David Ström

The Open Mapping Theorem for Analytic Functions

and some applications

Mathematics
D-level thesis, 20p

Date: 2006-05-23
Supervisor: Ilie Barza
Examiner: Alexander Bobylev
The Open Mapping Theorem for Analytic Functions and some applications

This thesis deals with the Open Mapping Theorem for analytic functions on domains in the complex plane: A non-constant analytic function on an open subset of the complex plane is an open map.

As applications of this fundamental theorem we study Schwarz’s Lemma and its consequences concerning the groups of conformal automorphisms of the unit disk and of the upper halfplane.

In the last part of the thesis we indicate the first steps in hyperbolic geometry.

Satsen om öppna avbildningar för analytiska funktioner och några tillämpningar

Denna uppsats behandlar satsen om öppna avbildningar för analytiska funktioner på domäner i det komplexa talplanet: En icke-konstant analytisk funktion på en öppen delmängd av det komplexa talplanet är en öppen avbildning.

Som tillämpningar på denna fundamentala sats studeras Schwarz’s lemma och dess konsekvenser för grupperna av konforma automorfismer på enhetsdisken och på det övre halvplanet.

I uppsatsens sista del antyds de första stegen inom hyperbolisk geometri.
Contents:

Introduction .................................................. 1
Definitions and terminology ............................... 2

Chapter 1

1.1 Laurent’s Theorem ......................................... 4
1.2 Cauchy’s Residue Theorem ............................... 4
1.3 The Argument Principle ................................... 5
1.4 Rouché’s Theorem ........................................ 7
1.5 The Identity Theorem for a Disk ....................... 8
1.6 The Open Mapping Theorem ............................ 9
1.7 The Maximum Modulus Principle ....................... 11
1.7.1 The Minimum Modulus Principle .................. 12
1.8 The Maximum Modulus Theorem ....................... 13
1.9 Schwarz’s Lemma ........................................ 14

Chapter 2

2.1 The Group of Möbius Transformations of \( \hat{\mathbb{C}} \) ........ 15
2.2 The Conformal Group of the Unit Disk ............... 16
2.3 The Conformal Group of the Upper Half-Plane ....... 19
2.4 The First Steps in Hyperbolic Geometry ............. 22
2.4.2 The Hyperbolic Distance Function ................. 27
2.4.3 Isometries of \( \mathbb{H} \) ................................. 28

References ....................................................... 30
MOTTO:
“Because of its simple and explicit formulation it is one of the most useful general theorems in the theory of functions. As a rule all proofs based on the maximum principle are very straightforward, and preference is quite justly given to proofs of this kind.”  Lars V Ahlfors

Introduction

The thesis is divided into two chapters.

In the first chapter we present the detailed proof of The Open Mapping Theorem, with its first major corollaries: The Maximum Modulus Principle for analytic functions, The Maximum Modulus Theorem and Schwarz’s Lemma.

In the second chapter we study some rudiments on general Möbius transformations of the Riemann sphere, and identify the groups of conformal automorphisms of the unit disk, $D_1$, and of the upper half-plane, $H$.

Finally, we present the fundamental elements of the hyperbolic geometry on $H$, following the model indicated by H. Poincaré.

I want to thank my supervisor Ilie Barza for guiding me through this fascinating area in complex analysis. I also want to thank my fellow students Fredrik Jonsson and Anna Persson for their support and encouragement.

David Ström
Skoghall, 23 maj 2006
**Definitions and terminology**

The *ball* with center $z_0 \in \mathbb{C}$ and radius $r > 0$ is the set of all $z \in \mathbb{C} : |z - z_0| < r$.

The ball is also called *disk* and equivalent notations are $B_r(z_0)$, $B(z_0, r)$ or $D_r(z_0)$.

The *punctured disk* excludes the center, $z_0$; $D_r'(z_0) = \{ z \in \mathbb{C} : 0 < |z - z_0| < r \}$

The *closed disk* includes the boundary, $|z - z_0| = r$; $\overline{D}_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| \leq r \}$

A subset of $\mathbb{C}$ is called a *neighborhood* of $z_0 \in \mathbb{C}$, if it contains a ball $B(z_0, r)$.

A set is *open* if it is a neighborhood of each of its elements, i.e.
the set $X \subset \mathbb{C}$ is open if $(\forall ) z \in X, (\exists ) r > 0 : D_r(z) \subset X$.

The set $X \subset \mathbb{C}$ is *closed* if its complement $\mathbb{C} \setminus X$ is open.

The *interior* of $X$, denoted $\overset{0}{\text{X}}$ is the largest open set contained in $X$.

The *closure* of $X$, denoted $\overline{\text{X}}$ is the smallest closed set containing $X$.

The *boundary* of $X$, denoted $\partial X$ is $\overline{\text{X}} \setminus \overset{0}{\text{X}}$.

The set $X \subset \mathbb{C}$ is *bounded* if $(\exists ) r > 0 : X \subset D_r(0)$.

A subset of $\mathbb{C}$ is *compact*, if it is closed and bounded.

A subset of $\mathbb{C}$ is *connected* if it cannot be represented as the union of two disjoint, non-empty, relatively open sets.

**Theorem:** A non-empty open set in the plane is connected if and only if any two of its points can be joined by a polygon which lies in the set. (See [1], p.56)

A non-empty, open, connected subset of $\mathbb{C}$ is called a *region* or a *domain*.

The closure of a region is called a *closed region*.

A region $G$ is defined as *simply connected* if every closed path in $G$ is homotopic to a null path in $G$, i.e. it can be shrunk (inside $G$) to a point in $G$. (See [2], p.143)
In the following, \( f \) is a complex valued function on the region \( G \), i.e. \( f : G \to \mathbb{C} \).

\( f \) is called \textit{continuous in} \( z_0 \in G \) if 
\[
\lim_{z \to z_0} f(z) = f(z_0),
\]
and for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( z \in G \) with \( |z - z_0| < \delta \) we have \( |f(z) - f(z_0)| < \varepsilon \).

\( f \) is called \textit{continuous}, if it is continuous in every point of \( G \).

\( f \) is called \textit{derivable in} \( z_0 \in G \) if 
\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]
exists in \( \mathbb{C} \).

The limit is denoted \( f'(z_0) \) and is called the \textit{derivative} of \( f \) in \( z_0 \).

\( f \) is called \textit{analytic in} \( z_0 \in G \) if \( f'(z) \) exists for every \( z \) in some \textit{neighborhood} of \( z_0 \).

\( f \) is called \textit{analytic}, if it is analytic in every point of \( G \).

The curve \( \gamma \) with parametrization \( z : [a,b] \to \mathbb{C} \) is called \textit{smooth} if the function \( z \) is derivable in every point \( t \in [a,b] \) and \( z' : [a,b] \to \mathbb{C} \) is continuous.

The curve \( \gamma \) is called \textit{piecewise smooth} if \( z \) is continuous and there exists a partition \( \Delta : a = t_0 < t_1 < \ldots < t_{k-1} < t_k < \ldots < t_n = b \) such that the restrictions of \( z \) to each interval \( [t_{k-1}, t_k] \), \( 1 \leq k \leq n \), is smooth i.e. it is derivable and its derivative is continuous.

The curve \( \gamma \) is \textit{closed} if \( z(a) = z(b) \).

The curve \( \gamma \) is \textit{simple} if \( z : [a,b] \to \mathbb{C} \) is injective.

The curve \( \gamma \) is \textit{simple closed} if \( \gamma \) is closed and the restriction of \( z \) to \([a,b]\) is injective.

\textbf{The Jordan Curve Theorem:}
Let \( \gamma \) be a simple closed curve in \( \mathbb{C} \). Then the set \( \mathbb{C} \setminus \gamma \) has exactly two connected components, one bounded and one unbounded, and \( \gamma \) is their common boundary.

We say that a piecewise smooth, simple closed curve is \textit{positively oriented} if the bounded component lies to the left of \( z'(t) \) when \( t \) varies from \( a \) to \( b \), except for the points where \( z'(t) \) does not exist.
1.1 Laurent's Theorem

A function $f$, analytic in the annulus $D = \{ z : r < | z - z_0 | < R \}$ can be written

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

for all $z \in D$, with

$$a_n = \frac{1}{2\pi i} \int_{C} \frac{f(w)}{(w - z_0)^{n+1}} \, dw$$

for all $n \in \mathbb{Z}$,

where $C$ is any piecewise smooth, positively oriented, simple closed curve in $D$, surrounding $z_0$.

(See [3], p.327)

For an isolated singular point $z_0$, the annular domain with $r = 0$ is a punctured disk $D_R'(z_0)$ and the coefficient $a_{-1}$ in the Laurent series is called the residue of $f(z)$ at $z_0$.

We see that, for $n = -1$, Laurent’s Theorem leads to:

$$\text{Res} \{ f(z), z_0 \} := a_{-1} = \frac{1}{2\pi i} \int_{C} f(w) \, dw$$

(See [3], p.327)

1.2 Cauchy's Residue Theorem

Let $D$ be a simply connected domain and $C$ a piecewise smooth, positively oriented, simple closed curve lying entirely within $D$. If $f$ is analytic on and within $C$, except at a finite number of isolated singular points $z_1, z_2, z_3, \ldots, z_n$ within $C$, then

$$\int_{C} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} \{ f(z), z_k \} .$$

(See [3], p.347)
Remarks (See [3], pp.337-339)

An analytic function $f$, has a **zero of order** $n$ in a point $z_0$ $\iff$ $f(z_0) = f'(z_0) = f''(z_0) = \ldots = f^{(n-1)}(z_0) = 0$ and $f^n(z_0) \neq 0$.

A function $f$, analytic in some disk $D_r(z_0)$, has a zero of order $n$ at $z_0$ $\iff$ $f$ can be written $f(z) = (z - z_0)^n \Phi(z)$, where $\Phi$ is analytic at $z_0$ and $\Phi(z_0) \neq 0$.

An isolated singular point $z_p$ is a **pole of order** $m$ $\defeq c_m$, $m \geq 1$, is the first non-zero coefficient in the Laurent series expansion of the function around $z_p$.

A function $f$, analytic in a punctured disk $D_r'(z_p)$, has a pole of order $m$, $m \geq 1$, at $z_p$ $\iff$ $f$ can be written $f(z) = (z - z_p)^m g(z)$, where $g$ is analytic at $z_p$ and $g(z_p) \neq 0$.

1.3 The Argument Principle

Suppose that $f$ is analytic in a domain $D$, except at a **finite number of poles**.

Let $C$ be a piecewise smooth, positively oriented, simple closed curve in $D$, which does not pass through any pole or zero of $f$ and whose inside lies in $D$.

Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} \, dz = N_0 - N_p$$

$N_0$ = total number of zeros of $f$ inside $C$.

$N_p$ = total number of poles of $f$ inside $C$.

All zeros and poles are counted with multiplicities (orders).

Fig 1.3 The zeros $z_0$ and poles $z_p$ of $f$, inside $C$:

- $z_{0,1}$ is of order $n_1$
- $z_{0,2}$ is of order $n_2$
- $z_{0,r}$ is of order $n_r$
- $z_{p,1}$ is of order $m_1$
- $z_{p,2}$ is of order $m_2$
- $z_{p,s}$ is of order $m_s$

$$\sum_{k=1}^{r} n_k = N_0$$

$$\sum_{k=1}^{s} m_k = N_p$$
PROOF (After [3], pp.363-364)

The integrand \( \frac{f'(z)}{f(z)} \) is analytic on and inside \( C \), except at the zeros and poles of \( f \).

Case I: If \( z_0 \) is a zero of order \( n \), \( f \) can be written \( f(z) = (z - z_0)^n \Phi(z) \), where \( \Phi \) is analytic at \( z_0 \) and \( \Phi(z_0) \neq 0 \).

We get

\[
\frac{f'(z)}{f(z)} = \frac{(z - z_0)^n \Phi'(z) + n(z - z_0)^{n-1} \Phi(z)}{(z - z_0)^n \Phi(z)} = \frac{\Phi'(z)}{\Phi(z)} + \frac{n}{z - z_0}.
\]

Thus \( \text{Res} \left\{ \frac{f'(z)}{f(z)}, z_0 \right\} = n \), the multiplicity of \( z_0 \) as a zero of \( f \).

Case II: If \( z_p \) is a pole of order \( m \), \( f \) can be written \( f(z) = (z - z_p)^{-m} g(z) \), where \( g \) is analytic at \( z_p \) and \( g(z_p) \neq 0 \).

In this case

\[
\frac{f'(z)}{f(z)} = \frac{(z - z_p)^{-m} g'(z) - m(z - z_p)^{-m-1} g(z)}{(z - z_p)^{-m} g(z)} = \frac{g'(z)}{g(z)} - \frac{m}{z - z_p}
\]

and \( \text{Res} \left\{ \frac{f'(z)}{f(z)}, z_p \right\} = -m \), the negative multiplicity of \( z_p \) as a pole of \( f \).

According to Cauchy’s Residue Theorem, for \( r \) zeros and \( s \) poles of \( f \), we get

\[
\int_C \frac{f'(z)}{f(z)} \, dz = 2\pi i \sum_{k=1}^{r} \text{Res} \left\{ \frac{f'(z)}{f(z)}, z_{0k} \right\} + 2\pi i \sum_{k=1}^{s} \text{Res} \left\{ \frac{f'(z)}{f(z)}, z_{pk} \right\} = 2\pi i \sum_{k=1}^{r} n_k + 2\pi i \sum_{k=1}^{s} -m_k = 2\pi i (N_0 - N_p).
\]

Thus \( \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = N_0 - N_p \) as stated. ☺
1.4 Rouché's Theorem

Let $f$ and $g$ be analytic functions defined in a simply connected domain $D$. Suppose $|f(z) - g(z)| < |f(z)|$ for every $z \in \gamma$, where $\gamma$ is a piecewise smooth, simple closed curve in $D$. Then $f$ and $g$ have the same number of zeros (counting multiplicities) inside $\gamma$.

**PROOF** (After [3], pp.365-366)

$|f(z) - g(z)| < |f(z)|$ for every $z \in \gamma \Rightarrow |f(z)| > 0$ and $g(z) \neq 0$ for every $z \in \gamma$.

Division by $|f(z)|$ gives $|F(z) - 1| < 1$, where $F(z) = \frac{g(z)}{f(z)}$.

The image of $\gamma$ under the map $w = F(z)$ is a closed, piecewise smooth curve (need not be simple) lying in the open disk $|w - 1| < 1$.

The function $\frac{1}{w}$ is analytic for $\text{Re}(w) > 0$.

Thus, by Cauchy's theorem, we have $\int_{F(\gamma)} \frac{dw}{w} = 0$.

Since $w = F(z)$, we get $0 = \int_{F(\gamma)} \frac{dw}{w} = \int_{\gamma} \frac{F'(z)}{F(z)} dz$,

where $\frac{F'(z)}{F(z)} = \frac{g'(z)f(z) - g(z)f'(z)}{[f(z)]^2} = \frac{g'(z)}{g(z)} \frac{f'(z)}{f(z)}$.

Thus $\int_{\gamma} \frac{F'(z)}{F(z)} dz = \int_{\gamma} \left( \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right) dz = 0 \Leftrightarrow$

$\Leftrightarrow \int_{\gamma} \frac{f'(z)}{f(z)} dz = \int_{\gamma} \frac{g'(z)}{g(z)} dz \Leftrightarrow$

$\Leftrightarrow N_0(f) - N_p(f) = N_0(g) - N_p(g)$, by the Argument Principle (1.3).

But $N_p(f) = N_p(g) = 0$, since both $f$ and $g$ are analytic, and thus we have $N_0(f) = N_0(g)$ as stated. ☺
Suppose that \( f \) is analytic in the disk \( D_r(a) \) and that \( f(a) = 0 \).
Then either \( f \) is identically zero in the disk or the zero at \( a \) is isolated, i.e.
\[
(\exists) \; \delta > 0 \text{ such that } 0 < |z - a| < \delta \implies f(z) \neq 0.
\]

**PROOF** (After [2], p.179)

Since \( f \) is analytic in the disk \( D_r(a) \), by Taylor’s Theorem we can write
\[
f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad \text{with} \quad c_n = \frac{f^n(a)}{n!} \quad \text{for every } z \in D_r(a).
\]

Either the coefficients \( c_n = 0 \), for every \( n \geq 0 \), and \( f \) is identically zero in the disk \( D_r(a) \)
or, in the second case, there exists a smallest integer \( m \geq 1 \) with \( c_m \neq 0 \) and
\[
f(z) = (z - a)^m \sum_{n=m}^{\infty} c_n (z - a)^{n-m} = (z - a)^m g(z) \quad \text{for every } z \in D_r(a).
\]

The Taylor series for \( g \) has a radius of convergence at least \( r \), and it follows that \( g \) is
analytic in the disk \( D_r(a) \).

\( g \) is continuous at \( a \), i.e.
for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( |z - a| < \delta \implies |g(z) - g(a)| < \varepsilon \).

Since \( g(a) = c_m \neq 0 \), we can choose an \( \varepsilon \leq |g(a)| \) and get \( |g(z) - g(a)| < |g(a)| \) for every \( z \in D_\delta(a) \).

The inequality implies that \( g(z) \neq 0 \) for every \( z \in D_\delta(a) \).
Thus \( 0 < |z - a| < \delta \implies f(z) = (z - a)^m g(z) \neq 0 \).

So either \( f \) is identically zero in the disk, or the zero at \( a \) is isolated. ☺
1.6 The Open Mapping Theorem

Suppose that $f$ is analytic and non-constant in an open set $G$. Then $f(G)$ is open.

**Proof** (After [2], p.190)

Choose an arbitrary $a \in G$. Since $f$ is analytic and non-constant in $G$, $f - f(a)$ has an isolated zero at $a$, by the Identity Theorem (1.5).

Choose a radius $r$, such that the closed disk $\overline{D}_r(a) = \{ z : |z - a| \leq r \}$ is a subset of $G$ and $f - f(a)$ is non-zero for every $z$ on the circle $\Gamma = \{ z : |z - a| = r \}$.

Let $m := \inf \{ |f(z) - f(a)| : z \in \Gamma \}$

Since $\Gamma$ is a compact set and $|f - f(a)|$ is a continuous, real-valued function, according to a theorem of Weierstrass there exists a $z_0 \in \Gamma$ such that $m = |f(z_0) - f(a)|$.

$z_0 \in \Gamma \Rightarrow f(z_0) - f(a) \neq 0 \Rightarrow m > 0$

For every $w \in D_m(f(a))$ and $z \in \Gamma$ we have

$$|f(z) - f(a)| \geq m > |f(a) - w| = |(f(a) - f(z)) + (f(z) - w)| \Rightarrow$$

$$\Rightarrow |(f(z) - f(a)) - (f(z) - w)| < |f(z) - f(a)|$$

Rouché’s Theorem states that $f - f(a)$ and $f - w$ have the same number of zeros, counted according to their multiplicities, inside $\Gamma$.

Since $f - f(a)$ has a zero at $a$, $f - w$ has at least one zero inside $\Gamma$, i.e. in $D_r(a)$. If this zero is at $b \in D_r(a)$, we have $w = f(b) \in f(D_r(a))$. It follows that $D_m(f(a)) \subseteq f(D_r(a))$ since $w$ was an arbitrary element of $D_m(f(a))$.

For every $a \in G$, we have $f(a) \in D_m(f(a)) \subseteq f(D_r(a)) \subseteq f(G)$.

Thus $f(G)$ is a neighborhood to each of its elements, and by definition $f(G)$ is open.
Fig 1.6a The open set $G$, an arbitrary element $a$, one circle $\Gamma$, where $f - f(a) \neq 0$ and a point $b$ such that $f(b) = w$.

Fig 1.6b The images under $f$ of $G$ and of $D_r(a)$, and an arbitrary point $w \in D_m(f(a))$, where $m = \inf \{|f(z) - f(a)| : z \in \Gamma\}$.
1.7 The Maximum Modulus Principle

If $f$ is a non-constant, analytic function in a domain $D$, then $|f|$ can have no local maximum in $D$.

**PROOF**

Suppose there exists a $z_0 \in D$, local maximum of $|f|$, i.e.

$(\exists) \ r > 0 : D_r(z_0) \subset D$ with $|f(z)| \leq |f(z_0)|$ $(\forall) \ z \in D_r(z_0)$.†

By the Open Mapping Theorem $w_0 = f(z_0)$ is an inner point of $f(D_r(z_0))$, i.e.

$(\exists) \ \rho > 0 : D_{\rho}(w_0) \subset f(D_r(z_0))$.

In the disk $D_{\rho}(w_0)$ there are points $w$, such that $|w| > |w_0|$, e.g.

for $w = w_0 + \frac{\rho}{2} e^{i \text{Arg} w_0}$ we get $|w| = |w_0 + \frac{\rho}{2} e^{i \text{Arg} w_0}| = |w_0| + \frac{\rho}{2} > |w_0|$.

But $w \in D_{\rho}(w_0) \subset f(D_r(z_0)) \Rightarrow (\exists) \ z \in D_r(z_0) : |f(z)| > |f(z_0)|$ in contradiction with †.

Thus our supposition was false and consequently $|f|$ cannot have a local maximum in $D$. ☺

---

Fig 1.7a A supposed local maximum of $|f|$, i.e. there exists a $r > 0$ such that $|f(z)| \leq |f(z_0)|$ for every $z$ in $D_r(z_0)$.†

Fig 1.7b The image under $f$ of $D$ and $D_r(z_0)$.

$w_0 = f(z_0)$ and $w = f(z) \in f(D_r(z_0))$.

$|w| > |w_0| \iff |f(z)| > |f(z_0)|$

in contradiction with †.
1.7.1 Corollary 1: The Minimum Modulus Principle

If $f$ is a **nowhere zero**, non-constant, analytic function in a domain $D$, then $|f|$ can have no local minimum in $D$.

**PROOF**

Suppose there exists a $z_0 \in D$, local minimum of $|f|$, then

$(\exists) \ r > 0 : D_r(z_0) \subset D$ with $|f(z)| \geq |f(z_0)| \ (\forall) \ z \in D_r(z_0)$.

Define the function $g : D \to \mathbb{C}$, $g(z) = \frac{1}{f(z)}$, $g$ is analytic since $f(z) \neq 0$ in $D$.

A local minimum of $|f|$ corresponds to a local maximum of $|g|$, since $|f(z)| \geq |f(z_0)| \iff |g(z)| \leq |g(z_0)| \ (\forall) \ z \in D_r(z_0)$.

$g$ is a non-constant, analytic function in the domain $D$ and by the Maximum Modulus Principle, $|g|$ can have no local maximum in $D$.

Consequently $|f|$ can have no local minimum in $D$. ☺

1.7.2 Corollary 2

If $f$ is a non-constant, analytic function in a domain $D$, then $\text{Re}(f)$ has no local maxima and no local minima in $D$.

**PROOF** (After [4], p.192)

Define the functions $g : D \to \mathbb{C}$, $g(z) = e^{f(z)}$,

$u : D \to \mathbb{R}$, $u(z) = \text{Re}(f(z))$.

Because $f$ is non-constant and analytic in $D$, so is $g$ and also $\frac{1}{g} : D \to \mathbb{C}$, since $g \neq 0$.

By the Maximum Modulus Principle, neither $|g|$ nor $\left|\frac{1}{g}\right|$ can have local maxima in $D$.

$|g| = |e^{f(z)}| = e^{\text{Re}(f(z))} = e^u$ and $\left|\frac{1}{g}\right| = e^{-u}$, and it follows that $u = \text{Re}(f)$ can have no local maximum and no local minimum in $D$. ☺
1.8 The Maximum Modulus Theorem

A non-constant function $f$ is defined and continuous on a bounded, closed region $K$. If $f$ is analytic in the interior of $K$, then the maximum value of $|f(z)|$ in $K$ must occur on the boundary of $K$.

(See [5], p.192)

PROOF

By a theorem of Weierstrass, since $K$ is compact, there exists $z_0 \in K : |f(z)| \leq |f(z_0)|$ ( $\forall z \in K$).

Suppose that this maximum is attained in an interior point, i.e. $z_0 \in \text{Int } K$.

It would then be the global maximum of the restriction of $f$ to $\text{Int } K$, in contradiction with the Maximum Modulus Principle.

Thus we have $z_0 \in K \setminus \text{Int } K = \partial K$, the boundary of $K$.

Remark: A maximum at an interior point is possible if we drop the condition of $f$ being non-constant.

Then $f(z) = f(z_0) \text{ for each } z \in K$.

By continuity of $f$, $f(z) = f(z_0)$ also on $\partial K$, and thus $f$ is constant for every $z \in K$.

Fig 1.8 Example of a bounded closed region $K$, and its image under a non-constant function $f$, analytic on the interior of $K$ and continuous on $K$. The maximum of $|f|$ is attained at $z_0$. 
1.9 Schwarz's Lemma

If \( f \) is analytic in the disk \( D_1 = \{ z \in \mathbb{C} : |z| < 1 \} \) and satisfies the conditions \( |f(z)| \leq 1 \) and \( f(0) = 0 \), then \( |f(z)| \leq |z| \) and \( |f'(0)| \leq 1 \).

If \( |f(z)| = |z| \) for some \( z \neq 0 \), or if \( |f'(0)| = 1 \), then there exists a constant \( c \in \mathbb{C} \) such that \( f(z) = cz \) for every \( z \in D_1 \).

(See [1], p.135)

**PROOF**

Define the function \( g : D_1 \to \mathbb{C} \), \( g(z) = \frac{f(z)}{z} \).

The singularity at \( z = 0 \) is removable and \( g(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0) \).

Thus the function \( g : D_1 \to \mathbb{C} \), \( g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases} \) is analytic.

On the circle \( |z| = r \), \( 0 < r < 1 \), we have \( |g(z)| = \frac{|f(z)|}{|z|} = \frac{|f(z)|}{r} \leq \frac{1}{r} \).

In the closed disk \( |z| \leq r \) the maximum of \( |g(z)| \) is attained on the boundary \( |z| = r \), by the Maximum Modulus Theorem, i.e. \( |z| \leq r \implies |g(z)| \leq \frac{1}{r} \).

Let \( r_n := 1 - \frac{1}{2n} \) (\( n \geq 1 \)). We have \( r_n \to 1 \) as \( n \to \infty \).

\( (\forall) z \in D_1, (\exists) m \geq 1 : |z| \leq r_m \leq r_n < 1 \) (\( \forall \) \( n \geq m \)) \implies |g(z)| \leq \frac{1}{r_n} (\forall) n \geq m.

\( n \to \infty \implies \frac{1}{r_n} \to 1 \) and we have \( |g(z)| \leq \lim_{n \to \infty} \frac{1}{r_n} = 1 \) for every \( z \in D_1 \).

\( |g(z)| = \frac{|f(z)|}{|z|} \leq 1 \) and \( f(0) = 0 \), so \( |f(z)| \leq |z| \) (\( \forall \) \( z \in D_1 \)) and \( |f'(0)| = |g(0)| \leq 1 \).

If \( |f(z)| = |z| \) for some \( z \neq 0 \) or if \( |f'(0)| = 1 \), then \( |g(z)| = 1 \) inside the disk \( |z| < 1 \), i.e. the maximum is attained at an interior point of \( D_1 \).

Then, by the Maximum Modulus Principle, \( g \) is a constant function: \( g(z) = c, c \in \mathbb{C}, |c| = 1 \).

In \( D_1 \) we get \( f(z) = cz = e^{i\theta}z, \theta \in \mathbb{R} \). ☺
2.1 The Group of Möbius Transformations of \( \hat{C} \)

A Möbius transformation is a function, \( m : \hat{C} \rightarrow \hat{C} = \mathbb{C} \cup \{ \infty \} \),

\[
m : z \rightarrow m(z) = \frac{az + b}{cz + d}; \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0, \quad M_m := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M_m \neq 0.
\]

\( c = 0 \Rightarrow m(z) = az + \beta ; \quad \alpha, \beta \in \mathbb{C} \text{ and } m(\infty) := \infty \)

\( c \neq 0 \Rightarrow m\left(\frac{d}{c}\right) := \infty \text{ and } m(\infty) := \lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c} \).

The set of all Möbius transformations is denoted \( \text{Möb}^+ \).

(See [6], pp.22-23, pp.36-37)

We will show that \( \text{Möb}^+ \) is a group under composition of functions:

First we check that \( \text{Möb}^+ \) is stable under \( \circ \), i.e. \( m_1, m_2 \in \text{Möb}^+ \Rightarrow m_1 \circ m_2 \in \text{Möb}^+ \).

Let \( m_1(z) := \frac{az + b}{cz + d}; \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0, \quad M_{m_1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det M_{m_1} \neq 0. \)

and \( m_2(z) := \frac{ez + f}{gz + h}; \quad e, f, g, h \in \mathbb{C} \text{ and } eh - fg \neq 0, \quad M_{m_2} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad \det M_{m_2} \neq 0. \)

We get \( (m_1 \circ m_2)(z) = m_1(m_2(z)) = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)}, \)

\[
M_{m_1 \cdot m_2} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = M_{m_1} \times M_{m_2} \quad \text{and}
\]

\( \det M_{m_1 \cdot m_2} = \det (M_{m_1} \times M_{m_2}) = \det M_{m_1} \cdot \det M_{m_2} \neq 0. \)

We see that \( m_1 \circ m_2 \in \text{Möb}^+ \) and since \( m_1 \) and \( m_2 \) was arbitrary elements of \( \text{Möb}^+ \), the set of Möbius transformations is stable under composition of functions.

The neutral element of \( \text{Möb}^+ \) is \( n(z) = z, \quad M_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

It remains to show that each element of \( \text{Möb}^+ \) is invertible.

Let \( w := m(z) = \frac{az + b}{cz + d}; \quad a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0. \)

Then \( z = m^{-1}(w) = \frac{-dw - b}{-cw + a}; \quad da - bc \neq 0, \quad M_{m^{-1}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det M_{m^{-1}} \neq 0. \)

We have \( m^{-1}(w) = \frac{-dw + b}{-cw + a} \in \text{Möb}^+ \) for every \( m \in \text{Möb}^+ \).

Thus \( (\text{Möb}^+, \circ) \) is a group. ☺
2.2 The Conformal Group of the Unit Disk

Let us consider the set \( G \) of all Möbius transformations \( g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of the form

\[
g(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}
\]

where \( \theta \in \mathbb{R} \), \( a \in \mathbb{C} \), \( |a| < 1 \).

We can formulate the following important statements:

\[ (G, \circ) \text{ is a subgroup of } (\text{Möb}^+, \circ). \] (2.2.1)

If \( g \in G \), then \( g(D_1) = D_1 \). (2.2.2)

If \( h : D_1 \to D_1 \) is analytic and bijective, then there exists a \( g \in G \) such that \( h = g|_{D_1} \), i.e. \( h \) is the restriction to \( D_1 \) of an element in \( G \). (2.2.3)

**PROOF** of 2.2.1

\( G \) contains the neutral element of \( \text{Möb}^+ \), \( n(z) = z \), corresponding to \( \theta = 0 \), \( a = 0 \).

We must check that \( G \) is stable under \( \circ \), i.e. \( (\forall) \ g_1, g_2 \in G : g_1 \circ g_2 \in G \).

\[
(g_1 \circ g_2)(z) = e^{i\theta_1} \frac{z-a_2}{1-\bar{a}_2 z} \cdot e^{i\theta_2} \frac{z-a_1}{1-\bar{a}_1 z} = e^{i(\theta_1 + \theta_2)} \frac{z-a_2 \bar{a}_1 e^{i\theta_1}}{1-\bar{a}_1 \bar{a}_2 e^{i\theta_2}}
\]

Note that \( \gamma \neq 0 \). We check the coefficients:

\[
\alpha = \frac{\bar{a}_2 + a_1 \bar{a}_2 e^{i\theta_2}}{1 + a_1 \bar{a}_2 e^{-i\theta_2}} = \frac{w + z}{1 + wz} := b, \text{ where } w = a_2 \text{ and } z = a_1 e^{-i\theta_2}
\]

\[
\beta = \frac{\bar{a}_2 + a_1 \bar{a}_2 e^{i\theta_2}}{e^{i(\theta_1 + \theta_2)} + a_1 \bar{a}_2 e^{i\theta_1}} = \frac{w + z}{1 + wz} := b
\]

We must check that \( |b| < 1 \).
\[ b = \frac{a_2 + a_1 e^{-i\theta}}{1 + a_1 a_2 e^{-i\theta}} = \frac{w + z}{1 + wz}. \] We see that \(|w| < 1\) and \(|z| < 1\).

We get successively: 
\[
0 < (1 - |w|^2)(1 - |z|^2) \iff |w|^2 + |z|^2 < 1 + |w|^2 |z|^2 \iff 
\]
\[
ww + zz + wz + wz < 1 + wzzz + wz + wz \iff (w + z)(\bar{w} + \bar{z}) < (1 + w\bar{z})(1 + w\bar{z}) \iff 
\]
\[
(w + z)(\bar{w} + \bar{z}) < (1 + wz)(1 + wz) \iff |w + z|^2 < |1 + wz|^2 \iff |w + z| < |\bar{w} + \bar{z}| = |b| < 1.
\]

We have \((g_1 \circ g_2)(z) = e^{i\phi} \frac{z - b}{1 - b\bar{z}}\), \(\phi \in \mathbb{R}\), \(b \in \mathbb{C}\), \(|b| < 1\), for arbitrary \(g_1, g_2 \in G\).

g_1 \circ g_2 \in G\) and thus \(G\) is stable under composition of functions.

Finally for every \(g \in G\) its inverse function \(g^{-1}: \mathring{\mathbb{C}} \to \mathring{\mathbb{C}}\) must belong to \(G\).

Let \(w := g(z) = e^{i\theta} \frac{z - a}{1 - a\bar{z}}\) where \(\theta \in \mathbb{R}\), \(a \in \mathbb{C}\), \(|a| < 1\).

Then \(z = g^{-1}(w) = e^{-i\theta} \frac{w + ae^{i\theta}}{1 + ae^{-i\theta}w}\) where \(-\theta \in \mathbb{R}\), \(b = -ae^{i\theta} \in \mathbb{C}\),

\(|b| = |a||e^{i\theta}| < 1\), so \(g^{-1} \in G\) for each \(g \in G\).

Thus \((G, \circ)\) is a subgroup of \((\text{Möb}^+, \circ)\). ☺

2.2.2 If \(g \in G\), then \(g(D_1) = D_1\).

**PROOF** of 2.2.2 (After [4], p.197)

We want to check that \(|z| < 1 \iff |g(z)| < 1\) for every \(g \in G\).

\[ |g(z)|^2 = \left| e^{i\theta} \frac{z - a}{1 - az} \right|^2 = \left| \frac{z - a}{1 - az} \right|^2 = \frac{(z - a)(\bar{z} - \bar{a})}{(1 - az)(1 - az)} = \frac{|z|^2 - az - z\bar{a} + |a|^2}{1 - az - \bar{a}z + |a|^2 |z|^2}. \]

We have \(|g(z)| < 1 \iff |g(z)|^2 = \frac{|z|^2 - az - z\bar{a} + |a|^2}{1 - az - \bar{a}z + |a|^2 |z|^2} < 1 \iff 
\]
\[
|z|^2 - az - z\bar{a} + |a|^2 < 1 - az - \bar{a}z + |a|^2 |z|^2 \iff 
\]
\[
0 < 1 - |z|^2 - |a|^2 + |a|^2 |z|^2 = (1 - |z|^2)(1 - |a|^2). \]

Since \(|a| < 1\), we have \(1 - |a|^2 > 0 \Rightarrow 1 - |z|^2 > 0 \iff |z|^2 < 1 \iff |z| < 1\).

We get \(|g(z)| < 1 \iff |z| < 1\) or \(g(z) \in D_1 \iff z \in D_1\).

That is \(g(D_1) = D_1\). ☺
2.2.3 If \( h : D_1 \to D_1 \) is analytic and bijective, then (\( \exists \)) \( g \in G : h = g|_{D_1} \).

PROOF of 2.2.3 (After [4], p.211)

We shall denote the group of conformal representations of the unit disk by \((K, \circ)\). 

\[ K := \{ h : D_1 \to D_1 | h \text{ analytic and bijective} \} \]

For an arbitrary \( h \in K \), let \( b := h(0) \) and define \( f : D_1 \to D_1, f(z) := \frac{h(z) - b}{1 - b h(z)} \).

Define \( \Phi : D_1 \to D_1, \Phi(z) := \frac{z - b}{1 - b z} \) where \( |b| < 1 \), since \( b = h(0) \in D_1 \).

\( \Phi \) is analytic since the denominator is never zero for \( |z| < 1 \).

Since both \( h \) and \( \Phi \) are analytic in \( D_1 \), so is their composite \( f = \Phi \circ h \).

Furthermore we know that \( |f(z)| < 1 \) in \( D_1 \) and \( f(0) = \Phi(h(0)) = \Phi(b) = 0 \), so we get \( |f(z)| \leq |z| \) by Schwarz's Lemma.\(^{\dagger}\)

\( f : z \to w = f(z) \) is bijective. \( \Rightarrow (\exists) f^{-1} : D_1 \to D_1, w \to z = f^{-1}(w). \)

\( f^{-1}(w) \) is analytic, \( |f^{-1}(w)| < 1 \) and \( f^{-1}(0) = 0 \) so, by Schwarz's Lemma, \( |f^{-1}(w)| \leq |w| \) and since \( w = f(z) \), we have \( |z| \leq |f(z)| \).\(^{\dagger\dagger}\)

From \(^{\dagger}\) and \(^{\dagger\dagger}\) we get \( |f(z)| = |z| \) for each \( z \in D_1 \) and thus, by Schwarz's Lemma, \( f(z) = cz \), where \( c \) is a complex constant with \( |c| = 1 \).

\[ f(z) = \frac{h(z) - b}{1 - b h(z)} = cz \iff h(z) = c \frac{z + \bar{c}b}{1 + c \bar{b} z} := e^{i \theta} \frac{z - a}{1 - a z} \] with \( \theta \in \mathbb{R} \), \( a := -\bar{c}b \), \( |a| = |c||b| < 1 \).

Thus every conformal representation of the unit disk, \( h \in K \), is the restriction to \( D_1 \) of an element in \( G \). ☺
2.3 The Conformal Group of the Upper Half-Plane

The upper half-plane $\mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$.

The map $f : \mathbb{H} \to \mathbb{H}$ is a conformal representation (analytic and bijective)

if and only if $f$ has the form $f(z) = \frac{az + b}{cz + d}$ where $a,b,c,d \in \mathbb{R}$ and $ad - bc > 0$.

**PROOF**

$(\Leftarrow)$

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $f(z) := \frac{az + b}{cz + d}$, where $a,b,c,d \in \mathbb{R}$, $ad - bc > 0$.

$f(\mathbb{R} \cup \{ \infty \}) = \mathbb{R} \cup \{ \infty \}$, i.e. $f$ maps the extended real axis onto itself, since $a,b,c,d \in \mathbb{R}$.

For every $z = x + iy : x,y \in \mathbb{R}$, $f(z) = \frac{(ax + b)(cx + d) + acy^2}{|cz + d|^2} + i\frac{(ad - bc)y}{|cz + d|^2}$ and since $\text{Im}(z) > 0 \iff \text{Im}(f(z)) > 0$, we see that $f(\mathbb{H}) = \mathbb{H}$.

The restriction $f : \mathbb{H} \to \mathbb{H}$ is bijective and analytic.

$(\Rightarrow)$

Let $f : \mathbb{H} \to \mathbb{H}$ be an arbitrary conformal representation.

Let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $T(z) := \frac{z - i}{z + i}$. We know that $T(\mathbb{H}) = D_1$.

Thus the restriction $T : \mathbb{H} \to D_1$ is a conformal representation.

Its inverse is $T^{-1} : D_1 \to \mathbb{H}$, $T^{-1}(z) = \frac{-iz - i}{z - 1}$.

We get the following mapping scheme:

$\begin{align*}
\mathbb{H} \xrightarrow{f} \mathbb{H} & \quad \text{T and f are conformal representations.} \\
\mathbb{H} \xrightarrow{T} D_1 & \quad \text{Thus } T \circ f \circ T^{-1} := h : D_1 \to D_1 \text{ is also a conformal representation.}
\end{align*}$

According to (2.2.3), there exists $a,c \in \mathbb{C}$, $|a| < 1$, $|c| = 1$,

such that $h(z) = \frac{az - a}{1 - az}$ for every $z \in D_1$.

Thus every conformal representation $f : \mathbb{H} \to \mathbb{H}$ can be written as $f = T^{-1} \circ h \circ T$. 

- 19 -
Since the matrices of $T^{-1}$, $h$ and $T$ are

\[
M_{T^{-1}} = \begin{pmatrix} -i & -i \\ 1 & 1 \end{pmatrix}, \quad M_h = \begin{pmatrix} c & -ac \\ -\overline{a} & 1 \end{pmatrix} \quad \text{and} \quad M_T = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},
\]

we get

\[
M_f = M_{T^{-1}} \times M_h \times M_T = -i \begin{pmatrix} c(1-a) + (1-\overline{a}) & -ic(1+a) + i(1+\overline{a}) \\ ic(1-a) - i(1-\overline{a}) & c(1+a) + (1+\overline{a}) \end{pmatrix} = -i \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.
\]

If $\alpha \neq 0$, $f(z) = \frac{z + \beta/\alpha}{(\gamma/\alpha)z + \delta/\alpha}$. Checking the coefficients yields:

\[
\beta = \frac{\beta \overline{\alpha}}{\alpha \overline{\alpha}} = \frac{-(ic-iac+i+\overline{a})(\overline{c}-\overline{ac}+1-a)}{|\alpha|^2} = \frac{2i(\overline{a}-a) + i(\overline{c}-c) - i(a^2c-a^2c)}{|\alpha|^2} = \frac{4\text{Im}(a) + 2\text{Im}(c) - 2\text{Im}(a^2c)}{|\alpha|^2} \in \mathbb{R},
\]

\[
\gamma = \frac{\gamma \overline{\alpha}}{\alpha \overline{\alpha}} = \frac{(ic-iac-i+\overline{a})(\overline{c}-\overline{ac}+1-a)}{|\alpha|^2} = \frac{2i(ac-ac) - i(\overline{c}-c) - i(a^2c-a^2c)}{|\alpha|^2} = \frac{4\text{Im}(ac) - 2\text{Im}(c) - 2\text{Im}(a^2c)}{|\alpha|^2} \in \mathbb{R},
\]

\[
\delta = \frac{\delta \overline{\alpha}}{\alpha \overline{\alpha}} = \frac{(c+ac+1+a)(\overline{c}-\overline{ac}+1-a)}{|\alpha|^2} = \frac{2 - 2(\overline{a}a) + (c + \overline{c}) - (a^2c + a^2\overline{c})}{|\alpha|^2} = \frac{2 - 2|\alpha|^2 + 2\text{Re}(c) - 2\text{Re}(a^2c)}{|\alpha|^2} \in \mathbb{R}.
\]

If $\alpha = 0$, we get $f(z) = \frac{\beta}{\gamma z + \delta} = \frac{1}{(\gamma/\beta)z + \gamma/\delta}$. Checking the coefficients yields:

\[
\beta = \frac{\beta \overline{\beta}}{|\beta|^2} = \frac{(ic-iac-i+\overline{a})(ic+iac-i-\overline{a})}{|\beta|^2} = \frac{-2(\overline{a}a) + (c + \overline{c}) - (a^2c + a^2\overline{c})}{|\beta|^2} = \frac{-2 + 2|\alpha|^2 + 2\text{Re}(c) - 2\text{Re}(a^2c)}{|\beta|^2} \in \mathbb{R},
\]
\[
\frac{\delta}{\beta} = \frac{\delta\bar{\beta}}{\beta\bar{\beta}} = \frac{(c + ac + 1 + ai)(ic + iac - i - ia)}{|\beta|^2} = \frac{2(ac - ac) + i(c - c) + i(a^2c - a^2c)}{|\beta|^2} = \frac{4\text{Im}(ac) + 2\text{Im}(c) + 2\text{Im}(a^2c)}{|\beta|^2} \in \mathbb{R}.
\]

Thus every conformal representation \( f : \mathbb{H} \to \mathbb{H} \) can be written \( f(z) = \frac{az + b}{cz + d} \) with real coefficients \( a, b, c, d \).

Finally \( ad - bc > 0 \), since we have \( \text{Im}(z) = y > 0 \iff \text{Im}(f(z)) = \frac{(ad - bc)y}{|cz + d|^2} > 0 \).

\( \smiley \)

**Remark:**
The elements of \( \text{Möb}^+ \) with real coefficients and \( ad - bc > 0 \) form a subgroup denoted \( \text{Möb}^+(\mathbb{H}) \).
2.4 The First Steps in Hyperbolic Geometry

Hyperbolic geometry is the non-Euclidean geometry obtained by replacing Euclides 5th axiom, the parallel postulate, with the hyperbolic postulate:

For every line $L$ and every point $P$ not on $L$, at least two distinct lines exist which pass through $P$ and do not intersect $L$.

Hyperbolic geometry was discovered by N.I. Lobachevsky, J. Bolyai and C.F. Gauss in the early 19th century. E. Beltrami in 1868 and later F. Klein were able to prove that the 5th axiom is not derivable from the other four.

Several models for the hyperbolic plane are in use. The half-plane model presented here was created by Henri Poincaré around 1880:

The hyperbolic plane is the upper half-plane: $H = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$.

The hyperbolic points are the elements of $H$.

The hyperbolic lines are either Euclidean half-lines parallel to the $0y$-axis, i.e.
$L = \{ z = x_0 + iy : x_0 \in \mathbb{R}, y > 0 \}$
or Euclidean half-circles centered on the $0x$-axis, i.e.
$L = \{ z \in H : |z - a| = r > 0, a \in \mathbb{R} \}$.

Fig 2.4a The two types of hyperbolic lines.
The element of arc-length considered by Poincaré is:

\[ ds := \frac{dz}{y} = \frac{\sqrt{dx^2 + dy^2}}{y} \]

(See [6], pp.62-68)

For a smooth curve \( \gamma \), defined by \( \gamma : [a,b] \to \mathbb{H} \), \( \gamma(t) = x(t) + iy(t) \), we define its hyperbolic length by

\[ l_H(\gamma) = \int_{\gamma} ds := \int_{a}^{b} \frac{\sqrt{x'^2(t) + y'^2(t)}}{y(t)} dt . \]

Example: \( z_1 = x_0 + iy_1 \), \( z_2 = x_0 + iy_2 \), \( y_1 \leq y_2 \)

\[ \gamma_0 : [0,1] \to \mathbb{H}, \]
\[ \gamma_0(t) = (1 - t) z_1 + t z_2 = x_0 + i [(1 - t) y_1 + ty_2] = x_0 + i [y_1 + t (y_2 - y_1)] \]

\[ l_H(\gamma_0) = \int_{0}^{1} \frac{\sqrt{(y_2 - y_1)^2}}{y(t)} dt = \int_{0}^{1} \frac{y_2 - y_1}{y_1 + t (y_2 - y_1)} dt = \ln (y_1 + t (y_2 - y_1))|_{t=0}^{t=1} = \ln y_2 - \ln y_1 = \]
\[ = \ln \frac{y_2}{y_1} . \]

Thus the H-length of the line segment with endpoints \( z_1 = x_0 + iy_1 \) and \( z_2 = x_0 + iy_2 \), \( z_1, z_2 \in \mathbb{H} \), is \( \ln \left| \frac{y_2}{y_1} \right| \).
2.4.1 Theorem

The line segment $\gamma_0 : [0,1] \to \mathbb{H}, \gamma_0(t) = (1 - t) z_1 + t z_2$ where $z_1 = x_0 + iy_1$ and $z_2 = x_0 + iy_2$ is the curve with the least hyperbolic length among the curves $\gamma$, which are piecewise smooth and join $z_1$ with $z_2$.

PROOF

First consider every smooth curve $\gamma : [0,1] \to \mathbb{H}, \gamma(t) = x(t) + iy(t)$ such that $\gamma(0) = z_1 = x_0 + iy_1$, $\gamma(1) = z_2 = x_0 + iy_2$, suppose that $y_1 \leq y_2$.

$$l_{\mathbb{H}}(\gamma) = \int_0^1 \frac{x^2(t) + y^2(t)}{y(t)} \, dt \geq \int_0^1 \frac{y^2(t)}{y(t)} \, dt = \int_0^1 \frac{|y(t)|}{y(t)} \, dt \geq \int_0^1 \frac{y(t)}{y(t)} \, dt = \ln \frac{y_2}{y_1} = l_{\mathbb{H}}(\gamma_0).$$

In the case $y_1 > y_2$, interchange the points $z_1$ and $z_2$.

For the piecewise smooth curve $\gamma : [0,1] \to \mathbb{H}, \gamma(t) = x(t) + iy(t)$,
project each smooth segment $\gamma_n : [t_{n-1},t_n] \to \mathbb{H}$, onto the $0y$-axis.

$$\Gamma_n : [t_{n-1},t_n] \to \mathbb{H}, \Gamma_n(t) = i \, \text{Im} (\gamma_n(t)) = iy(t).$$

Rename start and end-points, $z_0 = x_0 + iy_0$ and $z_k = x_0 + iy_k$, respectively.

With this notation $\Gamma_1(0) = iy_0$, $\Gamma_n(t_n) = iy_n$, for every $n, 1 \leq n \leq k$, and $\Gamma_k(1) = iy_k$.

For each segment $l_{\mathbb{H}}(\gamma_n) = \int_{t_{n-1}}^{t_n} \frac{x^2(t) + y^2(t)}{y(t)} \, dt \geq \ln \left| \frac{y_n}{y_{n-1}} \right| = l_{\mathbb{H}}(\Gamma_n)$.

We get $l_{\mathbb{H}}(\gamma) = l_{\mathbb{H}}(\gamma_1) + l_{\mathbb{H}}(\gamma_2) + \ldots + l_{\mathbb{H}}(\gamma_k) \geq l_{\mathbb{H}}(\Gamma_1) + l_{\mathbb{H}}(\Gamma_2) + \ldots + l_{\mathbb{H}}(\Gamma_k) \geq l_{\mathbb{H}}(\gamma_0)$.

Thus $l_{\mathbb{H}}(\gamma) \geq l_{\mathbb{H}}(\gamma_0)$ for all piecewise smooth curves joining $z_1$ and $z_2$, with equality if and only if $x'(t)$ is strictly zero and either $y'(t) \geq 0$ or $y'(t) \leq 0$ for every $t \in [0,1]$.

☺
One defines the hyperbolic distance, $d_H : H \times H \to \mathbb{R}$, as usual in Riemannian geometry; $(\forall \ z_1, z_2 \in H)$

$$d_H(z_1, z_2) := \inf \{ l_H(\gamma) : \gamma \text{ is a piecewise smooth curve joining } z_1 \text{ and } z_2 \}. $$

For the cross-ratio $[z, z_1, z_2, z_3] := \frac{z - z_2}{z - z_1} \frac{z_3 - z_1}{z_3 - z_2}$, we have $[z, z_1, z_2, \infty] = \frac{z - z_2}{z - z_1}$.

Particularly: $[x_0, x_0 + iy_1, x_0 + iy_2, \infty] = \frac{x_0 - (x_0 + iy_2)}{x_0 - (x_0 + iy_1)} = \frac{y_2}{y_1}$.

$$d_H(x_0 + iy_1, x_0 + iy_2) = \left| \ln \frac{y_2}{y_1} \right| = \left| \ln \left[ x_0, x_0 + iy_1, x_0 + iy_2, \infty \right] \right|, $$ according to (2.4.1).

What about $d_H(z_1, z_2)$ with arbitrary $z_1, z_2 \in H$?
Suppose that $\text{Re}(z_1) < \text{Re}(z_2)$.

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ with $x_1 < x_2$,
we have the Euclidean line of the points equidistant to $z_1$ and $z_2$
passing through $z_0 = x_0 + iy_0 = \frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2}$ with the slope $k = -\frac{x_2 - x_1}{y_2 - y_1}$ and
intersecting the 0x-axis at the point $a = x_0 - \frac{y_0}{k} = \frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2k} = \frac{1}{2} \frac{|z_1|^2 - |z_2|^2}{x_1 - x_2}$.

Thus $(\exists!) \ a \in \mathbb{R}$ such that $|z_1 - a| = |z_2 - a| := r$.
The Euclidean circle $\Gamma$, with center $a$ and radius $r$ intersects the 0x-axis in $a_1$ and $a_2$.

If $y = \Gamma \cap H$, then $y$ is the H-line through $z_1$ and $z_2$ and we can find a $m \in \text{Möb}^+$ such that $m(a_1) = 0$, $m(z_1) = i\alpha$, $m(z_2) = i\beta$ and $m(a_2) = \infty$; $\alpha, \beta \in \mathbb{R}$, $\beta > \alpha > 0$.
Namely $m(z) = -\frac{z - a_1}{z - a_2}$. 

- 25 -
m(z) has real coefficients, \( M_m = \begin{pmatrix} -1 & a_1 \\ 1 & -a_2 \end{pmatrix} \), \( \det M_m = a_2 - a_1 > 0 \).

Thus, according to (2.3), we have \( m \in \text{Mob}^+(\mathbb{H}) \).

\( m(\gamma) = 0 \)-axis, since \( m(\mathbb{R} \cup \{ \infty \}) = \mathbb{R} \cup \{ \infty \} \) and \( \gamma \perp 0 \)-axis.

\[ \begin{align*}
&0 = m(a_1) \\
&i\alpha = m(z_1) \\
&m(\gamma_0) \\
i\beta = m(z_2)
\end{align*} \]

The curve with least H-length joining \( m(z_1) \) and \( m(z_2) \) is \( m(\gamma_0) \), from (2.4.1).

\[ d_H(m(z_1), m(z_2)) = d_H(i\alpha, i\beta) = |\ln[0, i\alpha, i\beta, \infty]| = |\ln[m(a_1), m(z_1), m(z_2), m(a_2)]| = |\ln[a_1, z_1, z_2, a_2]|. \]

We get the following fundamental formula:

\[ d_H(z_1, z_2) := |\ln[a_1, z_1, z_2, a_2]| \text{ for } z_1, z_2 \in \mathbb{H}. \]

**Remark:** If \( \text{Re}(z_1) = \text{Re}(z_2) \) we get the cross-ratio \( = \frac{y_2}{y_1} \).
2.4.2 Theorem

\(d_H : H \times H \to \mathbb{R}, (z_1, z_2) \mapsto d_H(z_1, z_2)\) is a distance function on \(H\), i.e.

(\(\forall\) \(z_1, z_2, z_3 \in H\))

1. \(d_H(z_1, z_2) \geq 0, d_H(z_1, z_2) = 0 \iff z_1 = z_2\)

2. \(d_H(z_1, z_2) = d_H(z_2, z_1)\)

3. \(d_H(z_1, z_2) \leq d_H(z_1, z_3) + d_H(z_3, z_2)\) with equality if and only if \(z_3\) lies on the \(H\)-line through \(z_1\) and \(z_2\), between \(z_1\) and \(z_2\).

PROOF

1. If \(z_1 \neq z_2\) we have \(l_H(y_0) > 0\). Thus \(z_1 \neq z_2 \implies d_H(z_1, z_2) > 0\).

   If \(z_1 = z_2\) we have \(d_H(z_1, z_2) = 0\).

   Thus \(d_H(z_1, z_2) = 0 \iff z_1 = z_2\).

2. \(d_H(z_2, z_1) = |ln[a_2, z_2, z_1, a_1]| = |ln[a_1, z_1, z_2, a_2]| = d_H(z_1, z_2)\)

3. Take three arbitrary points \(z_1, z_2, z_3 \in H\).

   Map \(z_1\) and \(z_2\) on the 0y-axis by \(w = m(z)\), \(m \in \text{M"ob}^+(H)\).

   By (2.4.1) \(d_H(w_1, w_2) \leq d_H(w_1, w_3) + d_H(w_3, w_2)\).

   Thus \(d_H(z_1, z_2) \leq d_H(z_1, z_3) + d_H(z_3, z_2)\), by (2.4.2) and we have equality if and only if \(w_3\) lies on the 0y-axis between \(w_1\) and \(w_2\), i.e. \(z_3\) lies on the \(H\)-line through \(z_1\) and \(z_2\), between \(z_1\) and \(z_2\).

This completes the proof that \(d_H(z_1, z_2)\) is a distance function on \(H\). ☺
2.4.3 Theorem

The conformal representations \( f : \mathbb{H} \rightarrow \mathbb{H} \), \( f(z) = \frac{az + b}{cz + d} \), \( a,b,c,d \in \mathbb{R} \), \( ad – bc > 0 \) are isometries of the hyperbolic plane, i.e.
\[ d_H(f(z_1),f(z_2)) = d_H(z_1,z_2) \] for every \( z_1,z_2 \in \mathbb{H} \).

**PROOF**
If \( z_1 = z_2 \) we have \( d_H(f(z_1),f(z_2)) = d_H(z_1,z_2) = 0 \).
If \( z_1 \neq z_2 \), \( z_1 \) and \( z_2 \) both lie on an unique hyperbolic line with limits on the extended real axis in \( a_1 \) and \( a_2 \). If \( \text{Re}(z_1) = \text{Re}(z_2) \) we have \( a_2 = \infty \).

The extension of \( f \) to \( f : \mathbb{C} \rightarrow \mathbb{C} \), takes \( a_1 \) and \( a_2 \) onto the extended real axis. \( f \) maps \( \mathbb{H} \)-lines onto \( \mathbb{H} \)-lines, so \( f(z_1) \) and \( f(z_2) \) both lie on the hyperbolic line with limits in \( f(a_1) \) and \( f(a_2) \).
\[ d_H(f(z_1),f(z_2)) = \left| \ln \left[ f(a_1), f(z_1), f(z_2), f(a_2) \right] \right| = \left| \ln \left[ a_1, z_1, z_2, a_2 \right] \right| = d_H(z_1,z_2). \]

\( \smiley \)

**Remark 1:**
The indirectly conformal maps, \( g : \mathbb{H} \rightarrow \mathbb{H} \), \( g(z) = \frac{a\bar{z} + b}{c\bar{z} + d} \), \( a,b,c,d \in \mathbb{R} \), \( ad – bc < 0 \) are also isometries of the hyperbolic plane.

**Remark 2:**
The group of isometries of \( \mathbb{H} \) consists precisely of the two previous classes of maps.
A geodesic is a locally length-minimizing curve. The geodesics of the hyperbolic plane are the H-lines, i.e. the shortest path between two points in the H-plane is along the H-line connecting them.

\[ z_1, z_2 \in L, \text{ suppose } \Re(z_1) > \Re(z_2). \]
\[ \gamma_0 : \text{the arc of } L \text{ from } z_1 \text{ to } z_2. \]

\[ \gamma : [a, b] \to H, \gamma(a) = z_1, \gamma(b) = z_2 \]
and \( \gamma \) is piecewise smooth

\[ \gamma_0 : [\Arg(z_1 - a_0), \Arg(z_2 - a_0)] \to H, \gamma_0(t) = a_0 + re^{it} \]

\( \gamma_0(t) \) is called the distance realizing path.

\[ d_H(z_1, z_2) = \inf \{ l_H(\gamma) \} = l_H(\gamma_0) \]

\[ l_H(\gamma_0) \leq l_H(\gamma) \text{ for every } \gamma \text{ connecting } z_1 \text{ and } z_2. \]
References:


