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THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

Embedding Theorems for Mixed Norm Spaces and Applications

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Abstract

This thesis is devoted to the study of mixed norm spaces that arise in connection with embeddings of Sobolev and Besov type spaces. The work in this direction originates in a paper due to Gagliardo (1958), and was continued by Fournier (1988) and by Kolyada (2005).

We consider fully anisotropic mixed norm spaces. Our main theorem states an embedding of these spaces into Lorentz spaces. Applying this result, we obtain sharp embedding theorems for anisotropic fractional Sobolev spaces and anisotropic Sobolev-Besov spaces. The methods used are based on non-increasing rearrangements and on estimates of sections of functions and sections of sets. We also study limiting relations between embeddings of spaces of different type. More exactly, mixed norm estimates enable us to get embedding constants with sharp asymptotic behaviour. This gives an extension of the results obtained for isotropic Besov spaces B_p^α by Bourgain, Brezis, and Mironescu, and for Besov spaces $B_p^{\alpha_1,\dots,\alpha_n}$ by Kolyada.

We study also some basic properties (in particular the approximation properties) of special weak type spaces that play an important role in the construction of mixed norm spaces and in the description of Sobolev type embeddings.

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Contents

1. Introduction

This work is devoted to the study of mixed norm spaces that arise in connection with embeddings of Sobolev and Besov spaces.

A function $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, is said to belong to the Sobolev space $W_p^1(\mathbb{R}^n)$ if f has weak derivatives $D_k f \in L^p(\mathbb{R}^n)$ for all $1 \le k \le n$. In 1938 Sobolev proved the following, now classical, theorem.

Theorem 1.1. *Let $n \ge 2$, $1 < p < n$, and $q = np/(n-p)$. If $f \in W_p^1(\mathbb{R}^n)$ then $f \in L^q(\mathbb{R}^n)$ and*

$$\|f\|_q \le c \sum_{k=1}^{n} \|D_k f\|_p. \tag{1.1}$$

It was first in 1958 that this theorem was extended to the case $p = 1$. This was done independently by Gagliardo and Nierenberg. The next lemma was the central part of Gagliardo's approach (see [10]). We use the notation $\hat{x}_k$ for the vector in $\mathbb{R}^{n-1}$ obtained from a given vector $x \in \mathbb{R}^n$ by removing its kth coordinate.

Lemma 1.2. *Let $n \ge 2$. Assume that the functions $g_k \in L^1(\mathbb{R}^{n-1})$, $k = 1, ..., n$, are non-negative. Then*

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} g_k(\hat{x}_k)^{1/(n-1)} dx \le \Big(\prod_{k=1}^{n} \int_{\mathbb{R}^{n-1}} g_k(\hat{x}_k) d\hat{x}_k \Big)^{1/(n-1)}.$$

Let $f \in W_1^1(\mathbb{R}^n)$. For almost every $x \in \mathbb{R}^n$,

$$|f(x)| \le \frac{1}{2} \int_{\mathbb{R}} |D_k f(x)| dx_k \equiv g_k(\hat{x}_k), \quad k = 1, ..., n. \tag{1.2}$$

Applying Lemma 1.2, we obtain (n' denotes the conjugate of n)

$$\|f\|_{n'} \le \frac{1}{2} \Big(\prod_{k=1}^{n} \|D_k f\|_1 \Big)^{1/n}.$$

This implies Theorem 1.1 for $p = 1$. However, one can obtain a stronger statement from Lemma 1.2. Let

$$V_k \equiv L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})], \quad 1 \le k \le n,$$

be the space with the mixed norm

$$\|f\|_{V_k} \equiv \|\Psi_k\|_{L^1(\mathbb{R}^{n-1})},$$

where

$$\Psi_k(\hat{x}_k) = \operatorname{ess\,sup}_{x_k \in \mathbb{R}} |f(x)|.$$

We say that the L^1-norm is the "exterior" norm of V_k and the L^∞-norm is the "interior" norm. Applying Lemma 1.2 to the functions Ψ_k gives the following theorem.

Theorem 1.3. *Let $n \geq 2$. If $f \in \cap_{k=1}^n V_k$, then $f \in L^{n'}(\mathbb{R}^n)$ and*

$$\|f\|_{n'} \leq \Big(\prod_{k=1}^n \|f\|_{V_k} \Big)^{1/n}.$$

For $f \in W_1^1(\mathbb{R}^n)$, inequality (1.2) gives

$$\|f\|_{V_k} \leq \frac{1}{2}\|D_k f\|_1. \tag{1.3}$$

This estimate and Theorem 1.3 implies inequality (1.1) for $p = 1$.

Let $S_0(\mathbb{R}^n)$ be the class of all measurable functions f on $\mathbb{R}^n$ such that the distribution function $\lambda_f(y)$ is finite for all $y > 0$. Let f^* denote the non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$. If $0 < q, p < \infty$, then the Lorentz space $L^{q,p}(\mathbb{R}^n)$ is defined as the class of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{q,p} = \Big(\int_0^\infty \big[t^{1/q} f^*(t)\big]^p \frac{dt}{t} \Big)^{1/p} < \infty.$$

For any fixed q, the Lorentz spaces increase as the secondary index p increases (see Section 2.3 below).

It is well known that the left-hand side in (1.1) can be replaced by the stronger Lorentz norm (see [8], [27], [28], and [29]). That is, the following theorem holds.

Theorem 1.4. *Let $n \geq 2$ and $1 \leq p < n$. Set $q = np/(n-p)$. If $f \in W_p^1(\mathbb{R}^n)$, then $f \in L^{q,p}(\mathbb{R}^n)$ and*

$$\|f\|_{q,p} \leq c \sum_{k=1}^n \|D_k f\|_p. \tag{1.4}$$

In [9], Fournier proved this theorem for $p = 1$, using the following refinement of Theorem 1.3.

Theorem 1.5. *Let $n \geq 2$. If $f \in \cap_{k=1}^n V_k$, then $f \in L^{n',1}(\mathbb{R}^n)$ and*

$$\|f\|_{n',1} \leq n' \Big(\prod_{k=1}^n \|f\|_{V_k} \Big)^{1/n}. \tag{1.5}$$

Observe that for the characteristic function of the unit cube in $\mathbb{R}^n$ we have equality in (1.5). Thus, the constant n' is optimal.

Some extensions of Theorem 1.5 were obtained in the paper [5] due to Blei and Fournier. In particular, it was proved that for any $1 < r \leq \infty$

$$\|f\|_{q,1} \leq c \sum_{k=1}^n \|f\|_{V_k^{(r)}}, \tag{1.6}$$

where $q = nr/(nr - r + 1)$ and

$$V_k^{(r)} = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^r_{x_k}(\mathbb{R})] \quad (k = 1, \dots, n).$$

It was shown in [9], [25] that the preceding results give a sharpening of some inequalities for bilinear forms proved by Hardy and Littlewood.

In view of (1.3), Theorem 1.5 immediately implies Theorem 1.4 for $p = 1$. Fournier [9, p. 66] observed that it was not clear how the methods based on mixed norm estimates could be applied to obtain (1.4) also for $1 < p < n$. This problem was studied by Kolyada in [19]. He introduced a scale of more general mixed norm spaces in which the interior norms are defined by conditions on the rearrangements with respect to specific variables. These conditions are expressed in terms of the "weak" spaces Λ^σ. Let $\sigma \in \mathbb{R}$. Denote by $\Lambda^\sigma(\mathbb{R})$ the class of all functions $f \in S_0(\mathbb{R})$ such that

$$\|f\|_{\Lambda^\sigma} = \sup_{t>0} t^\sigma (f^*(t) - f^*(2t)) < \infty. \tag{1.7}$$

If $0 < \sigma < \infty$ and $r = 1/\sigma$, then $\Lambda^\sigma = L^{r,\infty}$ (where $L^{r,\infty}$ is the Marcinkiewicz space weak-L^r). If $\sigma = 0$, then Λ^σ coincides with the space weak-L^∞ introduced in [2]. If $\sigma < 0$, then (1.7) is a weak version of Lipschitz condition for the rearrangement (see Section 3).

The main result in [19] is the following theorem.

Theorem 1.6. *Let $n \geq 2$. Assume that $1 \leq p < \infty$ and that α_k, $k = 1, ..., n$, are positive numbers such that*

$$\alpha \equiv n \Big(\sum_{k=1}^n \frac{1}{\alpha_k} \Big)^{-1} < \frac{n}{p}. \tag{1.8}$$

Set

$$\sigma_k = \frac{1}{p} - \alpha_k, \quad V_k \equiv L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})],$$

and $q = np/(n - \alpha p)$. Suppose that $f \in S_0(\mathbb{R}^n)$ and $f \in \cap_{k=1}^n V_k$. Then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|_{q,p} \leq c \prod_{k=1}^n \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{1.9}$$

where

$$c = c_n \Big(\prod_{k=1}^n (n\alpha_k - \alpha)^{\alpha/(n\alpha_k)} \Big)^{-1/p} \tag{1.10}$$

and c_n depends only on n.

Observe that Theorem 1.6 remains true for $\alpha = n/p$ (the space $L^{\infty,p}$ is defined in Section 2.3).

It follows from Theorem 1.6 that for any $1 < r < \infty$, the interior L^r-norm on the right-hand side of (1.6) can be replaced by the weaker $L^{r,\infty}$-norm.

It was proved in [19] (see Lemma 5.5 and Remark 5.6 below) that if a function $f \in L^p(\mathbb{R}^n)$ has a weak derivative $D_k f \in L^p(\mathbb{R}^n)$, then

$$\|f\|_{L^p_{\hat{x}_k}[\Lambda^{1/p-1}_{x_k}]} \leq 4\|D_k f\|_p, \quad 1 \leq p < \infty.$$

Hence, there holds the embedding

$$W^1_p(\mathbb{R}^n) \subset \bigcap_{k=1}^{n} L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p-1}_{x_k}(\mathbb{R})]. \tag{1.11}$$

We now obtain Theorem 1.4 in two steps. The first (and simplest) step is (1.11) and the second step is Theorem 1.6 with $\alpha_1 = \cdots = \alpha_n = 1$.

In [19], Theorem 1.6 was also applied to study estimates involving certain Besov norms.

In Theorem 1.4 all derivatives $D_k f$ belong to the same space $L^p(\mathbb{R}^n)$. Nevertheless, it is quite reasonable to suppose that the functions $D_k f$, $k = 1, \dots, n$, belong to different spaces $L^{p_k}(\mathbb{R}^n)$. Such conditions naturaly appear in embedding theory as well as in applications. Furthermore, many authors have studied Sobolev and Besov spaces whose construction involves, instead of L^p-norms, norms in more general spaces - first of all, in the Lorentz spaces. There are many important problems in Analysis that lead to spaces of this type.

Therefore it is natural to study mixed norm spaces which are anisotropic not only with respect to interior norms, but also with respect to exterior norms. The main problem considered in this work is to extend Theorem 1.6 to these, more general, mixed norm spaces. Our main result is Theorem 4.5, it states in particular the following.

Theorem 1.7. *Let $n \geq 2$, $1 \leq p_1, \dots, p_n, s_1, \dots, s_n < \infty$, and $\alpha_1, \dots, \alpha_n > 0$. Put*

$$\alpha = n\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k}\Big)^{-1}, \; p = \frac{n}{\alpha}\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1}, \; \textit{and } s = \frac{n}{\alpha}\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k s_k}\Big)^{-1}.$$

Assume that $p < n/\alpha$ and put $q = np/(n - \alpha p)$. Set

$$\sigma_k = \frac{1}{p_k} - \alpha_k, \; \textit{and } V_k = L^{p_k, s_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})],$$

and assume that

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \sigma_k > 0,$$

for $k = 1, \dots, n$. *Suppose that*

$$f \in S_0(\mathbb{R}^n) \text{ and } f \in \bigcap_{k=1}^{n} V_k.$$

Then $f \in L^{q,s}(\mathbb{R}^n)$ *and*

$$\|f\|_{q,s} \le c \prod_{k=1}^{n} \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{1.12}$$

where c *depends only on* $p_1, \dots, p_n, s_1, \dots, s_n$, $\alpha_1, \dots, \alpha_n$, *and* n.

We have obtained the constant in (1.12) explicitly. This explicit value is used in Section 5, where we consider applications of Theorem 1.7.

As we will show, Theorem 1.7 holds in the case $p = n/\alpha$ as well.

The proof of Theorem 1.7 is based on the approach given in the works of Kolyada [19] and Kolyada and Péres [21].

Applying Theorem 1.7, we obtain sharp embedding theorems for anisotropic Sobolev-Liouville and anisotropic Sobolev-Besov spaces. We also study limiting relations between embeddings of spaces of different type. More exactly, mixed norm estimates enable us to get embedding constants with sharp asymptotic behaviour. This gives an extension of the results obtained for isotropic Besov spaces B_p^α by Bourgain, Brezis, and Mironescu [7], and for Besov spaces $B_p^{\alpha_1,\dots,\alpha_n}$ by Kolyada [19].

The use of mixed norm estimates clarifies the role of smoothness conditions in the embedding theorems for Sobolev and Besov-type spaces and provides a method which is sufficiently flexible to be applied in both of these settings. We stress that estimates of the interior Λ^σ-norms play a crucial role in these methods. This is why it is important to study the basic properties of the spaces Λ^σ.

In Section 3 we will see that approximation in the "norm" on Λ^σ behaves badly. However, we have obtained some positive results on approximation of functions f in this space. Our main result in this direction is the following theorem.

Let $C_0(\mathbb{R})$ denote the class of all continuous functions with bounded support in $\mathbb{R}$.

Theorem 1.8. *Let* $f \in \Lambda^\sigma$ $(\sigma \in \mathbb{R})$. *Then there exists a sequence* $\{f_k\}$, $f_k \in C_0(\mathbb{R})$, *such that* $\{f_k\}$ *converges to* f *in measure and* $\|f_k\|_{\Lambda^\sigma} \to \|f\|_{\Lambda^\sigma}$.

Observe that this theorem is similar to known results for approximation in variation (see [32], [14, Section 9.1]).

As follows from the exposition given above, the spaces Λ^σ have a relevant role in the description of Sobolev-type embeddings. We emphasize also that the use of Lorentz norms as exterior norms in the definition of mixed norm spaces is natural and important. We have already obtained some preliminary results (they are not included in this work) which show that spaces defined in terms of L^p-norms are embedded to spaces defined in terms of *Lorentz* norms. Moreover, these "intermediate" embeddings are sharp.

In the continuation of this work we plan to apply Theorem 1.7 to extend the results by Hardy and Littlewood concerning bilinear forms to forms bounded in Lorentz spaces.

This thesis is organized as follows. Section 2 contains main definitions and some auxilliary propositions. In Section 3 we study some basic properties of the space Λ^σ, in particular the approximation properties of these spaces. In Section 4 we prove our main result, Theorem 1.7. This section also includes some relevant lemmas. In Section 5 we study applications of Theorem 1.7 to embeddings of Sobolev- and Besov-type spaces.

2. Definitions and auxiliary propositions

This section contains definitions and known results. In Section 2.1 we state some known inequalities that we need. In Section 2.2 we define the non-increasing rearrangement of a function and give some of its basic properties. This definition was first given by G. Hardy and J. Littlewood [12]. Estimates in terms of rearrangements will be important in the following sections. In Section 2.3 we introduce the Lorentz spaces.

2.1. **Inequalities.** For $E \subset \mathbb{R}^n$ and $k = 1, \dots, n$ we let $\Pi_k E \subset \mathbb{R}^{n-1}$ be the orthogonal projection of E onto the hyperplane $x_k = 0$. If E is measurable, then we let $\mathrm{mes}_n E$ denote the Lebesgue measure of E in $\mathbb{R}^n$. The following theorem was proved by L. H. Loomis and H. Whitney [23].

Theorem 2.1. *For any F_σ-set $E \subset \mathbb{R}^n$ there holds the inequality*

$$(\mathrm{mes}_n E)^{n-1} \le \prod_{k=1}^{n} \mathrm{mes}_{n-1} \Pi_k E. \tag{2.1}$$

The next theorem was proved by G. Hardy (see e.g. [3, p. 124]).

Theorem 2.2. *Let $\alpha > 0$ and $1 \le p < \infty$. If f is a non-negative measurable function on $\mathbb{R}_+ \equiv (0, \infty)$ then*

$$\Big(\int_0^\infty t^{\alpha-1}\Big(\int_t^\infty f(u)du\Big)^p dt\Big)^{1/p} \le \frac{p}{\alpha}\Big(\int_0^\infty t^{p+\alpha-1} f(t)^p dt\Big)^{1/p} \tag{2.2}$$

and

$$\Big(\int_0^\infty t^{-\alpha-1}\Big(\int_0^t f(u)du\Big)^p dt\Big)^{1/p} \le \frac{p}{\alpha}\Big(\int_0^\infty t^{p-\alpha-1} f(t)^p dt\Big)^{1/p}. \tag{2.3}$$

If, as in the above theorem, f is a non-negative measurable function on $\mathbb{R}_+$ and $\alpha > 0$, there hold the obvious inequalities

$$\sup_{t>0} t^{\alpha} \int_t^{\infty} f(u)du \le \frac{1}{\alpha} \sup_{t>0} t^{1+\alpha} f(t) \tag{2.4}$$

and

$$\sup_{t>0} t^{-\alpha} \int_0^{t} f(u)du \le \frac{1}{\alpha} \sup_{t>0} t^{1-\alpha} f(t). \tag{2.5}$$

The next inequality is statement (iv) in Theorem 2 in [22]. It is similar to Hardy's inequality (2.2), but for the case $0 < p < 1$ and for non-increasing functions. The calculation of the constant can be found in [19, p. 150].

Theorem 2.3. *Let f be a non-negative non-increasing function on $\mathbb{R}_+$. Suppose that $\alpha > 0$ and $0 < p < 1$. Then*

$$\int_0^{\infty} t^{\alpha-1} \Big(\int_t^{\infty} f(u)du \Big)^p dt \le e\Big(1 + \frac{p}{\alpha}\Big) \int_0^{\infty} t^{\alpha+p-1} f(t)^p dt. \tag{2.6}$$

2.2. **The non-increasing rearrangement.** Let f be a measurable function on $\mathbb{R}^n$. For $y \ge 0$ we define the distribution function of f by

$$\lambda_f(y) = \text{mes}_n\{x \in \mathbb{R}^n : |f(x)| > y\}.$$

Observe that λ_f may take the value ∞. Recall that $S_0(\mathbb{R}^n)$ denotes the class of all measurable almost everywhere finite functions f on $\mathbb{R}^n$ for which $\lambda_f(y) < \infty$ for all $y > 0$. A non-negative and non-increasing function f^* on $\mathbb{R}_+$ which is equimeasurable with f, i.e. which satisfies

$$\text{mes}_1\{t > 0 : f^*(t) > y\} = \lambda_f(y),$$

for all $y \ge 0$, is said to be a non-increasing rearrangement of the function $f \in S_0(\mathbb{R}^n)$. We will also assume that f^* is left-continuous on $\mathbb{R}_+$. Under this condition f^* is defined uniquely by (see [17, p. 142])

$$f^*(t) = \sup\{\inf_{x\in E} |f(x)| : \text{mes}_n E = t\}, \tag{2.7}$$

where the supremum is taken over all measurable sets $E \subset \mathbb{R}^n$ having measure t.

We now give some basic properties of the rearrangement that will come to use in what follows. Let $f \in S_0(\mathbb{R}^n)$ and put

$$A_t = \{x : |f(x)| > f^*(t)\},$$

$t > 0$. By the definition of f^* it holds that

$$\text{mes}_n A_t \le t. \tag{2.8}$$

It is also a consequence of the definition of f^* that the measure of the set $B_t = \{x : |f(x)| \ge f^*(t)\}$ satisfies

$$\text{mes}_n B_t \ge t. \tag{2.9}$$

For each $f \in S_0(\mathbb{R}^n)$ and every scalar $a \in \mathbb{R}$ it is immediate that $af \in S_0(\mathbb{R}^n)$ (the distribution function of af is $y \mapsto \lambda_f(y/a)$, so it is finite). It follows directly from (2.7) that

$$(af)^*(t) = |a|f^*(t), \tag{2.10}$$

for all $t > 0$.

For $f, g \in S_0(\mathbb{R}^n)$ and $t, s > 0$ it holds that (see [17, p. 142])

$$(f+g)^*(t+s) \le f^*(t) + g^*(s). \tag{2.11}$$

Let $f \in S_0(\mathbb{R}^n)$ and fix $\varepsilon > 0$. Since f^* and f are equimeasurable we have

$$\mathrm{mes}_1\{t > 0 : f^*(t) > \varepsilon\} = \lambda_f(\varepsilon) < \infty.$$

Since f^* is non-increasing it follows that $f^*(t) \le \varepsilon$ for all $t > \lambda_f(\varepsilon)$. Thus,

$$\lim_{t\to\infty} f^*(t) = 0. \tag{2.12}$$

We also have

$$\lim_{t\to 0+} f^*(t) = \|f\|_\infty. \tag{2.13}$$

Indeed, let y_0 denote this limit. By (2.8) it holds that

$$\mathrm{mes}_n\{x : |f(x)| > y_0\} \le \mathrm{mes}_n\{x : |f(x)| > f^*(t)\} \le t,$$

for all $t > 0$. Thus $\mathrm{mes}_n\{x : |f(x)| > y_0\} = 0$, so that $\|f\|_\infty \le y_0$. Furthermore, (2.9) implies that $\|f\|_\infty \ge f^*(t)$ for all $t > 0$, and therefore $\|f\|_\infty \ge y_0$.

We also mention the following result [17, p. 143].

Proposition 2.4. *If the sequence $\{f_k\} \subset S_0(\mathbb{R}^n)$ converges in measure to the function $f \in S_0(\mathbb{R}^n)$, then $f_k^* \to f^*$ at every point of continuity of f^*.*

Let $C(\mathbb{R}^n)$ denote the class of all bounded continuous functions on $\mathbb{R}^n$.

Lemma 2.5. *Let $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then, for every $t_0 > 0$ there exists a point $x_0 \in \mathbb{R}^n$ such that $f^*(t_0) = |f(x_0)|$.*

Proof. Fix $t_0 > 0$. It is immediate from the definition of f^* that $0 \le f^*(t_0) \le \|f\|_\infty$. First we assume that $f^*(t_0) = 0$. Suppose $|f(x)| > 0$ for all $x \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a compact set having measure t_0. Since $f \in C(\mathbb{R}^n)$ there exists $x_1 \in E$ where

$$f^*(t_0) \ge \inf_{x\in E} |f(x)| = |f(x_1)| > 0,$$

which is a contradiction.

Next we suppose that $f^*(t_0) = \|f\|_\infty$. According to (2.9), it holds that

$$\mathrm{mes}_n\{x : |f(x)| = \|f\|_\infty\} = \mathrm{mes}_n\{x : |f(x)| \ge f^*(t_0)\} \ge t_0 > 0,$$

so there exists $x_0 \in \mathbb{R}^n$ where $|f(x_0)| = \|f\|_\infty = f^*(t_0)$.

The remaining case is when $0 < f^*(t_0) < \|f\|_\infty$. Since $f \in S_0(\mathbb{R}^n)$ we can not have $|f(x)| > f^*(t_0) > 0$ for all $x \in \mathbb{R}^n$. So there exists $x' \in \mathbb{R}^n$ such that

$$0 \le |f(x')| \le f^*(t_0). \tag{2.14}$$

Clearly there also exists a point $x'' \in \mathbb{R}^n$ where

$$f^*(t_0) \le |f(x'')| \le \|f\|_\infty. \tag{2.15}$$

Since f has the intermediate value property it follows from (2.14) and (2.15) that there exists some x_0 along the line segment from x' to x'' for which $|f(x_0)| = f^*(t_0)$. □

Lemma 2.6. *Let $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then f^* is continuous on $\mathbb{R}_+$.*

Proof. Fix $t_0 > 0$. Assume that f^* is discontinuous at t_0. Since f^* is left-continuous and non-increasing, it follows that

$$y_0 \equiv \lim_{t \to t_0+} f^*(t) < f^*(t_0).$$

So, f^* takes no values in $(y_0, f^*(t_0))$. Let $\tau \in (y_0, f^*(t_0))$ and suppose $|f(x_0)| = \tau$ for some $x_0 \in \mathbb{R}^n$. Since f is continuous, there exists some $\delta > 0$ such that if $x_1 \in \mathbb{R}^n$ and $|x_0 - x_1| < \delta$ then

$$\big|\tau - |f(x_1)|\big| = \big||f(x_0)| - |f(x_1)|\big| < f^*(t_0) - y_0.$$

Therefore

$$\mathrm{mes}_n\{x : |f(x)| \in (y_0, f^*(t_0))\} > 0.$$

But, f and f^* are equimeasurable so

$$\mathrm{mes}_n\{x : |f(x)| \in (y_0, f^*(t_0))\} = \mathrm{mes}_1\{s > 0 : f^*(s) \in (y_0, f^*(t_0))\} = 0,$$

which is a contradiction. Thus, if f^* is discontinuous at t_0, then $|f|$ takes no values in the interval $(y_0, f^*(t_0))$. By (2.9)

$$\mathrm{mes}_n\{x : |f(x)| \ge f^*(t_0)\} \ge t_0 > 0.$$

Again by (2.9) and the equimeasurability of f and f^*,

$$\mathrm{mes}_n\{x : f^*(t_0 + 1) \le |f(x)| \le y_0\} =$$

$$= \mathrm{mes}_n\{x : |f(x)| \ge f^*(t_0 + 1)\} - \mathrm{mes}_1\{s > 0 : f^*(s) > y_0\} \ge 1,$$

so $|f|$ takes values greater than $f^*(t_0)$ and values less than y_0. Since f has the intermediate value property, it follows that the whole interval $(y_0, f^*(t_0))$ is in the range of $|f|$. Thus, the assumption that f^* is discontinuous at some point t_0 leads to a contradiction. □

Let f be continuous on a set $E \subset \mathbb{R}^n$. The modulus of continuity of f is the function $\delta \mapsto \omega(f;\delta)$, which is defined for all $\delta > 0$ by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in E, |x - y| \leq \delta\}.$$

The supremum is over all x and y in the domain E of f such that $|x-y| < \delta$. For all $\alpha > 0$ it holds that (see [17, p. 123])

$$\omega(f;\alpha\delta) \leq (\alpha + 1)\omega(f;\delta). \tag{2.16}$$

The inequality stated by the next proposition is known, but we give a simpler proof of it. Similar estimates can be found e.g. in [13], [26] and [15].

Proposition 2.7. *Let $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then*

$$\omega(f^*;\delta) \leq c\,\omega(f;\delta^{1/n}), \tag{2.17}$$

for all $\delta > 0$, where $c = 2v_n^{-1/n} + 1$ and v_n is the measure of the unit ball in $\mathbb{R}^n$.

Proof. By the triangle inequality we have $\omega(|f|;\delta) \leq \omega(f;\delta)$, so we may assume that $f \geq 0$. Fix $0 < t' < t''$ and estimate $f^*(t') - f^*(t'')$. We can assume that $f^*(t'') < f^*(t')$. Let

$$A' = \{x : f(x) = f^*(t')\} \quad \text{and} \quad A'' = \{x : f(x) = f^*(t'')\}.$$

Since $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, the sets A' and A'' are nonempty by Lemma 2.5. Fix $N \geq 3$. We will show that there exist points $x' \in A'$ and $x'' \in A''$ such that

$$|x' - x''| < 2\frac{N+1}{N-1}v_n^{-1/n}(t'' - t')^{1/n}. \tag{2.18}$$

Let d be the distance from A' to A'', i.e.

$$d = \inf\{|x' - x''| : x' \in A', x'' \in A''\}.$$

If $d = 0$ then $|x' - x''|$ can be choosen arbitrarily small, in particular so small that (2.18) is satisfied. Assume that $d > 0$. Then there exists $x' \in A'$ and $x'' \in A''$ such that $|x' - x''| < (1 + 1/N)d$. Let these points be choosen so that the function $\tau \mapsto f(x'\tau + (1-\tau)x'')$ only takes values in $(f^*(t''), f^*(t'))$ for $\tau \in (0, 1)$. Set $\lambda_N = N/(N+1) - 1/2 > 0$. Let B be the ball in $\mathbb{R}^n$ centered at $p = (x' + x'')/2$ of radius $\lambda_N|x' - x''|$. Then $B \cap A' = \emptyset$. Indeed, suppose there exist a point $y' \in B \cap A'$. Then

$$|y' - x''| \leq |y' - \frac{x' + x''}{2}| + |\frac{x' + x''}{2} - x''| <$$

$$< (\lambda_N + \frac{1}{2})|x' - x''| < (\lambda_N + \frac{1}{2})(1 + \frac{1}{N})d = d,$$

which is a contradiction. Similarly $B \cap A'' = \emptyset$.

Let

$$E = \{x : f^*(t'') < f(x) < f^*(t')\}.$$

We will prove that $B \subset E$. By choice of x' and x'' we know that

$$f^*(t'') < f(p) < f^*(t').$$

Suppose there exists a point $q \in B$ where $f(q) < f^*(t'')$. Since f has the intermediate value property there exists a point r along the line segment from p to q where $f(r) = f^*(t'')$. Thus $r \in B \cap A''$, which is a contradiction. In the same way the assumption that $f(x) > f^*(t')$ for some $x \in B$ leads to a contradiction. This proves that $B \subset E$. By our observations (2.8) and (2.9) we then obtain

$$\text{mes}_n B \leq \text{mes}_n E \leq t'' - t'.$$

This gives inequality (2.18). Now

$$f^*(t') - f^*(t'') = f(x') - f(x'') \leq \omega\big(f; 2\frac{N+1}{N-1} v_n^{-1/n}(t'' - t')^{1/n}\big).$$

Since N is arbitrary, we obtain

$$f^*(t') - f^*(t'') \leq \omega\big(f; 2v_n^{-1/n}(t'' - t')^{1/n}\big). \tag{2.19}$$

By (2.16), this implies (2.17). □

Remark 2.8. *Let* $n = 1$. *Then we have* $c = 2$ *in* (2.17). *However, in this case* (2.19) *gives*

$$\omega(f^*; \delta) \leq \omega(f; \delta), \tag{2.20}$$

that is, (2.17) *holds with* $c = 1$. *It is possible to give a shorter proof of* (2.20). *Indeed, let* $0 < t < t+h$. *Assume that* $f^*(t) > f^*(t+h)$. *By Lemma 2.5, there exists* $x', x'' \in \mathbb{R}$ *such that* $|f(x')| = f^*(t)$, $|f(x'')| = f^*(t+h)$, *and* $f^*(t+h) < |f(x)| < f^*(t)$ *for all* x *between* x' *and* x''. *It is clear that* $|x' - x''| \leq h$ *since otherwise we would have*

$$\text{mes}_1\{x : f^*(t+h) < |f(x)| < f^*(t)\} > h,$$

which is a contradiction (*by* (2.8) *and* (2.9), *this set has measure at most* h). *Thus,*

$$f^*(t) - f^*(t+h) = |f(x')| - |f(x'')| \leq \omega(f; h).$$

This implies inequality (2.20).

2.3. **Lorentz spaces.** The Lorentz spaces $L^{q,p}$ form a two parameter family of spaces that contains the Lebesgue spaces L^p. We give here the definition and some basic properties.

We observe first that the rearrangement preserves the L^p-norm. Indeed it holds that (see [31, p. 191-192])

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_0^\infty [f^*(t)]^p dt, \tag{2.21}$$

for all $0 < p < \infty$, and

$$\|f\|_\infty = \|f^*\|_\infty.$$

It follows from Lemma 3.17 on page 201 in [31] that given $f \in S_0(\mathbb{R}^n)$ and $t > 0$, there exists a measurable set $E_t \subset \mathbb{R}^n$ having measure t such that

$$\int_{E_t} |f(x)|dx = \sup_{|E|=t} \int_E |f(x)|dx = \int_0^t f^*(u)du, \tag{2.22}$$

where $|E|$ denotes the measure of E and the supremum is over all measurable sets $E \subset \mathbb{R}^n$ having measure t.

In what follows we set

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(u)du.$$

It follows from (2.22) that the operator $f \mapsto f^{**}$ is subadditive, that is,

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t). \tag{2.23}$$

As was already mentioned in Section 1, when $0 < q, p < \infty$, the space $L^{q,p}(\mathbb{R}^n)$ is defined as the class of all $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{q,p} \equiv \Big(\int_0^\infty [t^{1/q} f^*(t)]^p \frac{dt}{t} \Big)^{1/p} < \infty.$$

By (2.21) we have that $L^{p,p}$ coincides with the space L^p, $0 < p < \infty$. For $0 < q < \infty$ we let $L^{q,\infty}(\mathbb{R}^n)$ be the space of all $f \in S_0(\mathbb{R}^n)$ for which

$$\|f\|_{q,\infty} \equiv \sup_{t>0} t^{1/q} f^*(t) < \infty.$$

We also set $L^{\infty,\infty}(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n)$. When $0 < p \le s \le \infty$, $0 < q < \infty$, there holds the inequality (see [3, Proposition 4.2])

$$\|f\|_{q,s} \le c\|f\|_{q,p}, \tag{2.24}$$

where c only depends on p, s, and q. The last range of the parameters for which we define the Lorentz space is when $q = \infty$, $0 < p < \infty$. Then we let $L^{\infty,p}(\mathbb{R}^n)$ consist of all $f \in S_0(\mathbb{R}^n)$ such that (see [1], [24])

$$\|f\|_{\infty,p} \equiv \Big(\int_0^\infty [f^{**}(t) - f^*(t)]^p \frac{dt}{t} \Big)^{1/p} < \infty.$$

If $1 \le q, p < \infty$ and $f \in L^{q,p}(\mathbb{R}^n)$, then by (2.24)

$$f^*(t) = O(t^{-1/q}), \tag{2.25}$$

as $t \to 0+$ and as $t \to \infty$.

For any function $f \in S_0(\mathbb{R}^n)$, we will use the notation

$$\Delta_f(t) \equiv f^*(t) - f^*(2t),$$

for $t > 0$. This difference will play an important role in the sequel. We now define the modified Lorentz norm, denoted $\|\cdot\|^*_{q,p}$, which will be equivalent to the Lorentz norm of f but which is defined in terms of Δ_f. This modified Lorentz norm was introduced in [19]. When $1 \le q < \infty$ we set

$$\|f\|^*_{q,p} = \begin{cases} \Big(\displaystyle\int_0^\infty [t^{1/q}\Delta_f(t)]^p \frac{dt}{t}\Big)^{1/p}, & 1 \le p < \infty \\ \sup\limits_{t>0} t^{1/q}\Delta_f(t), & p = \infty. \end{cases}$$

Clearly, $\|f\|^*_{q,p} \le \|f\|_{q,p}$. To show that $\|f\|_{q,p} \le c\|f\|^*_{q,p}$ for some constant c, we use the inequality:

$$f^*(2t) \le \frac{1}{\ln 2}\int_t^\infty \Delta_f(u)\frac{du}{u}. \tag{2.26}$$

To verify that (2.26) holds, fix $t > 0$ and take $N > 2t$. Then

$$\int_t^N \Delta_f(u)\frac{du}{u} = \int_t^{2t} f^*(u)\frac{du}{u} - \int_N^{2N} f^*(u)\frac{du}{u} \ge f^*(2t)\ln 2 - f^*(N).$$

Now (2.26) follows if we let N tend to ∞ and use (2.12). By (2.26), Hardy's inequality (2.2), and (2.4) we obtain that

$$\|f\|_{q,p} \le \frac{2^{1/q}q}{\ln 2}\|f\|^*_{q,p}, \quad 1 \le q < \infty, 1 \le p \le \infty. \tag{2.27}$$

We define the modified Lorentz norm also when $q = \infty$ and $1 \le p < \infty$. In this case we set

$$\|f\|^*_{\infty,p} \equiv \Big(\int_0^\infty (\Delta_f(t))^p \frac{dt}{t}\Big)^{1/p}.$$

To prove the equivalence between $\|\cdot\|_{\infty,p}$ and $\|\cdot\|^*_{\infty,p}$ we will use the following inequalities

$$\frac{1}{2}\Delta_f\big(\frac{t}{2}\big) \le f^{**}(t) - f^*(t) \le \frac{2}{t}\int_0^t \Delta_f(u)du. \tag{2.28}$$

The left inequality in (2.28) is immediate,

$$f^{**}(t) - f^*(t) \ge \frac{1}{t}\int_0^{t/2} [f^*(u) - f^*(t)]du \ge \frac{1}{2}[f^*(t/2) - f^*(t)].$$

To prove the right inequality in (2.28) we take $0 < \varepsilon < t/2$ and observe that

$$2\int_\varepsilon^t \Delta_f(u)du \geq \int_\varepsilon^t f^*(u)du - \int_t^{2t} f^*(u)du \geq \int_\varepsilon^t f^*(u)du - tf^*(t).$$

The left inequality in (2.28) immediately implies that $\|f\|_{\infty,p}^* \leq 2\|f\|_{\infty,p}$. By the right inequality in (2.28) and Hardy's inequality (2.3) we have that

$$\|f\|_{\infty,p} \leq 2\|f\|_{\infty,p}^*. \tag{2.29}$$

3. The space Λ^σ

In this section we consider a one parameter family of spaces denoted Λ^σ. These spaces were introduced in [19]. Let $\sigma \in \mathbb{R}$. Recall from Section 1 that a function $f \in S_0(\mathbb{R})$ belongs to Λ^σ if

$$\|f\|_{\Lambda^\sigma} \equiv \sup_{t>0} t^\sigma \Delta_f(t) < \infty.$$

Propositions 3.1, 3.2, and 3.3 below state embeddings of Λ^σ for different values of σ. These results where obtained in [19]. Theorems 3.6 and 3.8 show how functions in Λ^σ can be approximated by simple functions (defined below) and by continuous functions with compact support.

First we determine to what extent $\|\cdot\|_{\Lambda^\sigma}$ satisfies the properties of a norm. We have $\|f\|_{\Lambda^\sigma} \geq 0$ for all $f \in \Lambda^\sigma$ since Δ_f is non-negative. Moreover, $\Delta_f = 0$ on $\mathbb{R}_+$ if and only if $f^* = 0$ on $\mathbb{R}_+$. Therefore $\|f\|_{\Lambda^\sigma} = 0$ if and only if $f = 0$ a.e. on $\mathbb{R}^n$. Furthermore, by (2.10), we have $\|\lambda f\|_{\Lambda^\sigma} = |\lambda|\|f\|_{\Lambda^\sigma}$, for all $\lambda \in \mathbb{R}$. However, we will show that if $\sigma \leq 0$ then there is no constant c such that the "triangle inequality",

$$\|f+g\|_{\Lambda^\sigma} \leq c(\|f\|_{\Lambda^\sigma} + \|g\|_{\Lambda^\sigma}), \tag{3.1}$$

holds for all $f, g \in \Lambda^\sigma$. Set $f_n = n\chi_{(0,1]}$, $h_2 = \chi_{(1,2]}$, and $h_{n+1} = h_n + \chi_{(1,2^n]}$, $n \geq 2$. Using induction we prove that

$$\Delta_{f_n} = n\chi_{(1/2,1]},$$

$$\Delta_{f_n+h_n} = \chi_{(1/2,2^{n-1}]},$$

and

$$\Delta_{h_n} = \sum_{k=1}^{n-1} \chi_{((2^k-1)/2,2^k-1]}.$$

So, if $\alpha \geq 0$, then $\|f_n\|_{\Lambda^{-\alpha}} = n2^\alpha$, $\|f_n + h_n\|_{\Lambda^{-\alpha}} = 2^\alpha$, and $\|h_n\|_{\Lambda^{-\alpha}} = 2^\alpha$. Clearly there is no constant c for which

$$\|f_n\|_{\Lambda^{-\alpha}} \leq c(\|f_n + h_n\|_{\Lambda^{-\alpha}} + \|h_n\|_{\Lambda^{-\alpha}}),$$

for all $n \geq 2$, so (3.1) is not satisfied when $\sigma \leq 0$. For $\sigma > 0$, (3.1) holds with $c = 4^\sigma/(\sigma \ln 2)$. To prove this we will use the following proposition.

Proposition 3.1. *Let $\sigma > 0$ and set $r = 1/\sigma$. Then $\Lambda^\sigma = L^{r,\infty}(\mathbb{R})$ and*

$$\|f\|_{\Lambda^\sigma} \le \|f\|_{r,\infty} \le \frac{2^\sigma}{\sigma \ln 2}\|f\|_{\Lambda^\sigma}. \tag{3.2}$$

Proof. The first inequality in (3.2) is immediate for all $f \in L^{r,\infty}(\mathbb{R})$. Let $f \in \Lambda^\sigma(\mathbb{R})$. By (2.26),

$$\|f\|_{r,\infty} \le \frac{2^\sigma}{\ln 2} \sup_{t>0} t^\sigma \int_t^\infty \Delta_f(u)\frac{du}{u}.$$

The second inequality in (3.2) now follows by inequality (2.4). □

Let $\sigma > 0$ and set $r = 1/\sigma$. Suppose $f, g \in L^{r,\infty}(\mathbb{R})$. By (2.11) we have

$$\|f+g\|_{r,\infty} \le \sup_{t>0} t^{1/r}(f^*(t/2) + g^*(t/2)) \le 2^{1/r}(\|f\|_{r,\infty} + \|g\|_{r,\infty}).$$

This inequality and Proposition 3.1 now give

$$\|f+g\|_{\Lambda^\sigma} \le \frac{4^\sigma}{\sigma \ln 2}(\|f\|_{\Lambda^\sigma} + \|g\|_{\Lambda^\sigma}), \tag{3.3}$$

for all $f, g \in \Lambda^\sigma$, i.e. (3.1) holds when $\sigma > 0$.

Define the space W, called weak-L^∞, as the class of all $f \in S_0(\mathbb{R})$ such that

$$\|f\|_W = \sup_{t>0}[f^{**}(t) - f^*(t)] < \infty.$$

This space was introduced in [2] by Bennett, DeVore, and Sharpely.

Proposition 3.2. *The spaces Λ^0 and W coincide and*

$$\frac{1}{2}\|f\|_{\Lambda^0} \le \|f\|_W \le 2\|f\|_{\Lambda^0}.$$

Proof. Let $f \in W$. The first inequality follows immediately from the first inequality in (2.28). Therefore $W \subset \Lambda^0$. Suppose $f \in \Lambda^0$. Fix $t > 0$. By the second inequality in (2.28) we have

$$f^{**}(t) - f^*(t) \le \frac{2}{t}\int_0^t \Delta_f(u)du \le 2\|f\|_{\Lambda^0}.$$

The second inequality now follows. This gives $\Lambda^0 \subset W$. □

Recall that $C(\mathbb{R})$ denotes the class of all bounded continuous functions on $\mathbb{R}$. For $0 < \alpha \le 1$ we define $\operatorname{Lip}\alpha$ to be the space of all functions $f \in C(\mathbb{R})$ for which

$$\|f\|_{\operatorname{Lip}\alpha} \equiv \sup_{\delta>0} \frac{\omega(f;\delta)}{\delta^\alpha} < \infty. \tag{3.4}$$

Proposition 3.3. *Let $0 < \alpha \le 1$. If $f \in S_0(\mathbb{R}) \cap \operatorname{Lip}\alpha$ then $f \in \Lambda^{-\alpha}$ and*

$$\|f\|_{\Lambda^{-\alpha}} \le \|f\|_{\operatorname{Lip}\alpha},$$

Proof. Fix $t > 0$. By inequality (2.20) in Remark 2.8 we have

$$\Delta_f(t) \leq \omega(f^*; t) \leq \omega(f; t)$$

and then

$$t^{-\alpha} \Delta_f(t) \leq \|f\|_{\mathrm{Lip}\,\alpha}.$$

Taking supremum over all $t > 0$ we obtain the inequality stated in the proposition. □

The next proposition gives an equivalent definition of the space Λ^σ, when $\sigma < 0$.

Proposition 3.4. *Let $\sigma < 0$. Then $f \in \Lambda^\sigma$ if and only if there exists a constant A such that for all $t > 0$*

$$\|f\|_\infty \leq f^*(t) + At^{-\sigma}. \tag{3.5}$$

Moreover, if $A_0 \geq 0$ is the smallest constant such that inequality (3.5) *holds for all $t > 0$, then*

$$(1 - 2^\sigma) A_0 \leq \|f\|_{\Lambda^\sigma} \leq 2^{-\sigma} A_0. \tag{3.6}$$

Proof. Suppose (3.5) holds. Then

$$\Delta_f(t) \leq \|f\|_\infty - f^*(2t) \leq (2t)^{-\sigma} A,$$

and thus

$$\|f\|_{\Lambda^\sigma} \leq 2^{-\sigma} A.$$

So, $f \in \Lambda^\sigma$ and the right-hand side inequality in (3.6) follows. Let now $f \in \Lambda^\sigma$. For any $N > 0$,

$$f^*(2^{-N} t) - f^*(t) = \sum_{k=1}^{N} \Delta_f(2^{-k} t) \leq t^{-\sigma} \|f\|_{\Lambda^\sigma} \sum_{k=1}^{N} 2^{k\sigma}.$$

Let $N \to \infty$. By (2.13) we obtain

$$\|f\|_\infty \leq f^*(t) + t^{-\sigma} \frac{\|f\|_{\Lambda^\sigma}}{1 - 2^\sigma}.$$

Thus, (3.5) holds.

If $A_0 = 0$, then (3.6) follows immediately. Suppose $A_0 > 0$ and fix $\varepsilon \in (0, A_0)$. By definition of A_0 there exists $t_0 > 0$ such that

$$\|f\|_\infty > f^*(t_0) + (A_0 - \varepsilon) t_0^{-\sigma}.$$

Take $N > 0$ such that $f^*(2^{-N} t_0) > \|f\|_\infty - \varepsilon$. We then have

$$A_0 - \varepsilon < t_0^\sigma (f^*(2^{-N} t_0) - f^*(t_0) + \varepsilon) = \varepsilon t_0^\sigma + \|f\|_{\Lambda^\sigma} \sum_{k=1}^{N} 2^{k\sigma}.$$

Since $\varepsilon \in (0, A_0)$ was arbitrary, it follows that

$$A_0 \leq \|f\|_{\Lambda^\sigma} \sum_{k=1}^{N} 2^{k\sigma}$$

which implies the left-hand side inequality in (3.6). □

Corollary 3.5. *Let $\sigma < 0$. Then $\Lambda^\sigma \subset L^\infty(\mathbb{R})$.*

Proof. Let $f \in \Lambda^\sigma$. By Proposition 3.4, there exists a constant $A > 0$ for which inequality (3.5) holds. This implies that $f \in L^\infty(\mathbb{R})$. □

Proposition 3.6. *Let $\sigma > 0$. Then the space Λ^σ is not separable.*

Proof. We need only find an uncountable subfamily $S \subset \Lambda^\sigma$ with the property that

$$\|f - g\|_{\Lambda^\sigma} \geq 1, \tag{3.7}$$

for all functions $f, g \in S$ such that f is not equivalent to g. Indeed, suppose $S = \{f_\xi\}_{\xi \in I}$ is such a family and let $\{g_n\}_{n=1}^\infty$ be any countable sequence in Λ^σ. Set

$$r = \frac{\sigma \ln 2}{2^{2\sigma+1}}.$$

Then the balls $B_\xi = B(f_\xi, r)$ are pairwise disjoint. Indeed, suppose $g \in B_\xi \cap B_\eta$ for some $\xi, \eta \in I$, $\xi \neq \eta$. By inequality (3.3) we would then have

$$\|f_\xi - f_\eta\|_{\Lambda^\sigma} \leq \frac{4^\sigma}{\sigma \ln 2}(\|f_\xi - g\|_{\Lambda^\sigma} + \|g - f_\eta\|_{\Lambda^\sigma}) < 1,$$

which is a contradiction. Since S is uncountable there must then exists balls B_ξ which does not contain any of the functions g_n. Therefore the sequence $\{g_n\}_{n=1}^\infty$ can not be dense in Λ^σ.

To construct such a family $S \subset \Lambda^\sigma$ we set

$$f_\xi(t) = (t - \xi)^{-\sigma}\chi_{(\xi,1]}(t),$$

for $0 < \xi < 1$ and $t \neq \xi$. Then

$$f_\xi^*(t) = t^{-\sigma}\chi_{(0,1-\xi]}$$

so that

$$\sup_{t>0} t^\sigma \Delta_{f_\xi}(t) \leq \sup_{t>0} t^\sigma f_\xi^*(t) = 1,$$

and therefore $f_\xi \in \Lambda^\sigma$. Let $0 < \xi < \eta < 1$. By (3.2)

$$\frac{2^{\sigma+1}}{\sigma}\|f_\xi - f_\eta\|_{\Lambda^\sigma} \geq \sup_{t>0} t^\sigma (f_\xi - f_\eta)^*(t) \geq \sup_{t>0} t^\sigma \big((f_\xi - f_\eta)\chi_{(0,\eta]}\big)^*(t) = 1,$$

so we can let S consist of the functions $2^{\sigma+1}\sigma^{-1}f_\xi$, $\xi \in (0,1)$. □

Observe that if $\sigma \le 0$, it makes no sense to speak about approximation "in the norm" $\|\cdot\|_{\Lambda^\sigma}$. Indeed, if $\sigma < 0$ set $\alpha \equiv -\sigma$ and $f_n = \chi_{(0,n]}$. Then

$$\|f_n\|_{\Lambda^{-\alpha}} = \sup_{t>0} t^{-\alpha}\chi_{(n/2,n]}(t) = \Big(\frac{2}{n}\Big)^\alpha.$$

So, $f_n \in \Lambda^{-\alpha}$ and $\|f_n\|_{\Lambda^{-\alpha}} \to 0$, as $n \to \infty$. If $\sigma = 0$ we set

$$g_n(x) = \Big(1 - \frac{\log_2(1+x)}{n}\Big)\chi_{(0,2^n-1]}(x).$$

For $0 < t \le (2^n-1)/2$ we have

$$\Delta_{g_n}(t) = \frac{1}{n}\log_2\Big(\frac{1+2t}{1+t}\Big) < \frac{1}{n}$$

and for $((2^n-1)/2) < t \le 2^n - 1$ we have

$$\Delta_{g_n}(t) = g_n^*(t) \le g_n((2^n-1)/2) < 1 - \frac{1}{n}\log_2(2^{n-1}) = \frac{1}{n}.$$

So, $\|g_n\|_{\Lambda^0} < 1/n$. Thus $g_n \in \Lambda^0$ and $\|g_n\|_{\Lambda^0} \to 0$, as $n \to \infty$. These examples shows that even if $\|f\|_{\Lambda^\sigma}$ is small, it can still happen that f is "big".

Let w be a positive continuous function on $\mathbb{R}_+$. We say that a function $f \in S_0(\mathbb{R})$ belongs to the space $\Lambda(w)$ if

$$\|f\|_{\Lambda(w)} \equiv \sup_{t>0} w(t)\Delta_f(t) < \infty.$$

If $w(t) = t^\sigma$, then $\|\cdot\|_{\Lambda(w)} = \|\cdot\|_{\Lambda^\sigma}$. We will give two theorems on how a function $f \in \Lambda(w)$ can be approximated by a function g. The approximation will not be in the sence that $\|f-g\|_{\Lambda(w)}$ is small. As the above example shows, this does not imply that g is "close" to f. Instead we will ensure that g approximates f in measure and at the same time that $\|g\|_{\Lambda(w)}$ approximates $\|f\|_{\Lambda(w)}$. Observe that these results are similar to those obtained for functions of bounded variation (see [32, p. 225]). There is no additional complication of the proofs resulting from the replacement of Λ^σ by $\Lambda(w)$.

By a simple function we mean a real-valued measurable and everywhere finite function f on $\mathbb{R}$ which takes only finitely many values and which has the property that for every $c \ne 0$, the level set $\{x \in \mathbb{R} : f(x) = c\}$ has finite measure. It is well known that bounded measurable functions can be uniformly approximated by simple functions. We will use this property in the following form.

Lemma 3.7. *Let $f \in S_0(\mathbb{R})$. Suppose that $|f(x)| \le M$ for all $x \in \mathbb{R}$,*

$$|\{x : |f(x)| = M\}| > 0, \tag{3.8}$$

and

$$|\{x : f(x) \neq 0\}| < \infty. \tag{3.9}$$

Then for every $\varepsilon > 0$ *there exists a simple function* g *such that:*

(i) $|g(x)| \leq M$*, for all* $x \in \mathbb{R}$*;*
(ii) $\{x : |g(x)| = M\} = \{x : |f(x)| = M\}$*;*
(iii) $\{x : g(x) \neq 0\} = \{x : f(x) \neq 0\}$*;*
(iv) $|f(x) - g(x)| < \varepsilon$*, for all* $x \in \mathbb{R}$*.*

Proof. Fix $\varepsilon > 0$. We can assume that $M/\varepsilon \in \mathbb{N}$. Set $g(x) = f(x)$ if $f(x) = 0$ or $|f(x)| = M$. Let $E = \{x : 0 < |f(x)| < M\}$ and set

$$g(x) = \Big(\Big[\frac{f(x)}{\varepsilon}\Big] + \frac{1}{2}\Big)\varepsilon,$$

for all $x \in E$ (here $[a]$ denotes the integral part of a number a). Then for all $x \in E$

$$f(x) - \frac{\varepsilon}{2} < g(x) \leq f(x) + \frac{\varepsilon}{2}.$$

This implies statement (iv). Furthermore, $-M < f(x) < M$ on E and therefore

$$-\frac{M}{\varepsilon} \leq \Big[\frac{f(x)}{\varepsilon}\Big] \leq \frac{M}{\varepsilon} - 1,$$

for all $x \in E$. It follows that

$$-M + \frac{\varepsilon}{2} \leq g(x) \leq M - \frac{\varepsilon}{2}$$

on E. Thus $|g(x)| < M$ on E, and statements (i) and (ii) hold. Finally, $g(x) \neq 0$ on E which implies statement (iv). $\square$

Theorem 3.8. *Let* $f \in \Lambda(w)$*. For every* $\varepsilon > 0$ *there exists a simple function* g *on* $\mathbb{R}$ *which satisfies:*

(i) $|\{x \in \mathbb{R} : |f(x) - g(x)| > \varepsilon\}| < \varepsilon$*;*
(ii) $\big|\|f\|_{\Lambda(w)} - \|g\|_{\Lambda(w)}\big| < \varepsilon$*.*

Proof. We can assume that $\|f\|_\infty > 0$. Then we have $\|f\|_{\Lambda(w)} > 0$. Fix $0 < \varepsilon < \min(\|f\|_{\Lambda(w)}, \|f\|_\infty)$. We will construct a function f_1 that approximates f and which has certain good properties that allow us to approximate it with a simple function g. To construct f_1 we first define the function f_0 as follows. Take $t_* > 0$ such that

$$|w(t_*)\Delta_f(t_*) - \|f\|_{\Lambda(w)}| < \frac{\varepsilon}{4}. \tag{3.10}$$

Take $t_0 \in (0, \min(t_*, \varepsilon/2))$ and define f_0 as

$$f_0(x) = \begin{cases} f^*(t_0), & f(x) > f^*(t_0) \\ f(x), & -f^*(t_0) \leq f(x) \leq f^*(t_0) \\ -f^*(t_0), & f(x) < -f^*(t_0). \end{cases}$$

By (2.12), there exists $t_1 > 2t_*$ such that $\lambda \equiv f_0^*(t_1) < \min(\varepsilon/2, f^*(t_0))$. Define f_1 as

$$f_1(x) = \begin{cases} f_0(x) - \lambda, & f_0(x) > \lambda \\ 0, & -\lambda \le f_0(x) \le \lambda \\ f_0(x) + \lambda, & f_0(x) < -\lambda. \end{cases}$$

We will show that f_1 approximates f. If $f(x) = f_0(x)$ then $|f(x) - f_1(x)| = |f_0(x) - f_1(x)| \le \lambda < \varepsilon/2$, so

$$|\{x : |f(x) - f_1(x)| > \frac{\varepsilon}{2}\}| \le |\{x : f(x) \ne f_0(x)\}| =$$

$$= |\{x : |f(x)| > f^*(t_0)\}| \le t_0 \le \frac{\varepsilon}{2}, \tag{3.11}$$

where the second inequality holds by (2.8). By considering the three cases $t \in (0, t_0/2]$, $t \in (t_0/2, t_0]$, and $t \in (t_0, \infty)$ one can verify that $\Delta_{f_0}(t) \le \Delta_f(t)$, for all $t > 0$. Moreover, by considering the three cases $t \in (0, t_1/2]$, $t \in (t_1/2, t_1]$, and $t \in (t_1, \infty)$ one can also verify that $\Delta_{f_1}(t) \le \Delta_{f_0}(t)$ for all $t > 0$. Thus

$$\|f_1\|_{\Lambda(w)} \le \|f\|_{\Lambda(w)}. \tag{3.12}$$

Observe that $f_0^*(t) = \min(f^*(t), f^*(t_0))$. Since $t_0 \le t_*$ we then have

$$f_0^*(t_*) = f^*(t_*) \text{ and } f_0^*(2t_*) = f^*(2t_*). \tag{3.13}$$

We also note that $f_1^*(t) = \max(0, f_0^*(t) - \lambda)$. Since $t_1 \ge 2t_*$ we have $f_0^*(2t_*) \ge f_0^*(t_1) = \lambda$ and then

$$f_1^*(t_*) = f_0(t_*) - \lambda \text{ and } f_1^*(2t_*) = f_0(2t_*) - \lambda.$$

By these two equalities and (3.13) we see that

$$\Delta_{f_1}(t_*) = \Delta_f(t_*). \tag{3.14}$$

By (3.14) and (3.10) we obtain

$$\|f\|_{\Lambda(w)} \le \frac{\varepsilon}{4} + \|f_1\|_{\Lambda(w)}. \tag{3.15}$$

It remains only to approximate f_1 by a simple function. First we observe that $\|f_1\|_\infty < \infty$. Moreover,

$$m \equiv |\{x : f_1(x) = \|f_1\|_\infty\}| > 0. \tag{3.16}$$

Indeed, since $\lambda < f^*(t_0)$ we see that

$$\{x : |f_1(x)| = \|f_1\|_\infty\} = \{x : |f_0(x)| = f^*(t_0)\} = \{x : |f(x)| \ge f^*(t_0)\}.$$

So (3.16) holds by (2.9). We also note that by (2.8),

$$M \equiv |\{x : f_1(x) \ne 0\}| = |\{x : |f_0(x)| > \lambda\} \le t_1 < \infty. \tag{3.17}$$

Fix $\varepsilon_1 \in (0, \varepsilon/2)$ such that for all $t \in [m/2, M]$,

$$8\varepsilon_1 w(t) < \varepsilon. \tag{3.18}$$

By Lemma 3.7 there exists a simple function g on $\mathbb{R}$ such that

$$|f_1(x) - g(x)| \leq \varepsilon_1 \tag{3.19}$$

for a.e. $x \in \mathbb{R}$,

$$\{x : |g(x)| = \|g\|_\infty\} = \{x : |f_1(x)| = \|f_1\|_\infty\} \tag{3.20}$$

and

$$\{x : g(x) \neq 0\} = \{x : f_1(x) \neq 0\}. \tag{3.21}$$

By the triangle inequality

$$|\{x : |f(x) - g(x)| > \varepsilon\}| \leq$$
$$\leq |\{x : |f(x) - f_1(x)| > \frac{\varepsilon}{2}\}| + |\{x : |f_1(x) - g(x)| > \frac{\varepsilon}{2}\}| \leq \frac{\varepsilon}{2},$$

where the last inequality holds by (3.11) and (3.19). Thus, statement (i) is true. According to (3.19) it holds that

$$g(x) - \varepsilon_1 \leq f_1(x) \leq g(x) + \varepsilon_1,$$

for a.e. $x \in \mathbb{R}$. It follows that

$$g^*(t) - \varepsilon_1 \leq f_1^*(t) \leq g^*(t) + \varepsilon_1,$$

which in turn implies that

$$|\Delta_{f_1}(t) - \Delta_g(t)| \leq 2\varepsilon_1, \tag{3.22}$$

for all $t > 0$.

By (3.12), (3.10), and (3.14) we have

$$\|f_1\|_{\Lambda(w)} \leq \|f\|_{\Lambda(w)} \leq \frac{\varepsilon}{4} + w(t_*)\Delta_{f_1}(t_*).$$

Applying (3.22) gives

$$\|f_1\|_{\Lambda(w)} \leq \frac{\varepsilon}{4} + 2\varepsilon_1 w(t_*) + w(t_*)\Delta_g(t_*) \leq \frac{\varepsilon}{4} + 2\varepsilon_1 w(t_*) + \|g\|_{\Lambda(w)}. \tag{3.23}$$

We want to apply (3.18) to estimate $2\varepsilon_1 w(t_*)$, so we must check that $t* \in [m/2, M]$. It is clear that $\Delta_f(t_*) > 0$, indeed if $\Delta_f(t_*) = 0$ then by (3.10) we would have $\|f\|_{\Lambda(w)} < \varepsilon/4$ which contradicts our choice of ε. So by (3.14) we know that $\Delta_{f_1}(t_*) > 0$. However, by (3.16) and (3.17) it holds that $\Delta_{f_1}(t) = 0$ for all $t \in (0, m/2) \cup (M, \infty)$. Thus we conclude that $t_* \in [m/2, M]$, so (3.18) holds for $t = t_*$, i.e. we have

$$8\varepsilon_1 w(t_*) \leq \varepsilon$$

This inequality and (3.23) gives

$$\|f_1\|_{\Lambda(w)} \leq \frac{\varepsilon}{2} + \|g\|_{\Lambda(w)}.$$

By (3.20) and (3.16) we have $\Delta_g = 0$ on $(0, m/2)$ and by (3.21) and (3.17) we know that $\Delta_g = 0$ on (M, ∞). Therefore

$$\|g\|_{\Lambda(w)} = \sup\{w(t)\Delta_g(t) : \frac{m}{2} \le t \le M\} \le \frac{\varepsilon}{4} + \|f_1\|_{\Lambda(w)},$$

where the inequality holds by (3.22) and (3.18). By (3.12), (3.15), and the two preceding inequalities, we obtain (ii). □

We let $C_0(\mathbb{R})$ denote the class of all continuous functions on $\mathbb{R}$ with compact support.

Lemma 3.9. *Let f be a simple function on $\mathbb{R}$. For every $\delta > 0$ there exists a function $g \in C_0(\mathbb{R})$ such that:*

(i) $|\{x \in \mathbb{R} : f(x) \ne g(x)\}| < \delta$;
(ii) $\|g\|_\infty = \|f\|_\infty$.

Proof. Since f is a simple function we know that $\|f\|_\infty < \infty$ and

$$M \equiv |\{x \in \mathbb{R} : f(x) \ne 0\}| < \infty. \tag{3.24}$$

We can assume that f is not equivalent to 0 and then

$$m \equiv |\{x \in \mathbb{R} : |f(x)| = \|f\|_\infty\}| > 0. \tag{3.25}$$

Fix $\delta \in (0, m)$. By (3.24) there exists $N > 0$ such that

$$|\{x \in \mathbb{R} : f(x) \ne 0,\ |x| > N\}| < \frac{\delta}{4}. \tag{3.26}$$

Simple functions are finite and measurable. Lusin's theorem then ensures the existence of a closed set $F \subset [-N, N]$ such that f is continuous relative to F and

$$|[-N, N] \setminus F| < \frac{\delta}{4}. \tag{3.27}$$

So, by the extension theorem there exists a function $g \in C_0(\mathbb{R})$ such that

$$g(x) = f(x), \tag{3.28}$$

for all $x \in F$,

$$g(x) = 0, \tag{3.29}$$

if $|x| > N + \delta/4$, and

$$\|g\|_\infty \le \|f\|_\infty. \tag{3.30}$$

By (3.28) and (3.29) we have the inclusion

$$\{x \in \mathbb{R} : f(x) \ne g(x)\} \subset$$

$$([-N, N] \setminus F) \cup [-N - \frac{\delta}{4}, -N] \cup [N, N + \frac{\delta}{4}] \cup \{x \in \mathbb{R} : f(x) \ne 0,\ |x| > N\}.$$

By this inclusion and inequalities (3.27) and (3.26) we obtain statement (i). Since $\delta < m$, statement (i) and (3.25) implies that g attains the value $\|f\|_\infty$

on some set of positive measure. Thus, $\|f\|_\infty \le \|g\|_\infty$ which together with (3.30) give statement (ii). $\square$

Theorem 3.10. *Let $f \in \Lambda(w)$. For every $\varepsilon > 0$ there exists a function $g \in C_0(\mathbb{R})$ such that:*

(i) $|\{x \in \mathbb{R} : |f(x) - g(x)| > \varepsilon\}| < \varepsilon$;
(ii) $\big|\|f\|_{\Lambda(w)} - \|g\|_{\Lambda(w)}\big| < \varepsilon$.

Proof. Fix $\varepsilon > 0$. By Theorem 3.8 we can assume that f is a simple function. Let $c_1 > \cdots > c_N > 0$ be the positive values of $|f|$. We may assume that $c_1 = 1$. For $k = 1, \dots, N$ we put

$$E_k = \{x \in \mathbb{R} : f(x) = c_k\}.$$

We can assume that $|E_k| > 0$ for all $k = 1, \dots, N$. Indeed, if there is some k for which $|E_k| = 0$ then we replace the value of f by 0 on E_k. This does not change the value of f^* at any point so $\|f\|_{\Lambda(w)}$ remains the same. Put $t_0 = 0$ and

$$t_k = \sum_{i=1}^{k} |E_i|,$$

for all $k = 1, \dots, N$. Then $0 < t_1 < t_2 < \cdots < t_N$. Choose $\delta_1 \in (0, \varepsilon)$ such that

$$8\delta_1 < \min\{t_k - t_{k-1} : 1 \le k \le N\} \tag{3.31}$$

and the condition

$$4|w(t') - w(t'')| < \varepsilon, \tag{3.32}$$

holds for all $t', t'' \in [t_1/8, t_N]$ such that $|t' - t''| < \delta_1$ (this is possible since w is uniformly continuous on $[t_1/8, t_N]$).

First we will show that we can assume that $2t_k \ne t_l$ for all $1 \le k < l \le N$. We prove this by constructing a simple function h which has this property and which approximates f. Define

$$\eta' \equiv \frac{1}{2} \min\{|2t_k - t_l| : 1 \le k, l \le N,\ 2t_k \ne t_l\},$$

and set $\eta \equiv \min(\delta_1, \eta')$. Choose in E_1 any measurable subset of measure η and replace the value of f by 0 on this subset. Denote the new function by h. We then have

$$h^*(t) = f^*(t + \eta), \tag{3.33}$$

for all $t > 0$. Let $t'_0 \equiv 0$ and $t'_k \equiv t_k - \eta$, $k = 1, \dots, N$. The intervals of constancy of h^* are $(t'_{k-1}, t'_k]$, $k = 1, \dots, N$. Furthermore, for all $1 \le k, l \le N$ the numbers t'_k and t'_l satisfy

$$|2t'_k - t'_l| \ge \eta. \tag{3.34}$$

Indeed, fix $1 \leq k, l \leq N$. By the definition of t'_k and t'_l we have

$$2t'_k - t'_l = 2t_k - t_l - \eta, \tag{3.35}$$

so if $2t_k = t_l$ then (3.34) holds. On the other hand, if $2t_k \neq t_l$ then by the definition of η

$$0 < \eta \leq \frac{1}{2}|2t_k - t_l|.$$

From this and (3.35) we get (3.34).

Next we will show that

$$\|f\|_{\Lambda(w)} - \varepsilon \leq \|h\|_{\Lambda(w)} \leq \|f\|_{\Lambda(w)} + \varepsilon. \tag{3.36}$$

We start with the proof of the right-hand side of (3.36). Fix $t \in [t_1/4, t_N]$. By (3.33) it holds that

$$\Delta_h(t) = f^*(t+\eta) - f^*(2t+\eta) \leq \Delta_f(t+\eta).$$

From this and the fact that $\Delta_h = 0$ on $(0, t_1/4) \cup (t_N - \eta, \infty)$ we see that

$$\|h\|_{\Lambda(w)} \leq \sup\{w(s)\Delta_f(s+\eta) : \frac{t_1}{4} \leq s \leq t_N - \eta\} \tag{3.37}$$

Since $\eta \leq \delta_1$ we know from (3.32) that

$$w(s)\Delta_f(s+\eta) \leq$$
$$\leq w(s+\eta)\Delta_f(s+\eta) + \frac{\varepsilon}{4}\Delta_f(s+\eta) \leq \|f\|_{\Lambda(w)} + \frac{\varepsilon}{4}\|f\|_\infty,$$

for all $s \in [t_1/4, t_N - \eta]$. By this, (3.37), and the assumption that $\|f\|_\infty = 1$ we now obtain the right-hand side inequality in (3.36).

To obtain the left-hand side inequality in (3.36) we will first show that

$$\Delta_f(t) \leq \max(\Delta_h(t-\eta), \Delta_h(t-\eta/2)), \tag{3.38}$$

for all $t \in [t_1/4, t_N]$. To prove this estimate we will consider the three cases $t \in [t_1/4, t_N/2]$, $t \in (t_N/2, t_N/2 + \eta/2]$, and $t \in (t_N/2 + \eta/2, t_N]$. Suppose first that $t \in (t_N/2 + \eta/2, t_N]$. Then $f^*(2t) = 0$ and using (3.33) we see that also $h^*(2t - 2\eta) = 0$. Thus

$$\Delta_f(t) = f^*(t) = h^*(t-\eta) = \Delta_h(t-\eta),$$

where the second equality is (3.33). So, (3.38) holds in this case. Next we suppose that $t \in (t_N/2, t_N/2 + \eta/2]$. Take $k \in \{1, \ldots, N\}$ such that $t \in (t_{k-1}, t_k]$. Then $t \in (t_{k-1}, t_k - \eta/2]$. Indeed, if $t \in (t_k - \eta/2, t_k]$ then

$$2t \in (t_N, t_N + \eta] \cap (2t_k - \eta, 2t_k]. \tag{3.39}$$

So we would have $|2t_k - t_N| < 2\eta$, but this contradicts the definition of η (to see this, note that $2t_k \neq t_N$ by (3.39) so by definition $\eta \leq |2t_k - t_N|/2$). Thus, $t \in (t_{k-1}, t_k - \eta/2]$ and then $f^*(t) = f^*(t+\eta/2)$. From this and (3.33)

$$\Delta_f(t) = f^*(t+\eta/2) - f^*(2t) = h^*(t-\eta/2) - h^*(2t-\eta) = \Delta_h(t-\eta/2),$$

and thus (3.38) holds also in this case. The last case in the proof of (3.38) is when $t \in [t_1/4, t_N/2]$. Then there exist $k, l \in \{1, 2, \ldots, N\}$ such that $t \in (t_{k-1}, t_k]$ and $2t \in (t_{l-1}, t_l]$. We then have either

$$t \in (t_{k-1}, t_k - \frac{\eta}{2}], \tag{3.40}$$

or

$$2t \in (t_{l-1} + \eta, t_l]. \tag{3.41}$$

Indeed, suppose neither (3.40) nor (3.41) holds. Then we have

$$2t \in (2t_k - \eta, 2t_k] \cap (t_{l-1}, t_{l-1} + \eta]. \tag{3.42}$$

Therefore,

$$|2t_k - t_{l-1}| < 2\eta, \tag{3.43}$$

which contradicts the definition of η (to see this, observe that $2t_k \neq t_{l-1}$ by (3.42), so by definition $\eta \leq |2t_k - t_{l-1}|/2$). If (3.40) holds then $f^*(t) = f^*(t + \eta/2)$. Using (3.33) then gives

$$\Delta_f(t) = \Delta_h(t - \frac{\eta}{2}), \tag{3.44}$$

so (3.38) is satisfied. In the case (3.41), we have $f^*(2t) = f^*(2t - \eta)$. Applying again (3.33), we obtain

$$\Delta_f(t) = \Delta_h(t - \eta), \tag{3.45}$$

and thus (3.38) holds. The proof of (3.38) is now complete.

Since $\Delta_f = 0$ on $(0, t_1/4) \cup (t_N, \infty)$, we have

$$\|f\|_{\Lambda(w)} = \sup\{w(t)\Delta_f(t) : \frac{t_1}{4} \leq t \leq t_N\}.$$

Applying (3.38), we get

$$\|f\|_{\Lambda(w)} \leq \sup\{w(t)\max(\Delta_h(t - \eta/2), \Delta_h(t - \eta)) : \frac{t_1}{4} \leq t \leq t_N\}. \tag{3.46}$$

But by (3.32),

$$w(t)\Delta_h(t - \eta) \leq \frac{\varepsilon}{4}\|h\|_\infty + w(t - \eta)\Delta_h(t - \eta) \leq \frac{\varepsilon}{4} + \|h\|_{\Lambda(w)},$$

and similarly, $w(t)\Delta_h(t - \eta/2) \leq \varepsilon/4 + \|h\|_{\Lambda(w)}$, for all $t \in [t_1/4, t_N]$, so (3.46) implies the left hand side of inequality (3.36). The proof of (3.36) is then complete. We have now proved that we can assume that

$$2t_k \neq t_l,$$

for all $1 \leq k < l \leq N$.

We now choose $\delta \in (0, \delta_1)$ such that

$$8\delta < \min\{|2t_k - t_l| : 1 \leq k \leq l \leq N\}. \tag{3.47}$$

Since f is a simple function on $\mathbb{R}$, Lemma 3.9 ensures the existence of a function $g \in C_0(\mathbb{R})$ such that

$$\|g\|_\infty = \|f\|_\infty \tag{3.48}$$

and

$$|\{x \in \mathbb{R} : f(x) \neq g(x)\}| < \delta. \tag{3.49}$$

By this inequality we have statement (i) and the equality

$$(f-g)^*(\delta) = 0. \tag{3.50}$$

It only remains to check that also statement (ii) holds. First we will verify that

$$\|f\|_{\Lambda(w)} \leq \varepsilon + \|g\|_{\Lambda(w)}. \tag{3.51}$$

From (3.50) and the subadditivity (2.11) of the rearrangement we get that

$$f^*(t) \leq g^*(t-\delta) \text{ and } g^*(2t-2\delta) \leq f^*(2t-3\delta),$$

for all $t > 3\delta/2$. Set $\Psi(t) = f^*(t) - f^*(2t-3\delta)$, $t > 3\delta/2$. By the two preceding inequalities and (3.32) we obtain

$$w(t)\Psi(t) \leq w(t)\Delta_g(t-\delta) \leq \frac{\varepsilon}{4}\|g\|_\infty + \|g\|_{\Lambda(w)} = \frac{\varepsilon}{4} + \|g\|_{\Lambda(w)}, \tag{3.52}$$

for all $t > 3\delta/2$ (we use here that $\|g\|_\infty = 1$). To obtain (3.51) we only need to show that

$$w(t)\Delta_f(t) \leq \varepsilon + \|g\|_{\Lambda(w)}, \tag{3.53}$$

for all $t \in [t_1/4, t_N]$, since $\Delta_f = 0$ outside this intervall. To prove (3.53) we will consider the three cases $t \in [t_1/4, t_N/2]$, $t \in [t_N/2, t_N/2 + 3\delta/2]$, and $t \in [t_N/2 + 3\delta/2, t_N]$. Suppose first that $t \in [t_1/4, t_N/2]$. In this case there exists $k, l \in \{1, \ldots, N\}$ such that $t \in (t_{k-1}, t_k]$ and $2t \in (t_{l-1}, t_l]$. By choice of δ we have that either

$$t \in (t_{k-1}, t_k - 2\delta] \tag{3.54}$$

or

$$2t \in (t_{l-1} + 3\delta, t_l]. \tag{3.55}$$

Indeed, if neither (3.54) nor (3.55) holds then

$$2t \in (2t_k - 4\delta, 2t_k] \cap (t_{l-1}, t_{l-1} + 3\delta]$$

and then we would have

$$|2t_k - t_{l-1}| \leq |2t_k - 2t| + |2t - t_{l-1}| < 7\delta,$$

which contradicts the definition of η. In the case of (3.54) we have

$$\Delta_f(t) = f^*(t+2\delta) - f^*(2t) \leq f^*(t+2\delta) - f^*(2t+\delta) = \Psi(t+2\delta).$$

By (3.32) and (3.52) we then get

$$w(t)\Delta_f(t) \leq (w(t+2\delta) + \frac{\varepsilon}{4})\Psi(t+2\delta) \leq \frac{\varepsilon}{4}\Psi(t+2\delta) + \frac{\varepsilon}{4} + \|g\|_{\Lambda(w)}.$$

Since Ψ is bounded by $\|f\|_\infty = 1$, inequality (3.53) follows in this case. If instead (3.55) holds, then $f^*(2t) = f^*(2t - 3\delta)$ so

$$\Delta_f(t) = \Psi(t).$$

In this case we immediately get inequality (3.53) from (3.52). Thus (3.53) holds when $t \in [t_1/4, t_N/2]$. Next we suppose that $t \in (t_N/2, t_N/2 + 3\delta/2]$. Then there exists $k \in \{1, \dots, N\}$ such that $t \in (t_{k-1}, t_k]$. As above, the definition of δ implies that $t \in (t_{k-1}, t_k - 2\delta]$. As in the case (3.54) above, we obtain (3.53). The last case is when $t \in (t_N/2+3\delta/2, t_N]$. Then $f^*(2t) = f^*(2t - 3\delta) = 0$, so

$$\Delta_f(t) = \Psi(t)$$

and then (3.53) follows directly from (3.52). We have now proved inequality (3.53) for all $t \in [t_1/4, t_N]$. This implies (3.51).

To obtain statement (ii) we must also show that

$$\|g\|_{\Lambda(w)} \leq \varepsilon + \|f\|_{\Lambda(w)}. \tag{3.56}$$

Since $\delta < t_1/8$, by (3.49) and (3.48) we see that $\Delta_g = 0$ outside the interval $[t_1/2 - \delta/2, t_N + \delta]$, so (3.56) follows if we prove

$$w(t)\Delta_g(t) \leq \varepsilon + \|f\|_{\Lambda(w)}, \tag{3.57}$$

for all $t \in [t_1/2 - \delta/2, t_N + \delta]$.

Fix $t \in [t_1/2 - \delta/2, t_N + \delta]$ and prove (3.57). By (3.50) and the subadditivity (2.11) of the rearrangement we have

$$g^*(t) \leq f^*(t - \delta) \text{ and } g^*(2t) \geq f^*(2t + \delta),$$

Set $\Phi(t) = f^*(t) - f^*(2t + 3\delta)$. By the two preceding inequalities

$$\Delta_g(t) \leq f^*(t - \delta) - f^*(2t + \delta) = \Phi(t - \delta).$$

By (3.32) it holds that $w(t) \leq \varepsilon/4 + w(t - \delta)$ so the above estimate gives

$$w(t)\Delta_g(t) \leq \frac{\varepsilon}{4} + w(t - \delta)\Phi(t - \delta)$$

(we use here that $\Phi \leq \|f\|_\infty = 1$). Thus, for all $t \in [t_1/2 - \delta/2, t_N + \delta]$ it holds that

$$w(t)\Delta_g(t) \leq \frac{\varepsilon}{4} + \sup\{w(s)\Phi(s) : s \in [t_1/4, t_N]\}. \tag{3.58}$$

So, (3.57) follows from (3.58) if we prove that

$$w(s)\Phi(s) \leq \frac{\varepsilon}{2} + \|f\|_{\Lambda(w)}, \tag{3.59}$$

for all $s \in [t_1/4, t_N]$. Fix $s \in [t_1/4, t_N]$ and prove (3.59). Suppose first that $s \in (t_N/2, t_N]$. Then $\Phi(s) = \Delta_f(s)$ and so (3.59) follows immediately. Next we suppose that $s \in [t_1/4, t_N/2]$. Take $k, l \in \{1, \dots, N\}$ such that

$s \in (t_{k-1}, t_k]$ and $2s \in (t_{l-1}, t_l]$. As above, the definition of δ gives that either

$$s \in (t_{k-1}, t_k - \frac{3\delta}{2}], \tag{3.60}$$

or

$$2s \in (t_{l-1}, t_l - 3\delta]. \tag{3.61}$$

Suppose that (3.60) is true. Then $f^*(s) = f^*(s + 3\delta/2)$, so

$$\Phi(s) = f^*(s + \frac{3\delta}{2}) - f^*(2s + 3\delta) = \Delta_f(s + \frac{3\delta}{2}).$$

By (3.32) we then have

$$w(s)\Phi(s) = \frac{\varepsilon}{4} + w(s + \frac{3\delta}{2})\Delta_f(s + \frac{3\delta}{2})$$

which implies (3.59). In the case (3.61) we have $f^*(2s) = f^*(2s+3\delta)$ so that we again obtain $\Phi(s) = \Delta_f(s)$. So in this case (3.59) follows immediately. The proof of (3.59) is now complete. As we noted above, (3.59) together with (3.58) implies (3.56). Thus, statement (ii) holds. □

Remark 3.11. *Theorems 3.8 and 3.10 fail if one replaces statement (ii) in their formulations by the statement $\|f - g\|_{\Lambda(w)} < \varepsilon$. Indeed, let $w(t) = t^\sigma$ with $\sigma > 0$ and set*

$$f(x) = \begin{cases} x^{-\sigma} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then $\|f\|_{\Lambda^\sigma} = 1 - 2^{-\sigma}$, so $f \in \Lambda^\sigma(\mathbb{R})$. Let $g \in L^\infty(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$ and set $M = \|g\|_\infty$. Then $|f(x) - g(x)| \geq x^{-\sigma} - M > 0$ for $0 < x < M^{-1/\sigma}$. Thus $(f - g)^*(t) \geq t^{-\sigma} - M$ for $0 < t < M^{-1/\sigma}$. By Proposition (3.1) we then have

$$\|f - g\|_{\Lambda^\sigma} \geq \frac{\sigma \ln 2}{2^\sigma}.$$

So there is no function $g \in L^\infty(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$ such that $\|f - g\|_{\Lambda^\sigma} < (\sigma \ln 2)/2^\sigma$.

Applying Theorem 3.10 we obtain the following result.

Theorem 3.12. *Let $f \in \Lambda(w)$. Then there exists a sequence $\{f_n\}$, $f_n \in C_0(\mathbb{R})$, such that $\{f_n\}$ converges to f in measure and $\|f_n\|_{\Lambda(w)} \to \|f\|_{\Lambda(w)}$.*

Observe that by Riesz's theorem there exists a subsequence $\{f_{n_k}\}$ converging to f a.e.

As it was pointed out above, there is an analogy between Theorems 3.8 and 3.10 and the results concerning the so called approximation in variation [32], [14, Section 9.1].

4. Mixed norm spaces

This section contains our main result - a theorem on embedding of anisotropic mixed norm spaces into Lorentz spaces. As it was already pointed out in the introduction, the first results in this direction where obtained by Gagliardo [10] and Fournier [9] (see also [5]). These results were extended by V. Kolyada [19] to more general mixed norm spaces. Our main theorem is a follow-up of the work [19]. We consider fully anisotropic mixed norm spaces. Our study is based on the methods developed in the works by V. Kolyada [19] and V. Kolyada and J. Pérez [21].

In Section 4.1 we give the lemmas that we will use and in Section 4.2 we state and prove Theorem 4.5.

4.1. **Some lemmas.** First we give a simple lemma concerning the measurability of the part of a set $E \subset R^n$ lying "above" some subset of the projection of E onto a coordinate hyperplane.

Lemma 4.1. *Let $n \geq 2$ and $1 \leq k \leq n$. Assume that $E \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^{n-1}$ are measurable in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ respectively. Then the set*

$$E' = \{x \in E : \hat{x}_k \in D\}$$

is measurable in $\mathbb{R}^n$.

Proof. It is sufficient to consider the case $k = n$. In this case

$$E' = E \cap (D \times \mathbb{R}).$$

Since the Cartesian product of two measurable sets is measurable, the measurability of E' follows. □

Next we include the statement of a lemma which was proved by V. Kolyada in [16].

Lemma 4.2. *Let ψ be a measurable non-negative function on $\mathbb{R}^n$ and let $P \subset \mathbb{R}^n$ be a measurable set with $\mathrm{mes}_n P = \mu > 0$. Then for any $0 < \tau < \mu$ the set P can be decomposed into measurable disjoint subsets E' and E'' such that $\mathrm{mes}_n E' = \tau$,*

$$\sup_{x \in E''} \psi(x) \leq \inf_{x \in E'} \psi(x),$$

and

$$\int_{E''} \psi(x) dx \leq \int_{\tau}^{\mu} \psi^*(t) dt.$$

The following lemma was proved in [21] by V. Kolyada and F. Pérez. We give the proof in order to get an explicit value of the constant in statement (iii) in this lemma.

Lemma 4.3. *Let $\phi \in L^{p,s}(\mathbb{R}_+)$ $(1 \leq p, s < \infty)$ be a non-negative non-increasing function on $\mathbb{R}_+$. Then for any $\delta \in (0, 1/p)$ there exists a continuously differentiable function ψ on $\mathbb{R}_+$ such that:*

(i) $\phi(t) \leq \psi(t)$, $t \in \mathbb{R}_+$;

(ii) $\psi(t)t^{1/p-\delta}$ *decreases and* $\psi(t)t^{1/p+\delta}$ *increases on* $\mathbb{R}_+$;

(iii) $\|\psi\|_{p,s} \leq \frac{8}{\delta^2}\|\phi\|_{p,s}$.

Proof. Define

$$\phi_1(t) \equiv 2t^{\delta-1/p}\int_{t/2}^{\infty} u^{1/p-\delta-1}\phi(u)du,$$

for $t > 0$. Since $\phi \in L^{p,s}(\mathbb{R}_+)$ and $\phi^* = \phi$, we have by (2.25) that

$$\phi(u) = O(t^{-1/p}),$$

as $u \to \infty$. Therefore the integral in the definition of ϕ_1 converges, so ϕ_1 is well defined. Moreover, since ϕ is non-increasing on $\mathbb{R}_+$ it is easy to see that

$$\phi_1(t) \geq 2t^{\delta-1/p}\phi(t)\int_{t/2}^{t} u^{1/p-\delta-1}du \geq \phi(t), \tag{4.1}$$

for all $t > 0$. Since $\delta < 1/p$, then ϕ_1 is decreasing on $\mathbb{R}_+$ and thus $\phi_1^* = \phi_1$. By this observation and Hardy's inequality (2.2) we have

$$\|\phi_1\|_{p,s} = 2^{1+\delta}\Big(\int_0^{\infty} t^{\delta s-1}\Big(\int_t^{\infty} u^{1/p-\delta-1}\phi(u)du\Big)^s dt\Big)^{1/s} \leq \frac{4}{\delta}\|\phi\|_{p,s}. \tag{4.2}$$

Thus, $\phi_1 \in L^{p,s}(\mathbb{R}_+)$, so by (2.25) we obtain that

$$\phi_1(u) = O(t^{-1/p}),$$

as $u \to 0+$ (here we again use that $\phi_1^* = \phi_1$). Therefore the function

$$\psi(t) \equiv (\delta + \frac{1}{p})t^{-1/p-\delta}\int_0^t \phi_1(u)u^{1/p+\delta-1}du$$

is well defined on $\mathbb{R}_+$, since the integral converges. The function ψ is continuously differentiable on $\mathbb{R}_+$ since ϕ_1 is continuous on $\mathbb{R}_+$. Since ϕ_1 decreases on $\mathbb{R}_+$ it holds that

$$\psi(t) \geq (\delta + \frac{1}{p})t^{-1/p-\delta}\phi_1(t)\int_0^t u^{1/p+\delta-1}du = \phi_1(t).$$

This estimate and (4.1) gives statement (i).

Clearly, $\psi(t)t^{1/p+\delta}$ increases on $\mathbb{R}_+$. To obtain statement (ii) we must also show that $\psi(t)t^{1/p-\delta}$ decreases on $\mathbb{R}_+$. We make the change of variables $u \mapsto u^{2\delta}$ to see that

$$\psi(t)t^{1/p-\delta} = \frac{\delta p + 1}{2\delta p}t^{-2\delta}\int_0^{t^{2\delta}} \eta(v^{1/(2\delta)})dv$$

for all $t > 0$, where $\eta(u) \equiv u^{1/p-\delta}\phi_1(u)$. Differentiating with respect to t in the preceding equality gives

$$\frac{d}{dt}(\psi(t)t^{1/p-\delta}) = \frac{1+\delta p}{p}\, t^{-1}\big(\eta(t) - t^{-2\delta}\int_0^{t^{2\delta}} \eta(v^{1/(2\delta)})dv\big),$$

for all $t > 0$. Clearly η is non-increasing on $\mathbb{R}_+$, so by the preceding equality

$$\frac{d}{dt}(\psi(t)t^{1/p-\delta}) \le 0,$$

and thus the function $\psi(t)t^{1/p-\delta}$ is non-increasing on $\mathbb{R}_+$. So, statement (ii) holds.

The function ψ is decreasing on $\mathbb{R}_+$ since $\psi(t)t^{1/p-\delta}$ is non-increasing and $\delta < 1/p$. Therefore $\psi^* = \psi$. By this observation and Hardy's inequality (2.3) we have

$$\|\psi\|_{p,s} = (\delta + \frac{1}{p})\Big(\int_0^\infty t^{-\delta s-1}\Big(\int_0^t u^{1/p+\delta-1}\phi_1(u)\Big)^s dt\Big)^{1/s} \le$$
$$\le (1+\frac{1}{\delta p})\|\phi_1\|_{p,s} \le \frac{2}{\delta}\|\phi_1\|_{p,s}$$

(here we again use that $\phi_1^* = \phi_1$ and that $\delta < 1/p$). From this inequality and (4.2) we obtain statement (iii). $\square$

The following lemma is similar to Lemma 2.2 in [21] and the proof is based on the same reasonings.

Lemma 4.4. *Let $n \ge 2$, $1 \le p_1, \dots, p_n, s_1, \dots, s_n < \infty$, and $\alpha_1, \dots, \alpha_n > 0$. Put*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1},\ p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1},\ \textit{and}\ s = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k s_k}\Big)^{-1}.$$

Assume that $p \le n/\alpha$. Let

$$q = \begin{cases} np/(n-\alpha p), & \alpha p < n \\ \infty, & \alpha p = n. \end{cases}$$

For all $k = 1, \dots, n$ we set $\sigma_k = 1/p_k - \alpha_k$ and assume that

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \sigma_k > 0. \tag{4.3}$$

Denote

$$R = \max_{k=1,\dots,n} \frac{r_k}{\alpha_k} \max_{k=1,\dots,n} \frac{1}{r_k} \tag{4.4}$$

and

$$c_k = \frac{\alpha_k}{r_k}, \tag{4.5}$$

$k = 1, \dots, n$. For each $k = 1, \dots, n$ we let $\phi_k \in L^{p_k,s_k}(\mathbb{R}_+)$ be a non-increasing and non-negative function on $\mathbb{R}_+$ and define

$$\eta_k(z,t) = \left(\frac{z}{t}\right)^{\sigma_k} \phi_k(z), \quad z, t > 0.$$

Set also

$$w(t) = \inf\{\max_{k=1,\dots,n} \eta_k(z_k, t) : \prod_{k=1}^{n} z_k = t^{n-1}, z_k > 0\},$$

for $t > 0$. Then there holds the inequality

$$\left(\int_0^\infty t^{s/q-1} w(t)^s dt\right)^{1/s} \le c \prod_{k=1}^{n} \|\phi_k\|_{p_k,s_k}^{\alpha/(n\alpha_k)}, \tag{4.6}$$

where

$$c = K_n \prod_{k=1}^{n} (c_k^{1/s_k} \max(R^2, p_k^2))^{\alpha/(n\alpha_k)}, \tag{4.7}$$

and K_n only depends on n.

Proof. Fix $t > 0$. By (4.3) we see that $R > 0$. Set $\delta = 1/(2R)$. For each k we set $\delta_k = \min(\delta, 1/(2p_k))$ and apply Lemma 4.3 to the function ϕ_k. This way we obtain continuously differentiable functions ψ_k, $k = 1, \dots, n$, on $\mathbb{R}_+$ such that:

$$\phi_k(z) \le \psi_k(z), \text{ for all } z \in \mathbb{R}_+; \tag{4.8}$$

$$\psi_k(z) z^{1/p_k - \delta} \text{ decreases on } \mathbb{R}_+; \tag{4.9}$$

$$\psi_k(z) z^{1/p_k + \delta} \text{ increases on } \mathbb{R}_+; \tag{4.10}$$

$$\|\psi_k\|_{p_k,s_k} \le \frac{8}{\delta_k^2} \|\phi_k\|_{p_k,s_k} = 32 \max(R^2, p_k^2) \|\phi_k\|_{p_k,s_k}. \tag{4.11}$$

For $z > 0$ we define

$$G_k(z) = z^{\sigma_k} \psi_k(z)$$

and

$$\xi_k(z,t) = t^{-\sigma_k} G_k(z),$$

$k = 1, \dots, n$. Observe that

$$\delta < \alpha_k, \tag{4.12}$$

for all k. Indeed,

$$\delta < \frac{1}{R} \le \frac{\alpha_l}{r_l} \min_{k=1,\dots,n} r_k,$$

for all $l = 1, \dots, n$ which implies (4.12). Write G_k as

$$G_k(z) = \frac{\psi_k(z) z^{1/p_k - \delta}}{z^{\alpha_k - \delta}}. \tag{4.13}$$

It follows from (4.13), (4.9), and (4.12) that

$$\lim_{z\to 0+} G_k(z) = \infty \text{ and } \lim_{z\to\infty} G_k(z) = 0. \tag{4.14}$$

Define

$$\mu_t(z_1,\dots,z_n) = \max_{k=1,\dots,n} \xi_k(z_k,t).$$

Set also

$$v(t) = \inf\{\max_{k=1,\dots,n} \xi_k(z_k,t) : \prod_{k=1}^n z_k = t^{n-1},\ z_k > 0\}.$$

The function μ_t is continuous on $\mathbb{R}^n_+$ and $v(t)$ is the infimum of μ_t over the set

$$E_t = \{(z_1,\dots,z_n) \in \mathbb{R}^n_+ : \prod_{k=1}^n z_k = t^{n-1}\}.$$

From the definition of E_t we see that by choosing $z = (z_1,\dots,z_n) \in E_t$ so that $|z|$ is sufficiently big, we can make $\min_{k=1,\dots,n} z_k$ arbitrarily small, i.e. there holds the relation

$$\lim_{|z|\to\infty,\ z\in E_t} \big(\min_{k=1,\dots,n} z_k\big) = 0.$$

Furthermore, (4.14) implies that for each $k = 1,\dots,n$

$$\lim_{z_k\to 0+} \xi_k(z_k,t) = \infty.$$

By the two preceding equalities we see that

$$\lim_{|z|\to\infty,\ z\in E_t} \mu_t(z) = \infty$$

and therfore the infimum in the definition of v need only be taken over some compact subset of E (we use here that E_t is closed in $\mathbb{R}^n$). This infimum is then attained at some point, i.e. there is a point $(u_1^*,\dots,u_n^*) \in E_t$ where

$$\mu_t(u_1^*,\dots,u_n^*) = v(t). \tag{4.15}$$

Differentiate in (4.13) to get

$$G_k'(z) = \frac{d}{dz}(\psi_k(z) z^{1/p_k-\delta}) z^{\delta-\alpha_k} - (\alpha_k-\delta) z^{\sigma_k-1}\psi_k(z),$$

for all $z > 0$. The first term on the right-hand side of this equality is non-positive by (4.9), so we have

$$G_k'(z) \le -(\alpha_k-\delta) z^{\sigma_k-1}\psi_k(z). \tag{4.16}$$

We may assume that each of the functions ϕ_k is positive at some point. Since ϕ_k is non-increasing it then follows from (4.8) and (4.10) that $\psi_k(z) > 0$ for all $z \in \mathbb{R}_+$. Using this observation and (4.12) in the estimate (4.16) gives

$$G_k'(z) < 0, \tag{4.17}$$

for all $z > 0$. So by (4.14) and (4.17) each G_k is a bijection of $\mathbb{R}_+$ onto $\mathbb{R}_+$ and G_k^{-1} is continuously differentiable. Since $v(t) \in \mathbb{R}_+$ we then have that for each $k = 1, \ldots, n$ there exists a unique number $u_k = u_k(t) > 0$ such that $G_k(u_k) = t^{\sigma_k} v(t)$, and then

$$\xi_k(u_k, t) = v(t). \tag{4.18}$$

We will now show that $u_k = u_k^*$ for all k. Observe that by (4.17),

$$\frac{\partial \xi_k}{\partial z}(z, t) = t^{-\sigma_k} G_k'(z) < 0 \tag{4.19}$$

so ξ_k is strictly decreasing with respect to z. So if $u_k > u_k^*$ for some k, then (4.18) gives

$$v(t) = \xi_k(u_k, t) < \xi_k(u_k^*, t) \leq \mu_t(u_1^*, \ldots, u_n^*),$$

but this contradicts (4.15). Thus $u_k \leq u_k^*$ for all k. Fix $k \in \{1, \ldots, n\}$ and suppose that $u_k < u_k^*$. Since $(u_1^*, \ldots, u_n^*) \in E_t$, we know that

$$\prod_{k=1}^{n} u_k^* = t^{n-1}. \tag{4.20}$$

By (4.20) and the assumption $u_k < u_k^*$ there are positive numbers d_l, $l = 1, \ldots, n$ such that $0 < u_k < d_k < u_k^*$ and $0 < u_l \leq u_l^* < d_l$ for $l \neq k$, which satisfies

$$\prod_{l=1}^{n} d_l = t^{n-1}.$$

Therefore $(d_1, \ldots, d_n) \in E_t$ so that

$$v(t) \leq \mu_t(d_1, \ldots, d_n). \tag{4.21}$$

As we observed above, the functions $z \mapsto \xi_l(z, t)$, $l = 1, \ldots, n$, are strictly decreasing on $\mathbb{R}_+$, and thus $\xi_l(d_l, t) < \xi_l(u_l, t)$ for all l. Therefore

$$\mu_t(d_1, \ldots, d_n) < \mu_t(u_1, \ldots, u_n).$$

This inequality and (4.21) gives

$$v(t) < \mu_t(u_1, \ldots, u_n) = \max_{k=1,\ldots,n} \xi_k(u_k, t).$$

which is a contradiction according to (4.18). Thus, $u_k = u_k^*$ for all $k = 1, \ldots, n$, so (4.20) becomes

$$\prod_{k=1}^{n} u_k(t) = t^{n-1}. \tag{4.22}$$

We will now show that $u_k \in C^1(\mathbb{R}_+)$, for all k. By (4.18) $v(t) = t^{-\sigma_k} G_k(u_k(t))$, and therefore

$$u_k(t) = G_k^{-1}(t^{\sigma_k} v(t)) \tag{4.23}$$

for all $t > 0$, $k = 1, \ldots, n$. Define

$$\Psi(z,t) = \prod_{k=1}^{n} G_k^{-1}(z t^{\sigma_k}),$$

for $z, t > 0$. Then, by (4.23) and (4.22), $\Psi(v(t), t) = t^{n-1}$. By (4.17) we also have that $(G_k^{-1})' < 0$ on $\mathbb{R}_+$. Therefore

$$\frac{\partial}{\partial z}\Psi(z,t) = \sum_{k=1}^{n} \frac{\Psi(z,t)}{G_k^{-1}(z t^{\sigma_k})} (G_k^{-1})'(z t^{\sigma_k}) t^{\sigma_k} < 0,$$

for all $z > 0$. By the implicit function theorem we obtain that $v \in C^1(\mathbb{R}_+)$. From (4.23) we then see that $u_k \in C^1(\mathbb{R}_+)$ for all $k = 1, \ldots, n$.

Next we will show that

$$\frac{u_k(t)}{t} \leq 4 c_k u_k'(t), \tag{4.24}$$

for all $t > 0$, where c_k is the constant defined in (4.5). Write (4.18) as $v(t) = t^{-\sigma_k} G_k(u_k(t))$ and differentiate to get

$$-\frac{v'(t)}{v(t)} = \frac{\sigma_k}{t} - u_k'(t) \frac{G_k'(u_k(t))}{G_k(u_k(t))}. \tag{4.25}$$

By (4.9) and (4.10) we know that $G_k(z) z^{\alpha_k - \delta}$ is decreasing and $G_k(z) z^{\alpha_k + \delta}$ is increasing on $\mathbb{R}_+$. Therefore

$$\frac{\alpha_k - \delta}{z} \leq -\frac{G_k'(z)}{G_k(z)} \leq \frac{\alpha_k + \delta}{z}. \tag{4.26}$$

By (4.25) and (4.26) we get

$$\frac{\sigma_k}{t} + (\alpha_k - \delta)\frac{u_k'(t)}{u_k(t)} \leq -\frac{v'(t)}{v(t)} \leq \frac{\sigma_k}{t} + (\alpha_k + \delta)\frac{u_k'(t)}{u_k(t)}, \tag{4.27}$$

for $k = 1, \ldots, n$. Differentiate (4.22) with respect to t to obtain

$$\sum_{k=1}^{n} \frac{u_k'(t)}{u_k(t)} = \frac{n-1}{t}. \tag{4.28}$$

Observe that
$$\sum_{k=1}^{n} \frac{r_k}{\alpha_k} = n-1.$$
Since $r_k, \alpha_k > 0$ (see (4.3)), this equality and (4.28) implies that there exists a number $m \in \{1, \dots, n\}$ such that
$$\frac{r_m}{\alpha_m t} \leq \frac{u_m'(t)}{u_m(t)}. \tag{4.29}$$
Take $k = m$ in the left-hand side inequality in (4.27) and apply (4.29). We then get
$$-\frac{v'(t)}{v(t)} \geq \frac{\sigma_m}{t} + (\alpha_m - \delta)\frac{u_m'(t)}{u_m(t)} \geq \frac{1}{t}(\sigma_m + (\alpha_m - \delta)\frac{r_m}{\alpha_m}).$$
Set
$$\gamma \equiv \max_{k=1,\dots,n} \frac{r_k}{\alpha_k}.$$
Using the latter inequality and (4.3), we get
$$-\frac{v'(t)}{v(t)} \geq \frac{1}{t}(\frac{1}{p} - \frac{\alpha}{n} - \delta\gamma).$$
The right-hand side inequality in (4.27) implies
$$(\alpha_k + \delta)\frac{u_k'(t)}{u_k(t)} \geq (r_k - \delta\gamma)\frac{1}{t}.$$
Thus,
$$\frac{u_k(t)}{t} \leq \frac{\alpha_k + \delta}{r_k - \gamma\delta}\, u_k'(t).$$
By (4.12) and by observing that $\gamma\delta \leq r_k/2$, we see that the constant in this inequality is smaller than $4c_k$, so (4.24) holds.

We are now ready to prove inequality (4.6). First we observe that by (4.8), $\eta_k(z,t) \leq \xi_k(z,t)$ for all $k = 1, \dots, n$ and all $z > 0$, and thus
$$w(t) \leq v(t).$$
This inequality together with (4.18), and the fact that $\sum_{k=1}^{n} \alpha/(n\alpha_k) = 1$ gives
$$w(t) \leq \prod_{k=1}^{n} \xi_k(u_k(t), t)^{\alpha/(n\alpha_k)}.$$
It follows that
$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt\Big)^{1/s} \leq$$
$$\leq \Big(\int_0^\infty t^{s/q-1} \prod_{k=1}^{n} \Big((\frac{u_k(t)}{t})^{\sigma_k} \psi_k(u_k(t))\Big)^{s\alpha/(n\alpha_k)} dt\Big)^{1/s} =$$

$$= \Big(\int_0^\infty \prod_{k=1}^n \Big(u_k(t)^{s_k/p_k - 1} \frac{u_k(t)}{t} \psi_k(u_k(t))^{s_k} \Big)^{s\alpha/(n\alpha_k s_k)} dt \Big)^{1/s}. \qquad (4.30)$$

Indeed, the equality in (4.30) can be proved by checking that

$$t^{s/q-1} \prod_{k=1}^n \big(\frac{u_k(t)}{t} \big)^{s\alpha\sigma_k/(n\alpha_k)} = \prod_{k=1}^n \Big(u_k(t)^{s_k/p_k - 1} \frac{u_k(t)}{t} \Big)^{s\alpha/(n\alpha_k s_k)},$$

which is equivalent to

$$t^a = \prod_{k=1}^n u_k(t)^{b_k}, \qquad (4.31)$$

where

$$a = \frac{s}{q} - 1 + \frac{s\alpha}{n} \sum_{k=1}^n \Big(\frac{1}{s_k \alpha_k} - \frac{\sigma_k}{\alpha_k} \Big)$$

and

$$b_k = \frac{s\alpha}{n\alpha_k p_k} - \frac{s\alpha\sigma_k}{n\alpha_k}.$$

But,

$$\frac{\sigma_k}{\alpha_k} = \frac{1}{p_k \alpha_k} - 1.$$

Thus,

$$a = \frac{s}{q} - 1 + \frac{s\alpha}{n} \sum_{k=1}^n \frac{1}{s_k \alpha_k} - \frac{s\alpha}{n} \Big(\sum_{k=1}^n \frac{1}{p_k \alpha_k} - n \Big) =$$
$$= \frac{s}{q} - \frac{s}{p} + s\alpha = \frac{s\alpha}{n}(n-1)$$

and

$$b_k = \frac{s\alpha}{n},$$

$k = 1, \ldots, n$. Thus, (4.31) reduces to (4.22).

Observe that $\sum_{k=1}^n s\alpha/(n\alpha_k s_k) = 1$. We can then apply Hölder's inequality with the parameters $n\alpha_k s_k/(s\alpha)$, $k = 1, \ldots, n$, in the last integral in (4.30) to get

$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt \Big)^{1/s} \leq$$

$$\leq \prod_{k=1}^n \Big(\int_0^\infty u_k(t)^{s_k/p_k - 1} \frac{u_k(t)}{t} \psi(u_k(t))^{s_k} dt \Big)^{\alpha/(n\alpha_k s_k)} \leq$$

$$\leq \prod_{k=1}^n \Big(4c_k \int_0^\infty u_k(t)^{s_k/p_k - 1} u_k'(t) \psi(u_k(t))^{s_k} dt \Big)^{\alpha/(n\alpha_k s_k)},$$

where the last inequality holds by (4.24). Make the change of variables $z = u_k(t)$. By (4.24), u_k increases on $\mathbb{R}_+$, so we obtain

$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt\Big)^{1/s} \le 4^n \prod_{k=1}^n (c_k^{1/s_k} \|\psi_k\|_{p_k,s_k})^{\alpha/(n\alpha_k)}.$$

Applying (4.11) we get inequality (4.6). The lemma is proved. $\square$

4.2. **The main theorem.** We will consider rearrangements with respect to specific variables. Let $f \in S_0(\mathbb{R}^n)$ and $1 \le k \le n$. Fix $\hat{x}_k \in \mathbb{R}^{n-1}$ and consider the function $f_{\hat{x}_k}(x_k) = f(\hat{x}_k, x_k)$. By Fubini's theorem, $f_{\hat{x}_k} \in S_0(\mathbb{R})$ for almost all $\hat{x}_k \in \mathbb{R}^{n-1}$. We define the rearrangement of f with respect to x_k, as the function

$$\mathcal{R}_k f(t, \hat{x}_k) \equiv f^*_{\hat{x}_k}(t).$$

This function is defined almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. Moreover, $\mathcal{R}_k f$ is a measurable function equimeasurable with $|f|$ (see [18]). In the proof of the next theorem we will derive inequalities involving sections of sets. For $E \subset \mathbb{R}^n$ and $\hat{x}_k \in \mathbb{R}^{n-1}$ we define the $\hat{x}_k$-section of E as the set

$$E(\hat{x}_k) = \{x_k \in \mathbb{R} : (\hat{x}_k, x_k) \in E\},$$

where $(\hat{x}_k, x_k) \equiv (x_1, \ldots, x_n)$.

Theorem 4.5. *Let $n \ge 2$, $1 \le p_1, \ldots, p_n, s_1, \ldots, s_n < \infty$, and $\alpha_1, \ldots, \alpha_n > 0$. Put*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1}, \; p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}, \; \text{and } s = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k s_k}\Big)^{-1}.$$

Assume that $p \le n/\alpha$ and put

$$q = \begin{cases} np/(n-\alpha p), & \alpha p < n \\ \infty, & \alpha p = n. \end{cases}$$

Set

$$\sigma_k = \frac{1}{p_k} - \alpha_k, \; \text{and } V_k = L^{p_k,s_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})],$$

and assume that

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \sigma_k > 0, \tag{4.32}$$

for $k = 1, \ldots, n$. Suppose that

$$f \in S_0(\mathbb{R}^n) \text{ and } f \in \bigcap_{k=1}^n V_k.$$

Then $f \in L^{q,s}(\mathbb{R}^n)$ *and*

$$\|f\|^*_{q,s} \le c \prod_{k=1}^{n} \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{4.33}$$

where

$$c = K_n c' \max_{k=1,\dots,n} 4^{n\alpha_k} \prod_{k=1}^{n} \Big(1 + \frac{1}{\alpha_k}\Big)^{\alpha/(n\alpha_k)}, \tag{4.34}$$

K_n *only depends on* n*, and* c' *is the constant from Lemma 4.4 defined in* (4.7).

Proof. We may assume that $f \ge 0$. Fix $t > 0$. We will give a non-negative upper bound on $\Delta_f(t)$ and can therefore assume that $\Delta_f(t) > 0$. Let

$$E_1 = \{x : f(x) \ge f^*(t)\} \quad \text{and} \quad E_2 = \{x : f(x) > f^*(2t)\}.$$

By (2.9), $\mathrm{mes}_n E_1 \ge t$ so there exists an F_σ-set $A \subset E_1$ such that $\mathrm{mes}_n A = t$. Moreover, by (2.8), $\mathrm{mes}_n E_2 \le 2t$ so there exists a G_δ-set $B \subset \mathbb{R}^n$ such that $E_2 \subset B$ and $\mathrm{mes}_n B = 2t$. Since F_σ-sets and G_δ-sets have measurable sections, the functions

$$a_k(\hat{x}_k) \equiv \mathrm{mes}_1 A(\hat{x}_k) \quad \text{and} \quad b_k(\hat{x}_k) \equiv \mathrm{mes}_1 B(\hat{x}_k),$$

$k = 1, \dots, n$, are defined for all $\hat{x}_k \in \mathbb{R}^{n-1}$. By Fubini's theorem, these functions are measurable on $\mathbb{R}^{n-1}$. Since $f(x) \ge f^*(t)$ for all $x \in A$ we have

$$f^*(t) \le \inf_{x_k \in A(\hat{x}_k)} f(x_k, \hat{x}_k) \le \mathcal{R}_k f(a_k(\hat{x}_k), \hat{x}_k), \tag{4.35}$$

for all $k = 1, \dots, n$ and all $\hat{x}_k$ such that $0 < a_k(\hat{x}_k) < \infty$. Moreover, if $0 < b_k(\hat{x}_k) < \infty$ then

$$\mathcal{R}_k f(2b_k(\hat{x}_k), \hat{x}_k) \le f^*(2t). \tag{4.36}$$

Indeed, suppose $0 < b_k(\hat{x}_k) < \infty$ and let $E \subset \mathbb{R}$ be a measurable set of measure $2b_k(\hat{x}_k)$. Then there exists a point $x_k \in E$ such that $(x_k, \hat{x}_k) \notin B$. But $E_2 \subset B$ so then $f(x_k, \hat{x}_k) \le f^*(2t)$ and thus

$$\inf_{x_k \in E} f(x_k, \hat{x}_k) \le f^*(2t).$$

Since E was an arbitrary measurable set of measure $2b_k(\hat{x}_k)$, (4.36) follows.

Observe that by our assumption $\Delta_f(t) > 0$, it holds that

$$A \subset E_1 \subset E_2 \subset B. \tag{4.37}$$

For each k we put

$$P_k = \{\hat{x}_k \in \Pi_k A : 0 < b_k(\hat{x}_k) \le 2^{n+1} a_k(\hat{x}_k) < \infty\}.$$

The sets P_k are measurable since $\Pi_k A$ is measurable (A is an F_σ-set) and since the functions a_k and b_k are measuurable. For all $\hat{x}_k \in P_k$ we have the inequalities (4.35) and (4.36), and from these we obtain

$$\begin{aligned}
\Delta_f(t) &\leq \mathcal{R}_k f(a_k(\hat{x}_k), \hat{x}_k) - \mathcal{R}_k f(2b_k(\hat{x}_k), \hat{x}_k)\\
&\leq \mathcal{R}_k f(a_k(\hat{x}_k), \hat{x}_k) - \mathcal{R}_k f(2^{n+2}a_k(\hat{x}_k), \hat{x}_k)\\
&= \sum_{l=0}^{n+1} \left(\mathcal{R}_k f(2^l a_k(\hat{x}_k), \hat{x}_k) - \mathcal{R}_k f(2^{l+1}a_k(\hat{x}_k), \hat{x}_k)\right)\\
&\leq (a_k(\hat{x}_k))^{-\sigma_k}\Psi_k(\hat{x}_k)\sum_{l=0}^{n+1} 2^{-l\,\sigma_k},
\end{aligned}$$

where $\Psi_k(\hat{x}_k) = \|f(\hat{x}_k, \cdot)\|_{\Lambda^{\sigma_k}}$. So for all $\hat{x}_k \in P_k$ and every $k = 1, \ldots, n$ it holds that

$$\Delta_f(t) \leq c a_k(\hat{x}_k)^{-\sigma_k}\Psi_k(\hat{x}_k), \tag{4.38}$$

where

$$c = \sum_{l=0}^{n+1} 2^{-l\sigma_k} \leq 2n \max(1, 2^{-(n+1)\sigma_k}).$$

For all $\hat{x}_k \in (\Pi_k A) \setminus P_k$ we have

$$2^{n+1}a_k(\hat{x}_k) \leq b_k(\hat{x}_k). \tag{4.39}$$

Indeed, take $\hat{x}_k \in (\Pi_k A) \setminus P_k$. By the definition of P_k, we have either $b_k(\hat{x}_k) = 0$, $2^{n+1}a_k(\hat{x}_k) < b_k(\hat{x}_k)$, or $a_k(\hat{x}_k) = \infty$. However, if $b_k(\hat{x}_k) = 0$ or $a_k(\hat{x}_k) = \infty$ then (4.39) holds by (4.37).

For $k = 1, \ldots, n$ we put

$$A_k = \{x \in A : \hat{x}_k \in P_k\}.$$

These sets are measurable by Lemma 4.1. Moreover,

$$\mathrm{mes}_n A_k \geq t(1 - 2^{-n}), \tag{4.40}$$

for all k. Indeed, $\Pi_k A_k = P_k$ and for all $\hat{x}_k \in P_k$ we have $\mathrm{mes}_1(A_k(\hat{x}_k)) = a_k(\hat{x}_k)$, so

$$\mathrm{mes}_n A_k = \int_{P_k} a_k(\hat{x}_k)d\hat{x}_k.$$

By (4.39) and this equality we get

$$\begin{gathered}
2t = \int_{\Pi_k B} b_k(\hat{x}_k)d\hat{x}_k \geq 2^{n+1}\int_{(\Pi_k A)\setminus P_k} a_k(\hat{x}_k)d\hat{x}_k =\\
= 2^{n+1}(\mathrm{mes}_n A - \mathrm{mes}_n A_k) = 2^{n+1}(t - \mathrm{mes}_n A_k),
\end{gathered}$$

and then (4.40) follows.

Let A^* be an F_σ-subset of $\cap_{k=1}^n A_k$ such that $\mathrm{mes}_n A^* = \mathrm{mes}_n(\cap_{k=1}^n A_k)$. By (4.40) we have $\mathrm{mes}_n(A \setminus A^*) \le n2^{-n}t$. This implies that

$$\mathrm{mes}_n A^* \ge \frac{t}{2}. \tag{4.41}$$

Let u_k, $k = 1, \dots, n$, be positive numbers such that

$$\prod_{k=1}^n u_k = t^{n-1}. \tag{4.42}$$

Put

$$\Omega = \{k \in \{1, \dots, n\} : \mathrm{mes}_{n-1} P_k \ge \frac{u_k}{2}\}.$$

Then $\Omega \ne \emptyset$. Indeed, suppose $\Omega = \emptyset$. By (4.41) and the Loomis-Whitney inequality (2.1),

$$\left(\frac{t}{2}\right)^{n-1} \le (\mathrm{mes}_n A^*)^{n-1} \le \prod_{k=1}^n \mathrm{mes}_{n-1} \Pi_k A^*. \tag{4.43}$$

Since $\Pi_k A^* \subset P_k$ and $\Omega = \emptyset$, it follows that

$$\left(\frac{t}{2}\right)^{n-1} < \frac{1}{2^n} \prod_{k=1}^n u_k,$$

but this is false by (4.42). Thus, $\Omega \ne \emptyset$.

Fix k in Ω. Assume that $\sigma_k \le 0$ and define

$$\tilde{P}_k = \{\hat{x}_k \in P_k : a_k(\hat{x}_k) \le \frac{4t}{u_k}\}.$$

Then $\tilde{P}_k$ is measurable since P_k and the function a_k are measurable. Since $a_k(\hat{x}_k) > 4t/u_k$ for all $\hat{x}_k \in P_k \setminus \tilde{P}_k$, we have

$$t = \mathrm{mes}_n A = \int_{\Pi_k A} a_k(\hat{x}_k) d\hat{x}_k \ge \int_{P_k \setminus \tilde{P}_k} a_k(\hat{x}_k) d\hat{x}_k \ge$$
$$\ge \frac{4t}{u_k} (\mathrm{mes}_{n-1} P_k - \mathrm{mes}_{n-1} \tilde{P}_k).$$

Since $k \in \Omega$, we know that $\mathrm{mes}_{n-1} P_k \ge u_k/2$, so by the preceding inequality

$$\mathrm{mes}_{n-1} \tilde{P}_k \ge \frac{u_k}{4}. \tag{4.44}$$

Since $\sigma_k \le 0$, (4.38) gives that

$$\Delta_f(t) \le 2n4^{n\alpha_k} \left(\frac{u_k}{t}\right)^{\sigma_k} \Psi_k(\hat{x}_k),$$

for all $\hat{x}_k \in \tilde{P}_k$. Taking infimum over the set $\tilde{P}_k$ and using (4.44), we get

$$\Delta_f(t) \le 2n4^{n\alpha_k} \left(\frac{u_k}{t}\right)^{\sigma_k} \Psi_k^*(\frac{u_k}{4}). \tag{4.45}$$

From here on we assume that $\sigma_k > 0$ for each k in Ω. We now partition the sets P_k, $k \in \Omega$, as follows. If $\mathrm{mes}_{n-1} P_k > u_k/2$, then we apply Lemma 4.2 to obtain disjoint measurable sets P'_k and P''_k such that $P_k = P'_k \cup P''_k$,

$$\mathrm{mes}_{n-1} P'_k = \frac{u_k}{2}, \tag{4.46}$$

and

$$\int_{P''_k} \Psi_k(\hat{x}_k)^{1/\sigma_k} d\hat{x}_k \le \int_{u_k/2}^{\infty} \Psi^*_k(z)^{1/\sigma_k} dz. \tag{4.47}$$

On the other hand, if $\mathrm{mes}_{n-1} P_k = u_k/2$ then we put $P'_k = P_k$ and $P''_k = \emptyset$. Clearly (4.46) and (4.47) are satisfied also in this case.

For each $k \in \Omega$ we put

$$A''_k = \{x \in A^* : \hat{x}_k \in P''_k\}.$$

These sets are measurable by Lemma 4.1. We will consider two cases. First we assume that there exists $k \in \Omega$ for which $\mathrm{mes}_n A''_k \ge t/(4n)$. Fix such an index k. Since $\mathrm{mes}_1(A''_k(\hat{x}_k)) \le a_k(\hat{x}_k)$ and $\Pi_k A''_k \subset P''_k$, it holds that

$$\frac{t}{4n} \le \mathrm{mes}_n A''_k \le \int_{P''_k} a_k(\hat{x}_k) d\hat{x}_k \tag{4.48}$$

(the first inequality is by our assumption on k). Since $\sigma_k > 0$, (4.38) gives

$$a_k(\hat{x}_k) \Delta_f(t)^{1/\sigma_k} \le (2n\Psi_k(\hat{x}_k))^{1/\sigma_k},$$

for all $\hat{x}_k \in P_k$. Integrating over P''_k and applying (4.47) we get

$$\Delta_f(t)^{1/\sigma_k} \int_{P''_k} a_k(\hat{x}_k) d\hat{x}_k \le (2n)^{1/\sigma_k} \int_{u_k/2}^{\infty} \Psi^*_k(z)^{1/\sigma_k} dz. \tag{4.49}$$

By this inequality and (4.48),

$$\Delta_f(t) \le 4n^2 t^{-\sigma_k} \Big(\int_{u_k}^{\infty} \Psi^*_k\big(\frac{z}{2}\big)^{1/\sigma_k} dz \Big)^{\sigma_k}. \tag{4.50}$$

Now we turn to the remaining case when

$$\mathrm{mes}_n A''_k < \frac{t}{4n}, \tag{4.51}$$

for all $k \in \Omega$. Put

$$D = A^* \setminus \bigcup_{k \in \Omega} A''_k.$$

By (4.41) and (4.51),

$$\mathrm{mes}_n D \ge \mathrm{mes}_n A^* - \sum_{k \in \Omega} \mathrm{mes}_n A''_k \ge \frac{t}{2} - \sum_{k=1}^{n} \frac{t}{4n} = \frac{t}{4}. \tag{4.52}$$

Fix $k \in \Omega$. Let the set S be defined by

$$S = \{x \in D : a_k(\hat{x}_k) \geq \frac{t}{4u_k}\}.$$

This set is measurable by Lemma 4.1. Let Q be an F_σ-subset of $D \setminus S$ such that

$$\mathrm{mes}_n Q = \mathrm{mes}_n (D \setminus S).$$

Now,

$$\mathrm{mes}_n Q \leq \int_{\Pi_k Q} \mathrm{mes}_1 D(\hat{x}_k) d\hat{x}_k \leq \int_{\Pi_k Q} a_k(\hat{x}_k) d\hat{x}_k \leq \frac{t}{4u_k} \mathrm{mes}_{n-1} \Pi_k Q.$$

But $\Pi_k Q \subset \Pi_k D \subset \Pi_k(A^* \backslash A_k'') \subset P_k'$ so we have by (4.46) and the preceding inequality that

$$\mathrm{mes}_n (D \setminus S) = \mathrm{mes}_n Q \leq \frac{t}{8}.$$

By this and (4.52),

$$\mathrm{mes}_n S = \mathrm{mes}_n D - \mathrm{mes}_n (D \setminus S) \geq \frac{t}{8}. \tag{4.53}$$

Let $\tilde{S}$ be an F_σ-subset of S such that $\mathrm{mes}_n \tilde{S} = \mathrm{mes}_n S$. Then

$$\mathrm{mes}_{n-1} \Pi_l \tilde{S} \leq \frac{u_l}{2}, \tag{4.54}$$

for all $l = 1, \ldots, n$. Indeed, if $l \in \Omega$, then we have

$$\tilde{S} \subset S \subset D \subset A^* \setminus A_l'',$$

so that

$$\Pi_l \tilde{S} \subset \Pi_l (A^* \setminus A_l'') \subset P_l'.$$

By (4.46) we then have (4.54), for all $l \in \Omega$. Suppose $\Omega \neq \{1, \ldots, n\}$ and fix $l \in \{1, \ldots, n\} \setminus \Omega$. Then $\mathrm{mes}_{n-1} P_l < u_l/2$. But

$$\tilde{S} \subset S \subset D \subset A^* \subset A_l$$

and then

$$\Pi_l \tilde{S} \subset \Pi_l A_l = P_l,$$

so we again obtain (4.54).

By (4.53) and the Loomis-Whitney inequality (2.1),

$$\left(\frac{t}{8}\right)^{n-1} \leq (\mathrm{mes}_n S)^{n-1} = (\mathrm{mes}_n \tilde{S})^{n-1} \leq \prod_{l=1}^{n} \mathrm{mes}_{n-1} \Pi_l \tilde{S}.$$

Applying (4.54) for each $l \in \{1, \ldots, n\}$, except for $l = k$ (recall that $k \in \Omega$ is fixed), we obtain

$$\left(\frac{t}{8}\right)^{n-1} \leq \frac{2^{-n+1}}{u_k} \mathrm{mes}_{n-1} \Pi_k \tilde{S} \prod_{l=1}^{n} u_l.$$

By (4.42), this implies

$$\frac{u_k}{4^{n-1}} \le \mathrm{mes}_{n-1}\Pi_k\tilde{S}. \tag{4.55}$$

Let $\hat{x}_k \in \Pi_k\tilde{S}$. Then inequality (4.38) holds and $a_k(\hat{x}_k) \ge t/(4u_k)$, so we have

$$\Delta_f(t) \le 8n\Big(\frac{u_k}{t}\Big)^{\sigma_k}\Psi_k(\hat{x}_k)$$

(here we used that $0 < \sigma_k < 1$ to estimate the constant). Taking infimum over all $\hat{x}_k \in \Pi_k\tilde{S}$ in the preceding inequality and using (4.55), we obtain

$$\Delta_f(t) \le 8n\Big(\frac{u_k}{t}\Big)^{\sigma_k}\Psi_k^*\Big(\frac{u_k}{4^{n-1}}\Big).$$

Since Ψ_k^* is non-increasing, it follows that

$$\Delta_f(t) \le 8nt^{-\sigma_k}\Big(\int_{u_k}^{\infty}\Psi_k^*(z/4^n)^{1/\sigma_k}dz\Big)^{\sigma_k}. \tag{4.56}$$

For each $k \in \{1,\dots,n\}$ and $z > 0$ we define the function

$$\phi_k(z) = \begin{cases} \Psi_k^*(z/4), & \sigma_k \le 0 \\ z^{-\sigma_k}\Big(\displaystyle\int_z^{\infty}\Psi_k^*(\tau/4^n)^{1/\sigma_k}d\tau\Big)^{\sigma_k}, & \sigma_k > 0, \end{cases}$$

and set $\eta_k(z,t) \equiv (z/t)^{\sigma_k}\phi_k(z)$. It holds that

$$\|\phi_k\|_{p_k,s_k} \le 4^{n+1}\Big(1 + \frac{1}{\alpha_k}\Big)\|\Psi_k\|_{p_k,s_k}. \tag{4.57}$$

Indeed, fix $k \in \{1,\dots,n\}$. Assume first that $\sigma_k \le 0$. Then

$$\|\phi_k\|_{p_k,s_k} = \Big(\int_0^{\infty} z^{p_k/s_k-1}\Psi_k^*(z/4)^{s_k}dz\Big)^{1/s_k}.$$

Making the change of variables $z \mapsto z/4$ we obtain (4.57), with 4 as the constant. Now we suppose that $\sigma_k > 0$. Then

$$\|\phi_k\|_{p_k,s_k} = \Big(\int_0^{\infty} z^{\alpha_k s_k-1}\Big(\int_z^{\infty}\Psi_k^*(\tau/4^n)^{1/\sigma_k}d\tau\Big)^{s_k\sigma_k}dz\Big)^{1/s_k}.$$

Make the change of variables $\tau \mapsto 4^{-n}\tau$ and $z \mapsto 4^{-n}z$ to get

$$\|\phi_k\|_{p_k,s_k} = 4^{n/p_k}\Big(\int_0^{\infty} z^{\alpha_k s_k-1}\Big(\int_z^{\infty}\Psi^*(\tau)^{1/\sigma_k}d\tau\Big)^{s_k\sigma_k}dz\Big)^{1/s_k}.$$

Assume first that $1 \le s_k\sigma_k$. By Hardy's inequality (2.2),

$$\|\phi_k\|_{p_k,s_k} \le 4^{n/p_k}\Big(\frac{\sigma_k}{\alpha_k}\Big)^{\sigma_k}\Big(\int_0^{\infty} t^{\alpha_k s_k+s_k\sigma_k-1}\Psi_k^*(t)^{s_k}dt\Big)^{1/s_k} \le$$

$$\le \frac{4^n}{\alpha_k}\|\Psi_k\|_{p_k,s_k},$$

where we used that $0 < \alpha_k$, $\sigma_k < 1$ to estimate the constant. Suppose now that $0 < s_k\sigma_k < 1$. Applying Theorem 2.3 gives inequality (4.57) with the constant $4^{n/p_k}(e/(\alpha_k p_k))^{1/s_k}$ (this estimate is similar to the previous case, so we omit the details). This constant is less then $4^{n+1}/\alpha_k$, since $0 < \alpha_k < 1$ and $1 \le p_k, s_k$. In each of these three cases, the constant we get is less than $4^{n+1}(1 + 1/\alpha_k)$, so (4.57) holds.

By (4.45), (4.50), and (4.56) there exists $k \in \{1, \dots, n\}$ such that

$$\Delta_f(t) \le 8n^2 4^{n\alpha_k}\eta_k(u_k, t). \tag{4.58}$$

Set

$$w(t) \equiv \inf\{\max_{k=1,\dots,n} \eta_k(z_k, t) : \prod_{k=1}^{n} z_k = t^{n-1},\ 0 < z_1, \dots, z_n\}.$$

The numbers u_k, $k = 1, \dots, n$, are arbitrary positive numbers satisfying (4.42), so it follows from (4.58) that

$$\Delta_f(t) \le dw(t),$$

where

$$d \equiv 8n^2 \max_{k=1,\dots,n} 4^{n\alpha_k}.$$

Since $t > 0$ was arbitrary, we get

$$\|f\|^*_{q,s} = \Big(\int_0^\infty t^{s/q-1}\Delta_f(t)^s dt\Big)^{1/s} \le d\Big(\int_0^\infty t^{s/q-1}w(t)^s dt\Big)^{1/s}.$$

Each of the functions ϕ_k is non-negative and non-increasing. Furthermore, by (4.57) we have that $\phi_k \in L^{p_k,s_k}(\mathbb{R}_+)$. Since we also assumed (4.32), we can apply Lemma 4.4 to get

$$\|f\|^*_{q,s} \le dc' \prod_{k=1}^{n} \|\phi_k\|^{\alpha/(n\alpha_k)}_{p_k,s_k},$$

where c' is the constant from Lemma 4.4, given by (4.7). By (4.57) we then get

$$\|f\|^*_{q,s} \le 4^{n+1}dc' \prod_{k=1}^{n} \Big[\Big(1 + \frac{1}{\alpha_k}\Big)\|f\|_{V_k}\Big]^{\alpha/(n\alpha_k)},$$

so we have proved (4.33). It follows from (4.33) that $f \in L^{q,p}(\mathbb{R}^n)$. Indeed, when $q = \infty$, we apply (2.29) and when $q < \infty$ we apply (2.27). $\square$

Remark 4.6. *As was mentioned above, for $p_k = s_k = p$, $k = 1, \dots, n$, Theorem 4.5 was proved in* [19]. *Note that in this case the condition* (4.32) *reduces to the inequality $\alpha_k > \alpha/n$ which is certainly true for any $k = 1, \dots, n$. Indeed,*

$$\frac{\alpha}{n} = \Big(\sum_{k=1}^{n} \frac{1}{\alpha_k}\Big)^{-1} < \alpha_k.$$

5. Applications

It was shown in [19] that estimates in terms of mixed norms provide a unified approach to embeddings of Sobolev spaces and Besov spaces and enable us to obtain optimal embedding constants. In this section we apply Theorem 4.5 to get similar results for anisotropic Sobolev-Liouville spaces and anisotropic Sobolev-Besov spaces.

5.1. **Anisotropic Sobolev-Liouville spaces.** Let $1 \le p < \infty$. A function $f \in L^p(\mathbb{R}^n)$ is said to belong to the *partial* Sobolev space $W^1_{p;k}(\mathbb{R}^n)$ if f has a weak derivative $D_k f \in L^p(\mathbb{R}^n)$. The norm in this space is defined as

$$\|f\|_{W^1_{p;k}} = \|f\|_p + \|D_k f\|_p.$$

Let $1 \le p_1, \dots, p_n < \infty$. The anisotropic Sobolev space $W^1_{p_1,\dots,p_n}(\mathbb{R}^n)$ is defined as the intersection

$$W^1_{p_1,\dots,p_n}(\mathbb{R}^n) = \bigcap_{k=1}^n W^1_{p_k;k}(\mathbb{R}^n),$$

with the norm

$$\|f\|_{W^1_{p_1,\dots,p_n}} = \sum_{k=1}^n \|f\|_{W^1_{p_k;k}}.$$

In the case $p_k = p$, $k = 1, \dots, n$, we easily see that the space $W^1_{p_1,\dots,p_n}(\mathbb{R}^n)$ coincides with $W^1_p(\mathbb{R}^n)$ and that the norms are equivalent.

Let $0 < \alpha < 1$. The Bessel kernel of order α on $\mathbb{R}$ is the function G_α defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi x^2/t} e^{-t/(4\pi)} t^{(\alpha-n)/2-1} dt, \quad x \in \mathbb{R} \tag{5.1}$$

(see [30, p. 132]). We have

$$\|G_\alpha\|_1 = 1. \tag{5.2}$$

For $1 \le p < \infty$ and $0 < \alpha < 1$, we say that a measurable function f on $\mathbb{R}^n$ belongs to the *partial* Sobolev-Liouville space $L^\alpha_{p;k}(\mathbb{R}^n)$ if there exists a function $g_k \in L^p(\mathbb{R}^n)$ such that

$$f(x) = \int_{\mathbb{R}} G_\alpha(x_k - y) g_k(\hat{x}_k, y) dy,$$

a.e. on $\mathbb{R}^n$. The norm in $L^\alpha_{p;k}(\mathbb{R}^n)$ is defined as

$$\|f\|_{L^\alpha_{p;k}} = \|g_k\|_p.$$

The function g_k is called the Bessel derivative of order α of f with respect to x_k, and we denote it by $D_k^\alpha f$. We extend the notation $L_{p;k}^\alpha$ to the case $\alpha = 1$. Namely, we agree that $L_{p;k}^1 = W_{p;k}^1$ $(1 \le p < \infty)$ and

$$\|f\|_{L_{p;k}^1} = \|f\|_{W_{p;k}^1}.$$

By Minkowski's inequality and (5.2), we easily obtain the following.

Proposition 5.1. *Let* $1 \le p < \infty$ *and* $0 < \alpha \le 1$. *Suppose that* $f \in L_{p;k}^\alpha(\mathbb{R}^n)$. *Then* $f \in L^p(\mathbb{R}^n)$ *and*

$$\|f\|_p \le \|f\|_{L_{p;k}^\alpha}.$$

Let $0 < \alpha_1, \ldots, \alpha_n \le 1$ and $1 \le p_1, \ldots, p_n < \infty$. We define the anisotropic Sobolev-Liouville space $L_{p_1,\ldots,p_n}^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n)$ as the intersection

$$L_{p_1,\ldots,p_n}^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n) = \cap_{k=1}^n L_{p_k;k}^{\alpha_k}(\mathbb{R}^n),$$

with the norm

$$\|f\|_{L_{p_1,\ldots,p_n}^{\alpha_1,\ldots,\alpha_n}} = \sum_{k=1}^n \|f\|_{L_{p_k;k}^{\alpha_k}}$$

We also write $L_p^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n)$ for the space $L_{p,\ldots,p}^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n)$. From the definition we have $L_{p_1,\ldots,p_n}^{1,\ldots,1}(\mathbb{R}^n) = W_{p_1,\ldots,p_n}^1(\mathbb{R}^n)$, and the norms are equal.

We will apply Theorem 4.5 to prove an embedding for $L_{p_1,\ldots,p_n}^{\alpha_1,\ldots,\alpha_n}(\mathbb{R}^n)$. To this end we need the following lemmas. The first lemma was proved in [17, p. 148]. We use here the notation

$$\Delta(h)\varphi(x) = \varphi(x+h) - \varphi(x). \tag{5.3}$$

Lemma 5.2. *Let* $\varphi \in L_{loc}^1(\mathbb{R}) \cap S_0(\mathbb{R})$ *and assume that* $0 < t < \infty$. *Then, for each* $x \in \mathbb{R}$

$$|\varphi(x)| \le \varphi^*(t) + \frac{1}{t}\int_0^{2t} |\Delta(h)\varphi(x)|dh.$$

The next lemma states two well known inequalities concerning G_α.

Lemma 5.3. *Let* $0 < \alpha < 1$. *The Bessel kernel on* $\mathbb{R}$ *satisfies the inequalities*

$$G_\alpha(x) \le c|x|^{\alpha-1} \tag{5.4}$$

and

$$|G_\alpha'(x)| \le c|x|^{\alpha-2}, \tag{5.5}$$

where c *depends only on* α.

Proof. Differentiating in the formula (5.1), we get

$$|G'_\alpha(x)| = c|x| \int_0^\infty e^{-\pi x^2/t} e^{-t/(4\pi)} t^{(\alpha-1)/2-2} dt \leq$$

$$\leq c|x| \int_0^\infty e^{-\pi x^2/t} t^{(\alpha-1)/2-2} dt.$$

Using the inequality $e^z > z^2$, $z > 0$, we have

$$\int_0^{x^2} e^{-\pi x^2/t} t^{(\alpha-1)/2-2} dt \leq \frac{1}{\pi^2 x^4} \int_0^{x^2} t^{(\alpha-1)/2} dt = \frac{2}{\pi^2(\alpha+1)} |x|^{\alpha-3}.$$

Moreover,

$$\int_{x^2}^\infty e^{-\pi x^2/t} t^{(\alpha-1)/2-2} dt \leq \int_{x^2}^\infty t^{(\alpha-1)/2-2} dt = \frac{2}{3-\alpha} |x|^{\alpha-3}.$$

Combining these estimates, we obtain (5.5).

To estimate G_α, we break the integral as above. Using the inequality $e^z > z$, $z > 0$, to estimate the integral over $(0, x^2)$, we similarly obtain (5.4). □

The next lemma is known (see e.g. [30, p. 158-159]), but we include the proof for completeness.

Lemma 5.4. *Let* $0 < \alpha < 1$. *The Bessel kernel on* $\mathbb{R}$ *satisfies*

$$\|\Delta(h) G_\alpha\|_1 \leq c|h|^\alpha, \quad \textit{for any } h \in \mathbb{R}. \tag{5.6}$$

Proof. We have

$$\|\Delta(h) G_\alpha\|_1 = \int_{|x|>2|h|} |\Delta(h) G_\alpha(x)| dx + \int_{|x|\leq 2|h|} |\Delta(h) G_\alpha(x)| dx. \tag{5.7}$$

Observe that

$$|\Delta(h) G_\alpha(x)| \leq \int_0^1 \left| \frac{d}{dt}(G_\alpha(x+th)) \right| dt = |h| \int_0^1 |G'_\alpha(x+th)| dt.$$

By this and inequality (5.5),

$$|\Delta(h) G_\alpha(x)| \leq c|h| \int_0^1 |x+th|^{\alpha-2} dt.$$

If $2|h| < |x|$, it follows that

$$|\Delta(h) G_\alpha(x)| \leq c|h||x|^{\alpha-2}.$$

This implies

$$\int_{|x|>2|h|} |\Delta(h) G_\alpha(x)| dx \leq c|h| \int_{2|h|}^\infty x^{\alpha-2} dx \leq c'|h|^\alpha. \tag{5.8}$$

By (5.4),

$$\int_{|x|\le 2|h|} |\Delta(h)G_\alpha(x)|dx \le 4\int_0^{3|h|} G_\alpha(x)dx \le$$

$$\le c\int_0^{3|h|} x^{\alpha-1}dx \le c'|h|^\alpha.$$

Applying this inequality, (5.7), and (5.8) we obtain (5.6). □

The next lemma states the embedding from anisotropic Sobolev-Liouville spaces to the mixed norm spaces considered in Theorem 4.5. In the case when $\alpha = 1$, it was proved in [19].

Lemma 5.5. *Let* $1 \le p < \infty$, $0 < \alpha \le 1$, $n \ge 2$, *and* $k \in \{1,\dots,n\}$. *If* $f \in L^\alpha_{p,k}(\mathbb{R}^n)$, *then*

$$f \in V_k \equiv L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p-\alpha}_{x_k}(\mathbb{R})] \tag{5.9}$$

and

$$\|f\|_{V_k} \le c\|D^\alpha_k f\|_p. \tag{5.10}$$

where c *depends only on* p, α, *and* n.

Proof. Fix $t > 0$. Write $f_{\hat{x}_k}(x_k) = f(\hat{x}_k, x_k)$. Then $f_{\hat{x}_k} \in L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$, for a.e. $\hat{x}_k \in \mathbb{R}^{n-1}$. Indeed, $L^\alpha_{p,k}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ by Proposition 5.1. By Fubini's theorem it follows that $f_{\hat{x}_k} \in L^p(\mathbb{R}) \subset L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$, for a.e. $\hat{x}_k \in \mathbb{R}^{n-1}$. Fix such $\hat{x}_k$. By Lemma 5.2,

$$|f_{\hat{x}_k}(x_k)| \le f^*_{\hat{x}_k}(2t) + \frac{1}{2t}\Phi(x_k) \tag{5.11}$$

for all $x_k \in \mathbb{R}$, where

$$\Phi(x_k) = \int_0^{4t} |\Delta(h)f_{\hat{x}_k}(x_k)|dh.$$

We have $\Phi \in S_0(\mathbb{R})$. Indeed, by Minkowski's inequality

$$\|\Phi\|_p \le \int_0^{4t} \|\Delta(h)f_{\hat{x}_k}\|_p dh \le 8t\|f_{\hat{x}_k}\|_p.$$

As we noted above, $f_{\hat{x}_k} \in L^p(\mathbb{R})$ and thus $\Phi \in L^p(\mathbb{R}) \subset S_0(\mathbb{R})$.

It follows from (5.11) that

$$f^*_{\hat{x}_k}(t) - f^*_{\hat{x}_k}(2t) \le \frac{1}{t}\Phi^*(t). \tag{5.12}$$

Set $g \equiv D_k^\alpha f$, that is, if $0 < \alpha < 1$ then g denotes the Bessel derivative, and if $\alpha = 1$ then g is the usual weak derivative. Write $g_{\hat{x}_k}(x_k) = g(\hat{x}_k, x_k)$. In the case $\alpha = 1$,

$$\Delta(h) f_{\hat{x}_k}(x_k) = \int_0^h g_{\hat{x}_k}(x_k + u)du.$$

So

$$\Phi(x_k) \leq 4t \int_0^{4t} |g_{\hat{x}_k}(x_k + u)|du \leq 16t \int_0^t g_{\hat{x}_k}^*(s)ds,$$

and thus

$$\Phi^*(t) \leq 16t \int_0^t g_{\hat{x}_k}^*(s)ds. \tag{5.13}$$

Assume now that $0 < \alpha < 1$. By definition of the Bessel derivative,

$$f_{\hat{x}_k}(x_k) = \int_{\mathbb{R}} G_\alpha(x_k - u) g_{\hat{x}_k}(u)du.$$

It follows that

$$\Delta(h) f_{\hat{x}_k}(x_k) = \int_{\mathbb{R}} g_{\hat{x}_k}(u) \Delta(h) G_\alpha(x_k - u)du.$$

Changing variables we get

$$\Delta(h) f_{\hat{x}_k}(x_k) = \int_{\mathbb{R}} g_{\hat{x}_k}(x_k - u) \Delta(h) G_\alpha(u)du.$$

So by Fubini's theorem,

$$\Phi(x_k) \leq \int_{\mathbb{R}} |g_{\hat{x}_k}(x_k - u)| \varphi(u)du,$$

where

$$\varphi(u) = \int_0^{4t} |\Delta(h) G_\alpha(u)|dh.$$

Let $E \subset \mathbb{R}$ be a measurable set having measure t. Integrating over E in the preceding inequality and using Fubini's theorem and (2.22), we obtain

$$\int_E \Phi(x_k)dx_k \leq \int_{\mathbb{R}} \varphi(u) \int_E |g_{\hat{x}_k}(x_k - u)|dx_k du \leq \|\varphi\|_1 \int_0^t g_{\hat{x}_k}^*(s)ds.$$

Since E was an arbitrary set of measure t, (2.22) then implies

$$\Phi^*(t) \leq \frac{1}{t} \int_0^t \Phi^*(s)ds = \frac{1}{t} \sup_{|E|=t} \int_E \Phi(x_k)dx_k \leq \frac{1}{t} \|\varphi\|_1 \int_0^t g_{\hat{x}_k}^*(s)ds.$$

By Fubini's theorem and Lemma 5.4 we have

$$\|\varphi\|_1 = \int_0^{4t} \|\Delta(h) G_\alpha\|_1 dh \leq ct^{\alpha+1}.$$

Thus,

$$\Phi^*(t) \le ct^{\alpha} \int_0^t g^*_{\hat{x}_k}(s)ds. \tag{5.14}$$

By (5.12), (5.13), (5.14) we obtain

$$f^*_{\hat{x}_k}(t) - f^*_{\hat{x}_k}(2t) \le ct^{\alpha-1} \int_0^t g^*_{\hat{x}_k}(s)ds,$$

for all $0 < \alpha \le 1$. For all $0 < \alpha \le 1$ and $1 \le p < \infty$ we then get (using Hölder's inequality in the case $1 < p < \infty$)

$$\|f_{\hat{x}_k}\|_{\Lambda^{1/p-\alpha}} \le c\|g_{\hat{x}_k}\|_{L^p(\mathbb{R})}.$$

Taking $L^p(\mathbb{R}^{n-1})$-norm with respect to $\hat{x}_k$ and using Fubini's theorem we obtain (5.10), and then also (5.9). □

Remark 5.6. *As we mentioned above, Lemma 5.5 is already known in the case $\alpha = 1$. The proof appeared in* [19] *with* 4 *as the constant in inequality* (5.10)*.*

We will now apply Theorem 4.5 and the preceding lemma to prove the aforementioned embedding of the spaces $L^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$.

Theorem 5.7. *Let $n \ge 2$, $1 \le p_1,\dots,p_n < \infty$, and $0 < \alpha_1,\dots,\alpha_n \le 1$. Put*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \quad \textit{and} \quad p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}.$$

Assume that $p \le n/\alpha$ and that

$$\frac{1}{p} - \frac{\alpha}{n} - \frac{1}{p_k} + \alpha_k > 0, \quad k = 1,\dots,n.$$

Put

$$q = \begin{cases} np/(n-\alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Suppose $f \in L^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$. Then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|_{q,p} \le c \prod_{k=1}^n \|D_k^{\alpha_k} f\|_{p_k}^{\alpha/(n\alpha_k)}, \tag{5.15}$$

where c only depends on $\alpha_1,\dots,\alpha_n,p_1,\dots,p_n$, and n.

Proof. As in Theorem 4.5 we set

$$V_k \equiv L^{p_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p_k-\alpha_k}_{x_k}(\mathbb{R})].$$

Since $f \in L^{\alpha_k}_{p_k}(\mathbb{R}^n)$, for $k = 1, \dots, n$, it follows from Lemma 5.5 that $f \in V_k$ and

$$\|f\|_{V_k} \le c\|D_k^{\alpha_k} f\|_{p_k} < \infty. \tag{5.16}$$

Thus $f \in \cap_{k=1}^n V_k$. We also have $f \in S_0(\mathbb{R}^n)$, by Proposition 5.1. So by Theorem 4.5, $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \le c \prod_{k=1}^n \|f\|_{V_k}^{\alpha/(n\alpha_k)}. \tag{5.17}$$

Now, (5.15) follows from (5.17), (5.16), (2.27), and (2.29). □

Remark 5.8. *Let $p < n/\alpha$. The following special cases of the preceding theorem are known:*

If $\alpha_1 = \cdots = \alpha_n = 1$, but the numbers p_k may be distinct, then the above theorem is a special case of Theorem 13.1 in [17].

If the numbers p_k all coincide, but the numbers α_k may be distinct, then the above theorem is a special case of Theorem 9.3 in [17].

5.2. **Limiting embeddings and anisotropic Sobolev-Besov spaces.** Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$. For $x, h \in \mathbb{R}^n$ we use the notation

$$\Delta(h)f(x) = f(x+h) - f(x),$$

and set

$$I_p(h) = \|\Delta(h)f\|_p. \tag{5.18}$$

The L^p-modulus of continuity of f is the function $t \mapsto \omega(f;t)_p$ defined for $t > 0$ by

$$\omega(f;t)_p = \sup_{|h|\le t} I_p(h).$$

Further, for $h \ge 0$ and $x \in \mathbb{R}^n$, we set

$$I_{p,k}(h) = \|\Delta(he_k)f\|_p$$

(e_k is the kth unit coordinate vector in $\mathbb{R}^n$). We define the partial L^p-modulus of continuity of f with respect to x_k as

$$\omega_k(f;t)_p = \sup_{0\le h\le t} I_{p,k}(h). \tag{5.19}$$

Let $0 < \alpha < 1$ and $1 \le p, \theta < \infty$. The following relation between $\omega(f;t)_p$ and $\omega_k(f;t)_p$ is easy to verify

$$\max_{k=1,\dots,n} \omega_k(f;t)_p \le \omega(f;t)_p \le \sum_{k=1}^n \omega_k(f;t)_p. \tag{5.20}$$

We define the Besov space $B^\alpha_{p,\theta}(\mathbb{R}^n)$ as consisting of all $f \in L^p(\mathbb{R}^n)$ for which

$$\|f\|_{b^\alpha_{p,\theta}} \equiv \Big(\int_0^\infty (t^{-\alpha}\omega(f;t)_p)^\theta \frac{dt}{t}\Big)^{1/\theta} < \infty.$$

The partial Besov space $B^{\alpha}_{p,\theta;k}(\mathbb{R}^n)$ is similarly defined as the class of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{b^{\alpha}_{p,\theta;k}} \equiv \Big(\int_0^{\infty} (t^{-\alpha}\omega_k(f;t)_p)^{\theta}\frac{dt}{t}\Big)^{1/\theta} < \infty.$$

We set $B^{\alpha}_p \equiv B^{\alpha}_{p,p}$ and $B^{\alpha}_{p;k} \equiv B^{\alpha}_{p,p;k}$. Let $0 < \alpha_1, \dots, \alpha_n < 1$ and $1 \le p_1, \dots, p_n < \infty$. The anisotropic Besov space $B^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$ is defined as the intersection

$$B^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n) = \cap_{k=1}^n B^{\alpha_k}_{p_k;k}(\mathbb{R}^n).$$

We set also

$$\|f\|_{b^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}} = \sum_{k=1}^n \|f\|_{b^{\alpha_k}_{p_k;k}}.$$

We let $B^{\alpha_1,\dots,\alpha_n}_p$ denote $B^{\alpha_1,\dots,\alpha_n}_{p,\dots,p}$. It follows from (5.20) that if $\alpha_k = \alpha$ and $p_k = p$, $k = 1, \dots, n$, then the spaces $B^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$ and $B^{\alpha}_p(\mathbb{R}^n)$ coincide and

$$\frac{1}{n}\|f\|_{b^{\alpha,\dots,\alpha}_{p,\dots,p}} \le \|f\|_{b^{\alpha}_p} \le \|f\|_{b^{\alpha,\dots,\alpha}_{p,\dots,p}} \tag{5.21}$$

The following theorem is well known (see [11]).

Theorem 5.9. *Let* $0 < \alpha_1, \dots, \alpha_n < 1$,

$$\alpha \equiv n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1},$$

and $1 \le p < n/\alpha$. *Set* $q = np/(n - \alpha p)$. *For every* $f \in B^{\alpha_1,\dots,\alpha_n}_p(\mathbb{R}^n)$ *we have* $f \in L^{q,p}(\mathbb{R}^n)$ *and*

$$\|f\|_{q,p} \le c\sum_{k=1}^n \|f\|_{b^{\alpha_k}_{p;k}}, \tag{5.22}$$

where c *only depends on* $\alpha_1, \dots, \alpha_n$, p, *and* n.

Suppose $\alpha_1 = \dots = \alpha_n = \alpha$. By (2.24), (5.22), and (5.21),

$$\|f\|_q \le c\|f\|_{b^{\alpha}_p}. \tag{5.23}$$

Bourgain, Brezis, and Mironescu [6] proved a limiting relation between the Besov norm and the Sobolev norm. They showed that for any $f \in W^1_p(\mathbb{R}^n)$ $(1 \le p < \infty)$ there holds the equality

$$\lim_{\alpha\to 1-}(1-\alpha)^{1/p}\|f\|_{b^{\alpha}_p} = \Big(\frac{1}{p}\Big)^{1/p}\|\nabla f\|_p. \tag{5.24}$$

The sharp asymptotic of the best constant in (5.23) as $\alpha \to 1-$ was found by Bourgain, Brezis, and Mironescu in [7]. Namely, they proved that if $1/2 < \alpha < 1$, $1 \le p < n/\alpha$, and $q = np/(n-\alpha p)$, then for any $f \in B_p^\alpha(\mathbb{R}^n)$,

$$\|f\|_q^p \le c_n \frac{1-\alpha}{(n-\alpha p)^{p-1}} \|f\|_{b_p^\alpha}^p. \tag{5.25}$$

They where the first to explicitly observe that embeddings for Sobolev spaces can be derived from embeddings of Besov spaces. Indeed, in view of (5.24), Sobolev's inequality (1.1) can be considered as a limiting case of (5.25).

The next theorem was obtained in [19] as a corollary of estimates via mixed norms (see Theorem 1.6 and Proposition 5.14 below).

Theorem 5.10. *Let $1 \le p < \infty$, $n \ge 2$, and $1/2 < \alpha_1, \dots, \alpha_n < 1$. Assume that*

$$\alpha \equiv n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \le \frac{n}{p}.$$

Let

$$q = \begin{cases} np/(n-\alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Then, for every $f \in B_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ we have that $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|_{q,p}^* \le c_n \prod_{k=1}^n [(1-\alpha_k)^{1/p} \|f\|_{b_{p;k}^{\alpha_k}}]^{\alpha/(n\alpha_k)}, \tag{5.26}$$

where c_n only depends on n.

Inequality (5.26) gives the sharp asymptotic behaviour of the best constant in (5.22) as some of the numbers α_k tend to 1.

We shall apply Theorem 4.5 to extend these results to the fully anisotropic space $B_{p_1,\dots,p_n}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$. First we will prove a version of the relation (5.24) for the partial Besov norm (this is Proposition 5.13 below). Observe that we follow here the approach in [20].

Lemma 5.11. *Let $1 \le p < \infty$, $1 \le k \le n$. Then $C_0^1(\mathbb{R}^n)$ is dense in $W_{p;k}^1(\mathbb{R}^n)$.*

The proof of this lemma is exactly the same as that of Proposition 1 in [30, p. 122].

Proposition 5.12. *Let $1 \le p < \infty$. For every $f \in W_{p;k}^1(\mathbb{R}^n)$ there holds the relation*

$$\lim_{t\to 0+} \frac{\omega_k(f;t)_p}{t} = \|D_k f\|_p.$$

Proof. For all $h \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^n$ we have

$$\Delta_k(h)f(x) \equiv f(x+he_k) - f(x) = \int_0^1 D_k f(x+the_k)hdt.$$

By Minkowskis inequality,

$$I_{p,k}(h) = \|\Delta_k(h)f\|_p \le$$

$$\le \Big(\int_{\mathbb{R}^n}\Big(\int_0^1 |D_k f(x+the_k)h|dt\Big)^p dx\Big)^{1/p} \le |h|\,\|D_k f\|_p.$$

Thus (recall (5.19)),

$$\omega_k(f,\delta)_p \le \delta\|D_k f\|_p \tag{5.27}$$

for all $\delta > 0$. Fix $\varepsilon > 0$. By Lemma 5.11, there exists a function $f_\varepsilon \in C_0^1(\mathbb{R}^n)$ such that

$$\|D_k(f - f_\varepsilon)\|_p \le \varepsilon. \tag{5.28}$$

It follows that

$$\omega_k(f_\varepsilon;\delta)_p \le \omega_k(f;\delta)_p + \omega_k(f - f_\varepsilon;\delta)_p \le \omega_k(f;\delta)_p + \delta\varepsilon, \tag{5.29}$$

where the last inequality holds by (5.27) and (5.28). Since f_ε has compact support, there exists a compact set $K \subset \mathbb{R}^n$ which contains the support of $\Delta_k(\delta)f_\varepsilon - \delta D_k f_\varepsilon$, for all $0 \le \delta \le 1$. Since this function is continuous on $\mathbb{R}^n$, it follows that

$$\frac{1}{\delta}\Delta_k(\delta)f_\varepsilon - D_k f_\varepsilon \to 0$$

uniformly on K, as $\delta \to 0$. Hence, by the uniform convergence theorem,

$$\mu_\varepsilon(\delta) \equiv \|\frac{1}{\delta}\Delta_k(\delta)f_\varepsilon - D_k f_\varepsilon\|_{L^p(K)} \to 0,$$

as $\delta \to 0$. Thus, there exists a number $\delta_\varepsilon > 0$ such that

$$\mu_\varepsilon(\delta) < \varepsilon, \tag{5.30}$$

for all $0 < \delta < \delta_\varepsilon$. Furthermore, by the triangle inequality

$$\|D_k f_\varepsilon\|_p \le \mu_\varepsilon(\delta) + \frac{1}{\delta}\omega_k(f_\varepsilon;\delta)_p. \tag{5.31}$$

So, for all $0 < \delta < \delta_\varepsilon$ we have

$$\|D_k f\|_p \le \varepsilon + \|D_k f_\varepsilon\|_p \le \varepsilon + \mu_\varepsilon(\delta) + \frac{1}{\delta}\omega_k(f_\varepsilon;\delta)_p \le$$

$$\le 3\varepsilon + \frac{1}{\delta}\omega_k(f;\delta)_p,$$

where the first inequality holds by (5.28), the second by (5.31), and the third by (5.30) and (5.29). By this and inequality (5.27) the proof is complete. $\square$

Proposition 5.13. *Let $1 \le p, \theta < \infty$. If $f \in W^1_{p;k}(\mathbb{R}^n)$, then*

$$\lim_{\alpha \to 1-} (1-\alpha)^{1/\theta} \|f\|_{b^\alpha_{p,\theta;k}} = \theta^{-1/\theta} \|D_k f\|_p. \tag{5.32}$$

Proof. Put $A_k = \|D_k f\|_p$. Fix $\varepsilon > 0$ and let $0 < \alpha < 1$. By Proposition 5.12 there exists $\delta > 0$ such that

$$\left| \left(\frac{\omega_k(f;t)_p}{t} \right)^\theta - A_k^\theta \right| < \varepsilon, \tag{5.33}$$

for all $0 < t < \delta$. We have

$$(1-\alpha)\|f\|^\theta_{b^\alpha_{p,\theta;k}} - \frac{A_k^\theta}{\theta} = I_\delta + J_\delta, \tag{5.34}$$

where

$$I_\delta = (1-\alpha) \int_0^\delta [t^{-\alpha} \omega_k(f;t)_p]^\theta \frac{dt}{t} - \frac{A_k^\theta}{\theta}$$

and

$$J_\delta = (1-\alpha) \int_\delta^\infty [t^{-\alpha} \omega_k(f;t)_p]^\theta \frac{dt}{t}.$$

To estimate I_δ we observe that

$$(1-\alpha) \int_0^\delta t^{(1-\alpha)\theta - 1} \left[\left(\frac{\omega_k(f;t)_p}{t} \right)^\theta - A_k^\theta \right] dt =$$

$$= (1-\alpha) \int_0^\delta [t^{-\alpha} \omega_k(f;t)_p]^\theta \frac{dt}{t} - \frac{A_k^\theta}{\theta} \delta^{(1-\alpha)\theta} = I_\delta + \frac{A_k^\theta}{\theta}(1 - \delta^{(1-\alpha)\theta}).$$

Applying (5.33) we now get

$$|I_\delta + \frac{A_k^\theta}{\theta}(1 - \delta^{(1-\alpha)\theta})| \le \frac{\varepsilon}{\theta} \delta^{(1-\alpha)\theta},$$

and thus

$$|I_\delta| \le \frac{\varepsilon}{\theta} \delta^{(1-\alpha)\theta} + \frac{A_k^\theta}{\theta} |1 - \delta^{(1-\alpha)\theta}|. \tag{5.35}$$

Moreover, since $\omega_k(f;t)_p \le 2\|f\|_p$ for all $t > 0$, we have

$$J_\delta \le \frac{2(1-\alpha)\|f\|_p}{\alpha\theta\delta^{\alpha\theta}}. \tag{5.36}$$

Combining (5.34), (5.35) and (5.36) we get

$$\left| (1-\alpha)\|f\|^\theta_{b^\alpha_{p,\theta;k}} - \frac{A_k^\theta}{\theta} \right| \le \frac{\varepsilon}{\theta} \delta^{(1-\alpha)\theta} + \frac{A_k^\theta}{\theta} |\delta^{(1-\alpha)\theta} - 1| + \frac{2(1-\alpha)\|f\|_p}{\alpha\theta\delta^{\alpha\theta}}.$$

Clearly there exists $0 < \sigma < 1$ such that the left-hand side in this inequality is less that 3ε for all $\alpha \in (1-\sigma, 1)$. □

As it was observed in [19], the constant in (5.26) has the sharp asymptotic behaviour as some of the numbers α_k tend to 1. Indeed, if a function $f \in B_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ has a weak derivative $D_k f \in L^p(\mathbb{R}^n)$ for some k, then for the corresponding factor in (5.26) we have by (5.32) (with $\theta = p$)

$$(1-\alpha_k)^{1/p}\|f\|_{b_p^{\alpha_k}} \to \Big(\frac{1}{p}\Big)^{1/p}\|D_k f\|_p, \quad \alpha_k \to 1- .$$

The next proposition was proved in [19].

Proposition 5.14. *Let $n \ge 2$, $0 < \alpha < 1$, $1 \le p < \infty$, and $1 \le k \le n$. Set*

$$V_k = L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p-\alpha}_{x_k}(\mathbb{R})].$$

Assume that $f \in B^\alpha_{p;k}(\mathbb{R}^n)$. Then $f \in V_k$ and

$$\|f\|_{V_k} \le 100[\alpha(1-\alpha)]^{1/p}\|f\|_{b^\alpha_{p;k}}.$$

Applying Theorem 4.5 and Proposition 5.14, we obtain the following.

Theorem 5.15. *Let $n \ge 2$, $1/2 < \alpha_1, \dots, \alpha_n < 1$, and $1 \le p_1, \dots, p_n < \infty$. Set*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \quad \text{and} \quad p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}.$$

Assume that $p \le n/\alpha$ and set

$$q = \begin{cases} np/(n-\alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Assume also that for $k = 1, \dots, n$,

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \frac{1}{p_k} + \alpha_k > 0.$$

If $f \in B^{\alpha_1,\dots,\alpha_n}_{p_1\dots,p_n}(\mathbb{R}^n)$, then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \le c_n \prod_{k=1}^n \Big[d_k(1-\alpha_k)^{1/p_k}\|f\|_{b^{\alpha_k}_{p_k;k}}\Big]^{\alpha/(n\alpha_k)}, \tag{5.37}$$

where c_n depends only on n,

$$d_k = r_k^{-1/p_k}\max(R, p_k)^2, \quad \text{and} \quad R = \max_{k=1,\dots,n}\frac{1}{r_k}\max_{k=1,\dots,n} r_k.$$

Proof. By Proposition 5.14,

$$f \in V_k \equiv L^{p_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p_k-\alpha_k}_{x_k}(\mathbb{R})]$$

and (since $\alpha_k < 1$ for all k)

$$\|f\|_{V_k} \le 100(1-\alpha_k)^{1/p_k}\|f\|_{b^{\alpha_k}_{p_k;k}}.$$

So, $f \in \cap_{k=1}^n V_k$. By assumption, $f \in B^{\alpha_1,\dots,\alpha_n}_{p_1\dots,p_n}(\mathbb{R}^n) \subset S_0(\mathbb{R}^n)$. Hence, by Theorem 4.5, $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \le c \prod_{k=1}^n \|f\|_{V_k}^{\alpha/(n\alpha_k)},$$

where c is the constant defined in (4.34). We have

$$c = c_n \max_{k=1,\dots,n} 4^{n\alpha_k} \prod_{k=1}^n \Big(1+\frac{1}{\alpha_k}\Big)^{\alpha/(n\alpha_k)} \prod_{k=1}^n \Big[\Big(\frac{\alpha_k}{r_k}\Big)^{1/p_k} \max(R', p_k)^2\Big]^{\alpha/(n\alpha_k)},$$

where $R' = \max_{k=1,\dots,n} r_k/\alpha_k \max_{k=1,\dots,n} 1/r_k$. Since $1/2 < \alpha_1, \dots, \alpha_n < 1$ it follows that

$$c \le c_n \prod_{k=1}^n d_k^{\alpha/(n\alpha_k)}.$$

Now (5.37) follows from the three preceding inequalities. □

We will now define the Sobolev-Besov space $WB^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$. Let $n \ge 2$, $0 \le m \le n$, $\alpha_1 = \cdots = \alpha_m = 1$, and $0 < \alpha_{m+1}, \dots, \alpha_n < 1$ (with the obvious interpretation if $m = 0$ or $m = n$). Also let $1 \le p_1, \dots, p_n < \infty$. A measurable function f on $\mathbb{R}^n$ belongs to the space $WB^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$ if $f \in W^1_{p_k;k}(\mathbb{R}^n)$ for $k = 1, \dots, m$ and $f \in B^{\alpha_k}_{p_k;k}(\mathbb{R}^n)$ for $k = m+1, \dots, n$. A number of embedding theorems have been obtained for these spaces by Gagliardo, Slobodeckii, Uspenskii, and other authors (see [4, Chapter 18.15]).

The next result coincides with the preceding theorem in the case $m = 0$.

Theorem 5.16. *Let $n \ge 2$, $1 \le p_1, \dots, p_n < \infty$, and $0 \le m \le n$. We also let $\alpha_1 = \cdots = \alpha_m = 1$ and $1/2 < \alpha_{m+1}, \dots, \alpha_n < 1$. Set*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \quad \text{and} \quad p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}.$$

Assume that $p \le n/\alpha$ and set

$$q = \begin{cases} np/(n-\alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Assume also that for $k = 1, \dots, n$,

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \frac{1}{p_k} + \alpha_k > 0.$$

If $f \in WB^{\alpha_1,\dots,\alpha_n}_{p_1\dots,p_n}(\mathbb{R}^n)$, then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \le c_n \prod_{k=1}^m \Big[d_k \|D_k f\|_{p_k}\Big]^{\alpha/n} \prod_{k=m+1}^n \Big[d_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}}\Big]^{\alpha/(n\alpha_k)},$$

where c_n *depends only on* n,

$$d_k = r_k^{-1/p_k} \max(R, p_k)^2, \quad \textit{and} \quad R = \max_{k=1,\dots,n} \frac{1}{r_k} \max_{k=1,\dots,n} r_k.$$

Theorem 5.16 can be proved in the same way as Theorem 5.15, using Theorem 4.5, Proposition 5.14, and Lemma 5.5 (see remark 5.6). As we will now show, Theorem 5.16 can also be obtained by using Theorem 5.15 and letting $\alpha_1, \dots, \alpha_m \to 1-$. Assume therefore that the conditions of Theorem 5.16 hold. For simplicity we shall only consider the case $m = 1$. Then $f \in W^1_{p_1;1}(\mathbb{R}^n)$. By Proposition 5.13, it follows that there exists an $\varepsilon \in (0,1)$ such that $f \in B^{1-\varepsilon}_{p_1;1}(\mathbb{R}^n)$. Put $\tilde{\alpha}_1 = 1 - \varepsilon$. Let $\tilde{\alpha}$, $\tilde{p}$, $\tilde{q}$, $\tilde{r}_k$, and $\tilde{d}_k$, $k = 1, \dots, n$, be defined by replacing α_1 by $\tilde{\alpha}_1$ in α, p, q, r_k, and d_k, respectively. By assumption we know that $r_k > 0$, for all k. Therefore we may assume that ε was choosen so small that also $\tilde{r}_k > 0$, for all k. Moreover, $\tilde{p} < n/\tilde{\alpha}$. Indeed,

$$\tilde{\alpha}\tilde{p} = n\Big(\frac{1}{\tilde{\alpha}_1 p_1} + \sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1} < n\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1} = \alpha p \le n.$$

Hence, the conditions of Theorem 5.15 are satisfied, and so we have that

$$\|f\|^*_{\tilde{q},\tilde{p}} \le c_n \prod_{k=1}^{n} \Big[\tilde{d}_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\tilde{\alpha}_k}_{p_k;k}}\Big]^{\tilde{\alpha}/(n\tilde{\alpha}_k)}. \tag{5.38}$$

By Fatou's lemma,

$$\int_0^\infty \big[t^{1/q} f^*(t)\big]^p \frac{dt}{t} \le \liminf_{\varepsilon \to 0+} \int_0^\infty \big[t^{1/\tilde{q}} f^*(t)\big]^{\tilde{p}} \frac{dt}{t}$$

Let $\varepsilon \to 0+$ in (5.38). By the preceding inequality and Proposition (5.13),

$$\|f\|^*_{q,p} \le c_n \Big[d_1 \Big(\frac{1}{p_1}\Big)^{1/p_1} \|D_1 f\|_{p_1}\Big]^{\alpha/n} \prod_{k=2}^{n} \Big[d_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}}\Big]^{\alpha/(n\alpha_k)}.$$

This proves Theorem 5.16.

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Embedding Theorems for Mixed Norm Spaces and Applications

This thesis is devoted to the study of mixed norm spaces that arise in connection with embeddings of Sobolev and Besov type spaces. The work in this direction originates in a paper due to Gagliardo (1958), and was continued by Fournier (1988) and by Kolyada (2005).

We consider fully anisotropic mixed norm spaces. Our main theorem states an embedding of these spaces into Lorentz spaces. Applying this result, we obtain sharp embedding theorems for anisotropic fractional Sobolev spaces and anisotropic Sobolev-Besov spaces. The methods used are based on non-increasing rearrangements and on estimates of sections of functions and sections of sets. We also study limiting relations between embeddings of spaces of different type. More exactly, mixed norm estimates enable us to get embedding constants with sharp asymptotic behaviour. This gives an extension of the results obtained for isotropic Besov spaces B_p^α by Bourgain, Brezis, and Mironescu, and for Besov spaces $B^{\alpha_1,\dots,\alpha_n}_p$ by Kolyada.

We study also some basic properties (in particular the approximation properties) of special weak type spaces that play an important role in the construction of mixed norm spaces and in the description of Sobolev type embeddings.

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