} = \|\varphi\|_C + h_\mu(\varphi),$$

$$\|\varphi\|_C = \sup_{t \in \gamma} |\varphi(t)|, \quad h_\mu(\varphi) = \sup_{t', t'' \in \gamma} \frac{|\varphi(t') - \varphi(t'')|}{|t' - t''|^\mu}.$$ 

Note that $H_\mu(\gamma)$ coincides with the space $C^{0,\mu}(\gamma)$ introduced in Definition 2.5.

Definition 3.2 $\varphi \in \tilde{H}_\mu(\gamma)$ if:

1) $\varphi$ can be represented in the form

$$\varphi(t) = \frac{\varphi^*(t)}{R(t)}, \quad R(t) = \sqrt{(b-t)(t-a)}, \quad \varphi^* \in H_\mu(\gamma);$$

2) the norm is

$$\|\varphi\|_{\tilde{H}_\mu} = \|R\varphi\|_{H_\mu} = \|\varphi^*\|_{H_\mu}.$$ 

Definition 3.3 $\varphi \in \tilde{H}_{\mu, a}(\gamma)$ if:

1) $\varphi \in H_\mu(\gamma)$;

2) $\varphi^*$ equals zero at the ends of segment $\gamma$. 


CHAPTER 3. METHOD OF INTEGRAL EQUATIONS

Note that the set $\tilde{H}_{\mu,0}(\gamma)$ is a closed subspace of $\hat{H}_{\mu}$.

**Definition 3.4** $\varphi \in \tilde{H}_{\mu,0}^1(\gamma)$ if:

1) $\varphi' \in \tilde{H}_{\mu,0}(\gamma)$;
2) the norm is $\|\varphi\|_{\tilde{H}_{\mu,0}^1} = \|\varphi\|_C + \|\varphi'\|_{\tilde{H}_{\mu}}$.

Assume that $f$ is a differentiable function. Then, differentiating both sides of equation (3.41) with respect to $t_0$, we obtain a singular integral equation with the Cauchy kernel

$$S\varphi = \frac{1}{\pi} \int_a^b \frac{\varphi(t)dt}{t-t_0} = f'(t_0),$$

where the integral on the left-hand side is understood in the sense of the principal value, i.e.,

$$\int_a^b \frac{\varphi(t)dt}{t-t_0} = \lim_{\varepsilon \to 0} \left[ \int_a^{t_0-\varepsilon} + \int_{t_0+\varepsilon}^b \right] \frac{\varphi(t)dt}{t-t_0}.$$

Assume that function $f'$ satisfies the Hölder condition. Then, according to [28], the general solution to (3.41) in the space $\tilde{H}_{\mu}(\gamma)$ is given by the formula

$$\varphi(t) = L^{-1}f = \frac{1}{\pi} \int_a^b \frac{\sqrt{(b-t_0)(t_0-a)}}{\sqrt{(b-t)(t-a)(t_0-t)}} f'(t_0)dt_0 \frac{C}{\sqrt{(b-t)(t-a)}} + C \sqrt{(b-t)(t-a)} (t-t_0), \quad (3.42)$$

where $C$ is an arbitrary constant. This constant $C$ must be chosen so that $\varphi$ would satisfy equation (3.41). Substituting expression (3.42) into (3.41) and taking into account that

$$\frac{1}{\pi} \int_a^b \frac{\ln \left| {t_0-t} \right|}{\sqrt{(b-t)(t-a)}} \frac{dt}{(b-t)(a-t)} = \ln \frac{4}{b-a}, \quad t_0 \in [a, b],$$

we obtain the desired value of $C$

$$C = \frac{1}{\ln \frac{4}{b-a}} \left[ f(t_0) + \frac{1}{\sqrt{b-a}} \int_a^b \frac{1}{\ln \left| {t_0-t} \right|} \frac{dt}{\sqrt{(b-t)(t-a)}} \right], \quad (3.43)$$

The right-hand side of (3.43) is a constant, which can be proved by differentiating this equality with respect to $t_0 \in [a, b]$ and applying the Poincaré-Bertrand formula that justifies the possibility of changing the order of integration in singular integrals [28].

Formula (3.42) for the solution to equation (3.41) is valid (and $\varphi$ belongs to $\tilde{H}_{\mu}$ with some $\mu$) when $f'$ belongs to a wider class $\tilde{H}_{\mu,0}$ rather than to $H_{\mu}$ with the same $\mu$.

**Proposition 3.4** [36] Integral operator $L$ is a continuous map of $\tilde{H}_{\mu}$ onto $\tilde{H}_{\mu,0}^1$ if $b-a \neq 4$ and $\mu < 1/2$. 
3.4. LOGARITHMIC INTEGRAL EQUATIONS

Theorem 3.4 \( L : \tilde{H}_\mu \rightarrow \tilde{H}^1_{\mu,0} \) is a Fredholm operator.

Proof. For \( b - a \neq 4 \), \( L \) is a Fredholm operator because \( L \) is invertible. For \( b - a = 4 \), we represent \( L \) in the form

\[
\int_a^b \left( \ln \frac{1}{|t_0 - t|} + C_0 \right) \varphi(t) dt - C_0 \int_a^b \varphi(t) dt,
\]

with a constant \( C_0 \neq 0 \). The first operator in (3.44) is invertible, which can be verified directly, and the second operator is one-dimensional. Therefore, \( L \) is a Fredholm operator because it is represented as a sum of invertible and one-dimensional operators. □

Consider a general form of the integral operator with a logarithmic singularity of the kernel 

\[
K \varphi = L \varphi + N \varphi \equiv \int_a^b \left( \frac{1}{\pi} \ln \frac{1}{|t_0 - t|} + N(t_0, t) \right) \varphi(t) dt, \quad t_0 \in (a, b),
\]  

where \( N(t_0, t) \) is a differentiable function with respect to \( t_0 \) whose derivative admits the representation

\[
\frac{\partial N}{\partial t_0} = \frac{n(t_0, t) - n(t, t)}{t - t_0}; \quad n(t_0, t) \in H_{\lambda,\lambda}([a, b] \times [a, b]), \quad 0 < \lambda < 1.
\]

The second condition in (3.46) means that, for the given function \( n(t_0, t) \), there exist two positive constants \( A_1 \) and \( A_2 \) such that, for any points \( x, y \in [a, b] \) and \( t_1, t_2 \in [a, b] \),

\[
|n(y, t_2) - n(x, t_1)| \leq A_1 |y - x|^\lambda + A_2 |t_2 - t_1|^\lambda.
\]

Condition (3.46) may be replaced by a weaker requirement that \( N(t_0, t) \) is continuously differentiable with respect to \( t \) and \( t_0 \) in \([a, b] \times [a, b] \).

Proposition 3.5 [36] Operator \( N : \tilde{H}_\mu \rightarrow \tilde{H}^1_{\mu,0} \) is completely continuous.

Theorem 3.5 \( K : \tilde{H}_\mu \rightarrow \tilde{H}^1_{\mu,0} \) is a Fredholm operator.

Proof. According to Theorem 3.4 and Proposition 3.5, operator \( K \) is the sum of the Fredholm and completely continuous operators. Therefore, it is a Fredholm operator. □

Proposition 3.6 [36] There is an \( w_0 > 0 \) such that integral equation (3.45) is uniquely solvable if \( w = b - a < w_0 \).

The proof of this statement is based on the semi-inversion of the integral operator \( K = L + N \) (i.e. the reduction of (3.45) to an equivalent equation \( \varphi + L^{-1}N \varphi = L^{-1}f \) using (3.42), (3.43)) and estimates of the type \( \|L^{-1}N\|_{H_{\mu,a,b}} < C(w) \), where \( C(w) < 1 \) if \( w \in (0, w_0) \) and \( w_0 \) is sufficiently small.
3.4.2 Several intervals of integration

The results obtained above for integral operators (3.41) and (3.45) can be extended to the case when the set of integration consists of several intervals.

Consider the equation

\[ L\varphi \equiv \frac{1}{\pi} \int_\gamma \frac{1}{|t_0 - t|} \varphi(t) \, dt = f(t_0), \quad t_0 \in \gamma, \]

where \( \gamma = \bigcup_{i=1}^n \gamma_i, \gamma_i = (a_i, b_i) \) is a set of segments on the real axis.

**Definition 3.5**

\[ \tilde{H}_\mu(\gamma) = \prod_{i=1}^n \tilde{H}_\mu(\gamma_i) \quad \text{and} \quad \tilde{H}^1_{\mu,0}(\gamma) = \prod_{i=1}^n \tilde{H}^1_{\mu,0}(\gamma_i) \]

are Banach spaces with the norms

\[ \| \cdot \|_{\tilde{H}_\mu(\gamma)} = \sum_{i=1}^n \| \cdot \|_{\tilde{H}_\mu(\gamma_i)} \quad \text{and} \quad \| \cdot \|_{\tilde{H}^1_{\mu,0}(\gamma)} = \sum_{i=1}^n \| \cdot \|_{\tilde{H}^1_{\mu,0}(\gamma_i)}. \]

**Theorem 3.6** Operator \( L: \tilde{H}_\mu(\gamma) \to \tilde{H}^1_{\mu,0}(\gamma) \) is continuous.

**Proof.** \( L \) can be treated as a matrix operator

\[
L\varphi = \begin{pmatrix}
L_{11} \varphi_1 + L_{12} \varphi_2 + \cdots + L_{1n} \varphi_n \\
L_{21} \varphi_1 + L_{22} \varphi_2 + \cdots + L_{2n} \varphi_n \\
\vdots & \ddots & \vdots \\
L_{n1} \varphi_1 + L_{n2} \varphi_2 + \cdots + L_{nn} \varphi_n
\end{pmatrix}
= \begin{pmatrix}
L_{11} & L_{12} & \cdots & L_{1n} \\
L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
L_{n1} & L_{n2} & \cdots & L_{nn}
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\vdots \\
\varphi_n
\end{pmatrix},
\]

where \( L_{ij} \) are the integral operators

\[ L_{ij}\varphi \equiv \frac{1}{\pi} \int_{\gamma_j} \frac{1}{|t_0 - t|} \varphi(t) \, dt, \quad t_0 \in \gamma_i. \]

According to Proposition 3.4, each diagonal operator \( L_{ii}, i = 1, \ldots, n \), is a continuous operator from \( \tilde{H}_\mu(\gamma_i) \) to \( \tilde{H}^1_{\mu,0}(\gamma_i) \). Nondiagonal operators \( L_{ij}, i \neq j \), are continuous from \( \tilde{H}_\mu(\gamma_j) \) to \( \tilde{H}^1_{\mu,0}(\gamma_i) \) because their kernels are infinitely smooth. Then, for any \( \varphi \in \tilde{H}_\mu(\gamma) \),

\[ \| L\varphi \|_{\tilde{H}^1_{\mu,0}(\gamma)} \leqslant \sum_{i=1}^n \sum_{j=1}^n \| L_{ij}\varphi_j \|_{\tilde{H}^1_{\mu,0}(\gamma_i)} \leqslant C \sum_{i=1}^n \sum_{j=1}^n \| \varphi_j \|_{\tilde{H}_\mu(\gamma_i)}, \]

which proves the theorem. \( \square \)
3.4. LOGARITHMIC INTEGRAL EQUATIONS

Theorem 3.7 \( L : \hat{H}_\mu(\gamma) \rightarrow \hat{H}^1_{\mu,0}(\gamma) \) is a Fredholm operator.

Proof. Represent operator \( L \) as

\[
L\varphi = L_0\varphi + L_C\varphi = \begin{pmatrix}
L_{11} & 0 & \ldots & 0 \\
0 & L_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & L_{nn}
\end{pmatrix} + \begin{pmatrix}
0 & L_{12} & \ldots & L_{1n} \\
L_{21} & 0 & \ldots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
L_{n1} & L_{n2} & \ldots & 0
\end{pmatrix}.
\]

(3.47)

\( L_0 \) is a Fredholm operator. Indeed, if \( b_i - a_i \neq 4 \) for all \( i \), operator \( L_0 \) is invertible, because all \( L_{ii} \) are invertible. If \( b_i - a_i = 4 \) for some \( i \), then we can transform \( L_0 \) to a sum of invertible and finite-dimensional operators, representing the corresponding operator \( L_i \) by formula (3.44) with \( C_0 \neq 1 \).

\( L_C \) is completely continuous because all nondiagonal operators \( L_{ij}, j \neq i \), have smooth kernels; therefore, they are completely continuous operators from \( \hat{H}_\mu(\gamma_j) \) to \( \hat{H}^1_{\mu,0}(\gamma_j) \). According to the Tikhonov theorem [15], the product of compact sets is compact in the product space. Hence, operator \( L_C \) maps any bounded set in \( \hat{H}_\mu(\gamma_i) \) into a compact set in \( \hat{H}^1_{\mu,0}(\gamma_i) \), which means that \( L_C \) is a completely continuous operator.

Then, \( L \) is a Fredholm operator, because it is represented as a sum of Fredholm and completely continuous operators. \( \square \)

Consider the integral operator

\[
K\varphi = L\varphi + N\varphi \equiv \int_\gamma \left[ \frac{1}{\pi} \frac{1}{|t_0-t|} + N(t_0, t) \right] \varphi(t) dt, \quad t_0 \in \gamma.
\]

(3.48)

where \( N(t_0, t) \) is represented in the form (3.46) with \( n(t_0, t) \in H_{\lambda \times \lambda}(\gamma \times \gamma) \).

Operator \( K \) can be written in the matrix form

\[
K\varphi = \begin{pmatrix}
K_{11} & K_{12} & \ldots & K_{1n} \\
K_{21} & K_{22} & \ldots & K_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
K_{n1} & K_{n2} & \ldots & K_{nn}
\end{pmatrix}.
\]

(3.49)

where

\[
K_{ij}\varphi = L_{ij}\varphi + N_{ij}\varphi \equiv \frac{1}{\pi} \int_{\gamma_j} \left[ \ln \frac{1}{|t_0-t|} + N(t_0, t) \right] \varphi(t) dt, \quad t_0 \in \gamma.
\]

Theorem 3.8 Operator \( N : \hat{H}_\mu(\gamma) \rightarrow \hat{H}^1_{\mu,0}(\gamma) \) is completely continuous.

Proof. Represent \( N \) in the matrix form. According to Proposition 3.5, each operator \( N_{ij} : \hat{H}_\mu(\gamma_j) \rightarrow \hat{H}^1_{\mu,0}(\gamma_j) \) is completely continuous. Literally repeating the proof of Theorem 3.6, we conclude that operator \( N \) is completely continuous in the product space. \( \square \)

Theorem 3.9 \( K : \hat{H}_\mu(\gamma) \rightarrow \hat{H}^1_{\mu,0}(\gamma) \) is a Fredholm operator.

Proof. Represent operator \( K \) as the sum \( K = L_0 + L_C + N \). The diagonal operator \( L_0 \) is a Fredholm operator, and the sum \( L_C + N \) is a completely continuous operator. Hence, \( K \) is a Fredholm operator. \( \square \)

Denote \( w = \text{mes}(\gamma) = \sum_{i=1}^n w_i \), where \( w_i = b_i - a_i, \quad i = 1, 2, \ldots, n \).
Theorem 3.10 There is an \( w_0 > 0 \) such that integral equation \( K\varphi = f \) is uniquely solvable if \( w \in (0, w_0) \).

Proof. Represent \( K \) in the form (3.49) and (3.47), perform the semi-inversion of the diagonal singular operator \( L_0 \) (3.47), apply componentwise Proposition 3.6 and estimates \( \| L_0^{-1} N_{ij} \| < C_i(w_i) < 1 \) which hold for sufficiently small \( w_i \) \( (i, j = 1, 2, \ldots, n) \), and write then the estimate \( \| L_0^{-1} N \|_{\tilde{H}_\mu(\gamma)} < C(w) \) using Definition 3.5 for the norm in the space \( \tilde{H}_\mu(\gamma) \), where \( C(w) < 1 \) if \( w \in (0, w_0) \) and \( w_0 \) is sufficiently small.

The statement of the theorem follows then from the fact that the solution to (3.48) can be represented in terms of the resolvent \( R = R(w) = (I + L_0^{-1} N)^{-1} \) as the convergent Neumann series \( \varphi = RL_0^{-1} f = \sum_{n=0}^\infty (L_0^{-1} N)^n L_0^{-1} f. \)

3.5 Solution using the Fourier-Chebyshev series

The main results for the integral equation (3.48) \( (L + N)\varphi = f \) with a logarithmic singularity of the kernel obtained in Section 3.4 are: existence of the inverse \( L^{-1} \) and complete continuity of the integral operator \( N \) with a smooth kernel defined on a pair of weighted Hölder spaces \( X = \tilde{H}_\mu(\gamma) \), \( Y = \tilde{H}_{1,0}^1(\gamma) \) on which the logarithmic operator \( L \) is invertible (and its inverse \( L^{-1} \) constructed explicitly in Section 3.6 is continuous).

In addition, from Proposition 3.6 it follows the unique solvability of the integral equation in the weighted Hölder space \( X \) if the diameter \( w = \text{mes}(\omega) \) of the integration set is sufficiently small. This statement is extended (Theorem 3.10) to the case when the set of integration consists of several intervals.

We have reduced BCP (3.1) to an integral equation (3.11) with a logarithmic singularity of the kernel

\[
K_q \equiv \int_{\omega} K\left(\frac{x_1 - \xi}{h}\right) q(\xi) d\xi = \pi E_0 f_2(x_1),
\]

where \( K(t) = -\ln |t| - \mathcal{F}(t) \) and the smooth term \( \mathcal{F}(t) \) (3.12) is a continuously differentiable function. Therefore we can apply the results obtained for general logarithmic equation of the form (3.45) and (3.48).

In fact, it is easy to see that (3.50) can be written in the form (3.45) or (3.48) where, according to Proposition 3.2, the smooth component \( N(x_1, \xi) = -\mathcal{F}(x_1 - \xi) \) satisfies condition (3.46). Therefore, integral operator \( K \) defined by (3.50) is a Fredholm operator from \( H_\mu \) to \( \tilde{H}_{1,0}^1 \) and equation (3.50) is uniquely solvable in \( \tilde{H}_\mu \) if the diameter \( \text{mes}(\omega) \) of the integration set is sufficiently small.

One can solve integral equation (3.50) by the method of the Fourier-Chebyshev decomposition developed in [36] and described in the next section.

An asymptotic series for the solution to (3.50) in powers of the characteristic small parameter of the problem based on Lemma 3.3 is obtained in Section 3.6.
3.5. SOLUTION USING THE FOURIER-CHEBYSHEV SERIES

3.5.1 Reduction to an infinite linear equation system

Assume that \( \omega = (-a, a) \), consider the integral equation (3.50), and use Proposition 3.2 to obtain

\[
- \int_{-a}^{a} q(\xi) \ln \left| \frac{\xi - x}{h} \right| d\xi = \pi E_0 f(x) + \int_{-a}^{a} q(\xi) \mathcal{F} \left( \frac{\xi - x}{h} \right) d\xi, \quad |x| < a. \tag{3.51}
\]

The solution to (3.51) can be represented in the form

\[ q(\xi) = p^0(\xi) + p(\xi), \]

where \( p^0(\xi) \) and \( p(\xi) \) solve the equations

\[
- \int_{-a}^{a} p^0(\xi) \ln \left| \frac{\xi - x}{h} \right| d\xi = \pi E_0 f(x) + d_0 \int_{-a}^{a} p^0(\xi) d\xi \tag{3.52}
\]

and

\[
- \int_{-a}^{a} p(\xi) \ln \left| \frac{\xi - x}{h} \right| d\xi = \pi E_0 f^*(x) + \int_{-a}^{a} \mathcal{F} \left( \frac{\xi - x}{h} \right) d\xi, \tag{3.53}
\]

with

\[ f^*(\xi) = \frac{1}{\pi K} \int_{-a}^{a} p^0(\xi) \mathcal{F} \left( \frac{\xi - x}{h} \right) d\xi, \quad |x| < a, \tag{3.54} \]

\( d_0 \) from Proposition 3.2, and \( K \) from (3.11).

Proposition 3.7 [39] If the function \( f'(x) \in L_p[-a, a], p > 4/3, \) then integral equation (3.52) has the solution \( p^0(x) \in L_p[-a, a], 1 < p < 4/3, \) and

\[ p^0(x) = \frac{1}{\pi K \sqrt{a^2 - x^2}} \left( E_0 - \int_{-a}^{a} f(t) dt - d_0 \int_{-a}^{a} \sqrt{a^2 - t^2} dt \right), \]

where \( \lambda = h/a. \)

Proposition 3.8 [41] If the function \( f'(x) \in L_p[-a, a], p > 4/3, \) and the solution to (3.53) exists then this solution may be represented in the form

\[ p(x) = \psi(x) (a^2 - x^2)^{-1/2}, \quad |x| < a, \]

where \( \psi(x) \) is a continuously differentiable function.

Represent the function \( \mathcal{F}(t) \) from Proposition 3.2 in the form of the Fourier-Chebyshev series

\[ \mathcal{F} \left( \frac{\xi - x}{h} \right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}(\lambda) T_{2i} \left( \frac{x}{a} \right) T_{2j} \left( \frac{\xi}{a} \right), \tag{3.55} \]
where
\[ \alpha_{ij}(\lambda) = \frac{\beta_{ij}}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{f(y - \eta)}{\sqrt{1 - y^2})(1 - \eta^2)} T_{2i}(y) T_{2j}(\eta) \, dy \, d\eta, \]
\[ \beta_{00} = 1, \quad \beta_{0j} = \beta_{ij} = 2, \quad \beta_{ij} = 4, \]
and \( T_i \) are Chebyshev polynomials. Applying the equality \([1], [25] \)
\[ (-1)^i \pi J_{2i}(a) = \int_{-1}^{1} T_{2i}(y) \cos ay \sqrt{1 - y^2} \, dy, \]
where \( J_i \) are the Bessel functions, we obtain the following representation for the coefficients \( \alpha_{ij} \)
\[ \alpha_{00} = \int_{0}^{\infty} (1 - k(y) J_{20}(y)) J_{20}(y/\lambda) - \cos y y \, dy, \]
\[ \alpha_{ij} = (-1)^{i+j} \beta_{ij} \int_{0}^{\infty} \frac{1 - k(y)}{y} \, J_{2i}(y/\lambda) J_{2j}(y/\lambda) \, dy. \]

Write the Fourier-Chebyshev expansions for the function \( f^*(x) \) from (3.54) using (3.55)
\[ f^*(x) = \sum_{i=0}^{\infty} f^*_i T_{2i} \left( \frac{x}{a} \right), \quad (3.56) \]
where
\[ f^*_0 = \sum_{j=0}^{\infty} \alpha_{0j}(\lambda) p^0_j - d_0 p^0_0, \quad f^*_i = \sum_{j=0}^{\infty} \alpha_{ij}(\lambda) p^0_j, \]
\[ p^0_j = \frac{1}{\pi \Delta} \int_{-\alpha}^{\alpha} \psi(\xi) T_{2j} \left( \frac{\xi}{\alpha} \right) d\xi. \]

To find coefficients \( p^0_j \) we use (3.52) and obtain
\[ p^0_0 = \frac{f_0}{\ln(2\lambda) - d_0}, \quad p^0_j = j f_j, \]
where \( f_j \) are the coefficients of the Fourier-Chebyshev expansions for the function
\[ f(x) = \sum_{j=0}^{\infty} f_j T_{2j} \left( \frac{x}{a} \right), \]
\[ f_0 = \frac{1}{\pi} \int_{-\alpha}^{\alpha} f(x) T_{20} \left( \frac{x}{a} \right) \frac{x}{\sqrt{a^2 - x^2}} \, dx, \quad f_j = \frac{2}{\pi} \int_{-\alpha}^{\alpha} f(x) T_{2j} \left( \frac{x}{a} \right) \frac{x}{\sqrt{a^2 - x^2}} \, dx. \]

The function \( \psi(x) \) can also be represented as a Fourier-Chebyshev series
\[ \psi(x) = \sum_{i=0}^{\infty} \psi_i T_{2i} \left( \frac{x}{a} \right) \quad (3.57) \]
3.5. SOLUTION USING THE FOURIER-CHEBYSHEV SERIES

Substitute representation for the function \( p(x) \) from Proposition 3.8 and (3.57), representation for the function \( f^*(x) \) from (3.56), and representation for the function \( \mathcal{F}(t) \) from (3.55) into (3.53) to obtain an infinite system of linear algebraic equations

\[
E_0 f^*_i + \psi_0 \alpha_0(\lambda) + \frac{1}{2} \sum_{j=1}^{\infty} \psi_j \alpha_j(\lambda) = \begin{cases} 
\psi_0 \ln(2\lambda), & i = 0, \\
\psi_i / 2, & i = 1, 2, \ldots \end{cases} \quad (3.58)
\]

Solving this system with respect to \( \psi_j \) we obtain the representation for \( p(x) \) from Proposition 3.8.

The definition of the solution to the infinite linear equation system, detailed description of the solution technique, and the method of approximate solution based on truncation, including the proof of convergence, can be found in [36].

### 3.5.2 Evaluation of the truncated system matrix

We propose an efficient version of the above method of solution to an integral equation with a logarithmic singularity of the kernel based on the truncation of infinite system (3.58), use of the Fourier-Chebyshev series, and matrix-algebraic determination of the entries in the resulting infinite-system matrix.

The method is an alternative to the one based on the calculation of infinite-matrix entries using integrals with rapidly oscillating integrands in the form of products of higher order Bessel functions of the type \( f(\cdot)J_m(\cdot)J_n(\cdot) \) (see [33]).

Consider the integral equation of the type (3.11)

\[
\int_{\omega} \mathcal{K}\left(\frac{x-y}{h}\right) q(y) \, dy = f(x), \quad (3.59)
\]

with the kernel of the form \( \mathcal{K}(t) = -\ln|t| - \mathcal{F}(t) \) on the interval \( \omega = (-1, 1) \). According to Proposition 3.8 and the results summarized in Section 3.4, every solution to integral equation (3.59) has the form

\[
q(y) = \frac{\varphi(y)}{\sqrt{1 - y^2}}, \quad \varphi(y) \in H_{\mu}(\omega).
\]

Write integral equation (3.59) in the operator form

\[
(\mathcal{L} + A)\varphi = f, \quad (3.60)
\]

where

\[
\mathcal{L}\varphi = \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|x-y|} \frac{\varphi(y)}{\sqrt{1 - y^2}} \, dy,
\]

\[
A\varphi = \frac{1}{\pi} \int_{-1}^{1} \mathcal{F}(x-y) \frac{\varphi(y)}{\sqrt{1 - y^2}} \, dy.
\]

Let \( \mathcal{F}(t) = \sum_{l=0}^{\infty} a_l t^{2l} \) be an analytic function in a circle \( |t| < t_0 \) defined by
Proposition 3.2. Write $\mathcal{F}(x-y) \approx \mathcal{F}^{(N)}(x-y)$, where

\[
\mathcal{F}^{(N)}(x-y) = \sum_{l=0}^{2N} d_l(x-y)^{2l} = \sum_{m=0}^{2N} (-1)^m y^{2N-m} \sum_{j=0}^{2N} d_{jm} x^j =
\]

\[
= \sum_{m=0}^{2N} \left( \sum_{l=0}^{2N} g_{im} T_l(x) \right) T_m(x) = \sum_{j=0}^{2N} \left( \sum_{m=0}^{2N} g_{jm} b_j^{(N)} \right) T_j(x).
\]

and $T_i$ are Chebyshev polynomials.

Integral equation (3.60) is equivalent to an infinite linear equation system of the form (3.58). The corresponding truncated system of linear algebraic equations is

\[
(L + A)^2 = \mathcal{T},
\]

where $a_{ij}$ are the entries of the resulting infinite-system matrix.

Define the coefficient matrix

\[
A = \|a_{ij}\|_{i,j=0}^{2N}.
\]

The following representation is valid

\[
A = \varepsilon D^{-,odd} D^T P_x^T,
\]

where $A$ is a $(2N + 1) \times (2N + 1)$ upper-left-triangular matrix,

\[
P_x = \begin{pmatrix}
1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 & \cdots \\
0 & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \cdots \\
0 & 0 & 1 & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \cdots \\
0 & 0 & 0 & 1 & 0 & \frac{1}{7} & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{9} & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]

is the $(2N + 1) \times (2N + 1)$ upper-right-triangular matrix of the operator which performs the coefficient transformation of a general algebraic polynomial into Chebyshev polynomials, the $(2N + 1) \times (2N + 1)$ upper-left-triangular matrix

\[
\varepsilon D^{-,odd} = \begin{pmatrix}
\cdots & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdot & \cdot & \cdot \\
\cdots & 0 & -\frac{1}{4} & 0 & -\frac{1}{2} & 0 & -1 & 0 & \cdot & \cdot \\
\cdots & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdot & \cdot & \cdot \\
\cdots & 0 & -\frac{1}{4} & 0 & -\frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot \\
\cdots & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\]
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is the ‘symmetrical’ matrix $P_ε$ (odd columns acquire negative sign) with a size $(2N + 1) \times (2N + 1)$, $D = \|d_{jm}\|_{j,m=0}^{2N}$, where

$$
d_{jm} = \begin{cases} 
0, & j > m, \\
0, & j = 2q, \ m = 2t + 1, \\
0, & j = 2q + 1, \ m = 2t, \\
c_2(N-1)-(2q+1)d_{N-t}, & j = 2q + 1, \ m = 2t + 1,
\end{cases}
$$

and $L$ is a $(2N + 1) \times (2N + 1)$ diagonal matrix

$$
L = \begin{pmatrix} 
\ln 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{1}{8} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
$$

For example, in the case $N = 3$ matrix $L + A$ has the form

$$
\begin{pmatrix} 
\frac{\sqrt{2}}{2} \ln \frac{2}{3} + \frac{\sqrt{2}}{3} d_2 + \frac{\sqrt{2}}{3} d_3 & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) \\
0 & \frac{\sqrt{2}}{2} \ln \frac{2}{3} + \frac{\sqrt{2}}{3} d_2 + \frac{\sqrt{2}}{3} d_3 & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) & 0 \\
0 & 0 & \frac{\sqrt{2}}{2} \ln \frac{2}{3} + \frac{\sqrt{2}}{3} d_2 + \frac{\sqrt{2}}{3} d_3 & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) \\
0 & 0 & 0 & \frac{\sqrt{2}}{2} \ln \frac{2}{3} + \frac{\sqrt{2}}{3} d_2 + \frac{\sqrt{2}}{3} d_3 & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \ln \frac{2}{3} + \frac{\sqrt{2}}{3} d_2 + \frac{\sqrt{2}}{3} d_3 & 0 & \left( \frac{2}{3} d_3 - \frac{1}{3} d_1 \right) \\
\end{pmatrix}
$$

We see that the matrix of the truncated linear equation system (3.61) of a given order $N$ can be formed using two matrix multiplications involving three constant matrices $P_ε$, $P^{-\text{odd}}$ and $D$ that contains only few nonzero diagonals.

3.6 Semi-inversion of integral operators

In this section we develop elements of the asymptotic theory of integral operators with a logarithmic singularity of the kernel defined on several intervals of integration and the methods of analytical semi-inversion and apply, following [30], [31], and [35], the techniques to solve integral equations with a logarithmic singularity associated with BCPs.
3.6.1 One interval of integration

Consider the integral operator with a logarithmic singularity of the kernel

\[ K\varphi = \alpha L\varphi + N\varphi \equiv \int_\gamma \left[ \frac{1}{\pi} \frac{1}{|x_0 - x|} + N(x_0, x) \right] \varphi(x) dx, \quad x_0 \in \gamma, \quad (3.62) \]

where \( \gamma = (a, b) = (d-w, d+w) \) and \( N(x_0, x) \) is once continuously differentiable on \( \gamma \times \gamma \). Set

\[ \beta = \left( \frac{1}{\pi} \ln \frac{1}{w} \right)^{-1} \]

and let \( \beta \) be a small parameter. Obtain approximate representations for the inverse operator \( K^{-1} \) as segments of asymptotic series in powers of \( \beta \) using the method of approximate semi-inversion.

Consider the integral operator

\[ L_1 \varphi \equiv \alpha L\varphi + g(\varphi, 1)1, \quad g = \frac{\alpha}{\beta} + M_0 = \text{const} \quad (\alpha \neq 0), \quad M_0 = N(d, d). \quad (3.63) \]

Let us obtain the inverse \( L_1^{-1}(\beta) \) explicitly. To this end, consider the integral equation of the first kind

\[ L_1 \varphi \equiv \alpha L\varphi + g(\varphi, 1)1 = f, \quad (3.64) \]

where the integral operator

\[ L\varphi \equiv \frac{1}{\pi} \int_{-1}^1 \ln \frac{1}{|x_0 - x|} \varphi(x) \, dx \quad (3.65) \]

and the inverse (cf. (3.42) with \((a, b) = (-1, 1)\))

\[ L^{-1} f = -\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1 - x_0^2} f(x_0) dx_0}{(x_0 - x) \sqrt{1 - x^2}} + \frac{C}{\sqrt{1 - x^2}} \quad (3.66) \]

with the constant

\[ C = \frac{1}{\ln 2} \left[ f(x_0) + \frac{1}{\pi} \int_{-1}^1 \ln \frac{1}{|x_0 - x|} \frac{dx}{\sqrt{1 - x^2}} \int_{-1}^1 \frac{\sqrt{1 - \xi^2} f'(\xi) d\xi}{\sqrt{1 - (\xi - x)^2}} \right]. \quad (3.67) \]

Lemma 3.5

\[ L_1^{-1} f = \frac{1}{\alpha} [L^{-1} f + B_0 (L^{-1} f, 1)p_0], \quad B_0 = -\frac{g}{\alpha + \pi g}. \quad (3.68) \]

Proof. Applying \( L^{-1} \) to both sides of the equation \( L_1 \varphi = f \) and using the formulas

\[ L^{-1}1 = p_0(x) = \frac{1}{\ln 2} \frac{1}{\sqrt{1 - x^2}}, \quad (L^{-1}1, 1) = \frac{\pi}{\ln 2}, \quad (3.69) \]

yields an equivalent equation of the second kind

\[ \varphi = \frac{1}{\alpha} [L^{-1} f - g(\varphi, 1)L^{-1}1]. \quad (3.70) \]
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Multiplying both sides of (3.70) by the constant 1 and integrating over \((-1,1)\),
we obtain a (linear) equation with respect to \((\varphi, 1)\)
\[
[\alpha + \tilde{g}](\varphi, 1) = (L^{-1}f, 1), \quad \tilde{g} = \frac{\pi}{\ln 2}g. \tag{3.71}
\]
If \(\alpha + \tilde{g} \neq 0\), then equation (3.71) and hence (3.64) are uniquely solvable for
arbitrary \(f \in \mathcal{H}^1_{\alpha}(\mathbb{R}), \mathcal{H}^1_{\alpha}(\mathbb{R})\) denotes the weighted Hölder
space (Definition 3.4). Dividing both sides of (3.71) by \(\alpha + \tilde{g}\) and substituting
(\(\varphi, 1\)) into (3.70) we obtain the expression for \(L_1^{-1}f\)
\[
\varphi = L_1^{-1}f = \frac{1}{\alpha}[L^{-1}f + B_0(L^{-1}f, 1)p_0], \quad B_0 = -\frac{\ln 2}{\pi} \frac{\tilde{g}}{\alpha + \tilde{g}}.
\]
If \(\alpha + \tilde{g} = 0\), then equations (3.71) and (3.64) are not solvable: the homoge-
neneous integral equation
\[
L_1\varphi \equiv \alpha L\varphi + g(\lambda)(\varphi, 1)1 = 0, \tag{3.72}
\]
where \(g(\lambda)\) is a (given) analytical or meromorphic function of complex variable
\(\lambda\), has a nontrivial solution (a characteristic element (eigenfunction) of \(L_1(\lambda)\))
\(\varphi^*(x) = C\varphi_0(x)\), where \(C_0\) is an arbitrary constant. When performing the
semi-inversion we will assume that the condition \(\alpha + \tilde{g} \neq 0\) holds. \(\diamondsuit\)

\(B_0\) can be expanded in powers of small parameter \(\beta\)
\[
B_0 = -1 + \frac{\ln 2}{\pi} \beta - \frac{\ln 2}{\pi} \beta^2 \left(\frac{M_0}{\alpha} + \frac{\ln 2}{\pi}\right) + O(\beta^3),
\]
which yields a similar expansion for \(L_1^{-1}\)
\[
L_1^{-1}(\beta)f = \frac{1}{\alpha}L^{-1}f - \frac{\ln 2}{\alpha \pi} \left(1 - \beta \frac{\ln 2}{\pi} + \beta^2 \frac{M_0}{\alpha} + \frac{\ln 2}{\pi}\right) (L^{-1}f, 1)1 + O(\beta^3). \tag{3.73}
\]

Consider the integral equation (3.62). Performing the change of variables
\(x = wt + d, \ x_0 = wt_0 + d\), we obtain the equivalent integral equation
\[
K\tilde{\varphi} = \tilde{f},
\]
\[
K\tilde{\varphi} = \int_{-1}^{1} \left(\alpha \frac{1}{\pi} \ln \frac{1}{|t_0 - t|} + \frac{1}{\pi} \ln \frac{1}{|w|} + N(wt_0 + d, wt + d)\right) \tilde{\varphi}(t) \, dt, \tag{3.74}
\]
\(t_0 \in [-1,1]\), where \(\tilde{\varphi}(t) = w\tilde{\varphi}(wt + d)\) and \(\tilde{f}(t_0) = f(wt_0 + d)\).

Assume that kernel \(N(x_0, x)\) admits the following estimate for \(x_0 - x \to 0\)
\[
|N(x_0, x) - N(0, 0)| \leq C_1(x_0 - x)^2 \frac{1}{|x_0 - x|} + C_2 x_0^2 + C_3 x^2, \tag{3.75}
\]
where \(C_1, C_2,\) and \(C_3\) are constants independent of \(x\) and \(x_0\).

As follows from (3.74) and (3.75), \(K\) can be represented in the form
\[
K\tilde{\varphi} = \alpha L\tilde{\varphi} + \alpha \frac{1}{\pi} \ln \frac{1}{w} (\tilde{\varphi}, 1)1 + M_0(\tilde{\varphi}, 1)1 + w^2 \ln w R(w)\tilde{\varphi}, \tag{3.76}
\]
where \(M_0 = N(0, 0)\) and \(R(w)\) is an operator uniformly bounded as \(w \to 0\).

We have proved
Lemma 3.6  The approximate representation
\[ K = L_1(\beta) + w^2 \ln w N \]  
(3.77)
holds, where integral operators \( L_1 \) and \( N \) are defined in (3.63) and (3.62).

Lemma 3.7  There exist a value \( \beta_0 \) and a constant \( C > 0 \) such that, for any \( \beta < \beta_0 \), the estimate \( \|L^{-1}_1(\beta)\| < C \) is valid.

Proof.  As follows from representation (3.73),
\[ \|L^{-1}_1 f\| \leq \frac{1}{\alpha} \|L^{-1}\| \|f\| + C_1 \left( \frac{1}{\beta} + \frac{M_0}{\alpha} \right) \|L^{-1}\|, \]
where \( C_1 \) is a constant independent of \( f \) and \( \beta \). Consider the function
\[ F(\beta) = \frac{\left( \frac{1}{\beta} + \frac{M_0}{\alpha} \right)}{1 + \frac{\pi}{\ln 2} \left( \frac{1}{\beta} + \frac{M_0}{\alpha} \right)}. \]

For small \( \beta \),
\[ |F(\beta) - \frac{\ln 2}{\alpha \pi} \left( 1 - \beta \frac{\ln 2}{\pi} + \beta^2 \frac{\ln 2}{\pi} \left( \frac{M_0}{\alpha} + \frac{\ln 2}{\pi} \right) \right)| < C_2 \beta^3, \quad C_2 = \text{const}. \]

Then, on a certain interval \((0, \beta_0)\),
\[ |F(\beta)| \leq \frac{\ln 2}{\alpha \pi} + \beta_0 \left( \frac{\ln^2 2}{\alpha \pi^2} + \beta_0 \frac{\ln^2 2}{\alpha \pi^2} \left( \frac{M_0}{\alpha} + \frac{\ln 2}{\pi} \right) \right) + C_2 \beta_0^2. \]

Thus, function \( F(\beta) \) is uniformly bounded on the interval \((0, \beta_0)\), which proves the required estimate. \( \Diamond \)

Theorem 3.11  For sufficiently small \( w \), operator \( \tilde{K} : \tilde{H}_\mu(-1,1) \rightarrow \tilde{H}_\mu^1(-1,1) \) is invertible.

Proof.  According to (3.76) and (3.64) operator \( \tilde{K} \) can be represented in the form \( \tilde{K} = L_1 + N \). The norm of \( N \) is bounded uniformly with respect to \( w \), and, according to Lemma 3.7, the norm of \( L^{-1}_1 \) is uniformly bounded on a certain interval \((0, \beta_0)\). Then, for sufficiently small \( w \), \( |w^2 \ln w| ||L^{-1}_1 N|| < 1 \) and operator \( \tilde{K} \) is invertible. \( \square \)

Theorem 3.12  The inverse operator \( \tilde{K}^{-1} \) admits the representation in the form of the Neumann series
\[ \tilde{K}^{-1} = \sum_{n=0}^{\infty} (-1)^n w^{2n} \ln^n w (L^{-1}_1 N)^n L^{-1}_1, \]  
(3.78)
which converges in the operator norm uniformly with respect to \( w \) on a certain interval \((0, w_0)\).
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Proof. We have
\[ \tilde{K}^{-1} = (L_1 + w^2 \ln w N)^{-1} = (I + w^2 \ln w L_1^{-1} N)^{-1} L_1^{-1}. \]

As follows from Lemma 3.7, for sufficiently small \( w \), \( \|w^2 \ln w L_1^{-1} N\| < 1 \), and operator \( (I + w^2 \ln w L_1^{-1} N)^{-1} \) can be expanded in a converging Neumann series. Therefore, expansion (3.78) is valid. \( \square \)

Retaining the first two terms of expansion (3.78) we obtain
\[ \tilde{K}^{-1} f = L_1^{-1} f - w^2 \ln w L_1^{-1} N L_1^{-1} f + O(w^4 \ln^2 w) \quad \text{as} \quad w \to 0. \]  \quad (3.79)

In many cases, the following simplified formula can be applied:
\[ \tilde{K}^{-1} f = L_1^{-1} f + O(w^2 \ln w) = \]
\[ = 1 + \frac{1}{\beta} \ln 2 + \frac{2}{\pi} \left( \frac{L^{-1} f, 1}{\beta + \frac{M_0}{\alpha}} \right) \quad \text{as} \quad \beta \to 0. \]  \quad (3.80)

If, in addition, the value of \( \beta \) is sufficiently small, then the following expansion in powers of \( \beta \) can be used:
\[ \tilde{K}^{-1} f = 1 - \frac{1}{\alpha} L^{-1} f - \frac{1}{\alpha} \frac{\ln 2}{\pi} \left( 1 - \frac{1}{\beta} \ln 2 + \frac{2}{\pi} \right) \]
\[ + \beta \frac{\ln 2}{\pi} \left( \frac{M_0}{\alpha} + \frac{2}{\pi} \right) \left( L^{-1} f, 1 \right) L^{-1} f + O(\beta^3) \quad \text{as} \quad \beta \to 0. \]  \quad (3.81)

Below by \( T_n \) we will denote the Chebyshev polynomials of the first kind, by \( L \) integral operator (3.65) and set \( \rho(t) = (1 - t^2)^{-1/2} \).

**Proposition 3.9** [25]

\[ L(T_0 \rho) = \ln 2 T_0, \quad L(T_n \rho) = \frac{1}{n} T_n, \quad n = 1, 2, \ldots. \]  \quad (3.82)

**Lemma 3.8** Let \( f(t) = \sum_{n=0}^{\infty} f_n T_n(t), \quad t \in (-1, 1) \) where \( \{n f_n\}_{n=0}^{\infty} \in l^2 \). Then
\[ (L^{-1} f)(t) = \rho(t) \left( \frac{1}{\ln 2} T_0 + \sum_{n=1}^{\infty} n f_n T_n(t) \right), \]  \quad (3.83)
\[ (L_1^{-1} f)(t) = \rho(t) \left( \frac{f_0}{\ln 2 + \frac{2}{\pi} M_0} + \sum_{n=1}^{\infty} n f_n T_n(t) \right), \]  \quad (3.84)

where \( M_0 \) is defined in (3.76).

**Proof.** In order to prove (3.83), we use the relations
\[ L^{-1} T_0 = \frac{1}{\ln 2} T_0 \rho, \quad L^{-1} T_n = n T_n \rho, \quad n = 1, 2, \ldots, \]
that follow from (3.82) and apply \( L^{-1} \) to both sides of the equation \( L \varphi = f \), where \( f = \sum_{n=0}^{\infty} f_n T_n \). Next we substitute representation (3.83) into the expression in (3.80) that defines \( L_1^{-1} \) and obtain (3.84). \( \Diamond \)
Lemma 3.9 The solution to the integral equation

\[ L_1^* \phi \equiv \int_{d-w}^{d+w} \left( \frac{\alpha}{\pi} \ln \frac{1}{|x_0 - x|} + M_0 \right) \varphi(x) dx = \tilde{f}(x_0), \quad x_0 \in (d-w, d+w), \]  

(3.85)

with \( M_0 = \text{const} \) and the right-hand side

\[ \tilde{f}(x_0) = \sum_{n=0}^{\infty} f_n T_n \left( \frac{x_0 - d}{w} \right), \quad \{ nf_n \} \in l^2, \]  

(3.86)

has the form

\[ \varphi(x) = \rho_1(x) \left( \frac{f_0}{\ln \frac{1}{w} + \frac{\pi}{2} M_0} + \sum_{n=1}^{\infty} f_n T_n \left( \frac{x - d}{w} \right) \right), \]  

(3.87)

where \( \rho_1(x) = w((d + w) - x)(x - (d - w))^{-1/2} \).

Proof. Making the change of variables \( t_0 = (x_0 - d)/w, \quad t = (x - d)/w \), \( -1 \leq t, t_0 \leq 1 \) in the integral (3.85) we obtain integral equation (3.64) with the right-hand side \( f(t_0) = \tilde{f}(wt_0 + d) = \sum_{n=0}^{\infty} f_n T_n(t_0) \) and \( \tilde{\varphi}(t) = \varphi(wt + d) \). Inverting operator \( L_1^* \) with the help of Lemma 3.5 we obtain (3.84) which yields (3.87). \( \diamond \)

3.6.2 Asymptotic expansion of the solution to the integral equation associated with BCP

In this section, we use expansions (3.79) and (3.81) to obtain asymptotic series for the solution to integral equation (3.50) with a logarithmic singularity of the kernel associated with BCP (3.2).

Consider the integral equation

\[ K \varphi \equiv \int_{\omega} K \left( \frac{x_0 - x}{h} \right) \varphi(x) \, dx = f(x_0), \quad x_0 \in \omega, \]  

(3.88)

over one interval \( \omega = (d-w, d+w) \). This equation can be written in the form (3.62) with \( \alpha = \pi \) and

\[ N(x_0, x) = \mathcal{F} \left( \frac{x_0 - x}{h} \right), \quad \mathcal{F}(t) = d_0 + \mathcal{F}_1(t), \quad \mathcal{F}_1(t) = \sum_{n=1}^{\infty} d_n t^{2n}, \]

where the coefficients \( d_n \) are specified in Proposition 3.2. Making the change of variables \( x = wt + d, \quad x_0 = wt_0 + d \) we transform (3.88) to an equation with the integration over \((-1,1)\) similar to (3.74) and (3.76)

\[ \tilde{K} \tilde{\varphi} = \int_{-1}^{1} \left( \ln \frac{1}{|t_0 - t|} + \ln \frac{1}{w} + d_0 + \mathcal{F}_1 \left( \tilde{w}(t_0 - t) \right) \right) \tilde{\varphi}(t) \, dt = \tilde{f}(t_0), \quad t_0 \in (-1,1), \]

(3.89)

where \( \tilde{w} = w/h, \quad \tilde{\varphi}(t) = \varphi(wt + d), \) and \( \tilde{f}(t_0) = f(wt_0 + d) \).
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Write (3.89) in the operator form \( \hat{K}\tilde{\varphi} = (\hat{L} + N_1)\tilde{\varphi} \), where

\[
\hat{L}_1\tilde{\varphi} = \int_{-1}^{1} \left( \ln \frac{1}{|t_0 - t|} + \ln \frac{1}{w} + d_0 \right) \tilde{\varphi}(t) \, dt
\]

(3.90)

coincides with \( L_1 \) defined by (3.77), (3.64) if to set \( \alpha = \pi \) and \( M_0 = d_0 \).

Consider the integral operator

\[
N_1\tilde{\varphi} = \int_{-1}^{1} \tilde{\varphi}(t_0 - t) \tilde{\varphi}(t) \, dt.
\]

We have

\[
\tilde{\varphi}(t_0 - t) = \sum_{n=1}^{\infty} d_n \bar{\omega}^{2n} \sum_{k=0}^{2n} C_k 2^{2n-k} t^k,
\]

where \( C_k \) are binomial coefficients, so that

\[
N_1\tilde{\varphi} = \sum_{n=1}^{\infty} d_n \bar{\omega}^{2n} \sum_{k=0}^{2n} C_k 2^{2n-k} t^k \int_{-1}^{1} t^k \tilde{\varphi}(t) \, dt.
\]

Obtain the representation

\[
\hat{L}_1^{-1} N_1 \hat{L}_1^{-1} f(t) = L_1^{-1} \left( \sum_{n=1}^{\infty} d_n \bar{\omega}^{2n} \sum_{k=0}^{2n} C_k 2^{2n-k} \right) \int_{-1}^{1} \xi^k \hat{L}_1^{-1} f(\xi) \, d\xi
\]

= \sum_{n=1}^{\infty} d_n \bar{\omega}^{2n} \sum_{k=0}^{2n} C_k 2^{2n-k} (t) \int_{-1}^{1} \xi^k \hat{L}_1^{-1} f(\xi) \, d\xi,

where \( p_m = \hat{L}_1^{-1} t_m^m \). Retaining the first term proportional to \( w^2 \), which is the asymptotic degree of the second term in asymptotic representation (3.79), we obtain

\[
\hat{L}_1^{-1} N_1 \hat{L}_1^{-1} f(t) = d_1 \bar{\omega}^2 \sum_{k=0}^{2} C_k 2^{2n-k} (t) \int_{-1}^{1} \xi^k \hat{L}_1^{-1} f(\xi) \, d\xi + O(w^4),
\]

which yields a formula for the asymptotic expansion of the solution to integral equation (3.88) in powers of the small parameter \( w^3\ln w \) according to (3.79)

\[
\tilde{\varphi}(t) = \hat{K}^{-1} f(t) = \hat{L}_1^{-1} f(t) - w^2 \ln w \hat{L}_1^{-1} N_1 \hat{L}_1^{-1} f(t) + O(w^4 \ln^2 w) = \hat{L}_1^{-1} f(t) + O(w^3 \ln w).
\]

Series (3.91) will be applied in Section 3.7 for the determination of the solution to BCP (3.1) using formulas (3.10), (3.13), and (3.6).

If the right-hand side in (3.88) has the form (3.86), that is,

\[
f(t) = \sum_{n=0}^{\infty} f_n T_n(t), \quad t = \frac{x_0 - d}{w} \in (-1, 1),
\]

then

\[
\tilde{\varphi}(t) = \hat{K}^{-1} f(t) = \hat{L}_1^{-1} f(t) - w^2 \ln w \hat{L}_1^{-1} N_1 \hat{L}_1^{-1} f(t) + O(w^4 \ln^2 w) = \hat{L}_1^{-1} f(t) + O(w^3 \ln w).
\]
and (3.91) to obtain an asymptotic series for the solution to (3.88)
\[ \varphi(x) = \rho_1(x) \left( \frac{f_0}{\ln z + d_0} + \sum_{n=1}^{\infty} n f_n T_n \left( \frac{z-d}{w} \right) \right) + O(w^4 \ln^2 w), \tag{3.92} \]
where \( \rho_1(x) = w((id+w)-(x-(d-w)))^{-1/2} \).

## 3.6.3 Several intervals of integration

First, we present, following [35], a detailed description of the semi-inversion in the case of two intervals of integration and then consider the case of an arbitrary number of intervals.

When the set of integration in (3.62) consists of two intervals, 
\[ \gamma = \omega = \bigcup_{j=1}^{2} \omega_j, \quad \omega_j = (d_j - w_j, d_j + w_j), \quad j = 1, 2, \]
we introduce the vector-function \( \mathbf{f} = (\varphi_1, \varphi_2)^T \) and \( \mathbf{F} = (f_1, f_2)^T \), where \( \varphi_i(x) = \varphi(x) \), \( f_i(t) = f(t) \), \( x \in \omega_j \), and write the integral operator (3.62) in the matrix form
\[ \mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \]
where
\[ K_{ij}\varphi_j = \alpha L_{ij}\varphi_j + N_{ij}\varphi_j \equiv \int_{d_i-w_i}^{d_i+w_i} \left[ \frac{\alpha}{\pi} \ln \left| \frac{1}{|x_0-x|} + N(x_0, x) \right| \right] \varphi_j(x) dx. \tag{3.93} \]
Using the change of variables
\[ x = w_j t + d_j, \quad x_0 = w_i t_0 + d_i, \quad i, j = 1, 2, \tag{3.94} \]
we transform (3.93) to the equivalent form with the integration in each operator \( K_{ij} \) over \((-1,1)\)
\[ K_{jj}\tilde{\varphi}_j = \int_{1}^{1} \left[ \frac{\alpha}{\pi} \ln \left| \frac{1}{|t_0-t|} + \frac{\alpha}{|w_j|} \ln \left| \frac{1}{w_j} + N(w_i t_0 + d_i, w_j t + d_j) \right| \right] \tilde{\varphi}_j(t) dt, \]
\[ t_0 \in (-1,1), \quad j = 1, 2, \]
\[ K_{ij}\tilde{\varphi}_j = \int_{-1}^{1} \left[ \frac{\alpha}{\pi} \ln \left| \frac{1}{w_i t_0 - w_j t + (d_i - d_j)} + N(w_i t_0 + d_i, w_j t + d_j) \right| \right] \tilde{\varphi}_j(t) dt, \]
\[ t_0 \in \gamma_i, \quad i \neq j, \quad i, j = 1, 2, \tag{3.95} \]
here \( \tilde{\varphi}_j(t) = w_j \varphi_j(w_j t + d_j), \quad j = 1, 2. \)

When \( w_j \) are taken as small parameters, the diagonal operators \( \tilde{K}_{jj} = \tilde{K}_{jj}(w_j) \) can be represented in the form
\[ \tilde{K}_{jj}\tilde{\varphi}_j = \alpha L_{jj}\tilde{\varphi}_j + \frac{\alpha}{|w_j|} \ln \left| \frac{1}{w_j} (\tilde{\varphi}_j, 1) + M_{jj}(\tilde{\varphi}_j, 1) + w_j^2 \ln w_j R_{jj}(w_j) \tilde{\varphi}_j, \tag{3.96} \]
where \( M_{ij} = N(d_i, d_j) \); for the nondiagonal operators we have

\[
K_{ij} \tilde{\varphi} = M_{ij}(\tilde{\varphi}, 1) 1 + w_i^2 \ln w_j R_{ij}, \quad i \neq j, \quad i, j = 1, 2, \tag{3.97}
\]

where

\[
M_{ij} = \frac{\alpha}{\pi} \ln \frac{1}{|d_i - d_j|} + N(d_i, d_j), \quad i \neq j, \quad i, j = 1, 2, \tag{3.98}
\]

and \( R_{ij} \) are integral operators uniformly bounded with respect to the Sobolev norm as \( w_j \to 0, j = 1, 2 \).

Combining (3.96) and (3.97) we can separate the principal part \( L_P \) of the operator (3.62) in the matrix form

\[
L_P = \left( \begin{array}{cc} L_{1,1} & M_{12} \cdot (1) \\ M_{21} \cdot (1) & L_{2,1} \end{array} \right) \tag{3.99}
\]

and write

\[
L_P \Phi = \left( \begin{array}{c} L_1 \varphi_1 + g_1(\varphi_1, 1) + M_{12} \varphi_2, 1) 1 \\ L_2 \varphi_2 + g_2(\varphi_2, 1) + M_{21} \varphi_1, 1) 1 \end{array} \right), \tag{3.100}
\]

where

\[
g_j = \frac{\alpha}{\beta_j} + M_{jj, \nu}, \quad j = 1, 2.
\]

In order to obtain an explicit representation for the inverse \( L_P^{-1} \) we repeat componentwise the proof (3.64)–(3.68). Apply \( L^{-1} \) given by (3.66) and (3.67) to both sides of every line in integral equation \( L_P \Phi = \Phi \) written in the form (3.100). As a result, we obtain an equivalent equation of the second kind. Using formulas (3.69), we write this equation componentwise

\[
\varphi_1(t_o) = \frac{1}{\alpha} \left\{ L^{-1} f_1 - [g_1(\varphi_1, 1) + M_{12}(\varphi_2, 1)] p_0(t) \right\},
\]

\[
\varphi_2(t_o) = \frac{1}{\alpha} \left\{ L^{-1} f_2 - [M_{21}(\varphi_1, 1) + g_2(\varphi_2, 1)] p_0(t) \right\}. \tag{3.101}
\]

Denoting

\[
g_j = \frac{\pi}{\ln 2} g_j, \quad M_{ij} = \frac{\pi}{\ln 2} M_{ij}, \quad i, j = 1, 2,
\]

and multiplying both sides of (3.101) by the constant 1 and integrating over \((-1, 1)\), we obtain a linear equation system with respect to \((\varphi_1, 1)\) and \((\varphi_2, 1)\).

The determinant of the system matrix

\[
\frac{1}{\alpha} \left( \begin{array}{cc} \alpha + \hat{g}_1 & M_{12} \\ M_{21} & \alpha + \hat{g}_2 \end{array} \right)
\]

is

\[
D_P = (\alpha + \hat{g}_1)(\alpha + \hat{g}_2) - M_{12} M_{21}.
\]

Solving the system under the assumption \( D_P \neq 0 \) and substituting the solution to (3.101) we obtain the componentwise expression for \( L_P^{-1} \Phi \) with \( \Phi = (f_1, f_2) \):

\[
\varphi_1(t_o) \equiv L_1^{-1} F = \frac{1}{\alpha} \left\{ L^{-1} f_1 + [B_{11}(L^{-1} f_1, 1) + B_{12}(L^{-1} f_2, 1)] p_0(t) \right\},
\]

\[
\varphi_2(t_o) \equiv L_2^{-1} F = \frac{1}{\alpha} \left\{ L^{-1} f_2 + [B_{21}(L^{-1} f_1, 1) + B_{22}(L^{-1} f_2, 1)] p_0(t) \right\}. \tag{3.102}
\]
where the quantities $B_{ij} = B_{ij}$ are defined using the matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \frac{1}{DP} \begin{pmatrix} M_{12}M_{21} - \tilde{g}_1(\alpha + \tilde{g}_2) & -\alpha M_{12} \\ -\alpha M_{21} & M_{21}M_{12} - g_2(\alpha + \tilde{g}_1) \end{pmatrix}. $$

Introducing the diagonal matrix integral operator

$$L = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix},$$

we write (3.102) in the vector notation and obtain the definition of the inverse $L^{-1}$

$$f = L^{-1}F = \frac{1}{\alpha}L^{-1}F + \frac{1}{\alpha}p_0(t)B < L^{-1}F, 1 >,$$

where the vectors

$$L^{-1}F = \begin{pmatrix} L^{-1}f_1 \\ L^{-1}f_2 \end{pmatrix}.$$

Write the integral operator $K$ in (3.62) with the principal part $L_P$ separated according to (3.93)–(3.99)

$$K = L_P + \hat{w}NP,$$  

(3.103)

where the small parameter $\hat{w} = \max_{j=1,2} w_j^2 |\ln w_j|$, $\hat{w}_0 > 0$.

**Theorem 3.13** The inverse $K^{-1}$ exists and admits the representation in the form of the Neumann series

$$K^{-1} = \sum_{n=0}^{\infty} (-1)^n \hat{w}^n (L_P^{-1}N)^n L_P^{-1},$$  

(3.104)

which converges in the operator norm uniformly with respect to $\hat{w}$ in the interval $(0, \hat{w}_0)$ for a certain $\hat{w}_0 > 0$.

**Proof.** We have

$$K^{-1} = (L_P + \hat{w}NP)^{-1} = (I + \hat{w}L_P^{-1}N)^{-1}L_P^{-1},$$

(3.105)

where $\|\hat{w}L_P^{-1}N\| < 1$ for sufficiently small $\hat{w}$ because $\|L_P^{-1}N\|$ is uniformly bounded for $\hat{w} \in (0, \hat{w}_0)$ (see Theorems 3.11, 3.12). □

In applications, it is often sufficient to use only the first two terms of expansion (3.104)

$$K^{-1}f = L_P^{-1}f - \hat{w}L_P^{-1}NL_P^{-1}f + O(\hat{w}^2) \quad \text{as} \quad \hat{w} \to 0.$$  

(3.106)

Representations (3.104) and (3.106) can be applied for obtaining an asymptotic expansion of the solution to integral equation (3.50) or (3.88) similar to (3.91) and determination of the solution to BCP (3.1) in the case when the set $\omega$ consists of two (or several) intervals (using formulas (3.10), (3.13), and (3.6)).

When the set of integration consists of $m$ intervals (3.24) we use (3.47), (3.49), and the change of variables

$$x = \omega_j t + d_j, \quad x_0 = \omega_i t_0 + d_i, \quad i,j = 1,2,\ldots,m$$
and write integral operator (3.62) in the form similar to (3.77) and (3.103)

\[ K = L_1 + \hat{w}N, \]

where \( \hat{w} = \max_{1 \leq j \leq m} w_j^2 |\ln w_j| \), \( L_1 = \text{diag}(L_1^{(j)})_{j=1}^{m} \), and \( L_1^{(j)} \) is the integral operator (3.77) associated with interval \( \omega_j \), \( j = 1, 2, \ldots, m \). Performing componentwise the inversion of \( L_1 \) in the integral equation \( K\varphi = f \) according to (3.79) we obtain

\[ \varphi + \hat{w}L_1^{-1}N\varphi = L_1^{-1}f, \quad \varphi = \{\varphi_j\}_{j=1}^{m}, \quad f = \{f_j\}_{j=1}^{m}. \]

The latter formula yields the Neumann series for the solution similar to (3.105)

\[ \varphi = K^{-1}f = (I + \hat{w}L_1^{-1}N)^{-1}L_1^{-1}f = L_1^{-1}f + \hat{w}(L_1^{-1}NL_1^{-1})f + O(\hat{w}^2), \quad (3.107) \]

which converges for sufficiently small \( \hat{w} \).

Let

\[ f_j(x) = \sum_{n=0}^{\infty} f_n^{(j)} T_n \left( \frac{x - d_j}{w_j} \right), \quad (3.108) \]

where \( \{nf_n^{(j)}\} \in l^2, x \in w_j, j = 1, 2, \ldots, m \). Considering the integral equations \( L_1^{(j)} \varphi_j = f_j, \quad j = 1, 2, \ldots, m \), and using Lemmas 3.8 and 3.9 we obtain

\[ \varphi_j(x) = (L_1^{(j)})^{-1}f_j = \rho_j(x) \left( \frac{f_j^{(j)}}{\ln \frac{x - d_j}{w_j}} + \sum_{n=1}^{\infty} n f_n^{(j)} T_n \left( \frac{x - d_j}{w_j} \right) \right), \quad (3.109) \]

where \( \rho_j(x) = w_j((d_j + w_j) - x)(x - (d_j - w_j))^{-1/2} \) and \( j = 1, 2, \ldots, m \).

Now we can obtain an asymptotic series of the type (3.92) for the solution of integral equation (3.88) when \( \omega \) consists of several intervals (3.24). To this end, assume that the right-hand side in (3.88) is given by (3.108) and the integral operator \( K \) in (3.107) is defined by (3.88) and write (3.107) componentwise using (3.109) and (3.91) as

\[ \varphi_j(x) = (L_1^{(j)})^{-1}f_j + O(\hat{w}^2), \quad j = 1, 2, \ldots, m, \]

or

\[ \varphi = L_1^{-1}f + O(\hat{w}^2), \]

where the first term on the right-hand side of (3.110) is defined by (3.109). Formula (3.110) gives the desired asymptotic solution to integral equation (3.88).

### 3.7 Asymptotic series solution to the BCP in two dimensions

Let us apply the results of Sections 3.5 and 3.6 to obtain asymptotic series for the solution of BCP (3.2).

**Theorem 3.14** Assume that the conditions of Theorem 3.3 are fulfilled and function \( f(x_1) \) specifying the third boundary condition in (3.2) is given in the form (3.108)

\[ f_j(x_1) = \sum_{n=0}^{\infty} f_n^{(j)} T_n \left( \frac{x_1 - d_j}{w_j} \right), \quad \{nf_n^{(j)}\} \in l^2, \]
where $f_j$ denotes the restriction of $f$ on an interval $\omega_j$, ($j = 1, 2, \ldots, m$). Then the solution $u = (u_1, u_2)$ of problem (3.2) is given by the asymptotic series

$$u_k(x_1, x_2) = \frac{A_0}{2} \sum_{j=1}^{m} w_j \sum_{n=0}^{\infty} q_{n,k}^{(j)} u_{n,k}^{(j)}(x_1, x_2) + O(\hat{w}^2) = \frac{A_0}{2} \sum_{j=1}^{m} \left( w_j q_{0,k}^{(j)} + i w_j q_{1,k}^{(j)} + i w_j q_{0,k}^{(2,j)} (q_{0,k}^{(j)} - q_{2,k}^{(j)}) \right) + O(\hat{w}^2),$$

(3.111)

where

$$u_{n,k}^{(j)} = \int_{-\infty}^{\infty} U_k^*(\lambda, x_2) J_n(w_j \lambda) e^{-i\lambda(x_1 - d_j)} d\lambda, \quad k = 1, 2,$$

$$q_{0,n}^{(j)} = -\ln w_j + d_j, \quad q_{n}^{(j)} = n f_n^{(j)}, \quad n = 1, 2, \ldots,$$

$$\mu_{0,k}^{(0,j)} = \int_{-\infty}^{\infty} v_k^{(j)}(\lambda; x_1, x_2) d\lambda, \quad \mu_{0,k}^{(2,j)} = -\frac{1}{2} \int_{-\infty}^{\infty} \lambda^2 v_k^{(j)}(\lambda; x_1, x_2) d\lambda,$$

$$\mu_{1,k}^{(1,j)} = \int_{-\infty}^{\infty} \lambda v_k^{(j)}(\lambda; x_1, x_2) d\lambda,$$

$$v_k^{(j)}(\lambda; x_1, x_2) = U_k^*(\lambda, x_2) e^{-i\lambda(x_1 - d_j)}, \quad j = 1, 2, \ldots, m,$$

$$\hat{w} = \max_{1 \leq j \leq m} w_j^2 |\ln w_j|,$$ and $U_k^*$, $k = 1, 2$, are defined in (3.14).

**Proof.** Series (3.111) are obtained by substituting into (3.13) series (3.27) using (3.38), in which coefficients $q_{n}^{(j)}$ are determined according to (3.109) and (3.110), and estimating the residual term using Lemma 3.2. □

The results obtained in this chapter enable one to solve BCPs in two-dimensional bands when the diameter of set $\omega$ (simulating screen dots) is sufficiently small.

In order to construct solutions in a wider range of the parameter variation and solve BCPs in the two- and three-dimensional cases we apply different techniques and develop a method called approximate decomposition.
Chapter 4

Method of approximate decomposition

The analysis of boundary value conditions (2.6) and (2.24) shows that they can be decomposed in a special way producing a sequence of BVPs which can be formally separated componentwise. The resulting simplification of the general statements (2.5)–(2.7) and (2.23)–(2.25) consists in formulating auxiliary problems for a ‘shifted’ Laplacian in long parallelepipeds and reducing them to a sequence of iterative problems such that each can be solved (explicitly) by the Fourier method; the solution sequence is obtained with the help of a contracting transfer operator constructed explicitly using, in particular, a rather general family of boundary hat functions (that simulate surface displacements); the rate of convergence of the resulting Fourier-series solutions is exponential. The solution to the initial BCP is obtained then as a limiting function using fixed-point iterations.

This approach constitutes the essence of the approximate decomposition method (AMD) developed in this section.

In Appendix, we present a comprehensive introduction to the method considering the solution of several BCPs for the Laplace and Poisson equations with different types of boundary conditions. We show that ADM enables one both to construct the solution and prove the unique solvability under certain restrictions imposed on the problem parameters.

The method considerably facilitates both the analytic and approximate solutions to boundary contact problems and can be generalized to more complicated mixed BVPs for semilinear partial differential operators.
CHAPTER 4. METHOD OF APPROXIMATE DECOMPOSITION

4.1 Two-dimensional case

4.1.1 Simplified boundary contact problems and the method of approximate decomposition

Consider a simplified version of problem (2.5)–(2.7) subject to the assumptions 1-4 formulated in Section 3.1

\[ \Delta u_j + k_0 \frac{\partial}{\partial x_j} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) = 0, \quad x = (x_1, x_2) \in S, \quad j = 1, 2, \]

\[ u_2 = 0 \quad \text{on} \ K_2, \]

\[ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \quad \text{on} \ K_1 \cup K_2, \]

\[ u_2 = f(x_1) \quad \text{on} \ \omega, \]

\[ (k_0 - 1) \frac{\partial u_1}{\partial x_1} + (k_0 + 1) \frac{\partial u_2}{\partial x_2} = 0 \quad \text{on} \ \omega^*. \]  

(4.1)

We see that under the assumptions made, the statements of Theorem 2.1 and Corollary 2.1 are valid for problem (4.1). Namely, (2.9) holds and there exists the unique (classical) solution \( u = (u_1, u_2) \) to (4.1) where \( u_j, j = 1, 2 \) are twice continuously differentiable in \( S \) and continuous up to the boundary in every closed rectangle \( \Pi_{ah} = \{ x = (x_1, x_2) : 0 \leq x_1 \leq a, \ 0 \leq x_2 \leq h \} \) (see [17], [32], [34]).

Consider the statement of problem (4.1) in a "long" rectangle \( \Pi = \Pi_{ah} = \{ x \in \mathbb{R}^2 : 0 < x_1 < a, \ 0 < x_2 < h \} \) bounded by the curve \( \Gamma = \hat{K}_1 \cup \hat{K}_2 \cup H_1 \cup H_2, \)

where \( \hat{K}_i = K_i \cap \{ 0 < x_1 < a \} \), \( (i = 1, 2), \hat{\omega}^* = \omega^* \cap \{ 0 < x_1 < a \}, \) \( H_1 = \{ x : x_1 = 0, 0 < x_2 < h \} \), and \( H_2 = \{ x : x_1 = a, 0 < x_2 < h \} \). Introduce operators \( L^{(1)} \) and \( L^{(2)} \) specifying the boundary conditions on \( \Gamma \)

\[ L^{(1)} u = \begin{pmatrix} l_{11}^{(1)} & 0 \\ 0 & l_{22}^{(1)} \end{pmatrix} u, \]

\[ l_{11}^{(1)} u_1 = \frac{\partial u_1}{\partial x_1} (x \in \Gamma), \quad l_{22}^{(1)} u_2 = u_2 \quad (x \in \omega \cup \hat{K}_2 \cup H_1 \cup H_2) \]

is the operator of the Neumann-Dirichlet boundary conditions, and

\[ L^{(2)} u = \begin{pmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{pmatrix} u, \]

\[ l_{11}^{(2)} u_1 = 0, \quad l_{12}^{(2)} u_2 = \frac{\partial u_2}{\partial x_1} \quad (x \in \Gamma), \]

\[ l_{21}^{(2)} u_1 = \alpha \frac{\partial u_1}{\partial x_1}, \quad l_{22}^{(2)} u_2 = \frac{\partial u_2}{\partial x_2} \quad (x \in \hat{\omega}^*), \]

where

\[ \frac{\partial}{\partial x_i} \begin{cases} \frac{\partial}{\partial x_1}, & x \in \hat{K}_1 \cup \hat{K}_2 \\ \frac{\partial}{\partial x_2}, & x \in \hat{K}_1 \cup \hat{K}_2 \end{cases}, \quad \frac{\partial}{\partial \nu} = \begin{cases} (k_0 - 1) \frac{\partial}{\partial x_1}, & x \in \hat{K}_i \cup \hat{K}_i \cup H_i \\ (k_0 + 1) \frac{\partial}{\partial x_1}, & x \in \hat{K}_i \cup \hat{K}_i \cup H_i, \quad \alpha = \frac{k_0 - 1}{k_0 + 1} \]

and \( i = 1, 2 \).
Taking into account that the last pair of boundary conditions in (4.1) are stated on complementary sets, we can write boundary conditions (4.1) in the form

\[ L u = L(1)u + L(2)u. \]

Now we can introduce matrix differential operators and write (4.1) as

\[ D u = 0, \quad x \in \Pi, \quad L u = f, \quad x \in \Gamma, \quad (4.3) \]

where

\[ D = \Delta + k_0 A, \quad \Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \]

\[ \Delta_1 u_1 = (k_0 + 1) \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}, \quad \Delta_2 u_2 = \frac{\partial^2 u_2}{\partial x_1^2} + (k_0 + 1) \frac{\partial^2 u_2}{\partial x_2^2}. \quad (4.4) \]

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f = \left( -\frac{\partial f}{\partial x_1}, f(x_1) \right), \quad (x \in \omega). \]

**Definition 4.1** A function \( u(x_1) \in C^m_0(\omega; \Gamma), m \geq 1 \) if \( u(x_1) \in C^m(\Gamma) \), where the space and the corresponding norm are introduced in Definition 2.4, and \( \text{supp} u \subseteq \omega \).

Assume that displacements \( u_2 \) are absent on the cliché base \( \omega^* \). In this case we can apply the statement (4.1) with a boundary function belonging to \( C^m_0(\omega; \Gamma) \). Namely, problem (4.1) with differential and boundary operators specified by (4.2), (4.3), and (4.4) reduces to

\[ D u = 0, \quad x \in \Pi, \quad L(1)u = f, \quad x \in \Gamma, \quad f = \left( -\frac{\partial f}{\partial x_1}, f \right), \quad (4.5) \]

where \( f \in C^m_0(\omega; \Gamma), m \geq 2 \).

In Subsection 4.1.3 we solve BVPs (4.3) and (4.5) using contraction mappings and fixed-point iterations.

**4.1.2 Solution by the Fourier method**

Consider the BVP

\[ \Delta u_0 = 0, \quad x \in \Pi, \quad (4.6) \]

\[ L(1)u_0 = f = \left( -\frac{\partial f}{\partial x_1}, f \right), \quad x \in \Gamma, \quad f \in C^m_0(\omega; \Gamma), \quad m \geq 2. \]

Note that componentwise \( u_0 = (u^{(0)}_1, u^{(0)}_2) \) is composed of the solutions to the Neumann \((u^{(0)}_1)\) and Dirichlet \((u^{(0)}_2)\) BVPs for the shifted Laplace equation in the rectangle \( \Pi_{x_1} \) bounded by \( \Gamma \) and can be determined explicitly by the Fourier method.

**Lemma 4.1** Solution \( u^{(0)}_i \) to problem (4.6) has the form

\[ u^{(0)}_1(x_1, x_2) = \sum_{n=1}^{\infty} \gamma_n^{(1)} \cos(\beta_n^{(1)} x_1) \cosh(\gamma_n^{(1)} x_2), \]

\[ u^{(0)}_2(x_1, x_2) = \sum_{n=1}^{\infty} \gamma_n^{(2)} \sin(\beta_n^{(0)} x_1) \sinh(\gamma_n^{(2)} x_2), \]
where
\[ \beta_n^{(0)} = \frac{\pi n}{a}, \quad \beta_n^{(1)} = \frac{\pi n \sqrt{k_0 + 1}}{a}, \quad \beta_n^{(2)} = \frac{\pi n}{a \sqrt{k_0 + 1}}, \]
and
\[ \gamma_n^{(1)} = \frac{1}{\sqrt{k_0 + 1} \sinh(\beta_n^{(1)} h)}, \quad \gamma_n^{(2)} = \frac{f_n}{\sinh(\beta_n^{(2)} h)}, \]

and
\[ f_n = \int_0^a f(x) \sin(\beta_n^{(0)} x) \, dx. \quad (4.7) \]

Define the sequence \( \{u_s\} \) of vector-functions according to
\[
\begin{align*}
\Delta u_0 &= 0, \quad x \in \Pi, \\
L^{(1)} u_0 &= f = \left( -\frac{\partial f}{\partial x_1}, f(x) \right), \quad x \in \Gamma, \\
\Delta u_{s+1} &= -k_0 A u_s, \quad x \in \Pi, \\
L^{(1)} u_{s+1} &= f, \quad x \in \Gamma, \quad s = 0, 1, 2, \ldots, \\
\end{align*}
\quad (4.8)
\]
where \( f \in C^m(\omega; \Gamma), \ m \geq 2. \)

Consider BVP (4.8) for \( u_{s+1} = (u_1^{(s+1)}, u_2^{(s+1)}). \) Componentwise, (4.8) consists of two inhomogeneous BVPs in the rectangle \( \Pi \)
\[
\begin{align*}
- \Delta_1 u_1^{(s+1)} &= F_1^{(s+1)}, \quad \frac{\partial u_1^{(s+1)}}{\partial \nu} \bigg|_{\Gamma} = \varphi_1, \\
- \Delta_2 u_2^{(s+1)} &= F_2^{(s+1)}, \quad u_2^{(s+1)} \bigg|_{\Gamma} = \varphi_2, \\
\end{align*}
\quad (4.9)
\]
where
\[
F_1^{(s+1)} = k_0 \frac{\partial^2 u_2^{(s)}}{\partial x_1 \partial x_2}, \quad F_2^{(s+1)} = k_0 \frac{\partial^2 u_1^{(s)}}{\partial x_1 \partial x_2} \quad (x \in \Pi_{ah}), \\
\varphi_1 = -\frac{\partial f}{\partial x_1}, \quad \varphi_2 = f \quad (x \in \Gamma), \quad \text{supp } \varphi_i \subseteq \omega, \quad i = 1, 2. \\
(4.10)
\]

Problems (4.9), (4.10) will be called intermediate BVPs.

We consider BVPs in a rectangle which makes it possible to obtain their solutions explicitly in the form of Fourier series; e.g. for \( s = 0 \) the solution to BVPs (4.9), (4.10) are, respectively,
\[
\begin{align*}
u_1^{(1)}(x_1) &= \sum_{n=1}^{\infty} \cos(\beta_n^{(0)} x_1) \left( \eta_n^{(1)} \cosh(\beta_n^{(1)} x_2) + \eta_n^{(2)} \cosh(\beta_n^{(2)} x_2) \right), \\
\end{align*}
\quad (4.11)
\]
where
\[
\begin{align*}
\eta_n^{(1)} &= -\frac{(k_0 + 3)}{(k_0 + 2) \sqrt{k_0 + 1} \sinh(\beta_n^{(1)} h)} f_n = \eta_n^{(1)} f_n, \\
\eta_n^{(2)} &= \frac{\sqrt{k_0 + 1}}{k_0 + 2} \frac{f_n}{\sinh(\beta_n^{(2)} h)} = \eta_n^{(2)} f_n, \\
\end{align*}
\quad (4.12)
\]
and
\[
\begin{align*}
u_2^{(1)}(x_1) &= \sum_{n=1}^{\infty} \sin(\beta_n^{(0)} x_1) \left( \zeta_n^{(1)} \sinh(\beta_n^{(1)} x_2) + \zeta_n^{(2)} \sinh(\beta_n^{(2)} x_2) \right), \\
\end{align*}
\quad (4.13)
where
\[
\zeta_n^{(1)} = \frac{1}{(k_0 + 2)} \frac{f_n}{\sinh(\beta_n^1 h)} = \tilde{\zeta}_n^{(1)} f_n, \\
\zeta_n^{(2)} = \frac{k_0 + 3}{k_0 + 2} \frac{f_n}{\sinh(\beta_n^1 h)} = \tilde{\zeta}_n^{(2)} f_n,
\]
and \(f_n\) are Fourier coefficients of the boundary function \(f \in C_a^{(m)}(\omega; \Gamma)\) defined in (4.7).

**Lemma 4.2** Assume that \(f \in C_a^{(m)}(\omega; \Gamma), m \geq 1\), and let \(f_n\) denote the Fourier coefficients (4.7) of \(f(x_1)\). Then
\[
|f_n| \leq \frac{2m}{n^m} \|f\|_{C^{(m)}(\Gamma)} \text{mes}(\omega), \quad \gamma_m = \frac{2}{a} \left(\frac{a}{\pi}\right)^m.
\]

Proof. In order to prove this lemma, we integrate by parts in (4.7), apply the conditions \(f^{(j)}(a_k) = f^{(j)}(b_k) = 0, j = 0, 1, \ldots, m - 1\) (where \(a_k\) and \(b_k\) are endpoints of the intervals forming the set of \(\omega\) and \(j\) denotes the order of the derivative), and estimate directly the remaining integral to obtain inequality (4.15).

**Lemma 4.3** Assume that \(f \in C_a^{(m)}(\omega; \Gamma), m \geq 3\), and let \(\Pi_{ab}^s = \{x : 0 \leq x_1 \leq a, 0 \leq x_2 \leq \delta\}, 0 < \delta < h\). Then the following estimates hold in \(\Pi_{ab}^s\):

for series (4.13)
\[
|\eta_n^{(1)} \cos(\beta_n^{(0)} x_1) \cosh(\beta_n^{(1)} x_2)| \leq \frac{M_1}{n^m} e^{-\alpha_1 n}, \\
|\eta_n^{(2)} \cos(\beta_n^{(0)} x_1) \cosh(\beta_n^{(2)} x_2)| \leq \frac{M_2}{n^m} e^{-\alpha_2 n},
\]

for series (4.11)
\[
|\zeta_n^{(1)} \sin(\beta_n^{(0)} x_1) \sinh(\beta_n^{(1)} x_2)| \leq \frac{M_1}{n^m} e^{-\alpha_1 n}, \\
|\zeta_n^{(2)} \sin(\beta_n^{(0)} x_1) \sinh(\beta_n^{(2)} x_2)| \leq \frac{M_2}{n^m} e^{-\alpha_2 n},
\]

where \(n = 1, 2, \ldots\) and \(\alpha_1, \alpha_2, M_1, M_2, \ t = 1, 2, \) are positive constants.

Proof. We prove inequalities (4.16) and (4.17) by estimating the right-hand sides of explicit expressions (4.12) and (4.14) directly using inequality (4.15).

From Lemma 4.3, it follows that series (4.11) and (4.13) converge absolutely and uniformly and admit termwise differentiation an arbitrary number of times in any closed subdomain of \(\Pi\) because the rate of convergence is exponential.

For \(s \geq 1\) relations of the type (4.11) and (4.13) can be written in the operator form
\[
u_{s+1} = \mathcal{T} \nu_s, \quad \mathcal{T} = \text{diag}(\mathcal{T}_1, \mathcal{T}_2).
\]

Indeed, using the fact that the rate of convergence of series (4.11) and (4.13) is exponential, substituting (4.7) into (4.11)–(4.14), and changing the order of integration and summation, we obtain
\[
u_{j}^{(s+1)} = \mathcal{T}_j \nu_{j}^{(s)} = \int_0^{a} \psi^{(s)}(\xi) g_j(x; \xi) d\xi, \quad j = 1, 2.
\]
where
\[ g_1(x; \xi) = 2 \sum_{n=1}^{\infty} \cos(\beta_n^0 x_1) \sin(\beta_n^0 x_2) \left( \eta_n^{(1)} \cosh(\beta_n x_2) + \eta_n^{(2)} \cosh(\beta_n x_2) \right), \]
\[ g_2(x; \xi) = 2 \sum_{n=1}^{\infty} \sin(\beta_n^0 x_1) \sin(\beta_n^0 x_2) \left( \zeta_n^{(1)} \sinh(\beta_n x_2) + \zeta_n^{(2)} \sinh(\beta_n x_2) \right), \]
and \( \psi^{(s)}(\xi) = u_2^{(s)}|_\Gamma \).

### 4.1.3 Solution by the fixed-point iterations

In what follows, we solve BCPs using the approximate decomposition described in Appendix for a family of BVPs for the Laplace and Poisson equations. The method is based on the use of fixed-point iterations and contraction mappings.

Consider the sequence \( \{ u_s \} \) of vector-functions defined according to (4.6) and (4.8)
\[
\Delta u_0 = 0, \quad x \in \Pi, \quad L^{(1)} u_0 = f = \left( -\frac{\partial f}{\partial x_1}, f(x_1) \right), \quad x \in \Gamma,
\]
\[
\Delta u_{s+1} = -k_0 A u_s, \quad x \in \Pi, \quad L^{(1)} u_{s+1} = f, \quad x \in \Gamma, \quad s = 0, 1, 2, \ldots,
\]
where \( f \in C^m(\omega; \Gamma), \ m \geq 2. \)

The limiting function (if exists) \( u = \lim_{s \to \infty} u_s \) (where the limit is determined with respect to an appropriate norm) satisfies (4.5). The BVP for the starting function \( u_0 \in C^2(\Pi) \cap C^1(\Pi) \) (which in general may be taken arbitrarily) is formulated on the basis of the assumption that longitudinal variations \( \frac{\partial u}{\partial y_1}(\Psi_1, \Psi_2) \) of transverse elongation \( \frac{\partial^2 u}{\partial y_2^2}(\Psi_1, \Psi_2) \) and transverse variations \( \frac{\partial^2 u}{\partial x_1 \partial y_1}(\Psi_1, \Psi_2) \) of longitudinal elongation \( \frac{\partial u}{\partial y_1} \) are negligible in the bulk region of the band \( S \). Thus already the zero iteration \( u_0 \) gives a good qualitative description of displacements in \( S \) (as well as strains and stresses according to (1.41)–(1.44)) because, actually, \( u_0 \) is obtained under the only assumption that bulk forces are absent inside \( S \). The latter may be justified by another assumption that the boundary pressure is weak, leading to small boundary deformations (defined in terms of \( f \)).

We will justify the approximate decomposition by reducing BVPs to integral relationships on the boundary (as in Appendix) rather than making use of iterations in differential form (4.8).

Consider BVP (4.5) in the rectangle \( \Pi = \Pi_{ah} \) corresponding to the case when displacements on \( \omega^- \) are ignored
\[
-\Delta_1 u_1 = k_0 D^{(12)} u_2, \quad x \in \Pi, \quad \frac{\partial u_1}{\partial y_1} = -\frac{\partial f}{\partial y_1}(x), \quad x \in \Gamma,
\]
\[
-\Delta_2 u_2 = k_0 D^{(12)} u_1, \quad x \in \Pi, \quad u_2 = f(x), \quad x \in \Gamma, \quad (4.19)
\]
where \( u_j \in X(\Pi) = C^2(\Pi) \cap C^1(\Pi), \)
\[
\Delta_1 u_1 = \beta \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}, \quad \Delta_2 u_2 = \beta \frac{\partial^2 u_2}{\partial x_1^2} + \beta \frac{\partial^2 u_2}{\partial x_2^2}, \quad D^{(12)} u_2 = \frac{\partial^2 u_2}{\partial x_1 \partial x_2}, \quad (4.20)
\]
4.1. TWO-DIMENSIONAL CASE

\( \beta = k_0 + 1 \) is a constant, \( \frac{\partial f}{\partial \nu} \) and \( \frac{\partial u_1}{\partial \nu} \) denote, respectively, the normal and tangential derivatives (note that \( \frac{\partial f}{\partial \tau} \bigg|_{\omega} = \frac{\partial f}{\partial x_1} \) and \( \frac{\partial u_1}{\partial \tau} \bigg|_{\omega} = \frac{\partial u_1}{\partial x_2} \)), and it is assumed that the boundary function \( f \in C^m_{\omega}(\omega; \Gamma), m \geq 2 \).

Using superposition and applying the second Green’s formula one can reduce BCP (4.19), (4.20) to an operator equation

\[ u = Ku, \quad Ku = K_1u + F, \tag{4.21} \]

where \( F = (F_1, F_2)^T \),

\[ F_1(x) = \int_\omega G_1^{(2)}(x, y^0)f'(x_1)dx_1 \quad (y^0 = (x_1, h)), \]

\[ F_2(x) = -\int_\omega \frac{\partial}{\partial \nu} G_2^{(1)}(x, y^0)f(x_1)dx_1, \tag{4.22} \]

\[ K_1 = \begin{pmatrix} 0 & \mathcal{K}_{12} \\ \mathcal{K}_{21} & 0 \end{pmatrix} \]

is a linear matrix integrodifferential operator with

\[ \mathcal{K}_{ij}u = k_0 \int_\Pi K_{ij}(x, y)(D^{(12)}u)(y)dy. \tag{4.23} \]

having the kernels

\[ K_{12}(x, y) = \Phi^{(1)}(x, y) + \int_\Gamma G_1^{(2)}(x, z)\frac{\partial}{\partial \nu_z}\Phi^{(1)}(z, y)dz, \]

\[ K_{21}(x, y) = \Phi^{(2)}(x, y) - \int_\Gamma \frac{\partial}{\partial \nu_z}G_2^{(1)}(x, z)\Phi^{(2)}(z, y)dz. \tag{4.24} \]

\( G_2^{(1)} \) and \( G_1^{(2)} \) are Green’s functions of the, respectively, Dirichlet and Neumann problems for the 'shifted' Laplace equations (4.19) in \( \Pi \), and \( \Phi^{(i)}, i = 1, 2 \), are their fundamental solutions.

Consider operators \( K \) and \( K_1 \) in the space \( X(\Pi) \times X(\Pi) \) of two-component vector-functions equipped with the norm \( \|u\|_X = \max_j \|u_j\|_{C(\Pi)} \). Using the properties of the volume and line potentials associated with the shifted Laplacians it is easy to see that \( F_j \in X(\Pi) \) and \( \mathcal{K}_{ij}u \in X(\Pi) \) if \( u \in X(\Pi) \) \((i, j = 1, 2)\). Applying the Schauder estimates for the solutions of elliptic equations, estimating integrals \( \mathcal{K}_{ij}u \) (taking into account that \( G_j^{(i)} \) and \( \Phi^{(j)} \) \((j = 1, 2)\) have logarithmic singularities), and using Lemmas 5.3, 5.4, and 5.7–5.9 proved in Appendix we obtain

\[ \|K_1u\|_X \leq M_0 mes(\Gamma)mes(\Pi)\|u\|_X, \tag{4.25} \]

where \( M_0 = const \) is independent of \( \Pi \). The next statements follow directly from estimate (4.25) and the Banach fixed point theorem.

**Lemma 4.4** Operator \( K_1 \) defined by (4.21)–(4.24) is a contraction in \( X(\Pi) \times X(\Pi) \) if \( p = mes(\Pi) \) is sufficiently small.
**Theorem 4.1** There is a $p_0 > 0$ such that BCP (4.19), (4.20) has one and only one solution $u \in X(\Pi) \times X(\Pi)$ if $p = \text{mes} (\Pi) \in (0, p_0)$. This solution is the fixed point of operator $K$ determined as the limit with respect to the $\| \cdot \|_X$-norm of the fixed-point iterations $u_{n+1} = Ku_n$, $n = 0, 1, \ldots$.

If we do not ignore displacements on the cliché base $\omega^*$ and assume that the boundary condition $\frac{\partial u_2}{\partial \nu} + \alpha \frac{\partial u_1}{\partial \tau} = 0$ holds on $\omega^*$ and $\Sigma_2$, then BCP (4.1) reads

$$-\Delta_1 u_1 = k_0 D^{(12)} u_2, \quad \forall x \in \Pi, \quad \frac{\partial u_1}{\partial \nu} = -\frac{\partial u_2}{\partial \tau} (x), \quad x \in \Gamma, \quad (4.26)$$

$$-\Delta_2 u_2 = k_0 D^{(12)} u_1, \quad \forall x \in \Pi, \quad u_2 = f(x), \quad x \in \Gamma,$$

$$\frac{\partial u_2}{\partial \nu} = -\alpha \frac{\partial u_1}{\partial \tau} (x), \quad x \in \Sigma_2, \quad (4.27)$$

where $\Gamma = \partial \Pi = \omega \cup \Sigma_2$ and $u_j \in X(\Pi) = C^2(\Pi) \cap C^1(\bar{\Pi})$, $j = 1, 2$.

The solution to BVP (4.26), (4.27) can be reduced, as well as in the case of BVP (4.19), (4.20), to an operator equation

$$u = S^D u, \quad S^D = S^D_1 u + F^D, \quad (4.28)$$

where $F^D = (F^D_1, F^D_2)^T$

$$F^D_1(x) = \int_{\omega} G^{(2)}_1(x, y)f'(y)(x_1)dy_1 \quad (y^0 = (x_1, h)), \quad$$

$$F^D_2(x) = 2 (H_{\omega, 1}f)(x), \quad$$

operators $H_{\omega, 1}$ and $H_{\omega, 2}$ are defined in Appendix by formulas (A.48) and (A.49), and

$$S^D_1 = \begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix}$$

is a linear matrix integrodifferential operator with

$$S_{12} u_2 = -\int_{\Sigma} G^{(2)}_1(x, y) (T_{\Gamma, 1} u_2)(y)dy + k_0 \int_{\Pi} G^{(2)}_1(x, y) \frac{\partial}{\partial \nu} \int_{\Pi} \Phi^{(1)}(y, z) \left(D^{(12)} u_2\right)(z)dz, \quad (4.29)$$

$$S_{21} u_1 = 2k_0 H_{\omega, 1} \left[ \int_{\Pi} \Phi^{(2)}(y, z) \left(D^{(12)} u_1\right)(z)dz \right](x) - \alpha H_{\omega, 2} [(T_{\Gamma, 1} u_1)](x) + k_0 H_{\omega, 2} \left[ \frac{\partial}{\partial y} \int_{\Pi} \Phi^{(2)}(y, z) \left(D^{(12)} u_1\right)(z)dz \right](x), \quad (4.30)$$

where $T_{\Gamma, 1} : C^1(\bar{\Pi}) \rightarrow C(\Gamma)$ defined according to $T_{\Gamma, 1} u = \frac{\partial u}{\partial \tau} |_{\Gamma}$ is the restriction associated with the Neumann boundary condition.
4.1. TWO-DIMENSIONAL CASE

Consider operators $S^D$ and $S^D_1$ in the space $X(\Pi) \times X(\Pi)$. Using the properties of the volume and line potentials associated with the shifted Laplacians, the Schauder estimates for the solutions of elliptic equations, and estimating the integrals in (4.29) and (4.30) with the help of inequalities (A.8), (A.10), and (A.50) (Lemma 5.8) proved in Appendix we obtain the estimate (cf. (4.25))

$$\|S_1^D u\|_X \leq M_1 \text{mes}(\Gamma)\text{mes}(\Pi)\|u\|_X,$$

where $M_1 = \text{const}$ is independent of $\Pi$.

The next statements follow from estimate (4.31) and the Banach fixed point theorem.

**Lemma 4.5** Operator $S^D_1$ defined by (4.28)–(4.30) is a contraction in $X(\Pi) \times X(\Pi)$ if $p = \text{mes}(\Pi)$ is sufficiently small.

**Remark.** Assuming, as in (A.11) (see Section A.2 in Appendix), that $a \in [a_0, a_1]$, $0 < a_0 < a_1$, $h \in (0, 1)$,

$$\begin{align*}
\|K_1 u\|_X &\leq M_0 \tilde{h}\|u\|_X, \\
\|S^D_1 u\|_X &\leq M_1 \tilde{h}\|u\|_X,
\end{align*}$$

where $M_0$ and $M_1$ are constants independent of $h$ that are uniform with respect to parameters $a$ and $\text{mes}(\omega)$.

**Theorem 4.2** There is a $p_0 > 0$ such that BCP (4.26), (4.27) has one and only one solution $u \in X(\Pi) \times X(\Pi)$ if $p = \text{mes}(\Pi) \in (0, p_0)$. This solution is the fixed point of operator $S^D$ determined as the limit with respect to the $\|\cdot\|_X$-norm of the fixed-point iterations $u_{n+1} = S^D u_n$, $n = 0, 1, \ldots$.

The conditions under which we prove Theorems 4.1 and 4.2 have a clear physical meaning: if initial loads are sufficiently weak and applied through small (narrow) openings $\omega$ and the sample is finite and very thin ($0 < h \ll a$, i.e. $\tilde{h}$ is sufficiently small, and $\text{mes} \omega \ll a$) then one can justify and use an approximate linear model setting defined above and determine with increasing accuracy (increasing $s$) longitudinal and transverse displacements in the sample using fixed-point iterations.

The iterations for BVP (4.26), (4.27) similar to (4.9) and (4.10) can be written componentwise as

$$\begin{align*}
-\Delta_{\Pi} u_1^{(s+1)} &= f_1^{(s+1)}, & x \in \Pi, & i = 1, 2, \\
\frac{\partial u_1^{(s+1)}}{\partial \nu} &= -\frac{\partial u_2^{(s)}}{\partial \nu}, & x \in \Gamma, \\
u_1^{(s+1)} &= f, & x \in \omega, \\
u_2^{(s+1)} &= -\alpha \frac{\partial u_1^{(s)}}{\partial x_1}, & x \in \Gamma \setminus \omega,
\end{align*}$$

(4.33)
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with \( f \) as in (4.1), \( F_i^{(s+1)} \) defined in (4.10), and \( u_0 \) being e.g. the solution to BVP (4.6).

The general inhomogeneous BVP (2.5)–(2.7) formulated in a rectangle \( \Pi_{ah} \) with appropriate boundary conditions on side walls \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) can also be solved by the method of approximate decomposition. To this end, write componentwise the corresponding iterative BVPs similar to (4.9), (4.18), and (4.33)

\[
-\Delta u_i^{(s+1)} = g_i^{(s+1)} \quad (x \in \Pi), \quad i = 1, 2,
\]

\[
d_i^{(s+1)} = k_0 \frac{\partial^2 u_i^{(s)}}{\partial x_1 \partial x_2} + \mathcal{F}_1, \quad g_2^{(s+1)} = k_0 \frac{\partial^2 u_2^{(s)}}{\partial x_1 \partial x_2} + \mathcal{F}_2,
\]

\[
\frac{\partial u_i^{(s+1)}}{\partial \nu} + \frac{\partial u_2^{(s+1)}}{\partial \tau} = \hat{f}_1 \quad (x \in \Gamma), \quad \frac{\partial u_1^{(s+1)}}{\partial x_1} + \frac{\partial u_2^{(s+1)}}{\partial x_2} = \hat{f}_3 \quad (x \in \mathcal{W}^*),
\]

\[
u^{(s+1)} = f_2 \quad (x \in \omega), \quad s = 0, 1, 2, \ldots,
\]

(4.34)

where \( u_0 \) is composed of solutions to two independent BVPs

\[
-\Delta u_i^{(0)} = 0 \quad (x \in \Pi), \quad i = 1, 2,
\]

\[
\frac{\partial u_1^{(0)}}{\partial \nu} + \frac{\partial u_2^{(0)}}{\partial \tau} = \hat{f}_1 \quad (x \in \Gamma),
\]

\[
\frac{\partial u_1^{(0)}}{\partial x_2} = \hat{f}_3(x_1) \quad (x \in \mathcal{W}^*), \quad u_2^{(0)} = f_2(x) \quad (x \in \omega),
\]

with the boundary functions

\[
\hat{f}_1(x) = \begin{cases} 
    f_1(x_1), & x \in \mathcal{H}_1 \cup \mathcal{H}_2, \\
    0, & x \in \mathcal{H}_1 \cup \mathcal{H}_2
\end{cases}
\]

and

\[
\hat{f}_3(x) = \begin{cases} 
    f_3(x_1), & x \in \mathcal{W}^*, \\
    0, & x \in \Gamma \setminus \mathcal{W}^*
\end{cases}
\]

The BVPs for \( u_1^{(0)} \) and \( u_1^{(s)} (s \geq 1) \) is a Neumann BVP in \( \Pi_{ah} \), and the BVPs for \( u_2^{(0)} \) and \( u_2^{(s)} (s \geq 1) \) is a BVP of mixed type in \( \Pi_{ah} \) with the Dirichlet condition on \( \Gamma \setminus \mathcal{W}^* \).

Using the same technique, one can prove that the corresponding “transfer” operators \( K \) defined as in (4.21) will be a contraction. In fact, the solution \( u^{(s+1)} \) to the mixed BVPs (4.33) or (4.34) can be componentwise represented, similar to (4.21)–(4.24), as sums of volume and Green’s line potentials and reduced to an operator equation (4.21) where operator \( K \) is a contraction in the space \( X(\Pi) \times X(\Pi) \) if \( \text{mes}(\Pi) \) is sufficiently small.

The next step is to apply the above method of approximate decomposition to the solution of BVPs of the type (2.5)–(2.7) or (4.1) for semilinear systems with the differential operators \( D u = \Delta u + \mathcal{F}(u, u_{x_1}, u_{x_2}, u_{x_1 x_2}) \), where \( \mathcal{F} \) is nonlinear with respect to \( u \) and \( u_{x_i} \). Constructing the iterations similar to (4.21) or (4.33) and showing or assuming that the corresponding transfer operator \( K_1 \) is contraction, we obtain an easy recursive procedure (4.21) to determine displacements \( u \).
4.1. **TWO-DIMENSIONAL CASE**

### 4.1.4 Hat functions

All solutions from the previous subsection are obtained as the sine and cosine Fourier series which may, in general, converge slowly close to or on the boundary. In this section we describe a method to improve the convergence of these series based on the choice of specific boundary functions in (4.1). The functions of this family can also be used as basis functions to approximate virtually arbitrary boundary displacements. In addition, their sine and cosine Fourier coefficients can be calculated explicitly.

**Figure 4.1.** Hat function.

Define a four-parameter family of **shifted hat functions** of order \( m \geq 2 \) with the support \( \gamma = [x_S - p, x_S + p] \)

\[
H_m(x) = H_m(x; x_S, p, Q, r) = \begin{cases} 
Q[p^2 - (x - x_S)^2]^m e^{-r(x-x_S)^2}, & |x - x_S| \leq p, \\
0, & |x - x_S| \geq p,
\end{cases}
\]

where \( p, Q, r \) are positive, and

\[
d^iH_m(x_i \pm p) = 0, \quad i = 0, 1, \ldots, m - 1.
\]

According to Definition 2.4, we write \( H_m \in C^{(m-1)}(R) \) and \( \text{supp } H_m = \gamma \), where \( R \) denotes the real line. Such functions describe well the screen dots geometry so that \( Q, p, \) and \( r \) specify, respectively, the height, width, and slope angle (see Figs. 4.1, 4.2).

Define a family of **multi-hat functions** with a compact support consisting of several nonoverlapping intervals \( L_j = [x_S, j - p_j, x_S, j + p_j], \; j = 1, \ldots, n, \; n \geq 2 \)

\[
H_m(x) = H(x) = H_m(x; x_S, p, Q, r) = \begin{cases} 
Q_j[p_j^2 - (x - x_{S,j})^2]^m e^{-r_j(x-x_{S,j})^2}, & |x - x_{S,j}| \leq p_j, \\
0, & x \notin \bigcup_{j=1}^n L_j, \quad j = 1, \ldots, n,
\end{cases}
\]

where the parameter vectors

\[
x_S = [x_{S,1}, x_{S,2}, \ldots, x_{S,n}], \quad p = [p_1, p_2, \ldots, p_n], \\
Q = [Q_1, Q_2, \ldots, Q_n], \quad r = [r_1, r_2, \ldots, r_n]
\]
have positive elements.

A multi-hat function $H_m(x)$ satisfies the properties

$$H_m(x_S) = [Q_1 p_1^{2m}, Q_1 p_1^{2m}, \ldots, Q_n p_n^{2m}], \quad H_m(x) \geq 0,$$

$$H_m(x_S \pm p) = 0, \quad \frac{d}{dx}^i H_m(x_S \pm p) = 0, \quad i = 1, 2, \ldots, m - 1$$

in the vicinities of the points $x_{S,j}$ and is therefore an $(m - 1)$-times continuously differentiable function on the line $R$ with compact support $\omega = \bigcup_{j=1}^{n} [x_{S,j} - p_j, x_{S,j} + p_j]$. We can write $H_m \in C^{m-1}(R)$, $\text{supp } H_m = \omega$.

We have proved

\textbf{Lemma 4.6} A hat function $H_m(x)$ defined by (4.36) is an element of the space $C^m_0(\gamma; \Gamma)$ introduced in Definition 4.1.

Let us go back to the solution of BCPs under study. Function $f_2(x_1)$ specifies the boundary condition (2.6) in the general mixed BVP (2.5)–(2.7), the boundary conditions in (4.1), and enters integral (4.7). If we set $f_2(x) = H_m(x)$ where $H_m(x)$ is a multi-hat function then this $f_2(x_1)$ satisfies condition 4 formulated in Subsection 4.1.1; namely, $f_2(x_1) \in C^{m-1}(\omega)$ with $k = m - 1$, $\omega = \bigcup_{j=1}^{n} [x_{S,j} - p_j, x_{S,j} + p_j]$, and a certain $\mu > 0$, and problem (4.1) has the unique classical solution.

The Fourier coefficients of the hat function $H_m(x)$

$$h_n = \frac{2}{a} \int_{x_S - p}^{x_S + p} Q|p|^2 - (x - x_S)^2 e^{-r(x-x_S)^2} \sin(\beta_n^0 x) dx =$$

$$= \frac{2Q}{a} p^{2m+1} \sin(\beta_n^0 x_S) \int_0^1 (1 - t^2)^m e^{-\rho t^2} \cos(\beta_n^0 \rho t) dt,$$

where $t = (x - x_S)/p$, $\beta_n^0 = \pi n/a$, and $\rho = rp^2$, admit the estimate

$$|h_n| \leq \frac{M}{r^m}, \quad m \geq 1,$$

where $M > 0$ is a constant, and can be calculated explicitly (see Subsection A.5 in Appendix).
Hat functions of order $m \geq 1$ possess the properties of B-splines (namely, they $(m - 1)$ times differentiable on the line $R$, have finite support and one maximum at every subinterval of $\omega$). This implies that one can approximate or interpolate a smooth function on the line $R$ with a finite support by a linear combination of hat functions. Thus one can apply the method of approximate decomposition with rapidly converging series solutions to BCPs with virtually arbitrary boundary functions where hat functions describe screen dots in the case of more than one dot of complicated geometry (see Fig. 4.3).

4.2 Three-dimensional case

4.2.1 Simplified boundary contact problems and the method of approximate decomposition

Consider a simplified version of problem (2.23)–(2.25) subject to the following assumptions:

1. Body forces $F_j = 0 \ (j = 1, 2, 3)$;

2. Shear stresses on the surface $\mathcal{K}_1$ $f_j = 0 \ (j = 1, 2)$;

3. Normal stresses on $\Omega$ $f_4 = 0$;

4. Normal displacements on $\Omega$ $f_3(x_1, x_2) = f(x_1, x_2) \in C^{k, \mu}(\Omega)$, $k \geq 2$, $\mu > 0$;

5. Boundaries $\Gamma_m$ belong to the class $C^{1, \mu}(\Gamma_m)$, $\mu > 0$. 

![Figure 4.3. Multi-hat function and composition of multi-hat functions.](image)
Under these conditions equations (2.23) and (2.24) take the form
\[ \Delta u_j + k_0 \frac{\partial}{\partial x_j} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = 0, \quad x = (x_1, x_2, x_3) \in V, \]
where \( u_3 = 0 \) on \( \mathcal{K}_2 \), \( \frac{\partial u_3}{\partial x_k} = 0 \) on \( \mathcal{X}_1 \cup \mathcal{K}_2 \), \( u_3 = f(x_1, x_2) \) on \( \Omega^* \),
\[ (k_0 - 1) \frac{\partial u_1}{\partial x_1} + (k_0 - 1) \frac{\partial u_2}{\partial x_2} = 0 \tag{4.37} \]
where \( j = 1, 2, 3 \) and \( k = 1, 2, 3 \).

We see that under the assumptions made, the statements of Theorem 2.2 and Corollary 2.12 are valid for problem (4.37). Namely, (2.27) holds and there exists the unique (classical) solution \( u = (u_1, u_2, u_3) \) to (4.37) where \( u_1 \), \( j = 1, 2, 3 \) are twice continuously differentiable in \( V \) and continuous up to the boundary in every closed parallelepiped \( \Pi_{abk} = \{ x = (x_1, x_2, x_3) : 0 \leq x_1 \leq a, \ 0 \leq x_2 \leq b, \ 0 \leq x_3 \leq h \} \).

Introduce the operator \( L = L^{(1)} + L^{(2)} \) specifying the boundary conditions in (4.37), where
\[ L^{(1)} = \text{diag} \left( l_{ii}^{(1)} \right), \]
\[ l_{ii}^{(1)} = \frac{\partial u_i}{\partial x_3} \quad i = 1, 2, \quad (x \in \mathcal{K}_1 \cup \mathcal{K}_2), \]
\[ l_{ii}^{(2)} = l_{ii}^{(1)}, \quad l_{ij}^{(2)} = 0, \quad i, j = 1, 2, \quad (x \in \mathcal{K}_1 \cup \mathcal{K}_2), \]
\[ l_{ij}^{(2)} = (k_0 - 1) \frac{\partial u_1}{\partial x_3}, \quad l_{ij}^{(2)} = (k_0 - 1) \frac{\partial u_2}{\partial x_3}, \quad l_{ij}^{(2)} = (k_0 - 1) \frac{\partial u_3}{\partial x_3} \quad (x \in \Omega^*), \tag{4.38} \]
and the matrix differential operator
\[ D = \Delta + k_0 A, \quad \Delta = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix}, \]
\[ \Delta_1 u_1 = (k_0 + 1) \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}, \]
\[ \Delta_2 u_2 = \frac{\partial^2 u_2}{\partial x_1^2} + (k_0 + 1) \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}, \]
\[ \Delta_3 u_3 = \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + (k_0 + 1) \frac{\partial^2 u_3}{\partial x_3^2} \tag{4.39} \]
\[ A = \| a_{ij} \|_{3 \times 3}, \quad a_{ij} = \begin{cases} \frac{\partial^2}{\partial x_i \partial x_j}, & i \neq j, \\ 0, & i = j. \end{cases} \]

Write BCP (4.37) as
\[ Du = 0, \quad x \in V, \quad Lu = f, \quad x \in \mathcal{K}_1 \cup \mathcal{K}_2, \tag{4.40} \]
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where \( f = \left( -\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, f \right) \) \( (x \in \Omega) \).

Let \( \Pi = \Pi_{abh} = \{ x : 0 < x_1 < a, \ 0 < x_2 < b, \ 0 < x_3 < h \} \) denote a parallelepiped bounded by the surface

\[ \Sigma = \bigcup_{k=1,2,3} H_k, \]

where

\[ H_1^1 = \{ x : x_1 = 0, \ 0 < x_2 < b, \ 0 < x_3 < h \}, \]
\[ H_2^1 = \{ x : x_1 = a, \ 0 < x_2 < b, \ 0 < x_3 < h \}, \]
\[ H_3^1 = \{ x : x_2 = 0, \ 0 < x_1 < a, \ 0 < x_3 < h \}, \]
\[ H_2^2 = \{ x : x_2 = b, \ 0 < x_1 < a, \ 0 < x_3 < h \}, \]
\[ H_1^3 = \{ x : x_3 = 0, \ 0 < x_1 < a, \ 0 < x_2 < b \}, \]
\[ H_2^3 = \{ x : x_3 = h, \ 0 < x_1 < a, \ 0 < x_2 < b \}. \]

**Definition 4.2** Let \( \Omega = \bigcup_{m=1}^{N} \Omega_m \) be a set of disjoint domains bounded by closed piecewise smooth contours \( \Gamma_m, \ m = 1, \ldots, N, \) and situated on one of the sides of parallelepiped \( \Pi \). A function \( u(x_1, x_2) \in C^m(\Omega; \Sigma), \ m \geq 1, \) if \( u(x_1, x_2) \in C^m(\Sigma), \) where the space and the corresponding norm are introduced in Definition 2.4, and \( \text{supp} \ u \subseteq \Omega. \)

Assume that displacements \( u_3 \) are absent on the cliché base \( \Omega^* \). In this case we can formulate the BCP in a parallelepiped \( \Pi \) with the boundary function belonging to \( C^m(\Omega; \Sigma) \). Taking into account the initial statement (1.37)–(1.39) of the BCP we will assume that \( \Omega \subseteq H_2^3. \) Thus, problem (4.37) with differential and boundary operators specified by (4.38) and (4.39) reduces to

\[ \Delta u = 0, \ \ x \in \Pi, \quad L^{(1)}u = f, \ \ x \in \Sigma, \quad f = \left( -\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, f \right), \]

where \( f(x_1, x_2) \in C^m(\Omega; \Sigma), \ m \geq 2. \)

In Section 4.2.3 we will solve BVP (4.41) using contraction mappings and fixed-point iterations.

### 4.2.2 Solution by the Fourier method

Consider the BVP

\[ \Delta u_0 = 0, \ \ x \in \Pi, \quad L^{(1)}u_0 = f = \left( -\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, f \right), \ \ x \in \Sigma, \]

where \( f \in C^m(\Omega; \Sigma), \ m \geq 2. \)

Note that componentwise \( u_0 = (u_0^1, u_0^2, u_0^3) \) is composed of the solutions to the Neumann \((u_0^1, u_0^2, u_0^3)\) and Dirichlet \((u_0^0)\) BVPs for the shifted Laplace equation in the parallelepiped \( \Pi_{abh} \) bounded by \( \Sigma \) and can be determined explicitly by the Fourier method.
Lemma 4.7  Solution $u_0$ to problem (4.42) has the form

$$u_1^0(x_1, x_2, x_3) = \sum_{k=1}^{\infty} \gamma_{kn} \cos(\beta_k x_1) \sin(\beta_k x_2) \cosh(\beta_k x_3),$$

$$u_2^0(x_1, x_2, x_3) = \sum_{k=1}^{\infty} \gamma_{kn} \sin(\beta_k x_1) \cos(\beta_k x_2) \cosh(\beta_k x_3),$$

$$u_3^0(x_1, x_2, x_3) = \sum_{k=1}^{\infty} \gamma_{kn} \sin(\beta_k x_1) \sin(\beta_k x_2) \sinh(\beta_k x_3),$$

where

$$\gamma_{kn}^{(1)} = \frac{\beta_k}{\beta_k^{(1)} \sinh(\beta_k^{(1)} h)}, \quad \gamma_{kn}^{(2)} = -\frac{\beta_k^{(2)}}{\beta_k^{(2)} \sinh(\beta_k^{(2)} h)}, \quad \gamma_{kn}^{(3)} = \frac{f_{kn}}{\sinh(\beta_k^{(2)} h)},$$

$$\beta_k^{(1)} = \pi \sqrt{(k_0 + 1) \frac{k^2}{a^2} + \frac{n^2}{b^2}}, \quad \beta_k^{(2)} = \pi \sqrt{\left(k_0 + 1\right) \frac{k^2}{a^2}}, \quad \beta_k^{(3)} = \pi \frac{k}{a}, \quad \beta_k^{(b)} = \pi \frac{n}{b}. \quad (4.43)$$

$f_{kn}$ are Fourier coefficients of the compactly supported boundary function $f \in C^0(\Omega; \Sigma)$ from problem (4.37)

$$f_{kn} = \frac{4}{ab} \int_0^a \int_0^b f(x_1, x_2) \sin(\beta_k^{(a)} x_1) \sin(\beta_k^{(b)} x_2) \, dx_1 \, dx_2. \quad (4.44)$$

Define the sequence $\{u_n\}$ of vector-functions according to

$$\Delta u_0 = 0, \quad L^{(i)} u_0 = f, \quad L^{(i)} u_{s+1} = f, \quad s = 0, 1, 2, \ldots. \quad (4.45)$$

Consider BVP (4.45) for $u_{s+1} = (u_{1}^{(s+1)}, u_{2}^{(s+1)}, u_{3}^{(s+1)})$. Componentwise, (4.45) consists of three inhomogeneous BVPs for Poisson equation in the parallelepiped

$$-\Delta u_j^{(s+1)} = f_j^{(s+1)}, \quad j = 1, 2, 3,$$

$$\frac{\partial u_j^{(s+1)}}{\partial \nu} \bigg|_\Sigma = \varphi_j, \quad u_3^{(s+1)} \bigg|_\Sigma = \varphi_3, \quad j = 1, 2, \quad (4.46)$$

where

$$f_j^{(s+1)} = k_0 \sum_{i=1}^{3} \frac{\partial^2 u_i^{(s)}}{\partial x_i \partial x_j}, \quad j = 1, 2, 3,$$

$$\varphi_j = \frac{\partial f}{\partial \nu}, \quad \varphi_3 = f, \quad j = 1, 2, \quad (4.47)$$

$$\frac{\partial u_j}{\partial \nu} \bigg|_{\Sigma} = \frac{\partial u_j}{\partial x_i} \bigg|_{\Sigma}, \quad i = 1, 2, 3, \quad i \neq j, \quad j = 1, 2.$$
4.2. THREE-DIMENSIONAL CASE

Problems (4.46) will be called intermediate BVPs.

We consider BVPs in a parallelepiped which makes it possible to obtain their solutions explicitly in the form of Fourier series; e.g. for \( s = 0 \) the solution to BVPs (4.46) is

\[
\begin{align*}
  u_1^{(1)}(x_1, x_2, x_3) &= \sum_{k,n=1}^{\infty} (\zeta_k^{(1)} \cos(\beta_k^{(1)} x_3)) + \\
  &+ \zeta_k^{(2)} \cosh(\beta_k^{(2)} x_3) + \zeta_k^{(3)} \cos(\beta_k^{(3)} x_3)) \cos(\beta_k^{(a)} x_1) \sin(\beta_k^{(b)} x_2), \quad (4.48)
\end{align*}
\]

where

\[
\begin{align*}
  \zeta_k^{(1)} &= \left( -1 - k_0 \frac{(\beta_k^{(b)})^2}{(\beta_k^{(1)})^2 - (\beta_k^{(2)})^2} - k_0 \frac{(\beta_k^{(3)})^2}{(\beta_k^{(1)})^2 - (\beta_k^{(2)})^2} \right) \frac{\beta_k^{(a)}}{\beta_k^{(1)} \sinh(\beta_k^{(1)} h)} f_k, \\
  \zeta_k^{(2)} &= k_0 \frac{\beta_k^{(a)}}{\beta_k^{(2)} ((\beta_k^{(1)})^2 - (\beta_k^{(2)})^2)} \frac{f_k}{\sinh(\beta_k^{(2)} h)}, \\
  \zeta_k^{(3)} &= k_0 \frac{\beta_k^{(a)} \beta_k^{(3)}}{((\beta_k^{(1)})^2 - (\beta_k^{(2)})^2)} \frac{f_k}{\sinh(\beta_k^{(3)} h)}. 
\end{align*}
\]

\[
\begin{align*}
  u_2^{(1)}(x_1, x_2, x_3) &= \sum_{k,n=1}^{\infty} (\eta_k^{(1)} \cos(\beta_k^{(1)} x_3)) + \\
  &+ \eta_k^{(2)} \cosh(\beta_k^{(2)} x_3) + \eta_k^{(3)} \cos(\beta_k^{(3)} x_3)) \sin(\beta_k^{(a)} x_1) \cos(\beta_k^{(b)} x_2), \quad (4.50)
\end{align*}
\]

where

\[
\begin{align*}
  \eta_k^{(1)} &= k_0 \frac{(\beta_k^{(a)})^2 \beta_k^{(b)}}{\beta_k^{(1)} ((\beta_k^{(1)})^2 - (\beta_k^{(2)})^2)} \frac{f_k}{\sinh(\beta_k^{(1)} h)}, \\
  \eta_k^{(2)} &= \left( -1 - k_0 \frac{(\beta_k^{(a)})^2}{(\beta_k^{(1)})^2 - (\beta_k^{(2)})^2} - k_0 \frac{(\beta_k^{(3)})^2}{(\beta_k^{(1)})^2 - (\beta_k^{(2)})^2} \right) \frac{\beta_k^{(b)}}{\beta_k^{(2)} \sinh(\beta_k^{(2)} h)} f_k, \\
  \eta_k^{(3)} &= k_0 \frac{\beta_k^{(b)} \beta_k^{(3)}}{((\beta_k^{(1)})^2 - (\beta_k^{(2)})^2)} \frac{f_k}{\sinh(\beta_k^{(3)} h)}. 
\end{align*}
\]

and

\[
\begin{align*}
  u_3^{(1)}(x_1, x_2, x_3) &= \sum_{k,n=1}^{\infty} (\zeta_k^{(1)} \sin(\beta_k^{(1)} x_3)) + \\
  &+ \zeta_k^{(2)} \sin(\beta_k^{(2)} x_3) + \zeta_k^{(3)} \sin(\beta_k^{(3)} x_3)) \sin(\beta_k^{(a)} x_1) \sin(\beta_k^{(b)} x_2), \quad (4.52)
\end{align*}
\]
Lemma 4.9

Proof.

and let $\Pi_{\text{sh}}^{\delta} = \{ x : 0 \leq x_1 \leq a, 0 \leq x_2 \leq b, 0 \leq x_3 \leq \delta \}, 0 < \delta < h$. Then the following estimates hold in $\Pi_{\text{sh}}^{\delta}$:

$$
|\zeta_{kn}^{(1)}| \leq \frac{M(1.1)}{k^{m_1} h^{m_2}} e^{-n_1(h_1)} k^{(h_1+h)},
$$

$$
|\zeta_{kn}^{(2)}| \leq \frac{M(1.2)}{k^{m_1} h^{m_2}} e^{-n_1(h_1)} k^{(h_1+h)},
$$

$$
|\zeta_{kn}^{(3)}| \leq \frac{M(1.3)}{k^{m_1} h^{m_2}} e^{-n_1(h_1)} k^{(h_1+h)},
$$

for series (4.50)

$$
|h_{kn}^{(1)}| \leq \frac{M(2.1)}{k^{m_1} h^{m_2}} e^{-n_2(h_1)} k^{(h_1+h)},
$$

$$
|h_{kn}^{(2)}| \leq \frac{M(2.2)}{k^{m_1} h^{m_2}} e^{-n_2(h_1)} k^{(h_1+h)},
$$

$$
|h_{kn}^{(3)}| \leq \frac{M(2.3)}{k^{m_1} h^{m_2}} e^{-n_2(h_1)} k^{(h_1+h)},
$$

for series (4.51)
for series (4.52)\)

\[
\begin{align*}
\left|\xi_k^{(1)} \sin(x_k) \sin(x_1) \sin(x_2)\right| & \leq \frac{M_1}{k^{m_1} m_2} e^{-\alpha_1(k+n)} , \\
\left|\xi_k^{(2)} \sin(x_k) \sin(x_1) \sin(x_2)\right| & \leq \frac{M_2}{k^{m_1} m_2} e^{-\alpha_2(k+n)} , \\
\left|\xi_k^{(3)} \sin(x_k) \sin(x_1) \sin(x_2)\right| & \leq \frac{M_3}{k^{m_1} m_2} e^{-\alpha_3(k+n)} ,
\end{align*}
\]

where \( m_1 + m_2 = m_1, m_2 > 1, n = 1, 2, \ldots \), and \( \alpha_{i,j} \) and \( M_{i,j} \), \( i, j = 1, 2, 3 \) are positive constants.

**Proof.** We prove inequalities (4.55)–(4.57), by estimating the right-hand sides of (4.49), (4.51), and (4.53) using (4.54) and explicit expressions for the coefficients. \( \square \)

From Lemma 4.9, it follows that series (4.48)–(4.52) converge absolutely and uniformly and admit termwise differentiation an arbitrary number of times in any closed subdomain of \( \Pi \) because the rate of convergence is exponential.

### 4.2.3 Solution by the fixed-point iteration

Consider the sequence \( \{u_n\} \) of vector-functions according to (4.45)

\[
\begin{align*}
\Delta u_0 &= 0, \quad L^{(1)} u_0 = f, \\
\Delta u_{s+1} &= -k_0 Au_s, \quad L^{(3)} u_{s+1} = f, \quad s = 0, 1, 2, \ldots ,
\end{align*}
\]

The limiting function (if exists) \( u = \lim_{s \to \infty} u_s \), (where the limit is determined with respect to an appropriate norm) satisfies (4.40). The BVP for the starting function \( u_0 \) (which in general may be taken arbitrary) is formulated on the basis of the assumption that longitudinal variations \( \frac{\partial u_1}{\partial x_2} \) and transverse variations \( \frac{\partial u_2}{\partial x_2} \), \( i = 1, 2 \) of longitudinal elongations \( u_1 \) and \( u_2 \) are negligible inside \( \Pi \) (i.e. bulk forces are ignored).

We will justify the approximate decomposition by reducing BVPs to boundary integral equations (as in Appendix) rather than making use of iterations in differential form (4.45).

Consider BCP (4.37) in the parallelepiped \( \Pi = \Pi_{ab} \) corresponding to the case when displacements on \( \Omega^* \) are ignored

\[
\begin{align*}
-\Delta_1 u_1 &= -k_0 (D^{(12)} u_2 + D^{(13)} u_3), \quad x \in \Pi, \\
\frac{\partial u_1}{\partial x_3} &= -\frac{\partial f}{\partial x_3} (x) \quad (x \in \Omega), \\
-\Delta_2 u_2 &= -k_0 (D^{(21)} u_1 + D^{(23)} u_3), \quad x \in \Pi, \\
\frac{\partial u_2}{\partial x_3} &= -\frac{\partial f}{\partial x_3} (x) \quad (x \in \Omega), \\
-\Delta_3 u_3 &= -k_0 (D^{(31)} u_1 + D^{(32)} u_2), \quad x \in \Pi, \\
u_3 &= f(x) \quad (x \in \Sigma),
\end{align*}
\]

(4.58)

(4.59)

(4.60)
where \( u_j \in X(\Pi) = C^2(\Pi) \cap C^1(\Pi) \), \( \Delta_j \) are defined in (4.39),
\[
D^{(ij)} u = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i, j = 1, 2, 3, \quad i \neq j.
\]
and it is assumed that the boundary function \( f \in C^m_0(\Omega; \Sigma) \), \( m \geq 2 \), which implies
\[
\frac{\partial u_i}{\partial x_i} = 0, \quad i = 1, 2, \quad x \in \Sigma_\Omega = \Sigma \setminus \bar{\Omega}.
\]

Using superposition and applying, as in the two-dimensional case, the second Green’s formula one can reduce BCP (4.58)–(4.60) to an operator equation similar to (4.21)
\[
u = K u, \quad K u = K_1 u + F,
\]
where \( F = (F_1, F_2, F_3)^T \).}

\[
F_i(x) = \int_\Omega \mathcal{G}_i^{(2)}(x, y^0) f_{i\alpha}(x) dy^0, \quad i = 1, 2,
\]
\[
F_3(x) = -\int_\Omega \frac{\partial}{\partial y^a} \mathcal{G}_3^{(1)}(x, y^0) f(y^0) dy^0, \quad y^0 = (x_1, x_2, h),
\]
\[
K_1 = \begin{pmatrix} 0 & \mathcal{K}_{12} & \mathcal{K}_{13} \\ \mathcal{K}_{21} & 0 & \mathcal{K}_{23} \\ \mathcal{K}_{31} & \mathcal{K}_{32} & 0 \end{pmatrix}
\]
is a linear matrix integrodifferential operator with
\[
\mathcal{K}_{ij} u = -k_0 \int_\Pi K_{ij}(x, y)(D^{(ij)} u)(y) dy,
\]
having the kernels
\[
K_{1j}(x, y) = \Phi^{(1)}(x, y) + \int_\Sigma \mathcal{G}_1^{(2)}(x, z) \frac{\partial}{\partial x_k} \Phi^{(1)}(z, y) dz, \quad j = 2, 3,
\]
\[
K_{2j}(x, y) = \Phi^{(2)}(x, y) + \int_\Sigma \mathcal{G}_2^{(2)}(x, z) \frac{\partial}{\partial x_k} \Phi^{(2)}(z, y) dz, \quad j = 1, 3,
\]
\[
K_{3j}(x, y) = \Phi^{(3)}(x, y) - \int_\Sigma \frac{\partial}{\partial x_k} \mathcal{G}_3^{(1)}(x, z) \Phi^{(3)}(z, y) dz, \quad j = 1, 2,
\]
\( \mathcal{G}_1^{(1)} \) and \( \mathcal{G}_2^{(2)} \) (\( j = 1, 2 \)) are Green’s functions of the, respectively, Dirichlet and Neumann problems in \( \Pi \) for the ‘shifted’ Laplace operators \( \Delta_j \) defined in (4.58)–(4.60) and \( \Phi^{(j)}(x, y) \) are their fundamental solutions (\( j = 1, 2, 3 \)).

Consider operators \( K \) and \( K_1 \) in the space \( X^3(\Pi) = X(\Pi) \times X(\Pi) \times X(\Pi) \) of three-component vector-functions equipped with the norm \( \| u \|_{X} = \max_{i,j} \| u_i \|_{C(\Pi)} \).

Using the properties of the volume and surface potentials associated with the shifted Laplacians it is easy to see that \( F_j \in X(\Pi) \) and \( \mathcal{K}_{ij} u \in X(\Pi) \) if \( u \in X(\Pi) \) (\( i, j = 1, 2, 3 \)). Applying the Schauder estimates for the solutions of elliptic equations, estimating integrals \( \mathcal{K}_{ij} u \) taking into account that \( \mathcal{G}_j^{(1)} \) and \( \Phi^{(j)} \) (\( j = 1, 2, 3 \)) have singularities of the type \( |x - y|^{-1} \), and using Lemmas 5.7–5.9 and 5.11 proved in Appendix we obtain
\[
\| K_1 u \|_X \leq M_0 \text{mes} (\Sigma) \text{mes} (\Pi) \| u \|_X,
\]
where $M_0 = \text{const}$ is independent of $\Pi$.

**Remark.** Assuming, as in (A.11) and (4.32), that

\[ a \in [a_0, a_1], \quad b \in [b_0, b_1], \quad 0 < a_0 < a_1, \quad 0 < b_0 < b_1, \quad h \in (0, 1), \]

and $h \ll \min(a, b)$ (a 'thin' parallelepiped) we can write inequality (4.66) in terms of the dimensionless small parameter $\hat{h} = h/\max(a, b)$

\[ \|K_1 u\|_X \leq M\hat{h}\|u\|_X, \]

where $M$ is independent of $a$ and $b$.

The next statements follow directly from estimate (4.66) and the Banach fixed point theorem.

**Lemma 4.10** Operator $K_1$ defined by (4.61)–(4.65) is a contraction in $X^3(\Pi)$ if $p = \text{mes} (\Pi)$ is sufficiently small.

**Theorem 4.3** (4.58)–(4.60) has one and only one solution $u \in X^3(\Pi)$ if $p = \text{mes} (\Pi) \in (0, p_0)$. This solution is the fixed point of operator $K$ determined as the limit with respect to the $\|\cdot\|_X$-norm of the fixed-point iterations $u_{n+1} = K u_n$, $n = 0, 1, \ldots$.

If we do not ignore displacements on the cliché base $\hat{\Omega}^*$, considering thus (4.37) in its initial statement in a rectangle $\Pi = \Pi_{a,b}$, then the iterations similar to (4.46), (4.47) can be written componentwise as

\[
\begin{align*}
-\triangle_j u_j^{(s+1)} &= F_j^{(s+1)}, & x \in \Pi, & j = 1, 2, 3, \\
\frac{\partial u_j^{(s+1)}}{\partial \nu} &= -\frac{\partial u_j^{(s)}}{\partial \nu}, & x \in \Sigma, & j = 1, 2, \\
\alpha_1 \frac{\partial u_1^{(s+1)}}{\partial x_1} + \alpha_2 \frac{\partial u_2^{(s+1)}}{\partial x_2} &= -\alpha_3 \frac{\partial u_3^{(s)}}{\partial x_3}, & x \in \hat{\Omega}^*,
\end{align*}
\]

with $F_j^{(s+1)}$ defined in (4.46) and $u_0$ being the solution to mixed BVP

\[
\begin{align*}
-\triangle_i u_i^{(0)} &= 0, & x \in \Pi, & i = 1, 2, 3, \\
\frac{\partial u_i^{(0)}}{\partial \nu} &= -\frac{\partial u_i^{(0)}}{\partial \nu}, & x \in \Sigma, & j = 1, 2, \\
u_3^{(0)} &= f(x), & x \in \hat{\Omega}, \\
\alpha_1 \frac{\partial u_1^{(0)}}{\partial x_1} + \alpha_2 \frac{\partial u_2^{(0)}}{\partial x_2} &= 0, & x \in \hat{\Omega}^*,
\end{align*}
\]

with $f(x)$ defined in (4.40).

The general inhomogeneous BVP (2.23), (2.24) formulated in a parallelepiped $\Pi = \Pi_{a,b}$ with appropriate boundary conditions on side walls $\mathcal{H}_{lj}$, $j, l = 1, 2$ can also be solved by the method of approximate decomposition. To this end, write componentwise the corresponding iterative BVPs similar to (4.46), (4.61), and (4.67)
CHAPTER 4. METHOD OF APPROXIMATE DECOMPOSITION

\[-\Delta_j u_j^{(s+1)} = g_j^{(s+1)} \quad (x \in \Pi),\]

\[g_j^{(s+1)} = k_0 \sum_{i=1}^{3} \frac{\partial^2 u_i^{(s)}}{\partial x_i \partial x_j} + \mathcal{F}_j, \quad j = 1, 2, 3,\]

\[\frac{\partial u_j^{(s+1)}}{\partial \nu} = -\frac{\partial u_3^{(s)}}{\partial \nu}, \quad x \in \Sigma, \quad j = 1, 2,\]

\[u_3^{(s+1)} = f, \quad x \in \bar{\Omega},\]

\[\alpha_1 \frac{\partial u_1^{(s+1)}}{\partial x_1} + \alpha_2 \frac{\partial u_2^{(s+1)}}{\partial x_2} = -\alpha_3 \frac{\partial u_3^{(s)}}{\partial x_3}, \quad x \in \bar{\Omega}^*,\]

where \(u_0\) is composed of solutions to three independent BVPs

\[-\Delta_i u_i^{(0)} = 0, \quad x \in \Pi, \quad i = 1, 2, 3,\]

\[\frac{\partial u_i^{(0)}}{\partial \nu} + \frac{\partial u_i^{(0)}}{\partial \nu} = \tilde{f}_j(x), \quad x \in \Sigma, \quad j = 1, 2,\]

\[u_3^{(0)} = \tilde{f}_3(x), \quad x \in \bar{\Omega},\]

\[\alpha_1 \frac{\partial u_1^{(0)}}{\partial x_1} + \alpha_2 \frac{\partial u_2^{(0)}}{\partial x_2} = \tilde{f}_4(x), \quad x \in \bar{\Omega}^*,\]

with the boundary functions

\[\tilde{f}_j(x) = \begin{cases} f_j(x_1, x_2), & x \in \bar{\Omega}^2, \\ 0, & x \in \Sigma \setminus \bar{\Omega}^2. \end{cases}\]

The BVPs for \(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}\), and \(u_2^{(s)}(s \geq 1)\) is a Neumann BVP in \(\Pi_{abh}\), and the BVPs for \(u_3^{(0)}\) and \(u_3^{(s)}(s \geq 1)\) is a BVP of mixed type in \(\Pi_{abh}\), with the Dirichlet condition on \(\Gamma_{\Omega}^*\).

Using the same technique, one can prove that, under certain (sufficient) conditions, the corresponding "transfer" operators \(K\) defined as in (4.61) will also be a contraction.

The next step is to apply the above method of approximate decomposition to the solution of BVPs of the type (2.23), (2.24) and (4.37) for semilinear systems with the differential operators \(\mathcal{D}u = \Delta u + \mathcal{F}(u, u_x, u_{xx})\), where \(\mathcal{F}\) is nonlinear with respect to \(u\) and \(u_x\), and \(u_{xx}\). Taking the initial \(u_0\) according to (4.45) or (4.67), constructing the iterations similar to (4.45), (4.61) or (4.67), and showing or assuming that the corresponding transfer operator \(K_1\) is contraction, we obtain an easy recursive procedure (4.61) to determine displacements \(u\).

4.2.4 Hat functions

All solutions from the previous subsection are obtained as the sine and cosine Fourier series which may, in general, converge slowly close to or on the boundary. In this subsection we show a method to improve the convergence of these Fourier series based on the choice of specific boundary functions in (4.37). The
functions of this family can also be used as basis functions to approximate virtually arbitrary boundary displacements. In addition, their Fourier coefficients can be calculated explicitly.

Define a four-parameter family of shifted hat functions of order \( m \geq 3 \) with the support \( \Omega = \{(x_1, x_2) : \sqrt{(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2} \leq p\} \)

\[
H_m(x) = H_m(x; x_S, p, Q, r) =
\begin{cases}
Q(p^2 - (x_1 - x_1^j)^2 - (x_2 - x_2^j)^2) & \times e^{-r((x_1-x_1^j)^2 + (x_2-x_2^j)^2)}, \\
0, & \sqrt{(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2} \leq p,
\end{cases}
\]

where \( p, Q, r \) are positive, and

\[
\frac{\partial^i H_m(x_S \pm p)}{\partial x_j^{i}} = 0, \quad i = 0, 1, \ldots, m - 1, \quad j = 1, 2.
\]

According to Definition 2.4, we write \( H_m \in C^{(m-1)}(R \times R) \) and \( \text{supp} \ H_m = \Omega \), where \( R \times R \) denotes the real plane.

Define a family of multi-hat functions with a compact support consisting of several nonoverlapping domains \( \Omega_j = \{(x_1, x_2) : \sqrt{(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2} \leq p_j\} \)

\[
H_m(x) = H(x) = H_m(x; x_S, p, Q, r) =
\begin{cases}
Q(p^2 + (x_1^j - x_1^j)^2 + (x_2 - x_2^j)^2) & \times e^{-r((x_1-x_1^j)^2 + (x_2-x_2^j)^2)}, \\
0, & \sqrt{(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2} \leq p_j,
\end{cases}
\]

where the parameter vectors

\[
x_S = [(x_1^1, x_2^1), \ldots, (x_1^n, x_2^n)], \quad p = [p_1, p_2, \ldots, p_n],
\]

\[
Q = [Q_1, Q_2, \ldots, Q_n], \quad r = [r_1, r_2, \ldots, r_n]
\]

have positive elements.

A multi-hat function \( H_m(x) \) satisfies the properties

\[
H_m(x_S) = [Q_1 p_1^{2m}, Q_1 p_2^{2m}, \ldots, Q_n p_n^{2m}], \quad H_m(x) \geq 0,
\]

\[
H_m(x_S \pm p) = 0, \quad \frac{\partial^i}{\partial x_j^{i}} H_m(x_S \pm p) = 0, \quad j = 1, 2, \quad i = 1, 2, \ldots, m - 1
\]

in the vicinities of the points \( x_{S,j} \) and is therefore an \( (m-1) \)-times continuously differentiable function on the plane \( R \times R \) with the compact support \( \Omega = \bigcup_{j=1}^{n} \Omega_j \).

We can write \( H_m \in C^{(m-1)}(R \times R), \text{supp} \ H_m = \Omega \).

Let us go back to the solution of BCPs under study. Function \( f_3(x_1, x_2) \) specifies the boundary condition in the general mixed BVP (2.23)-(2.25), the boundary conditions in (4.37), and enters integral (4.44). If we set \( f_3(x_1, x_2) = H_m(x_1, x_2) \), where \( H_m(x_1, x_2) \) is a multi-hat function, then this \( f_3(x_1, x_2) \) satisfies condition 4 formulated in Subsection 4.2.1, \( f_3(x_1, x_2) \in C^m(\Omega) \) with
\[ k = m - 1, \quad \Omega = \bigcup_{j=1}^{n} \Omega_j, \] and a certain \( \mu > 0 \), and problem (4.37) has the unique classical solution.

The Fourier coefficients of the hat function \( H_m(x_1, x_2) \) are

\[
h_{kn} = \frac{4}{ab} \int_{0}^{a} \int_{0}^{b} H_m(x_1, x_2) \sin(\beta_k x_1) \sin(\beta_n x_2) \, dx_1 \, dx_2
\]

and admit the estimate

\[
|h_{kn}| \leq \frac{M}{k^{m_1} n^{m_2}},
\]

where \( m_1, m_2 > 1, \quad m_1 + m_2 = m, \quad m > 3, \) and \( M > 0 \) is a constant.

Hat functions of order \( m \geq 1 \) possess the properties of two-dimensional B-splines (namely, they \( (m - 1) \) times differentiable on the plane \( \mathbb{R} \times \mathbb{R} \), have finite support and one maximum at every domain of \( \Omega \)). This implies that one can approximate or interpolate a smooth function on the plane \( \mathbb{R} \times \mathbb{R} \) with a finite support by a linear combination of hat functions and apply the method of approximate decomposition with rapidly converging series solutions to BVPs with virtually arbitrary boundary functions.

In Chapter 5, we apply hat functions to simulate boundary displacements when calculations are performed using both the approximate decomposition and the finite element method.

![Figure 4.4. Hat functions in the three-dimensional case.](image-url)
4.2. THREE-DIMENSIONAL CASE

**Figure 4.5.** Hat function dependence on $p$ in the three-dimensional case.

**Figure 4.6.** Hat function dependence on $Q$ in the three-dimensional case.

**Figure 4.7.** Hat function dependence on $r$ in the three-dimensional case.
Figure 4.8. Composition of multi-hat functions in the three-dimensional case.
Chapter 5

Numerical results and discussion

5.1 Computer simulation of the contact between the substrate and the printing plate

5.1.1 General remarks

In this chapter we present an example of the computer simulation of the contact between the substrate and the printing plate in flexographic printing calculated with the help of the approximate decomposition method (ADM). The simulation results represent a good qualitative picture of the processes in question and can give some practical indications for improving the printing quality.

The International System of Units (SI) is used: the length is expressed in meters (m), modulus of elasticity, shear modulus and stresses are expressed in Pascal (Pa). Some common prefixes used in the text are given in Table 5.1.

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Notation</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>micro-</td>
<td>µ</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>milli-</td>
<td>m</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>mega-</td>
<td>M</td>
<td>$10^{6}$</td>
</tr>
</tbody>
</table>

Table 5.1. Some prefixes used in the text.

For the purposes of clarity letters \( x \), \( y \), and \( z \) are used to denote the axial directions. The substrate (in our case a paperboard sheet) was modelled as a linear elastic orthotropic material. The axial directions \( x \) and \( y \) correspond to the machine (MD) and cross-machine (CD) directions, respectively, and \( z \)-direction corresponds to the thickness (out-of-plane) direction.

Since the radii of the cylinders in the flexography are of an order of $10^{-1}$ m, which is much greater than geometrical sizes of the screen dots ($\approx 10^{-3}$ m) and the surface irregularity of the sheet ($\approx 10^{-4}$ m), the curvatures of the cylinders may be neglected. Therefore, this problem can in this approximation be reduced to the planar problem (1.37)–(1.38) for the printing plate in the contact with the sheet lying on a plane stiff base.
The sheet was modelled as a long and thin parallelepiped $\Pi_{abh} = \{0 < x < a, 0 < y < b, 0 < z < h\}$ where $a, b \gg h$ (see Fig. 5.1). The printing plate was modelled using screen dot patterns which mimic well the typical geometry of printing plates used in flexography.

A weak pressure transferred by the screen dots is simulated in the form of given boundary displacements; the corresponding boundary conditions are described by the hat function

$$H(x, y) = \begin{cases} 
q_j \left(p_j^2 - (x - x_j^p)^2 - (y - y_j^p)^2\right)^4 \times 
& e^{-r \left((x-x_j^p)^2 + (y-y_j^p)^2\right)}, \\
0, & (x, y) / \in \bigcup_{j=1}^{n} \Omega_j, \quad j = 1, \ldots, n,
\end{cases}$$

where $j$ is the screen dot number, $x_j^p$ is the center point of the $j^{th}$ screen dot, $p_j$ and $q_j$ characterize the screen dot height and width, and, finally, $r$ is a ‘quality factor’ that specifies the form of the screen dot surface (the local rate of decrease).

The displacements of the points of the substrate are calculated by ADM; the components of the strain are then obtained from the relations

$$\epsilon_x = \frac{\partial u_x}{\partial x}, \quad \epsilon_y = \frac{\partial u_y}{\partial y}, \quad \epsilon_z = \frac{\partial u_z}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad \gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z},$$

and the components of the stress are determined according to

$$\sigma = D \epsilon,$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$, $\epsilon = (\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$,

$$D = \begin{bmatrix}
\frac{E}{1 - \nu_{xy}\nu_{yz}} & \frac{E}{2} (\nu_{yz} + \nu_{xz}) & \frac{E}{2} (\nu_{xz} + \nu_{yz}) & 0 & 0 & 0 \\
\frac{E}{2} (\nu_{yz} + \nu_{xz}) & \frac{E}{2} (1 - \nu_{xz}\nu_{yz}) & \frac{E}{2} (\nu_{xz} + \nu_{yz}) & 0 & 0 & 0 \\
\frac{E}{2} (\nu_{xz} + \nu_{yz}) & \frac{E}{2} (\nu_{yz} + \nu_{xz}) & \frac{E}{2} (1 - \nu_{yz}\nu_{xy}) & 0 & 0 & 0 \\
0 & 0 & 0 & 2G_{xy} & 0 & 0 \\
0 & 0 & 0 & 0 & 2G_{xz} & 0 \\
0 & 0 & 0 & 0 & 0 & 2G_{yz}
\end{bmatrix},$$

and

$$E = 1 - \nu_{xz}\nu_{yz} - \nu_{yz}\nu_{xz} - \nu_{xz}\nu_{xy}\nu_{yz} - \nu_{yz}\nu_{xy}\nu_{xz}.$$
5.1. COMPUTER SIMULATION

Two-dimensional cross-sections of the volume of the substrate were chosen for visual representation of the calculated quantities.

The calculating program was designed and run in MATLAB. MATLAB was chosen for its flexibility and because the code can be generated and debugged quickly. However, MATLAB does not show high-speed performance and has confined graphical possibilities. The details of the programming are presented in Section 5.4.

5.1.2 Calculation of the strain-stress state of the substrate

This section gives an example of simulation of the contact between a substrate consisting of three layers and a printing plate modelled using a five screen dot pattern (Fig. 5.2).

![Hat function H(x, y) for five-screen dot pattern used in the simulations.](image)

The parameters of the hat function $H(x, y)$ are given in Table 5.2. The value of $q = 4.44 \cdot 10^2$ corresponds to the depth $h = 4 \cdot 10^{-2}$ mm of the screen dot penetration into the substrate. The substrate length and width are taken equal to 2 mm and the substrate thickness (height) is 200 μm. Other substrate parameters are given in Table 5.3.

<table>
<thead>
<tr>
<th>Screen dot</th>
<th>$x^S$</th>
<th>$y^S$</th>
<th>$p$</th>
<th>$r$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.42 \cdot 10^{-3}</td>
<td>0.42 \cdot 10^{-3}</td>
<td>0.3 \cdot 10^{-3}</td>
<td>1</td>
<td>4.44 \cdot 10^2</td>
</tr>
<tr>
<td>2</td>
<td>0.42 \cdot 10^{-3}</td>
<td>1.46 \cdot 10^{-3}</td>
<td>0.3 \cdot 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.46 \cdot 10^{-3}</td>
<td>0.42 \cdot 10^{-3}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.46 \cdot 10^{-3}</td>
<td>1.46 \cdot 10^{-3}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.94 \cdot 10^{-3}</td>
<td>0.94 \cdot 10^{-3}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The other material parameters are obtained according to the symmetry conditions for the orthotropic material, namely

$$E_x\nu_{yz} = E_y\nu_{zx}, \quad E_y\nu_{yx} = E_z\nu_{yz}, \quad E_z\nu_{xz} = E_z\nu_{xz}.$$  

Figures 5.3–5.10 represent the calculated displacements, normal and shear strains and stresses. Figures labelled (a), (b), and (c) show the distribution...
Table 5.3. Parameters of the substrate used in the simulations.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Thickness [µm]</th>
<th>Modulus of elasticity [MPa]</th>
<th>Shear modulus [MPa]</th>
<th>Poisson’s ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$E_x$ $E_y$ $E_z$ $G_{xy}$ $G_{xz}$ $G_{yz}$ $\nu_{xy}$ $\nu_{xz}$ $\nu_{yz}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>50</td>
<td>100 100 20 50 10 20 0.1 0.45 0.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Middle</td>
<td>100</td>
<td>100 100 40 50 10 16 0.1 0.3 0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bottom</td>
<td>50</td>
<td>100 125 25 50 10 12.5 0.2 0.4 0.45</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

of the calculated quantities, respectively, on the surface of the substrate, in the vertical cross-section along the $y$-direction ($x = \text{const}$), and in the vertical cross-section along the $x$-direction ($y = \text{const}$).

Figure 5.3 demonstrates the directional displacement $u_x$ along the $x$-axis. The displacement along the $y$-axis, $u_y$, is quite similar taking into account the direction; therefore, these figures were not included here. It can be seen that on the surface of the substrate the displacements are directed towards the center of the respective contact point. Figure 5.3(a) shows the displacement $u_x$ on the surface ($H = 200 \, \mu\text{m}$), Figures 5.3(b) and (c) show the displacement $u_x$ in the cross-sections, respectively, at the length $L = 0.7 \, \text{mm}$ and at the width $W = 0.5 \, \text{mm}$. It can be observed that the displacements in the region between two contact points are not zero which is caused by interaction between the screen dots.

The directional displacement along the $z$-axis, $u_z$, are shown in Fig. 5.4. The values are one order of magnitude greater than those of the in-plane directions at the same points.

The normal strains at $x$- and $z$-directions, $\epsilon_x$ and $\epsilon_z$, are shown in Figs. 5.5 and 5.6. The minus sign corresponds to compression. A small tension can also be noticed between the regions of the interaction between the screen dots. A region of tension can be observed in Fig. 5.5(c) where the cross-section at the length $W = 0.5 \, \text{mm}$ is shown. This region appears as a result of the influence of the central contact point shown in Fig. 5.5(c). The positive values of the strain $\epsilon_z$ (Fig. 5.6(a)) reflect the fact that material bulges around the interaction region. The deformation in the $z$-direction is considerably greater than the in-plane deformation. The distortions of the angles can be seen in Fig. 5.7. The shear strain of the plane $(x, y)$, $\gamma_{xy}$, is concentrated close to the surface of the substrate.

The normal stresses $\sigma_x$ and $\sigma_z$ demonstrate similar distributions on the surface of the substrate (Figs. 5.8(a) and 5.9(a)). The magnitude is a little higher for $\sigma_z$. The shear stresses are presented in Fig. 5.10.
5.1. COMPUTER SIMULATION

Figure 5.3. ADM-simulation. Directional displacement (x-axis).

Figure 5.4. ADM-simulation. Directional displacement (x-axis).

Figure 5.5. ADM-simulation. Normal strain (x-axis).
CHAPTER 5. NUMERICAL RESULTS AND DISCUSSION

Figure 5.6. ADM-simulation. Normal strain ($z$-axis).

Figure 5.7. ADM-simulation. Shear strain (plane ($x, y$)).

Figure 5.8. ADM-simulation. Normal stress ($x$-axis).
5.1. COMPUTER SIMULATION

Figure 5.9. ADM-simulation. Normal stress ($z$-axis).

Figure 5.10. ADM-simulation. Shear stress (plane ($x, y$)).
5.2 Comparison of the results of simulations obtained by ADM and FEM

The finite element method (FEM) is commonly used for the numerical solution of boundary value problems for partial differential equations. Hence it is interesting to compare our results with the results of simulations by FEM. We have performed comparative analysis of the results calculated by ADM and a commercial software ANSYS; the latter has been verified many times in the course of numerical solutions of a great number of similar problems. The requirements concerning computer resources, such as CPU-time and memory, as well as the adequacy of the results can be discussed.

In the finite element solution of the problem considered in Subsection 5.1.2 (the data are given in Tables 5.2, 5.3), the model was meshed with 8-node fully integrated hexahedrons with three translational degrees of freedom per node. A commercial finite element code [3] was used. The total number of elements was 110000, the number of nodes was 122412, and the number of degrees of freedom was 367236. The problem was solved by an implicit integration in static (steady-state) mode. The system of linear equations generated by the finite element procedure was solved with a Pre-conditioned Conjugate Gradient iterative solver. This is a least demanding equation solver available in ANSYS. For solving fully in-core on a 32-bit computer it requires about 0.9 Kb of RAM per degree of freedom.

In the ADM solution, the total number of elements in the data arrays was 24000. The Fourier series were calculated for each element of the arrays. To store and treat all data arrays it is necessary to use 750 Kb of RAM. Hence ADM requires considerably less RAM compared to FEM which can be of crucial importance for simulation of larger systems. The details of the data representation in the program are presented in Section 5.4.

Both methods demonstrate almost the same speed of calculations (approximately 2 minutes on PC). However, ADM allows for code parallelization which can significantly increase the speed of calculations.

Figures 5.11–5.18 represent the displacements, normal and shear strains and stresses calculated by FEM with the same input parameters as in Tables 5.2 and 5.3. Figures labelled (a), (b), and (c) show the distributions of the calculated quantities, respectively, on the surface of the substrate, in the vertical cross-section along the y-direction ($x = \text{const}$), and in the vertical cross-section along the x-direction ($y = \text{const}$). The cross-sections are taken at the same points as in Figs. 5.3–5.10. It can be seen that the results are quite similar.

Figure 5.11 shows the directional displacements along the x-axis on the surface of the substrate. The displacements for both simulations (see Fig 5.3) have the magnitude of an order of $1 \times 10^{-6}$ m. The maximum and the minimum of the displacements calculated by ADM are $1.912 \times 10^{-6}$ and $-1.912 \times 10^{-6}$ while the respective values for FEM are $3.313 \times 10^{-6}$ and $-3.313 \times 10^{-6}$. We see that the ADM- and FEM-calculated results are also quite close in magnitude.

However, some differences can be noticed. In the case of FEM-simulations (see Fig. 5.11(a)) pronounced displacements are observed in a small vicinity of the contact point. The deformed regions calculated by ADM (see Fig. 5.3(a)) are wider, involving the regions between the neighboring screen dots.

Figure 5.11(c) illustrates the distribution of the directional displacements
5.2. COMPARISON OF ADM AND FEM

for FEM-simulation along the x-axis in the vertical cross-section \( y = 0.5 \times 10^{-3} \) m. Both for FEM and ADM (see Fig. 5.3(c)), higher deformations are located near the surface. Figure 5.11(c) for FEM-simulation demonstrates four large regions of deformation going through the thickness of the substrate. The distribution is uniform and no effect of the presence of different layers can be observed. On the other hand, the distribution of displacements calculated by ADM (see Fig. 5.3(c)) shows a more non-uniform distribution. The regions of deformation similar to those shown in Fig. 5.11(c) appear only on the top layer and the magnitude of displacements for ADM is lower \((1.23 \times 10^{-6})\) compared to FEM \((3.13 \times 10^{-6})\). The distribution is not monotonic along the layers of the substrate.

The results obtained for the directional displacements along the x-axis in the vertical cross-section \( x = 0.7 \times 10^{-3} \) m are analogous (see Figs. 5.11(b) and 5.3(b)).

The displacements along the z-axis calculated by FEM are shown in Fig. 5.12. One can see that the displacements according to both ADM and FEM have the same magnitude of an order of \(1 \times 10^{-5}\). The maximum and the minimum of displacements calculated by ADM (see Fig. 5.4(a)) are \(8.273 \times 10^{-7}\) and \(-3.993 \times 10^{-5}\), while the respective values for FEM (see Fig. 5.12(a)) are \(8.391 \times 10^{-7}\) and \(-4.0 \times 10^{-5}\). Figures 5.4(b) and 5.4(c) demonstrate a more pronounced effect of the interaction between the different screen dots and the influence of the layers on the displacement in z-direction calculated by ADM as compared with FEM (Figs.5.12(b) and 5.12(c)).

We may conclude that the results calculated by ADM and FEM are quite similar. The developed software implementing ADM demands less computer resources compared to ANSYS. It should be noted that a particular realization using MATLAB is far from being the most efficient. The procedure of ADM allows for the code parallelization; a C-code can speed up AMD calculations considerably.
Figure 5.11. FEM-simulation. Directional displacement (x-axis). Cross-sections:
(a) $H = 200 \mu m$, (b) $L = 0.7\ mm$, (c) $W = 0.5\ mm$.

Figure 5.12. FEM-simulation. Directional displacement (z-axis). Cross-sections:
(a) $H = 200 \mu m$, (b) $L = 1\ mm$, (c) $W = 0.5\ mm$.

Figure 5.13. FEM-simulation. Normal strain (x-axis). Cross-sections: (a) $H = 200 \mu m$, (b) $L = 1\ mm$, (c) $W = 0.5\ mm$. 
5.2. COMPARISON OF ADM AND FEM

Figure 5.14. FEM-simulation. Normal strain (z-axis). Cross-sections: (a) $H = 200 \text{ m\mu}$, (b) $L = 1 \text{ mm}$, (c) $W = 0.5 \text{ mm}$.

Figure 5.15. FEM-simulation. Shear strain (plane $(x, y)$). Cross-sections: (a) $H = 200 \text{ m\mu}$, (b) $L = 0.6 \text{ mm}$, (c) $W = 0.4 \text{ mm}$.

Figure 5.16. FEM-simulation. Normal stress (x-axis). Cross-sections: (a) $H = 200 \text{ m\mu}$, (b) $L = 1 \text{ mm}$, (c) $W = 0.5 \text{ mm}$.
Figure 5.17. FEM-simulation. Normal stress ($z$-axis). Cross-sections: (a) $H = 200$ m$\mu$, (b) $L = 1$ mm, (c) $W = 0.5$ mm.

Figure 5.18. FEM-simulation. Shear stress (plane ($x, y$)). Cross-sections: (a) $H = 200$ m$\mu$, (b) $L = 0.6$ mm, (c) $W = 0.4$ mm.
5.3 Interaction between the screen dots

As it was shown above, the results obtained by ADM indicate that the deformation of the region of the contact around a separate screen dot is influenced by the deformation of the contact regions related to other screen dots. Hence it is reasonable to assume that when the screen dots occupy certain positions, an interaction between the contact regions takes place. From the points of view of flexographic printing already small in-plane deformations can affect the ink transferring and worsen the printing quality. In commercial printing two types of half toning (screen dot positioning) are used to reproduce tone variation: Amplitude Modulate (AM) when the size of dots (radii) are varied and the distance between them is constant; and Frequency Modulate (FM) when the size is constant and the distance is varied. Therefore it is interesting to examine how the calculated results depend on the mutual disposition of the screen dots to find out a way of optimization of the configuration of the printing plate with respect to the in-plane deformation.

The field of the displacement vector $u(x, y)$ describes the process perfectly well (see Fig. 5.19). However, in order to be able to relate the configuration of the printing plate to the deformation it causes, a factor characterizing the in-plane displacements need to be introduced. The quantity

$$\chi = \iint_S u(x, y) \, ds,$$  \hspace{1cm} (5.1)

where $S$ is a surface of the substrate, was chosen as such a factor.

The four screen dot pattern was examined. The parameters of the hat function $H(x, y)$ which simulates the boundary displacement are given in Table 5.4. The value of $q = 1.25 \cdot 10^2$ corresponds to the depth $h = 5 \cdot 10^{-3}$ mm of the screen dot penetration into the substrate.

The distance $d$ between the screen dots and the point $(1.0, 1.0)$ was chosen as a parameter to study; $d$ was varied stepwise from 0.4 to 0.8 mm. The axial displacements were calculated for each new configuration, and the quantity $\chi$ was calculated and plotted against the distance $d$.
Table 5.4. Parameters of the hat function $H(x, y)$ used in the simulations.

<table>
<thead>
<tr>
<th>Screen dot</th>
<th>$x^S$</th>
<th>$y^S$</th>
<th>$p$</th>
<th>$r$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.72 \cdot 10^{-3}$</td>
<td>$0.72 \cdot 10^{-3}$</td>
<td>$0.3 \cdot 10^{-3}$</td>
<td>1</td>
<td>$1.25 \cdot 10^2$</td>
</tr>
<tr>
<td>2</td>
<td>$0.72 \cdot 10^{-3}$</td>
<td>$1.28 \cdot 10^{-3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$1.28 \cdot 10^{-3}$</td>
<td>$0.72 \cdot 10^{-3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$1.28 \cdot 10^{-3}$</td>
<td>$1.28 \cdot 10^{-3}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The substrate length and width were taken equal to 2 mm and the substrate thickness was 200 $\mu$m. Other substrate parameters are given in Table 5.3.

The graph of the quantity $\chi$ in Fig. 5.20(a) demonstrates the local maximum and minimum. To analyze the physical meaning of these extremum points we plotted the distribution of the in-plane displacement vector $u(x, y)$ for the respective printing plate configurations. The result is presented in Figs. 5.20(b)–(d). A more precise picture can be obtained by plotting the displacement fields using contour plot and calculating the displacement gradient (Fig. 5.21). These two figures show clearly that the maximum points (Fig. 5.21(a), points A and C) reflect the situation when the contact between the printing plate and the substrate creates small displacements virtually of the entire surface (Figs. 5.21(b), (d)), while the minimum (Fig. 5.21(a), point B) gives rise to the displacements which are greater in magnitude and are located in the vicinity of the contact points (Fig. 5.21(c)). The configuration corresponding to the point of minimum of the quantity $\chi$ looks more preferable from the viewpoint of printing quality since the ink will follow the direction of the gradient growth and hence stay around the contact point. On the other hand, the configurations corresponding to the points A and C reflect the worse possible scenario since the ink will be probably transferred to the large regions which may lead to mottling.

We note however that in order to make final conclusions concerning the effect of the interaction between screen dots and its dependence on the distance between them one has to perform further investigations, also taking into account the results of experimental studies. Such analysis can be carried out using the methods and software developed in this study.

In the same way we can analyze the dependence on other input data: geometrical parameters of the substrate and the cliché, material parameters of the substrate, etc. However, such investigations go beyond the scope of the dissertation volume and are not presented here.
5.3. INTERACTION BETWEEN THE SCREEN DOTS

Figure 5.20. The quantity $\chi$ and displacement field $u(x, y)$ for the printing plate configurations corresponding to the points of local extreme.
Figure 5.21. The quantity $\chi$, the displacement field $u(x, y)$ (contour plot), and displacement gradients (quiver plot) for the printing plate configurations corresponding to the points of local extreme.
5.4 Description of the software

The software was developed in MATLAB. It consists of two modules: the numerical module implementing ADM, and the module for graphical analysis of the calculated data.

Both modules are executed by the main program called Flexo. The dialog-window of the program is shown in Fig. 5.22. To design the dialog-window the special feature in MATLAB, the Graphical User Interface (GUI), was used. The dialog-window was designed to facilitate the description setting of the printing plate-substrate system. The program provides a number of tools which help to enter all necessary parameters and to change the configuration of the system.

![Figure 5.22. The main dialog-window of the program.](image)

The system consists of two separate parts, the substrate and the printing plate. The whole system is described by its length and width. The substrate is composed of several layers and the printing plate is described by the pattern of screen dots. The dialog-window contains two figures illustrating the current configuration of the parts of the system in interactive way. They show the number and the heights of the layers in the substrate and the form and the location of the screen dots in the printing plate.

The composed configuration of the system can be saved in the special data file `input.mat` and then be loaded by demand. The program reads the new values and displays them in the respective edit fields in the dialog-window. The figures are also adjusted to these new values.

The number of the greed points should be specified before the simulation is started. The button `Run` calls the routine implementing ADM. The calculated data is then written in the file named `result.mat`.

For the parameter study a special feature has been designed. The button Parameter Study calls a new dialog-window (Fig. 5.23) where the desired parameter can be selected. The range of variation of the parameter should be entered in the respective edit fields. The button Run starts a series of simulations by ADM, where the selected parameter varies according to given number of steps. The calculated data will be written in the file named result.ps.mat.

Figure 5.23. The dialog-window for the parameter study.

Another feature of the program is the possibility of visualizing the calculated data for future analysis. The button Show Results calls a new dialog-window Show Results (Fig. 5.24).

Figure 5.24. The dialog-window for the module for graphical analysis of the calculated data.

At the top of the dialog-window the list of files in the actual data folder is displayed. The full name of the data folder is shown above the list. The data folder contains two types of data files: a text file input_data.txt where all information about the respective calculation and the parameters is stored, and
the MATLAB data files: `input.mat`, storing the input information, and the files `result.mat` or `result_ps.mat`, storing the result of calculation. The list allows for navigating in the file system and for selecting an arbitrary data folder from the previous calculations. All files can be opened and viewed. Double click on the selected file will call the appropriate viewer for this particular file type.

The data file `result.mat` contains the arrays of calculated axial displacements $u_x$, $u_y$, and $u_z$. Since the data file is chosen (highlighted) the user may select a desired quantity to analyze from the menu below. These quantities are the axial displacements, the normal and shear strains and stresses in all directions. Selecting the quantity different from the displacement will initiate a subroutine which calculates the demanded quantity using corresponding displacements and material parameters stored in the respective data files `input.mat`.

Several plotting functions are available and can be selected from the menu on the right side of the dialog-window. They are the surface plot, the two-dimensional projection of the surface plot on the specified plane, and the curve plot. The desired cross-section or volume can be specified in the specially designed edit fields. The program automatically checks the correctness of the values for the selected plot function and displays an error message in the case of improper input. The button `Plot` calls the routine creating the demanded plot.

It is also possible to plot more than one quantity or to plot the same quantity from the different calculation sets on the same figure. Depending on the plotting function plots will be places as subplots on the same figure or shown as the curve family plot. This was made to simplify the comparison of results. The user may select a data folder, the desired quantity and then push the button `Add`. The names of the data files and chosen quantities will be displayed in the lists below. The button `Reset` allows one to modify the entered values. The plot will be drawn by pushing the button `Plot All`.

A single layer in the substrate as well as a single screen dot in the printing plate were programmed and as structures with the data fields. The whole substrate and the printing plate were programmed as arrays of such structures. This makes it easy to deal with the parameters of the layers or the screen dots and to add or to remove the layers or the screen dots. The respective variables of type structure are called `substrate` and `cliche`.

The program creates the data arrays $u_x$, $u_y$, and $u_z$ as three-dimensional arrays. The dimensions of the arrays are set according to the recalculated grid parameters and reflect the geometry of the substrate. It should be noted that the value of grid in $z$-direction will not be used in the calculation because it is not sensitive to the number of layers and their heights. This value will only be used to recalculate a more precise grid, where all layers will be divided by the grid points separately. In the beginning the arrays are empty and will be filled with the calculated values while the program is running.

The boundary conditions are set by calculating the hat functions according to the parameters of the structure `cliche`. The calculation is executed top-down layer-wise.
Conclusions

The main result of this work is creating a mathematical model and complete investigation of boundary-contact problems (BCPs) in layered structures arising in flexographic printing.

We elaborate a mathematical model of squeezing a thin elastic sheet placed on a stiff base without friction by weak loads through several openings on one of its boundary surfaces. We formulate and consider the corresponding BCPs in two- and three-dimensional layers, prove the existence and uniqueness of solutions and investigate their smoothness including the behavior at infinity and in the vicinity of critical points. The BCP in a two-dimensional layer is reduced to a Fredholm integral equation with a logarithmic singularity of the kernel which is solved by two methods based on (i) the use of the Fourier-Chebyshev series and matrix-algebraic determination of the entries in the resulting infinite system matrix and (ii) asymptotic expansion of the solution using semi-inversion of logarithmic integral operators. The solutions to BCPs in two-dimensional bands are obtained explicitly in the form of asymptotic series.

We propose the method of approximate decomposition for the solution to the BCPs under study. The main idea of this method is to simplify the general BCPs and reduce them to a chain of auxiliary problems for a ‘shifted’ Laplacian in long rectangles or parallelepipeds and then to a sequence of iterative problems such that each of them can be solved (explicitly) by the Fourier method. The solution to the initial BCP is then obtained as a limit using fixed-point iterations.

We elaborate a numerical method and algorithms based on the approximate decomposition and the computer codes and perform comprehensive numerical analysis of the BCPs including the simulation for problems of practical interest. A variety of computational results are presented and discussed and a comparison with a commercial software package ANSYS is performed.

The results obtained in the study form the basis for further applications to modelling and simulation of printing-plate contact systems and other structures of flexographic printing.
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Appendices

A.1 Introductory remarks

In Appendix, we describe a method of solution to a family of BVPs for the Laplace and Poisson equations based on the use of fixed-point iterations and contraction mappings. This method is applied to the solution of BCPs under study.

Let $\Omega$ be a domain in $\mathbb{R}^2$ bounded by a closed piecewise smooth curve $\Gamma$. We estimate the norms of solutions to BVPs considered in $\Omega$ with respect to $\text{mes}(\Omega)$ or $\text{mes}(\omega)$, where $\omega \subset \partial \Omega$ is the support of the boundary data, using the specifications of the Schauder estimates for the solutions of BVPs for elliptic partial differential equations (PDEs) summarized in [21]. An example (see [21], Ch.II, §2) is $\|u\|_{2,\Omega} \leq c_0 \text{mes}(\Omega) \|\nabla |\nabla u|^2\|_{2,\Omega}$, where $u$ is an element of the Sobolev space $W^{2,0}_2(\Omega)$. Such specifications are applied in [21] and [5] in the proofs of the BVP unique solvability in 'small' domains $\Omega$ and of the continuous dependence of solutions on $\Omega$ and boundary data (see [5], Ch.2, §2.3). We reduce BVPs with a smooth compactly supported boundary data to operator equations of the form $u = Ku$ where the kernel singularity of the nonlinear integral or integrodifferential operator $K$ does not exceed that of the fundamental solution to the Laplace equation. The latter enables us to prove that $K$ is a contraction e.g. in the space $C(\Omega)$ if $\text{mes}(\Omega)$ or $\text{mes}(\omega)$ is sufficiently small using direct specifications of the Schauder estimates based on the standard analysis of two- and three-dimensional potential-type integral operators. If $\Omega$ is a rectangle then all necessary estimates can be obtained directly using the Fourier series solution (see Lemmas 4.2 and 4.8).

A.2 Estimates for potential-type integral operators with logarithmic kernels

Let $\Omega$ be a domain in $\mathbb{R}^2$ bounded by a closed piecewise smooth curve $\Gamma$, $x = (x_1, x_2) \in \mathbb{R}^2$, and $\omega \subset \Gamma$ an open arc (or an interval) on a smooth part of $\Gamma$. We shall say that a function $\psi(x) \in C^m(\omega; \Gamma)$, $m \geq 1$ if $\psi(x) \in C^m(\Gamma)$, where $C^m(\Gamma)$ is the space of $m$ times continuously differentiable functions (the norm is defined in a usual manner) and $\text{supp} \psi \subseteq \omega$ (that is, $\psi(x) = 0$, $x \notin \omega$).

Let $G(x, y)$ denote Green’s function of the Neumann problem for the Laplace equation in $\Omega$ for which the representation

$$G(x, y) = \Phi_0(x-y) + N(x, y), \quad x, y \in \Omega, \quad \Phi_0(x-y) = -\frac{1}{2\pi} \ln|x-y|, \quad (A.1)$$

and the uniqueness condition \( \int_\Gamma G(x, y)dy = 0 \) hold \([40]\), where \( N(x, y) \) is a differentiable function in \( \Omega \).

Let us prove several estimates for potential-type integral operators with logarithmic kernels.

Lemma 5.1 Let \( \omega = (a, b) = (a, a + \delta) \) be an interval on the line \( x_2 = 0 \) with \( 0 < \delta < 1 \), and \( \omega \subset \Gamma_{AB} = (A, B) \). Then the following relationships hold:

\[
\int_0^\delta |\ln t|dt = \delta |\ln e|, \quad (A.2)
\]

\[
\int_\omega |\ln |t||dt \leq \delta \max(|\ln |a||, |\ln |a + \delta||), \quad 0 \not\in [a, a + \delta], \quad (A.3)
\]

\[
\int_\omega |\ln |t - s||dt \leq C \delta |\ln \delta||\ln (B - A)|, \quad s \in [A, B], \quad (A.4)
\]

where \( C \) is a constant independent of \( B - A, a, \) and \( \delta \).

Proof. We prove inequality (A.3) explicitly taking into account all possible mutual locations of the intervals \( (a, b) \) and \((-1, 1)\) and the cases \( sgn a = sgn b \) or \( sgn a \neq sgn b \). We also use the inequality \( |\delta \ln e^{-1}| \leq 2\delta |\ln \delta| \) if \( \delta \leq e^{-1} \).

Then we prove (A.4) by making the substitution \( t = s - t \), calculating the integral explicitly and estimating the results using the formulas obtained for integrals (A.2) and (A.3). \( \Box \)

Note that estimates (A.3) and (A.4) can be written in terms of the parameters \( mes(\omega) = b - a = \delta \) and \( mes(\Gamma_{AB}) = B - A > \delta \) as

\[
\int_\omega |\ln |t||dt \leq mes(\omega) \max(|\ln |a||, |\ln |b||), \quad (A.5)
\]

\[
\int_\omega |\ln |t - s||dt \leq C mes'(\omega)|\ln (mes(\Gamma_{AB}))|, \quad s \in [A, B],
\]

where \( mes'(\omega) = mes(\omega)|\ln (mes(\omega))| \).

Lemma 5.2 Let \( \Gamma \) be a closed piecewise smooth curve and \( \omega \subset \Gamma \) an open arc (or an interval) on \( \Gamma \). Then the following estimate is valid

\[
\int_\omega |\ln |x - y||dy \leq q_0 \cdot mes(\Gamma)mes'(\omega), \quad x \in \Gamma, \quad (A.6)
\]

where \( q_0 \) is a constant independent of \( mes(\omega) \) and \( mes(\Gamma) \), and \( mes(\omega) \) and \( mes(\Gamma) \) denote, respectively, the length of \( \omega \) and \( \Gamma \).

Proof. In order to prove this lemma we parametrize contour \( \Gamma \) in the line integral (A.6) by smooth functions \( \theta_j(s), \theta_j(t), j = 1, 2, s, t \in (s_0, s_1), \) so that \( x = \theta(s) = (\theta_1(s), \theta_2(s)) \) and \( y = \theta(t) = (\theta_1(t), \theta_2(t)) \), write

\[
|x - y| = \sqrt{\sum_{j=1}^2 (\theta_j(s) - \theta_j(t))^2} = |s - t| r(s, t),
\]

where \( r(s, t) = |\theta(s) - \theta(t)||s - t| \) is a continuous function at \( s = t \), and apply then Lemma 5.1 representing \( \ln |x - y| = \ln |s - t| + \ln |r(s, t)| \). \( \Box \)
A.2. POTENTIAL-TYPE INTEGRAL OPERATORS

Let \( \Pi = \Pi_{ab} = \{ x : 0 < x_1 < a, 0 < x_2 < h \} \) denote a rectangle. Applying
the calculations that lead to inequalities (A.3)–(A.6) when evaluating double
integrals over a rectangle one can obtain the following statements similar to
Lemmas 5.1 and 5.2.

Lemma 5.3

\[
\int_\Pi |\ln|y|| \, dy \leq \frac{ah}{2} \left( \ln \frac{a^2 + h^2}{e^2} + \pi \right), \tag{A.7}
\]

\[
\int_\Pi |x - y|| \, dy \leq M_1 \text{mes}'(\Pi), \quad x \in \Pi, \tag{A.8}
\]

where \( M_1 > 0 \) is a constant independent of \( \Pi \),

\[
\text{mes}'(\Pi) = \text{mes}(\Pi)|\ln(\text{mes}(\Pi))|,
\]

and \( \text{mes}(\Pi) = ah \) denotes the area of \( \Pi \).

Lemma 5.4

\[
\int_\Pi \left| \frac{\partial}{\partial y_2} \ln|y|| \, dy \right| \leq \pi h, \quad y = (y_1, y_2), \tag{A.9}
\]

\[
\int_\Pi \left| \frac{\partial}{\partial y_2} \Phi_0(x - y)|| \, dy \right| \leq M_2 h, \quad x \in \Pi, \tag{A.10}
\]

where \( M_2 > 0 \) is a constant independent of \( h \).

Assuming that

\[
a \in [a_0, a_1], \quad 0 < a_0 < a_1, \quad h \in (0, 1), \tag{A.11}
\]

we can write (A.7)–(A.10) as uniform estimates in terms of small parameters \( h \) or \( \text{mes}(\Pi) \) (for a ‘thin’ rectangle with \( h \ll a \)), e. g.

\[
\int_\Pi |\ln|y|| \, dy \leq M_0 \text{mes}(\Pi), \tag{A.12}
\]

\[
\max_{x \in \bar{\Omega}} \int_\omega |\Phi_0(x - y)|| \, dy \leq M_3 h, \tag{A.13}
\]

where \( M_0 = \frac{1}{2} \left( \left| \ln \frac{a^2 + 1}{e^2} \right| + \pi \right) \) and \( M_3 > 0 \) are constants independent of \( h \).

When applying estimates of the type (A.3)–(A.6) or (A.7)–(A.13) we will
omit the prime in the expressions involving \( \text{mes}'(\omega) \) and \( \text{mes}'(\Pi) \).

From Lemmas 5.1 and 5.2 the maximum principle for harmonic functions
[40] it follows an important estimate for integrals of the type of potentials (that
are harmonic functions in \( \Omega \)).

Lemma 5.5 Let \( \Omega \) be an open two-dimensional domain bounded by a closed
discrete smooth curve \( \Gamma, \omega \subset \Gamma \) an open arc (or an interval) on \( \Gamma, G(x, y) \)
Green’s function of the Neumann problem for the Laplacian in \( \Omega \) introduced
according to (A.1) and the uniqueness condition, and \( \psi(x) \in C^m(\omega; \Gamma), \ m \geq 1 \).
Then the following estimate is valid

\[
\max_{x \in \bar{\Omega}} \int_\omega G(x, y)\psi(y)dy \leq \max_{x \in \Gamma} \int_\omega G(x, y)\psi(y)dy \leq Q_0 \cdot \text{mes}(\omega)\|\psi\|_{C(\Gamma)}, \tag{A.14}
\]

where \( Q_0 = Q_0(\Omega) \) is a constant independent of \( \text{mes}(\omega) \).
A.3 Approximate decomposition: examples

Consider a BVP

\[ \Delta u = 0, \quad u = u(x), \quad x \in \Omega, \]
\[ \begin{array}{c}
\ell u \equiv \partial u / \partial \nu + h(x)u = f(x), \\
\end{array} \]
\[ x \in \Gamma, \]
\[ (A.15) \]

where \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \), \( \Delta \) and \( \partial / \partial \nu \) denote, respectively, the Laplacian and the normal derivative, and it is assumed that \( h \in C^m_0(\omega; \Gamma) \) with \( m \geq 1 \) and \( f(x) \in C(\Gamma) \).

From the second Green’s formula it follows that problem (A.15) is equivalent to a boundary integral equation

\[ u(x) = \int_\Gamma \mathcal{K}(x, y) u(y) dy + F(x), \quad x \in \Gamma, \]
\[ F(x) = - \int_\Gamma G(x, y) f(y) dy, \quad \mathcal{K}(x, y) = G(x, y) h(y), \]
\[ (A.16) \]

where \( G(x, y) \) is Green’s function of the Neumann problem for the Laplace equation in \( \Omega \) for which the representation (A.1) is assumed to be valid.

We will solve BVP (A.15) with the help of contraction and fixed-point iterations using boundary integral equation (A.16).

Write (A.16) in the operator form

\[ u = Ku, \quad Ku = K_1 u + F, \quad K_1 u = \int_\Gamma \mathcal{K}(x, y) u(y) dy. \]
\[ (A.17) \]

Since \( h \in C^m_0(\omega; \Gamma) \) it is clear that

\[ K_1 u = \int_\omega \mathcal{K}(x, y) u(y) dy. \]
\[ (A.18) \]

Considering integral operators \( K : X \to X \) and \( K_1 : X \to X \) in the space \( X = C(\Gamma) \), using (A.1) and (A.18), and applying Lemma 5.5 one can show that \( \|K_1\| < 1 \) if the parameter \( w = \text{mes } \omega \) is sufficiently small because the following estimates hold (we omit the lower index in the notation for the norm in \( C(\Gamma) \))

\[ \|K_1 u\| = \max_{x \in \Gamma} \left| \int_\omega \mathcal{K}(x, y) u(y) dy \right| \leq w Q_0 \|h\| \|u\|, \]
\[ (A.19) \]

where \( Q_0 \) is a constant that does not depend on \( w \). Namely, according to estimate (A.14), for every positive \( q_0 < 1 \) there is an \( w_0 > 0 \) such that \( \|K_1\| < q_0 \) for \( 0 < w < w_0 \) if \( w_0 Q_0 \|h\| \leq q_0 \) which yields

\[ w_0 \leq \frac{q_0}{Q_0 \|h\|}. \]
\[ (A.20) \]

Let us make an additional assumption that \( f \in C^m_0(\omega; \Gamma) \) with \( m \geq 1 \). Then

\[ F(x) = - \int_\omega G(x, y) f(y) dy \]

and

\[ \|F\| \leq Q_0 w \|f\| \]

in line with Lemma 5.5, where \( Q_0 \) is a constant that does not depend on \( w \).
A.3. APPROXIMATE DECOMPOSITION: EXAMPLES

We can use this assumption to establish a sufficient condition under which operator $K$ maps a given ball $S_p = \{ u \in X : ||u|| < p \}$ into itself. Indeed, if $u \in S_p$, then

$$
||Ku|| \leq ||K_1|| ||u|| + ||F|| \leq Q_0 w(||h|| ||u|| + ||f||) \leq Q_0 w(p ||h|| + ||f||) \quad (A.21)
$$

and operator $K$ maps a ball $S_p$ into itself if the condition $Q_0 w(p ||h|| + ||f||) < p$ is fulfilled which yields

$$
w < w_1(p) = \frac{p}{Q_0 (p ||h|| + ||f||)}. \quad (A.22)
$$

Thus the conditions $f \in C^m(\omega; \Gamma)$ ($m \geq 1$) and

$$
w < \min\{ w_0, w_1(p) \},
$$

where $w_0$ and $w_1(p)$ are determined according to (A.20) and (A.22), provide that operator $K$ is a contraction in the space $X$ and additionally maps a ball $S_p$ into itself.

If $\Omega$ is a rectangle then, remarkably, one can construct the solution to (A.15) and consequently the transition operator $K_1$ in the form of Fourier series having exponential rate of convergence in $\Omega$. The estimates for the norms similar to (A.19) and (A.21) can be performed in terms of the Fourier coefficients (see Lemmas 4.2 and 4.8).

**A.3.1 Solution by iterations**

Define a function sequence $u_n$ according to

$$
\Delta u_0 = 0, \quad x \in \Omega, \quad \frac{\partial u_0}{\partial \nu} = f(x), \quad x \in \Gamma,
$$

$$
\Delta u_{n+1} = 0, \quad x \in \Omega,
$$

$$
\frac{\partial u_{n+1}}{\partial \nu} = \phi_{n+1}(x), \quad x \in \Gamma,
$$

$$
\phi_{n+1}(x) = -h(x) u_n(x) + f(x), \quad n = 0, 1, 2, \ldots, \quad (A.25)
$$

(it is assumed that $f$ satisfies the condition that guarantees the unique solvability of the Neumann problems).

If $u_{n+1}$ ($n = 0, 1, 2, \ldots$) solves (A.24), (A.25) then $u_{n+1}(x), \quad x \in \Gamma$, can be determined, according to (A.17), from

$$
u_{n+1} = K u_n, \quad K u_n = K_1 u_n + F. \quad (A.26)
$$

Considering integral operators $K : X \to X$ and $K_1 : X \to X$ in the space $X = C(\Gamma)$, we assume that condition (A.20) holds, that is, $||K_1|| < 1$. From the Banach fixed point theorem it follows then that the equation $u = K u$ (integral equation (A.16)) has the unique solution $u = u^* \in C(\Gamma)$ (the fixed point of operator $K$) and $u^* = \lim_{n \to \infty} u_n$, where $u_n$ are the fixed-point iterations defined in (A.26). The solution to BVP (A.15) is obtained from the integral representation

$$
u(x) = \int_\Gamma \mathcal{K}(x, y) u^*(y) dy + F(x), \quad F(x) = -\int_\Gamma G(x, y) f(y), \quad x \in \Omega. \quad (A.27)
$$

Summarizing the results of the analysis above we can formulate the following
A.3.2 BVP for the Poisson equation

Consider a BVP for the Poisson equation

\[ -\Delta u = g(x), \quad x \in \Omega, \]  

\[ lu = \frac{\partial u}{\partial n} + \psi(x)u = \phi(x), \quad x \in \Gamma, \]  

where \( u, g \in C^2(\Omega) \cap C^1(\Omega) \), \( \Delta \) and \( \frac{\partial u}{\partial n} \) denote, respectively, the Laplacian and the normal derivative, and it is assumed that the data \( \psi \in C^m(\omega; \Gamma) \) with \( m \geq 1 \) and \( \phi \in C(\Gamma) \).

Using superposition, it is not difficult to extend the results obtained above for BVP (A.15) to BVP (A.28).

Recall that

\[ v_p(x) = \int_{\Omega} \Phi_0(x - y)v(y)dy, \quad x \in \bar{\Omega}, \]

where \( \Phi_0(x - y) \) is defined in (A.1), is called [40] the volume potential having the volume density \( v(y) \) associated with the Laplace operator in two dimensions.

**Proposition 5.1** [40] If \( v \in C^2(\Omega) \cap C^1(\Omega) \) then \( v_p \in C^2(\Omega) \cap C^1(\Omega) \) and

\[ \Delta v_p(x) = -v(x), \quad x \in \Omega. \]

Setting

\[ g_p(x) = \int_{\Omega} \Phi_0(x - y)g(y)dy, \quad u(x) = U(x) + g_p(x), \quad x \in \Omega, \]

and assuming that \( u \) is a solution to (A.28), we obtain an equivalent BVP of the type (A.15) for \( U \)

\[ \Delta U = 0, \quad x \in \Omega, \]

\[ lu = \tilde{\phi}(x), \quad \tilde{\phi}(x) = \phi(x) - \int [g_p](x), \quad x \in \Gamma. \]

From Proposition 5.1 it follows that \( [g_p](x) \in C(\Gamma) \) and hence \( \tilde{\phi}(x) \in C(\Gamma) \). Thus all the results obtained above for BVP (A.15) remain valid for BVP (A.30) if we replace in (A.15) \( f \) by \( \tilde{\phi} \) and \( h \) by \( \psi \), and we can formulate

**Theorem 5.2** Let \( \Omega \) be an open two-dimensional domain bounded by a closed piecewise smooth curve \( \Gamma, \omega \subset \Gamma \) an open arc (or an interval) on \( \Gamma, g(x) \in C^2(\Omega) \cap C^1(\Omega), \psi(x) \in C^m(\omega; \Gamma) \) (\( m \geq 1 \)), and \( \phi(x) \in C(\Gamma) \). Then there is an \( \omega_0 > 0 \) such that BVP (A.30) has one and only one solution \( U \in C^2(\Omega) \cap C^1(\Omega) \)
Lemma 5.5 the following estimate is valid

\[ u(x) = U(x) + g_p(x) = \int_\Omega \mathcal{K}(x, y)U^*(y)\,dy + \tilde{F}(x), \quad x \in \Omega, \]  

(A.31)

where

\[ \tilde{F}(x) = \int_\Gamma G(x, y)\tilde{\phi}(y)\,dy, \quad x \in \Omega, \]  

(A.32)

\[ u^* = U^* = \lim_{n \to \infty} U_n, \]  

\[ U_n \]  

are the fixed-point iterations determined according to (A.26).

Proof. We can apply the argument leading to the solution of BVP (A.30) as a limit of the fixed-point iterations (A.26). Indeed, (A.30) is equivalent to the boundary integral equation (A.16)

\[ U(x) = \int_\Gamma \mathcal{K}(x, y)U(y)\,dy + \tilde{F}(x), \quad x \in \Gamma, \]  

(A.33)

\[ \tilde{F}(x) = -\int_\Gamma G(x, y)\tilde{\phi}(y)\,dy, \quad \mathcal{K}(x, y) = G(x, y)h(y), \]

where \( G(x, y) \) is Green’s function of the Neumann problem for the Laplace equation in \( \Omega \). Define a function sequence \( \{U_n\} \) according to (A.23)–(A.25) with the data in (A.25) \( f = \tilde{\phi}(x) \). If \( U_{n+1} \) \( (n = 0, 1, 2, \ldots) \) solves (A.24), (A.25) with \( f = \tilde{\phi}(x) \), then \( U_{n+1}(x), x \in \Gamma \), can be determined from (A.26).

Considering integral operators \( K : X \to X \) and \( K_1 : X \to X \) in the space \( X = C(\Gamma) \), we assume that condition (A.20) holds providing \( \|K_1\| < 1 \). From the Banach fixed point theorem it follows then that the equation \( U = KU \) (integral equation (A.33)) has the unique solution \( U = U^* \in C(\Gamma) \) (the fixed point of operator \( K \)) and \( U^* = \lim_{n \to \infty} U_n \), where \( U_n \) are the fixed-point iterations defined in (A.26). The solution to BVP (A.30) is obtained, according to (A.29), from the integral representation

\[ u(x) = U^*(x) + g_p(x) = \int_\Gamma \mathcal{K}(x, y)U^*(y)\,dy + \tilde{F}(x), \]

where \( g_p(x) \) and \( \tilde{F}(x) \) are determined, respectively, from (A.29) and (A.32).

Consider a special case when the right-hand side of the Poisson equation in (A.28) admits the form of an integral representation as a single-layer logarithmic potential

\[ g(x) = \int_\Omega T(x, y)\eta(y)\,dy, \quad x \in \Omega, \]  

(A.34)

where \( \eta \in C^m(\omega; \Gamma) \) with \( m \geq 1 \), \( \omega \subset \Gamma \) is an open arc (or an interval) on \( \Gamma \), and \( T(x, y) \) has a logarithmic singularity,

\[ T(x, y) = -\frac{1}{2\pi \ln |x - y|} + T_1(x, y), \quad x, y \in \Omega, \]  

(A.35)

where \( T_1(x, y) \) is a continuously differentiable function in \( \Omega \). According to Lemma 5.5 the following estimate is valid

\[ \max_{x \in \Omega} |g(x)| \leq l_0 \cdot \text{mes} (\omega) \|\eta\|_{C(\Gamma)}, \]
where $t_0 = t_0(\Omega)$ is a constant independent of $\text{mes} (\omega)$. 

Denote by 
\[
\frac{\partial g}{\partial \nu_x}(x) = \frac{\partial}{\partial \nu_x} \int_\Gamma T(x, y) \eta(y) dy
\]
the normal derivative of a single-layer logarithmic potential.

**Proposition 5.2** [40] If $\eta \in C_m^0 (\omega; \Gamma)$ with $m \geq 1$ then 
\[
\lim_{x \to x^0 \in \Gamma} \frac{\partial}{\partial \nu_x} \int_\Gamma T(x, y) \eta(y) dy = \eta(x^0) + \int_\Gamma T_1(x^0, y) \eta(y) dy,
\]
where $T_1(x^0, y) = \frac{\partial}{\partial \nu_x} T(x^0, y)$, 
and $T(x^0, y)$ is continuous on $\Gamma$.

Using Proposition 5.2 and the properties of the volume potential (Proposition 5.1) it is easy to prove

**Lemma 5.6** Let $g(x)$ be a single-layer logarithmic potential defined by (A.34) and $\eta \in C_m^0 (\omega; \Gamma)$ with $m \geq 1$. Then 
\[
l(x) = \eta(x) + \int_\Gamma T_1(x, y) \eta(y) dy + h(x) \int_\Gamma T(x, y) \eta(y) dy, \quad x \in \Gamma.
\]

**Lemma 5.7** Let $\Omega$ be an open two-dimensional domain bounded by a closed piecewise smooth curve $\Gamma$, $\omega \subset \Gamma$ an open arc (or an interval) on $\Gamma$, $G(x, y)$ Green’s function of the Neumann problem for the Laplacian in $\Omega$ introduced according to (A.1) and the uniqueness condition, $\eta, \psi \in C_m^0 (\omega; \Gamma)$ with $m \geq 1$, 
\[
g(x) = \int_\Gamma T(x, y) \eta(y) dy, \quad g_T(x) = \int_\Omega \Phi_0(x - y) g(y) dy, \quad x \in \bar{\Omega};
\]
\[
\tilde{\phi}(x) = \phi(x) - l[g_T](x) = \phi(x) - \eta(x) - \int_\Gamma T_1(x, y) \eta(y) dy + h(x) \int_\Gamma T(x, y) \eta(y) dy, \quad x \in \Gamma, \quad (A.36)
\]
where $\Phi_0, T,$ and $T_1$ are introduced, respectively, in (A.1), (A.34), and (A.35). Then the following estimate is valid
\[
\max_{x \in \Omega} \left| \int_\Omega G(x, y) \tilde{\phi}(y) dy \right| \leq \max_{x \in \bar{\Omega}} \left| \int_\Omega G(x, y) \tilde{\phi}(y) dy \right| \leq \tilde{Q}_0 \cdot \text{mes} (\omega) (\|\phi\| + \|\eta\|), \quad (A.37)
\]
where $\tilde{Q}_0 = \tilde{Q}_0(\Omega)$ is a constant independent of $\text{mes} (\omega)$.

**Proof.** If $\eta(x), \psi(x) \in C_m^0 (\omega; \Gamma)$ with $m \geq 1$ then $\tilde{\phi}(y) \in C^1_0 (\omega; \Gamma)$ because $\int_\Gamma T_1(x, y) \eta(y) dy \in C^0_0 (\omega; \Gamma)$ according to the properties of logarithmic potentials [40] and every other term in (A.36) is a compactly supported function belonging to $C^0_0 (\omega; \Gamma)$. Applying estimate (A.14) with $\psi = \phi$ from Lemma 5.5 we obtain (A.37) and thus complete the proof. \(\diamondsuit\)
Let Ω, ω, φ, and ψ satisfy the conditions of Theorem 5.1 and the inequality

\[ Q_0 \| h \| < 1 \]

holds, where \( Q_0 \) and \( w \) are introduced in (A.19). Then operator \( K_1 \) (A.16), (A.17) is a contraction in \( C(\Gamma) \) (\( \| K_1 \| < 1 \)) and BVP (A.15) has one and only one solution \( u \in C^2(\Omega) \cap C^1(\Omega) \) which admits integral representation (A.27), where \( u^* = \lim_{n \to \infty} u_n \) and \( u_n \) are the fixed-point iterations determined according to (A.26) with an arbitrary \( u_0 \in C(\Gamma) \).

Theorem 5.5 Let Ω, ω, φ, and ψ satisfy the conditions of Theorem 5.2, \( g(\mathbf{x}) \) in (A.28) is given by (A.34), and the inequality

\[ Q_0 \| \psi \| < 1 \]

holds.

\[ \| \tilde{F} \| \leq Q_0 \cdot \text{mes} (\Omega)(\| \phi \| + \| \eta \|), \]

where \( Q_0 \) is a constant independent of \( \text{mes} (\Omega) \). If \( U \in S_p = \{ \phi \in C(\Gamma) : \| \phi \| < p \} \), then

\[ \| KU \| \leq \| K_1 \| \| U \| + \| \tilde{F} \| \leq Q_0 w(\| \psi \| \| U \| + (\| \phi \| + \| \eta \|)) \]

where operators \( K \) and \( K_1 \) are introduced by (A.16) with \( h = \psi, (A.17) \), and (A.26). Operator \( K \) maps a ball \( S_p \) into itself if the condition \( Q_0 w(\| \psi \| + (\| \phi \| + \| \eta \|)) < p \) is fulfilled which yields

\[ w < \tilde{q}_1(p) = \frac{Q \| \psi \|}{Q_0 (\| \psi \| + (\| \phi \| + \| \eta \|))} \]

Thus, if \( g(\mathbf{x}) \) in (A.28) is given by (A.34) with \( \eta(\mathbf{x}) \in C^m(\omega; \Gamma) \) (\( m \geq 1 \)) and the condition

\[ w < \min\{ w_0, \tilde{q}_1(p) \} \]

is fulfilled, where \( w_0 \) and \( \tilde{q}_1 \) are determined according to

\[ w_0 = \frac{Q_0}{Q_0 \| \psi \|} \]

(similar to (A.20)) and (A.38), then operator \( K \) (A.16), (A.17) with \( h = \psi \) is a contraction in the space \( X = C(\Gamma) \) and additionally maps a ball \( S_p \) into itself. \( \square \)

Using \( \| h \| \) or \( \| \psi \| \) as parameters and applying estimates (A.19) we can reformulate Theorems 5.4, 5.5.

Theorem 5.4 Let Ω, ω, f, and h satisfy the conditions of Theorem 5.1 and the inequality

\[ Q_0 \| f \| < 1 \]

holds, where \( Q_0 \) and \( w \) are introduced in (A.19). Then operator \( K_1 \) (A.16), (A.17) is a contraction in \( C(\Gamma) \) (\( \| K_1 \| < 1 \)) and BVP (A.15) has one and only one solution \( u \in C^2(\Omega) \cap C^1(\Omega) \) which admits integral representation (A.27), where \( u^* = \lim_{n \to \infty} u_n \) and \( u_n \) are the fixed-point iterations determined according to (A.26) with an arbitrary \( u_0 \in C(\Gamma) \).
holds, where $Q_0$ and $w$ are introduced in (A.19). Then BVP (A.28) has one and only one solution $u \in C^2(\Omega) \cap C^3(\Omega)$ which admits integral representation (A.31) where $U^*$ is determined according to Theorem 5.2.

Consider the Dirichlet BVP

$$\Delta u = 0, \quad x \in \Omega, \quad u = f(x), \quad x \in \Gamma,$$

where $u \in C^2(\Omega) \cap C(\Omega)$ and $f \in C^m_0(\omega; \Gamma)$ with $m \geq 1$.

From the second Green’s formula it follows that the solution to (A.39) is

$$u(x) = \int_{\Gamma} \frac{\partial G^{(1)}(x, y)}{\partial \nu} f(y) dy, \quad x \in \Omega,$$

where $G^{(1)}(x, y)$ is Green’s function of the Dirichlet problem for the Laplace equation in $\Omega$. Let $\Omega_2 \subset \Omega$ be a compact set. Then, according to Lemma 5.4, the following estimate holds

$$\|u\|_{C(\Omega_2)} \leq Q_3 w \|f\|_{C(\Gamma)},$$

where $Q_3$ is a constant that does not depend on $w$.

Let $B^m_{0,p} = \{f \in C^m_0(\omega, \Gamma): \|f\|_{C^m_0} \leq p \}$ denote a ball in the space $C^m_0(\omega, \Gamma)$. Then, taking into account that (i) $\|f\|_{C^m(\Gamma)} \leq pw$ if $f \in B^m_{0,p}$ and (ii) $u$ given in the form of a double-layer potential by integral representation (A.40) satisfies the boundary condition $u = f$ on $\Gamma$, one can extend estimate (A.41) to obtain

$$\|u\|_{C(\Omega)} \leq Q_1 w \|f\|_{C(\Gamma)},$$

where $Q_1$ is a constant that does not depend on $w$.

Using superposition and the argument similar to the proof of Theorem 5.3 one can show that an estimate of the type (A.42) is also valid for the solution to the BVP for the Poisson equation $-\Delta u = g, \ x \in \Omega, \ u = f(x), \ x \in \Gamma$.

### A.3.3 Mixed BVPs in a rectangle

Let $\Pi = \Pi_{ah} = \{x: 0 < x_1 < a, \ 0 < x_2 < h\}$ be a rectangle bounded by a (closed piecewise smooth) curve $\Gamma = \partial \Pi = \omega \cup \Gamma_\omega$ where $\omega = (d-w/2, d+w/2) \subset \{x: 0 < x_1 < a, \ x_2 = h\}$ with $\text{mes}(\omega) = w$ being an open interval.

Consider a mixed BVP for the Laplace equation

$$\Delta u = 0, \quad u = u(x), \quad x \in \Pi,$$

$$u = \phi(x_1), \quad x = x^0 = (x_1, h) \in \omega, \quad \frac{\partial u}{\partial \nu} = \psi(x), \quad x \in \Gamma_\omega,$$

$$u \in C^2(\Pi) \cap C^3(\Pi), \quad \phi(x_1) \in C^m(\omega) \quad (m \geq 1), \quad \psi(x) \in C(\Gamma_\omega).$$

Applying the second Green’s formula for the solution $u$ to problem (A.43) and Green’s function $G(x, y)$ of the Neumann problem for the Laplace equation in $\Pi$ (see (A.1)) we obtain an integral representation

$$u(x) = \int_{\omega} G(x, y^0) \psi(y_1) dy_1 + \Psi(x), \quad x \in \Pi \quad (y^0 = (y_1, h)), \quad$$

where

$$\psi(y_1) = \frac{\partial u}{\partial \nu}(y^0), \quad \Psi(x) = \int_{\Gamma_\omega} G(x, y) \psi(y) dy,$$
which yields an integral equation with respect to \( \psi \)

\[
\mathcal{K}_\omega \psi \equiv \int_\omega G(x^0, y^0) \psi_\omega(y_1) dy_1 = \Psi_1(x^0), \quad x^0 \in \omega,
\]

(\text{A.46})

According to (\text{A.1}), (\text{A.46}) is an integral equation with a logarithmic singularity of the kernel. If \( \text{mes} (\omega) \) is sufficiently small then there exists a (linear) bounded inverse \( \mathcal{K}_\omega^{-1} \) (see Theorems 3.11 and 3.12), equation (\text{A.46}) is uniquely solvable (note that \( \Psi_1(x^0) \in C^1(\omega) \)), and its solution

\[
\psi_\omega = \mathcal{K}_\omega^{-1} \Psi_1.
\]

(\text{A.47})

The solution to BVP (\text{A.43}) can be written using (\text{A.45})–(\text{A.47}) in the form

\[
u(x) = - H_{\omega,1} \phi + H_{\omega,2} \psi,
\]

(\text{A.48})

where

\[
(H_{\omega,1} \phi)(x) = \int_\omega G(x, y^0) (\mathcal{K}_\omega^{-1} \phi)(y^0) dy_1,
\]

(\text{H.49})

\[
(H_{\omega,2} \psi)(x) = - \int_\omega G(x, y^0) \left[ \mathcal{K}_\omega^{-1} \left( \int_{\Gamma_\omega} G(y^0, z) \psi(z) dz \right) \right] (y_1) dy_1 + \int_{\Gamma} G(x, y) \psi(y) dy.
\]

Lemma 5.8 The following estimate holds

\[
\|u\|_{C^0(\Pi)} \leq Q_0 (\text{mes} (\omega) \|\phi\|_{C^0(\Pi)} + \|\psi\|_{C^0(\Pi)}),
\]

(\text{A.50})

where \( Q_0 \) is a constant that does not depend on \( \text{mes} (\omega) \) and \( u \) is the solution to BVP (\text{A.43}).

Proof. Inequality (\text{A.50}) follows from estimate (\text{A.14}) (Lemma 5.5) applied to integral representation (\text{A.45}) and the fact that the inverse \( \mathcal{K}_\omega^{-1} \) is a bounded operator on \( C^1(\omega) \). \( \Box \)

From Lemma 5.8 it follows that BVP (\text{A.43}) is uniquely solvable and its solution is given by (\text{A.48}) if \( \text{mes} (\omega) \) is sufficiently small. Let us show that under this assumption one can also solve (\text{A.43}) by the fixed-point iterations. Representing the Dirichlet boundary condition in (\text{A.43}) as

\[
\nu(x) = -u(x) + 2\phi(x), \quad x \in \omega,
\]

we can rewrite (\text{A.48}) in the form

\[
u = Su, \quad Su = S_1 u + \Phi,
\]

(\text{A.51})

where

\[
(S_1 u)(x) = -H_{\omega,1} (T_{\omega,0} u), \quad \Phi(x) = 2H_{\omega,1} \phi + H_{\omega,2} \psi,
\]

(\text{A.52})

and \( T_{\omega,0} : C(\overline{\Pi}) \rightarrow C(\Gamma) \) is the restriction associated with the Dirichlet boundary condition [5].
Lemma 5.9 The following estimate holds
\[ \|S_1 u\|_{C(\Pi)} \leq Q_1 \text{mes}(\omega) \|u\|_{C(\Pi)}, \]
where \(Q_1\) is a constant that does not depend on \(\text{mes}(\omega)\) and \(u\) is the solution to BVP \((A.43)\).

Proof. Inequality \((A.53)\) follows from estimates \((A.50)\) (Lemma 5.8) and \((A.14)\) (Lemma 5.5) applied to the integral representation given by the right-hand side of \((A.52)\) and the boundedness of the restriction, \(T_{\omega,0}\). \(\Box\)

Theorem 5.6 Let \(\phi\) and \(\psi\) satisfy conditions \((A.44)\). Then there is an \(w_0 > 0\) such that BVP \((A.43)\) has one and only one solution \(u \in C^2(\Pi) \cap C^1(\bar{\Pi})\) if \(w = \text{mes}(\omega) \in (0, w_0)\). \(u = \lim_{n \to \infty} u_n\) (the convergence is with respect to the \(C(\Pi)\)-norm) where \(u_{n+1} = Su_n\) and operator \(S\) is defined by \((A.51)\) and \((A.52)\).

Proof. Introduce the function sequence \(\{u_n\}\) associated with BVP \((A.43)\) according to\n\[ \Delta u_{n+1} = 0, \ x \in \Pi, \]
\[ u_{n+1} = -u_n + 2\phi, \ x \in \omega, \ \frac{\partial u_{n+1}}{\partial \nu} = \psi, \ \ x \in \Gamma_{\omega}, \ n = 0, 1, \ldots, \]
where it is reasonable to choose \(u_0 \in C^2(\Pi) \cap C^1(\Pi)\) as the solution to the BVP
\[ \Delta u_0 = 0, \ x \in \Pi, \ u_0 = \phi, \ x \in \omega, \ \frac{\partial u_0}{\partial \nu} = 0, \ x \in \Gamma_{\omega}. \]

BVP \((A.54)\) is equivalent to the operator equation (cf. \((A.51)\))
\[ u_{n+1} = Su_n, \ Su_n = S_1 u_n + \Phi, \]
where operator \(S_1\) defined by \((A.51)\) and \((A.52)\) is a contraction in the space \(C(\Pi)\) according to Lemma 5.9 if \(w = \text{mes}(\omega) \in (0, w_0)\) where \(w_0\) is sufficiently small. This \(\{u_n\}\) converges with respect to the \(C(\Pi)\)-norm to the unique solution of BVP \((A.43)\) (the fixed point of operator \(S\)). The theorem is proved. \(\Box\)

Consider a mixed BVP for the Poisson equation
\[ -\Delta u = g(x), \ x \in \Pi, \]
\[ u = \phi(x_1), \ x = x^0 = (x_1, h) \in \omega, \]
\[ \frac{\partial u}{\partial \nu} = \psi(x), \ x \in \Gamma_{\omega}, \ u \in C^2(\Pi) \cap C^1(\Pi), \]
\[ g \in C(\Pi), \ \phi(x_1) \in C^m(\bar{\omega}) \ (m \geq 1), \ \psi(x) \in C(\Gamma_{\omega}). \]

Setting, as in \((A.29)\),
\[ g \rho(x) = \int_{\Pi} \Phi_0(x - y) g(y) dy, \ u(y) = U(y) + g \rho(y), \ x \in \Pi, \]
assuming that \(u\) is a solution to \((A.56)\) and applying superposition, we obtain an equivalent BVP for \(U\)
\[ \Delta U = 0, \ x \in \Pi, \ U = \tilde{\phi}(x_1), \ x \in \omega, \ \frac{\partial U}{\partial \nu} = \tilde{\psi}(x), \ x \in \Gamma_{\omega}. \]
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where
\[ \tilde{\phi} = (\phi - g_P)|_\omega, \quad \tilde{\psi} = \left( \psi - \frac{\partial g_P}{\partial \nu} \right) |_{\Gamma_\omega}, \]
(A.60)

\( \tilde{\phi} \in C^2(\omega) \cap C(\omega) \cap C(\Gamma) \) \((m \geq 1)\), and \( \tilde{\psi} \in C(\Gamma_\omega) \) according to the properties of the volume potential \( g_P \) [40].

Lemma 5.10 The following estimate holds
\[ \|u\|_{C(\Omega)} \leq Q_2(\text{mes}(\omega)) \|\phi\|_{C(\Gamma)} + \|g\|_{C(\Omega)} + \|\psi\|_{C(\Omega)}, \]
(A.61)

where \( Q_2 \) is a constant that does not depend on \( \text{mes}(\omega) \) and \( u \) is the solution to BVP (A.56).

Proof. Inequality (A.61) follows from estimates (A.37) (Lemma 5.7) and (A.50) (Lemma 5.8) applied to integral representation (A.45), (A.46) where \( \phi \) and \( \psi \) are replaced by \( \tilde{\phi} \) and \( \tilde{\psi} \) defined in (A.60).

\( \square \)

Theorem 5.7 Let \( \phi, \psi, \) and \( g \) satisfy conditions (A.57). Then there is an \( w_0 > 0 \) such that BVP (A.56) has one and only one solution \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \)
if \( w = \text{mes}(\omega) \in (0, w_0) \). \( u(x) = U(x) + g_P(x) \), where \( U = \lim_{n \to \infty} U_n \) (the convergence is with respect to the \( C(\Omega) \)-norm), \( U_{n+1} = SU_n \), and operator \( S \) is defined by (A.51) and (A.52) where \( \phi \) and \( \psi \) are replaced by \( \tilde{\phi} \) and \( \tilde{\psi} \) defined in (A.60).

Proof. Introduce the function sequence \( \{U_n\} \) associated with BVP (A.59), (A.60) according to (A.54),
\[ \Delta U_{n+1} = 0, \quad x \in \Omega, \quad U_{n+1} = -U_n + 2\tilde{\phi}, \quad x \in \omega, \quad \frac{\partial U_{n+1}}{\partial \nu} = \tilde{\psi}, \quad x \in \Gamma_\omega, \]
where \( n = 0, 1, \ldots \), \( \tilde{\phi} \) and \( \tilde{\psi} \) are specified in (A.60), and \( U_0 \in C^2(\Omega) \cap C^1(\Omega) \) may be chosen as the solution to the BVP (A.55). Repeating the proof of Theorem 5.6 and taking into account representation (A.58) we obtain the unique solution to BVP (A.56) in the form (A.31) (see Theorem 5.2)
\[ u(x) = U(x) + g_P(x), \]
where \( U = \lim_{n \to \infty} U_n \) (the fixed point of operator \( S \)). The theorem is proved.

\( \square \)

A.3.4 Approximate decomposition in three dimensions

Let us first formulate some definitions and introduce the necessary notations.

Let \( D \) be a domain in \( \mathbb{R}^3 \) bounded by a closed piecewise smooth Lyapunov surface \( \Sigma \) and \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Let \( \Omega \subset \Sigma \) be an open domain on a smooth part of \( \Sigma \) bounded by a closed piecewise smooth curve \( \Gamma \). A parallelepiped \( \Pi = \Pi_{abh} \) will be a domain of our particular interest.

We will assume that if \( D = \Pi \) then
(i) \( \Omega \) is situated on one of its side planes \( \Sigma_{j_0} \), \( 1 \leq j_0 \leq 6 \) (let us assume, for the sake of clarity, that \( \Sigma_{j_0} = \Sigma_0 = \{ x : 0 < x_1 < a, 0 < x_2 < b, x_3 = h \} \)), and
Figure 5.25. Approximate decomposition in three-dimensional case.

(ii) there is a ball \( S_w(x_0) = \{ x : |x - x_0| < w \} \) such that \( S_w(x_0) \subset H_0, \Omega \subset S_w(x_0), \) \( \text{diam} \ (\Omega) = \text{diam} \ (S_w(x_0)) = 2w, \) and, consequently, \( \text{mes} \ (\Omega) \leq \text{mes} \ (S_w(x_0)) = \pi w^2 \) (see Fig. 5.25).

We shall say that a function \( \psi(x) \in C^m_0(\Omega; \Sigma), m \geq 1, \) if \( \psi(x) \in C^m(\Sigma) \), where \( C^m(\Sigma) \) is the space of \( m \) times continuously differentiable functions (the norm is defined in a usual manner) and \( \text{supp} \psi \subseteq \Omega \) (that is, \( \psi(x) = 0, x \notin \Omega \)).

Let \( G(x, y) \) denote Green’s function of the Neumann problem for the Laplace equation in \( D \) for which the representation

\[
G(x, y) = \Phi_0(x - y) + N(x, y), \quad x, y \in \bar{D}, \quad \Phi_0(x - y) = \frac{1}{4\pi|x - y|}, \tag{A.62}
\]

and the uniqueness condition \([40]\int_{\Sigma} G(x, y) dy = 0\) hold, where \( N(x, y) \) is a differentiable function in \( \bar{D} \).

Applying the argument that leads to the proof of Lemma 5.3 it is easy to see that if \( D = \Pi \) then the following estimate holds

\[
\int_{\Omega} \Phi_0(x - y) dy \leq C_0 \text{mes} (\Sigma) w, \quad x \in \Sigma,
\]

where \( C_0 = \text{const} \) is independent of \( \Pi \).

Lemma 5.11. Let \( \Pi \) be a parallelepiped in \( \mathbb{R}^3 \) bounded by a closed piecewise smooth surface \( \Sigma_\Pi; \Omega \subset \Sigma \) an open domain bounded by a closed piecewise smooth curve \( \Gamma \) situated on a smooth part of \( \Sigma_\Pi \) and satisfying conditions (i) and (ii); \( G(x, y) \) Green’s function of the Neumann problem for the Laplacian in \( \Pi \) introduced according to \((A.62)\) and the uniqueness condition; and \( \psi(x) \in C^m_0(\Omega; \Sigma), m \geq 1. \) Then the following estimate is valid

\[
\max_{x \in D} \int_{\Omega} G(x, y)\psi(y) dy \leq \max_{x \in \Sigma} \int_{\Omega} G(x, y)\psi(y) dy \leq Q_0 \cdot w \|\psi\|_{C(\Sigma_\Pi)},
\]

where \( Q_0 = Q_0(D) \) is a constant independent of \( \text{mes} (\Omega) \).

It is not difficult to check that most of the results obtained above for the two-dimensional case remain valid in three dimensions. The proofs are based on the use of superposition and application of the statements and estimates of Lemmas 5.1-5.5 and 5.11.
A.3.5 Generalization

The method can be generalized, after appropriate changes in the notation and only with technical complications, to the case of an equation system (A.15) with \( u \) being a vector-function and more complicated boundary operators than in (A.15),

\[
\Delta u = 0, \quad u = (u_1(x), \ldots, u_m(x)), \quad x \in \Omega, \quad (A.63)
\]

or

\[
-\Delta u = g, \quad g = (g_1(x), \ldots, g_m(x)), \quad x \in \Omega,
\]

\[
L_1 u + L_2 u = f(x), \quad x \in \Gamma,
\]

where \( L_1 \) is the Dirichlet or Neumann boundary operator. The function sequence \( \{u_n\} \) similar to (A.24) is defined, for problem (A.63), according to

\[
\Delta u_{n+1} = 0, \quad x \in \Omega,
\]

\[
L_1 u_{n+1} = -L_2 u_n + f(x), \quad x \in \Gamma.
\]

A.4 Extension

Let us prove two lemmas (in two- and three-dimensional cases) which show how one can extend functions defined on a subdomain to the whole domain. First we formulate the lemma in the two-dimensional case.

Lemma 5.12 Let \( g(x_1) \in W^2_1(\omega) \). Then \( g \) can be extended to \( \tilde{g}(x_1, x_2) \) on \( S \) and the following conditions hold:

1) \( \tilde{g} \in W^2_1(S) \);
2) \( \tilde{g} = 0 \) outside some rectangle \( h_1 \leq x_2 \leq h, \ |x_1| \leq d; \)
3) \( \|\tilde{g}\|_{W^2_1(S)} \leq m \|g\|_{W^2_1(\omega)}, \) where \( m \) does not depend on \( g \).

Proof. Let \( g(x_1) \in C^1(\omega) \). Extend \( g(x_1) \) to \( \tilde{g}(x_1, x_2) \in C^1(S) \) which satisfies conditions 2) and 3) of Lemma 5.12.

First, we extend \( g \) onto \( X_1 \) [42]. To this end consider the interval \( \omega_k \) with the ends \( a_k, b_k \) and denote by \( l_k \) the length of \( \omega_k \), by \( d_k \) the distance between \( \omega_k \) and \( \omega_{k+1} \), and introduce the parameter

\[
\rho = \frac{1}{3} \min \{ \min_{k=1, \ldots, N} l_k, \min_{k=1, \ldots, N-1} d_k \}.
\]

Define the functions

\[
g_{a_k}(x_1) = \begin{cases} -3g(2b_k - x_1) + 4g(\frac{3}{2}b_k - \frac{1}{2}x_1), & b_k \leq x_1 \leq b_k + \rho, \\ g(x_1), & a_k \leq x_1 \leq b_k \end{cases}
\]

and

\[
g_{b_k}(x_1) = \begin{cases} -3g(2a_k - x_1) + 4g(\frac{3}{2}a_k - \frac{1}{2}x_1), & a_k - \rho \leq x_1 \leq a_k, \\ g(x_1), & a_k \leq x_1 \leq b_k. \end{cases}
\]
It is obvious that $g_{b_k}$ is continuous together with its derivative on $b_k \leq x_1 \leq b_k + \rho$ and $g_{a_k}$ is continuous together with its derivative on $a_k - \rho \leq x_1 \leq a_k$. Denote

\[
\tilde{g}(x_1, h) = \begin{cases} 
    g(x_1), & x_1 \in \omega, \\
    g_{b_k}(x_1) \cdot \chi(x_1 - b_k), & x_1 \in [b_k, b_k + \rho], \\
    g_{a_k}(x_1) \cdot \chi(a_k - x_1), & x_1 \in [a_k - \rho, a_k], \\
    0, & \text{elsewhere},
\end{cases}
\]

where $\chi$ is an infinitely differentiable function defined according to

\[
\chi(x) = \begin{cases} 
    1, & -\infty < x \leq D - \delta, \\
    0, & x \geq D, \\
    \text{any function}, & D - \delta < x \leq D,
\end{cases}
\]

(A.64)

Here $D = \rho$ and $\delta = 1/2\rho$. Finally denote

\[
\tilde{g}(x_1, x_2) = \tilde{g}(x_1, h) \cdot (1 - \chi(x_2)).
\]

It is obvious that $\tilde{g}(x_1, x_2) \in C^4(S)$ and $\tilde{g}(x_1, x_2) = 0$ outside the rectangle $h_1 \leq x_2 \leq h$, $|x_1| \leq d$, where $h_1 = 1/3h$ and $d = \max\{|b_N + \rho, |a_1 - \rho|\}$. The following chain of inequalities hold

\[
\begin{align*}
    \|\tilde{g}(x_1, x_2)\|_{W^2_2(\omega)} &\leq m_0 \|\tilde{g}(x_1, h)\|_{W^2_2(\omega_1)} \leq m_0 \|g(x_1)\|_{W^2_2(\omega)} + \\
    &+ \sum_{k=1}^N \|\tilde{g}(x_1, h)\|_{W^2_2([b_k, b_k + \rho])} + \sum_{k=1}^N \|\tilde{g}(x_1, h)\|_{W^2_2([a_k - \rho, a_k])} \\
    &\leq m_0 \|g(x_1)\|_{W^2_2(\omega)} + \sum_{k=1}^N m_0 \|\tilde{g}(x_1, h)\|_{W^2_2(\omega_1)} + \\
    &+ \sum_{k=1}^N m_0 \|\tilde{g}(x_1, h)\|_{W^2_2(\omega_k)} \leq m \|g(x_1)\|_{W^2_2(\omega)}.
\end{align*}
\]

We have proved that if $g(x_1) \in C^4(\omega)$ then $g(x_1)$ can be extended to $\tilde{g}(x_1, x_2) \in C^4(S)$ and conditions 2 and 3 hold.

Now let $g(x_1) \in W^2_2(\omega)$. Consider the sequence $g^{(n)}(x_1) \in C^4(\omega)$ such that $g^{(n)} \to g$ (convergence with respect to the norm) in $W^2_2(\omega)$. Extend every $g^{(n)}(x_1)$ into $C^4(\omega)$ to $\tilde{g}^{(n)}(x_1, x_2) \in C^4(S)$ onto $S$ and obtain the sequence $\tilde{g}^{(n)}(x_1, x_2)$ which converges in $W^2_2(\omega)$ to the desired limiting function $\tilde{g}(x_1, x_2)$. Lemma 5.12 is proved. $\Box$

In three-dimensional case we have the following

**Lemma 5.13** Let $g(x_1, x_2) \in W^2_2(\Omega)$ and curvature of $\Gamma_k$ be continuously differentiable. Then $g$ can be extended to $\tilde{g}(x_1, x_2, x_3)$ on $V$ and the following conditions hold:

1) $\tilde{g} \in W^2_2(V)$;

2) $\tilde{g} = 0$ outside some cylinder $0 < h_1 \leq x_3 \leq h$, $x_1^2 + x_2^2 \leq a^2$;

3) $\|\tilde{g}\|_{W^2_2(V)} \leq m \|g\|_{W^2_2(\Omega)}$, where $m$ does not depend on $g$.

**Proof.** The proof is similar to the one of Lemma 5.12. In this case

\[
\tilde{g}(x_1, x_2, x_3) = \tilde{g}(x_1, x_2, h) (1 - \chi(x_3)),
\]
where
\[ \hat{g}(x_1, x_2, h) = \begin{cases} \chi(x^{(n)})(4g(x^{(t)}, -1/2x^{(n)}) - 3g(x^{(t)}, -x^{(n)})), & (x_1, x_2) \in \tilde{\Omega}_k \setminus \Omega_k, \\ 0, & (x_1, x_2) \notin \tilde{\Omega}_k, \end{cases} \]
\[ \chi \text{ is from (A.64), and } \Omega_k \in \tilde{\Omega}_k. \] The contour \( \tilde{\Gamma}_k \) of the domain \( \tilde{\Omega}_k \) is at a distance \( \rho \) from the contour \( \Gamma_k \) of the domain \( \Omega_k \). The coordinates \( x^{(t)}, x^{(n)} \) are the components of a local Cartesian coordinate system with the origin at a point on \( \Gamma_k \); here \( x^{(t)} \) lies on the tangent to \( \Gamma_k \) and \( x^{(n)} \) lies on the normal to \( \Gamma_k \) (see Fig. 5.26). The parameter
\[ \rho = \frac{1}{3} \min \{ \alpha_k, \beta_k, \gamma_k, h \}, \]
where \( \alpha_k \) is the distance between contours \( \Gamma_t \) and \( \Gamma_k \), \( \beta_k \) is the minimal radius of curvature of \( \Gamma_k \), and \( \gamma_k \) is the diameter of \( \Omega_k \). The parameters of the function \( \chi \) are \( D = \rho \) and \( \delta = 0.5\rho \). 

Figure 5.26. Extension of \( f_3 \).

A.5 Fourier coefficients of a hat function

The solution by the Fourier method employs calculation (analytical or numerical) of the integrals
\[ \int_L H(x) \sin \alpha n x dx, \quad \int_L H(x) \cos \alpha n x dx \quad (n = 0, 1, 2, \ldots) \quad (A.65) \]
and then the Fourier coefficients of the boundary hat functions \( H(x) \) where \( L \) denotes the support of the boundary function. The basic integral to be determined is
\[ J_n(m; \rho; \alpha) = \int_0^1 (1 - x^2)^m e^{-\rho x^2} \cos \alpha n x dx, \quad m \geq 2. \quad (A.66) \]
Let \( m \geq 2 \) be a natural number. Set
\[ h(x) = Q(\rho^2 - x^2)^m e^{-\rho x^2}, \quad \text{where } Q \neq 0, \quad p \neq 0, \quad \text{and } r > 0. \quad (A.67) \]
Then the derivatives
\[ h^{(j)}(\pm p) = 0, \quad j = 1, \ldots, m - 1. \]

Using integration by parts and A.67 one can show that, e.g., in the case of the hat function
\[ H_m(x) = H_m(x; 1, 1, \rho) = \begin{cases} h_m(x) = (1 - x^2)^m e^{-\rho x^2}, & |x| \leq 1, \\ 0, & |x| \geq 1, \end{cases} \]
with the support \([-1, 1]\) and \(m \geq 2\), the integral (A.65) becomes
\[
\mathcal{I}_m = \frac{1}{2} \int_{-1}^1 h_m(x) \cos \alpha n x \, dx = \frac{1}{2} \int_0^1 h_m(x) \cos \alpha n x \, dx =
\]
\[
= \frac{\sin \alpha n x}{\alpha n} h_m(x) \bigg|_0^1 - \frac{1}{\alpha n} \int_0^1 h'_m(x) \sin \alpha n x \, dx =
\]
\[
= -\frac{1}{\alpha n} \left[ -\frac{\cos \alpha n x}{\alpha n} h'_m(x) \bigg|_0^1 + \frac{1}{\alpha n} \int_0^1 h''_m(x) \cos \alpha n x \, dx \right] =
\]
\[
= -\frac{1}{(\alpha n)^2} \int_0^1 h''_m(x) \cos \alpha n x \, dx = \ldots =
\]
\[
= \left\{ \begin{array}{ll}
(1)^{k} (\alpha n)^{-m} \int_0^1 h_m(x) \sin \alpha n x \, dx, & m = 2k + 1 (k = 1, 2, \ldots), \\
(1)^{k+1} (\alpha n)^{-m} \int_0^1 h_m(x) \cos \alpha n x \, dx, & m = 2k.
\end{array} \right.
\]
Thus the Fourier coefficients of the \(m - 1\) times continuously differentiable hat function (\(m \geq 2\)) are proportional to \((\alpha n)^{-m}\).

Consider the shifted hat function (4.35)
\[ \mathcal{H}_{m[S]}(x) = \mathcal{H}_{m[S]}(x; x_S, p, 1, \rho) = \]
\[ = \begin{cases} \mathcal{T}_{m,S}(x) = \rho^2 - (x - x_S)^2 \rho e^{-\rho^2}, & |x - x_S| \leq p, \\ 0, & |x - x_S| \geq p, \end{cases} \quad (A.68) \]
with the support \(L = [x_S - p, x_S + p]\) and \(m \geq 2\). Note that
\[ \mathcal{T}_{m,S}(x) = \rho^2 [1 - t^2] e^{-\rho^2}, \quad t = \frac{x - x_S}{\rho}, \quad \rho = rp^2. \quad (A.69) \]
The sine Fourier coefficients of (A.68) are expressed via the integral (A.66) with the help of the substitution \(t = (x - x_S)/\rho\) used in (A.69):
\[ I_m^n = \int_L \mathcal{H}_{m[S]}(x) \sin \alpha n x \, dx = \]
\[ = \int_{x_S-p}^{x_S+p} \rho^2 - (x - x_S)^2 \rho e^{-\rho^2} \sin \alpha n x \, dx =
\]
\[ = 2\rho^{2m+1} \sin(\alpha n x_S) J_n(m; \rho^2; \alpha). \]
Integrals \(J_n(m; \rho; \alpha)\) can be calculated by expanding the integrand in the McLau-
A.5. **FOURIER COEFFICIENTS OF A HAT FUNCTION**

For example, in the case \( m = 2 \) we have

\[
J_n(2; \rho; \alpha) = \int_0^1 (1 - x^2)^2 e^{-\rho x^2} \cos \alpha n x \, dx = \int_0^1 [1 - (\rho + 2)x^2 + \sum_{q=2}^{\infty} c_{2q} x^{2q}] \cos \alpha n x \, dx = Y_0 - (\rho + 2)Y_2 + \sum_{q=2}^{\infty} c_{2q} Y_{2q},
\]

where

\[
Y_{2q} = \frac{2q \cos \alpha n + \alpha n \sin \alpha n}{(\alpha n)^2} - \frac{2q(2q - 1)}{(\alpha n)^2} Y_{2q-2}, \quad q = 1, 2, \ldots,
\]

\[
Y_0 = \frac{\sin \alpha n}{\alpha n}.
\]

The case of arbitrary even number \( m \) can be considered in a similar manner, by expanding the function

\[
(1 - x^2)^m e^{-\rho x^2}
\]

in the Maclaurin series using the binomial formula.
Bibliography


