Noname manuscript No.
(will be inserted by the editor)

Niclas Bernhoff

Half-Space Problem for the Discrete Boltzmann Equation: Condensing Vapor Flow in the Presence of a Non-Condensable Gas

Received: date / Accepted: date

Abstract We consider a non-linear half-space problem related to the condensation problem for the discrete Boltzmann equation and extend some known results for a single-component gas to the case when a non-condensable gas is present. The vapor is assumed to tend to an assigned Maxwellian at infinity, as the non-condensable gas tends to zero at infinity. We assume that the vapor is completely absorbed and that the non-condensable gas is diffusively reflected at the condensed phase and that the vapor molecules leaving the condensed phase are distributed according to a given distribution. The conditions, on the given distribution, needed for the existence of a unique solution of the problem are investigated. We also find exact solvability conditions and solutions for a simplified six+four-velocity model, as the given distribution is a Maxwellian at rest, and study a simplified twelve+six-velocity model.

Keywords Boltzmann equation · boundary layers · discrete velocity models · half-space problem · non-condensable gas

Mathematics Subject Classification (2000) 82C40 · 76P05 · 35Q20

1 Introduction

In this paper we consider the condensation problem for a single-component gas or vapor when a non-condensable gas is present [15]. Formulation and motivation of the problem can be found in [15]. The vapor is assumed to tend to an assigned Maxwellian M_{∞}^{A}, with a flow velocity towards the condensed phase, at infinity, while the non-condensable gas tends to zero at infinity.

N. Bernhoff
Department of Mathematics, Karlstad University, 65188 Karlstad, SWEDEN
Tel.: +46-54-7002024
Fax: +46-54-7001851
E-mail: niclas.bernhoff@kau.se

Steady condensation of the vapor takes place at the condensed phase, which is held at a constant temperature. We assume that the vapor is completely absorbed and that the non-condensable gas is diffusively reflected at the condensed phase, i.e. there is no net flow across the condensed phase and the gas molecules leaving the condensed phase are distributed according to a non-drifting Maxwellian M_0^B at the condensed phase. The vapor molecules leaving the condensed phase are distributed according to a given distribution. The conditions, on the given distribution at the condensed phase, needed for the existence of a unique solution of the problem are investigated. We assume that the given distribution is sufficiently close to the Maxwellian M_∞^A at the infinity and that the total mass of the non-condensable gas relatively this distance is sufficiently small. The explicit number of conditions on the given distribution is given in Theorem 2, under some assumptions on the discrete velocity models for the gases.The typical case is that the given distribution is the Maxwellian at the condensed phase [15]. However, we can't be sure that there is any Maxwellian at rest close enough to the Maxwellian at infinity, but if there is, of course our results are valid also in this case.

Similar problems have been studied for the discrete Boltzmann equation for single species (a vapor in the absence of a non-condensable gas) [4],[3], and references therein, and binary mixtures of two vapors [5], as well as for the full Boltzmann equation for single species [19],[1],[20] and binary mixtures [18], and references therein. For the discrete Boltzmann equation, one obtain for binary mixtures of two vapors a similar structure as for single-component gases [5]. One can then extend results for half-space problems of single-component gases [2], [4] to yield also for binary mixtures of two vapors. However, though, both complete absorption and diffuse reflection conditions are considered (at least implicitly) for both single-component gases and binary mixtures of two vapors in [4] and [5], the situation will be different when one of the gases is non-condensable. The fact that the distribution function for the non-condensable gas tends to zero at infinity changes the situation. First of all we can not use the standard transformation used in [4] and [5], but we use instead a slight modification of it, which changes the structure of the obtained system. Secondly, the trivial case when the non-condensable gas is absent, i.e. the case of a single-component gas considered in [4] and [19], is a trivial solution of the system. Therefore, in difference to the case of a vapor, we need, in the case of a non-condensable gas, to have a free parameter, which will later be settled by fixing the amount of the non-condensable gas. Hence, even if our proof is influenced by the proof in [4] (and [19]) for single-component gases, we have to take these differences into account. To our knowledge, there is no corresponding results for the full Boltzmann equation up to now.

The paper is organized as follows. In Section 2 we present the discrete velocity model for binary mixtures and some of its properties. We make a transformation and obtain a transformed system, presented with some of its properties in Section 3. In Section 4 we present our assumptions and our main result in Theorem 2. The proof of our main result (Theorem 2) is presented in Section 5. In Section 6 we find an exact solvability condition and the solution for a simplified six+four-velocity model, for which the non-linear problem becomes linear. Here the vapor molecules leaving the condensed phase are distributed according to the Maxwellian at the condensed phase. In Section 7 we prove all the necessary conditions for existence, except one, which still is most likely to be fulfilled, for a simplified twelve+six-velocity model.

2 Discrete velocity models (DVMs) for binary mixtures

We first remind some properties of the discrete Boltzmann equation, or the general discrete velocity model (DVM), for binary mixtures [5].

The planar stationary discrete Boltzmann equation for a binary mixture of the gases A and B reads

$$\begin{cases} \xi_i^{A,1}\dfrac{dF_i^A}{dx} = Q_i^{AA}(F^A,F^A) + Q_i^{BA}(F^B,F^A),\ i=1,...,n_A, \\ \xi_j^{B,1}\dfrac{dF_j^B}{dx} = Q_j^{AB}(F^A,F^B) + Q_j^{BB}(F^B,F^B),\ j=1,...,n_B, \end{cases} \tag{1}$$

where $V_\alpha = \{\xi_1^\alpha, ..., \xi_{n^\alpha}^\alpha\} \subset \mathbb{R}^d$, $\alpha,\beta \in \{A,B\}$ are finite sets of velocities, $F_i^\alpha = F_i^\alpha(x) = F^\alpha(x,\xi_i^\alpha)$ for $i=1,...,n_\alpha$, and $F^\alpha = F^\alpha(x,\xi)$ represents the microscopic density of particles (of the gas α) with velocity ξ at position $x \in \mathbb{R}$. We denote by m_α the mass of a molecule of the gas α. Here and below, $\alpha,\beta \in \{A,B\}$.

For a function $g^\alpha = g^\alpha(\xi)$ (possibly depending on more variables than ξ), we will identify g^α with its restrictions to the set V^α, but also when suitable consider it like a vector function

$$g^\alpha = (g_1^\alpha, ..., g_{n_\alpha}^\alpha),\ \text{with}\ g_i^\alpha = g^\alpha(\xi_i^\alpha).$$

The collision operators $Q_i^{\beta\alpha}(F^\beta,F^\alpha)$ in (1) are given by

$$Q_i^{\beta\alpha}(F^\beta,F^\alpha) = \sum_{k=1}^{n_\alpha}\sum_{j,l=1}^{n_\beta} \Gamma_{ij}^{kl}(\beta,\alpha)\left(F_k^\alpha F_l^\beta - F_i^\alpha F_j^\beta\right) \text{ for } i=1,...,n_\alpha,$$

where it is assumed that the collision coefficients $\Gamma_{ij}^{kl}(\beta,\alpha)$, with $1 \leq i,k \leq n_\alpha$ and $1 \leq j,l \leq n_\beta$, satisfy the relations

$$\Gamma_{ij}^{kl}(\alpha,\alpha) = \Gamma_{ji}^{kl}(\alpha,\alpha) \text{ and } \Gamma_{ij}^{kl}(\beta,\alpha) = \Gamma_{kl}^{ij}(\beta,\alpha) = \Gamma_{ji}^{lk}(\alpha,\beta) \geq 0,$$

with equality unless the conservation laws

$$m_\alpha\xi_i^\alpha + m_\beta\xi_j^\beta = m_\alpha\xi_k^\alpha + m_\beta\xi_l^\beta \text{ and } m_\alpha\left|\xi_i^\alpha\right|^2 + m_\beta\left|\xi_j^\beta\right|^2 = m_\alpha\left|\xi_k^\alpha\right|^2 + m_\beta\left|\xi_l^\beta\right|^2$$

are satisfied. We denote

$$\begin{aligned} F &= \left(F^A,F^B\right) = \left(F^A(\xi),F^B(\xi)\right) \text{ and } Q(F,F) \\ &= \left(Q^{AA}(F^A,F^A) + Q^{BA}(F^B,F^A), Q^{AB}(F^A,F^B) + Q^{BB}(F^B,F^B)\right). \end{aligned}$$

Then the system (1) can be rewritten as

$$D\frac{dF}{dx}=Q\left(F,F\right),$$

where

$$D=\begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}, \text{ with } D_\alpha=\operatorname{diag}(\xi_1^{\alpha,1},...,\xi_{n_\alpha}^{\alpha,1}).$$

We consider the case of non-zero $\xi_i^{\alpha,1}$, $\xi_i^{\alpha,1}\neq 0$, and we can then (without loss of generality) assume that

$$D_\alpha=\begin{pmatrix} D_\alpha^+ & 0 \\ 0 & -D_\alpha^- \end{pmatrix},$$

where

$$D_\alpha^+=\operatorname{diag}(\xi_1^{\alpha,1},...,\xi_{n_\alpha^+}^{\alpha,1}) \text{ and } D_\alpha^-=-\operatorname{diag}(\xi_{n_\alpha^++1}^{\alpha,1},...,\xi_{n_\alpha}^{\alpha,1}), \text{ with}$$
$$\xi_1^{\alpha,1},...,\xi_{n_\alpha^+}^{\alpha,1}>0 \text{ and } \xi_{n_\alpha^++1}^{\alpha,1},...,\xi_{n_\alpha}^{\alpha,1}<0.$$

The collision operator $Q(f,f)$ can be obtained from the bilinear expressions

$$Q_i\left(F,G\right)=\frac{1}{2}\sum_{j,k,l=1}^{n_A}\Gamma_{ij}^{kl}(A,A)(F_k^AG_l^A+G_k^AF_l^A-F_i^AG_j^A-G_i^AF_j^A)$$
$$+\frac{1}{2}\sum_{k=1}^{n_A}\sum_{j,l=1}^{n_B}\Gamma_{ij}^{kl}(B,A)(F_k^AG_l^B+G_k^AF_l^B-F_i^AG_j^B-G_i^AF_j^B),\ i=1,...,n_A,$$

and

$$Q_{n_A+i}\left(F,G\right)=\frac{1}{2}\sum_{k=1}^{n_B}\sum_{j,l=1}^{n_A}\Gamma_{ij}^{kl}(A,B)(F_k^BG_l^A+G_k^BF_l^A-F_i^BG_j^A-G_i^BF_j^A)$$
$$+\frac{1}{2}\sum_{j,k,l=1}^{n_B}\Gamma_{ij}^{kl}(B,B)(F_k^BG_l^B+G_k^BF_l^B-F_i^BG_j^B-G_i^BF_j^B),\ i=1,...,n_B.$$

Denoting

$$Q(F,G)=\left(Q_1\left(F,G\right),...,Q_n\left(F,G\right)\right), \text{ with } n=n_A+n_B,$$

we see that, for arbitrary F and G

$$Q\left(F,G\right)=Q\left(G,F\right).$$

A vector $\phi=\left(\phi^A,\phi^B\right)$ is a collision invariant if and only if

$$\phi_i^\alpha+\phi_j^\beta=\phi_k^\alpha+\phi_l^\beta,$$

for all indices $1\leq i,k\leq n_\alpha$, $1\leq j,l\leq n_\beta$ and $\alpha,\beta\in\{A,B\}$, such that $\Gamma_{ij}^{kl}(\beta,\alpha)\neq 0$.

We consider below only DVMs, such that the DVMs for the gases A and B are normal, i.e. the only collision invariants of the forms $\phi = \left(\phi^A, 0\right)$ and $\phi = \left(0, \phi^B\right)$, respectively, fulfills

$$\phi^\alpha = \phi^\alpha(\xi) = a_\alpha + m_\alpha \mathbf{b} \cdot \xi + c m_\alpha \left|\xi\right|^2,$$

for some constant $a_\alpha, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. It is also preferable that any general collision invariant of our DVMs is of the form

$$\phi = \left(\phi^A, \phi^B\right), \text{ with } \phi^\alpha = \phi^\alpha(\xi) = a_\alpha + m_\alpha \mathbf{b} \cdot \xi + c m_\alpha \left|\xi\right|^2, \qquad (2)$$

for some constant $a_A, a_B, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$. In this case the equation

$$\langle \phi, Q\left(F, F\right) \rangle = 0$$

has the general solution (2). Here and below, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^n$. Such DVMs, being normal both considering the gases together as a mixture as well as considering them separately as single species, is called supernormal [7]. This property is fulfilled for the continuous Boltzmann equation. In the discrete case we can obtain so called spurious (unphysical) collision invariants. However, possible spurious collision invariants (for the mixture) don't seem to affect the qualitative properties of our results. We would also like our DVMs to fulfill that the equation

$$\left\langle \phi, Q^{AB}\left(F^A, F^B\right) \right\rangle = 0 \qquad (3)$$

has the general solution $\phi = a$, where a is constant. We call a supernormal DVM fulfilling condition (3) for optinormal. This property is fulfilled for the continuous Boltzmann equation [14], but not necessarily for a DVM. However, we will below see that we can relax this assumption a little.

Example 1 The DVM, with

$$m_A = 2m_B,$$

where the vapor, gas A, is modeled by the twelve-velocity model with velocities

$$(\pm 1, \pm 1), (\pm 1, \pm 3), \text{ and } (\pm 3, \pm 1),$$

and the non-condensable gas B is modeled by the six-velocity model with velocities

$$(\pm 2, 0) \text{ and } (\pm 2, \pm 4),$$

is optinormal.

A binary Maxwellian distribution (or just a bi-Maxwellian) is a function

$$M = \left(M^A, M^B\right),$$

such that

$$Q(M, M) = 0 \text{ and } M_i^\alpha \geq 0 \text{ for all } 1 \leq i \leq n_\alpha.$$

All bi-Maxwellians are of the form $M = e^\phi$, where ϕ is a collision invariant, i.e. for normal models we will have

$$M = \left(M^A, M^B\right), \text{ with } M^\alpha = e^{\phi^\alpha} = e^{a_\alpha + m_\alpha \mathbf{b} \cdot \xi + c m_\alpha |\xi|^2}. \qquad (4)$$

We will study distributions F, such that

$$F \to \left(M^A, 0\right) \text{ as } x \to \infty, \text{ where } M^A = e^{\phi^A} = e^{a_A + m_A \mathbf{b} \cdot \xi + c m_A |\xi|^2}. \qquad (5)$$

3 Transformed system

For a bi-Maxwellian
$$M = \left(M^A, \epsilon^2 M^B\right),$$
where $M^\alpha = e^{\phi^\alpha} = e^{a_\alpha + m_\alpha \mathbf{b}\cdot\xi + cm_\alpha|\xi|^2}$ and ϵ is a so far undetermined positive constant less or equal to 1, $0 < \epsilon \leq 1$, we obtain, by denoting
$$F = \left(M^A, 0\right) + \sqrt{M} f, \tag{6}$$
in Eq.(1), the system
$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA} f^A = -\epsilon L_{BA} f^B + S_{AA}(f^A, f^A) + \epsilon S_{BA}(f^B, f^A) \\ D_B \dfrac{df^B}{dx} + L_{AB} f^B = \epsilon S_{BB}(f^B, f^B) + S_{AB}(f^A, f^B) \end{cases},$$
where
$$\left(L_{AA} f^A\right)_i = -2 \sum_{k=1}^{n_A} \sum_{j,l=1}^{n_A} \sqrt{M_j^A} \Gamma_{ij}^{kl}(A, A)(\sqrt{M_l^A} f_k^A - \sqrt{M_j^A} f_i^A),$$
$$\left(L_{AB} f^B\right)_{i'} = -\sum_{k=1}^{n_B} \sum_{j,l=1}^{n_A} \sqrt{M_j^A} \Gamma_{i'j}^{kl}(A, B)(\sqrt{M_l^A} f_k^B - \sqrt{M_j^A} f_{i'}^B), \text{ and}$$
$$\left(L_{BA} f^B\right)_i = -\sum_{k=1}^{n_A} \sum_{j,l=1}^{n_B} \sqrt{M_j^B} \Gamma_{ij}^{kl}(B, A)(\sqrt{M_k^A} f_l^B - \sqrt{M_i^A} f_j^B),$$
for $i = 1, ..., n_A$ and $i' = 1, ..., n_B$,

and the quadratic parts $S_{\alpha\beta}$ are given by
$$\left(S_{\alpha\beta}(f^\alpha, f^\beta)\right)_i = \sum_{k=1}^{n_\beta} \sum_{j,l=1}^{n_\alpha} \sqrt{M_j^\alpha} \Gamma_{ij}^{kl}(\alpha, \beta)(f_k^\alpha f_l^\beta - f_i^\alpha f_j^\beta), \; i = 1, ..., n_\beta.$$

The matrices $L_{A\alpha}$ are symmetric and semi-positive. Furthermore,
$$L_{AB} f^B = 0 \text{ if } f^B \in \operatorname{span}(\sqrt{M^B}),$$
$$L_{AA} f^A = 0 \text{ if and only if } f^A = \sqrt{M^A}\phi^A,$$
where $\phi = \left(\phi^A, 0\right)$ is a collision invariant,
$$\left\langle L_{BA} f^B, \sqrt{M^A} \right\rangle = \left\langle S_{BA}(f^B, f^A), \sqrt{M^A} \right\rangle = \left\langle S_{\alpha B}(f^\alpha, f^B), \sqrt{M^B} \right\rangle = 0,$$
and
$$\left\langle S_{AA}(f^A, f^A), \sqrt{M^A}\phi^A \right\rangle = 0.$$
In the continuous case $\ker(L_{AB}) = \operatorname{span}(\sqrt{M^B})$ [14], so for an optimal model
$$N(L_{AB}) = \operatorname{span}(\sqrt{M^B}),$$

(cf. assumption (3)). We will, however, relax this assumption below. Here and below, we denote by $N(L_{\alpha\beta})$ the null-space of $L_{\alpha\beta}$.

By assumption (5)

$$f \to 0 \text{ as } x \to \infty.$$

We denote by $n_\alpha^\pm$, where $n_\alpha^+ + n_\alpha^- = n_\alpha$, and $m_\alpha^\pm$, with $m_\alpha^+ + m_\alpha^- = q_\alpha$, the numbers of positive and negative eigenvalues (counted with multiplicity) of the matrices D_α and $D_\alpha^{-1}L_{A\alpha}$ respectively, and by m_α^0 the number of zero eigenvalues of $D_\alpha^{-1}L_{A\alpha}$. Moreover, we denote by k_α^+, k_α^-, and l_α, with $k_\alpha^+ + k_\alpha^- = k_\alpha$, where $k_\alpha + l_\alpha = p_\alpha$, the numbers of positive, negative, and zero eigenvalues of the $p_\alpha \times p_\alpha$ matrix K_α, with entries $k_{ij}^\alpha = \left\langle y_i^\alpha, y_j^\alpha \right\rangle_{D_\alpha} = \left\langle y_i^\alpha, D_\alpha y_j^\alpha \right\rangle$, such that $\left\{ y_1^\alpha, ..., y_{p_\alpha}^\alpha \right\}$ is a basis of the null-space of $L_{A\alpha}$, i.e. in our case,

$$\begin{aligned} p_A &= d+2,\ p_B \geq 1, \text{ and span} \left(y_1^A, ..., y_{d+2}^A \right) = N(L_{AA}) \\ &= \text{span}(\sqrt{M^A}, \sqrt{M^A}\xi^{A,1}, ..., \sqrt{M^A}\xi^{A,d}, \sqrt{M^A} \left|\xi^A\right|^2). \end{aligned}$$

We remind that we by $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product on $\mathbb{R}^n$ and below we also denote

$$\langle \cdot, \cdot \rangle_{D_\alpha} = \langle \cdot, D_\alpha \cdot \rangle .$$

We now remind a result by Bobylev and Bernhoff in [6] (see also [2]) and apply it in a specific case of interest for us.

Theorem 1 *The numbers of positive, negative and zero eigenvalues of* $D_\alpha^{-1}L_{A\alpha}$ *are given by*

$$\begin{cases} m_\alpha^+ = n_\alpha^+ - k_\alpha^+ - l_\alpha \\ m_\alpha^- = n_\alpha^- - k_\alpha^- - l_\alpha \\ m_\alpha^0 = p_\alpha + l_\alpha \end{cases} .$$

In the proof of Theorem 1 bases

$$u_1^\alpha, ..., u_{q_\alpha}^\alpha, y_1^\alpha, ..., y_{k_\alpha}^\alpha, z_1^\alpha, ..., z_{l_\alpha}^\alpha, w_1^\alpha, ..., w_{l_\alpha}^\alpha \tag{7}$$

of $\mathbb{R}^{n_\alpha}$, $\alpha \in \{A, B\}$, such that

$$y_i^\alpha, z_r^\alpha \in N(L_{A\alpha}),\ D_\alpha^{-1}L_{A\alpha}w_r^\alpha = z_r^\alpha \text{ and } D_\alpha^{-1}L_{A\alpha}u_\tau^\alpha = \lambda_\tau^\alpha u_\tau^\alpha, \tag{8}$$

and

$$\begin{gathered} \langle u_\tau^\alpha, u_\nu^\alpha \rangle_{D_\alpha} = \lambda_\tau^\alpha \delta_{\tau\nu}, \text{ with } \lambda_1^\alpha, ..., \lambda_{m_\alpha^+}^\alpha > 0 \text{ and } \lambda_{m_\alpha^+ + 1}^\alpha, ..., \lambda_{q_\alpha}^\alpha < 0, \\ \left\langle y_i^\alpha, y_j^\alpha \right\rangle_{D_\alpha} = \gamma_i^\alpha \delta_{ij}, \text{ with } \gamma_1^\alpha, ..., \gamma_{k_\alpha^+}^\alpha > 0 \text{ and } \gamma_{k_\alpha^+ + 1}^\alpha, ..., \gamma_{k_\alpha}^\alpha < 0, \\ \langle u_\tau^\alpha, z_r^\alpha \rangle_{D_\alpha} = \langle u_\tau^\alpha, w_r^\alpha \rangle_{D_\alpha} = \langle u_\tau^\alpha, y_i^\alpha \rangle_{D_\alpha} = \langle w_r^\alpha, y_i^\alpha \rangle_{D_\alpha} = \langle z_r^\alpha, y_i^\alpha \rangle_{D_\alpha} = 0, \\ \langle w_r^\alpha, w_s^\alpha \rangle_{D_\alpha} = \langle z_r^\alpha, z_s^\alpha \rangle_{D_\alpha} = 0 \text{ and } \langle w_r^\alpha, z_s^\alpha \rangle_{D_\alpha} = \delta_{rs}, \end{gathered} \tag{9}$$

are constructed.

If we assume that

$$n_B^- = n_B^+, \text{ or equivalently } n_B = 2n_B^+,$$

and that

$$\xi^B_{i+n_B^+} = (-\xi_i^{B,1}, \xi_i^{B,2}, ..., \xi_i^{B,d}),\ \xi_i^{B,1} > 0, \text{ for } i = 1, ..., n_B^+, \tag{10}$$

then

$$D_B^- = D_B^+.$$

Let b be the first component of $\mathbf{b}$ in Eqs.(4). If we assume that $b < 0$, then

$$k_B^- \geq 1,$$

since

$$\left\langle \sqrt{M^B}, \sqrt{M^B} \right\rangle_{D_B} = \sum_{i=1}^{n_B^+} \xi_i^{B,1}(1 - e^{-2b\xi_i^{B,1}})M_i^B < 0.$$

For an optimal model

$$k_B^- = 1$$

(cf. condition (3)) and hence,

$$k_B^+ = l_B = 0 \text{ and } m_B^+ = n_B^+. \tag{11}$$

We will relax condition (3) by assuming

$$k_B^- = p_B \geq 1.$$

Then the conditions (11) are still satisfied.

4 Main result

We consider the non-linear system

$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA}f^A = -\epsilon L_{BA}f^B + S_{AA}(f^A, f^A) + \epsilon S_{BA}(f^B, f^A) \\ D_B \dfrac{df^B}{dx} + L_{AB}f^B = \epsilon S_{BB}(f^B, f^B) + S_{AB}(f^A, f^B) \end{cases}, \tag{12}$$

where the solution tends to zero at infinity, i.e.

$$f^A(x) \to 0 \text{ and } f^B(x) \to 0 \text{ as } x \to \infty, \tag{13}$$

and

$$L_{BA}f^B, S_{BA}(f^B, f^A) \in \operatorname{span}(\sqrt{M^A})^\perp,\ S_{AA}(f^A, f^A) \in N(L_{AA})^\perp, \\ \text{and } S_{\alpha B} \in N(L_{AB})^\perp.$$

We define the projections

$$R_+^\alpha : \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha^+} \text{ and } R_-^\alpha : \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha^-}, n_\alpha^- = n_\alpha - n_\alpha^+,$$

by

$$R_+^\alpha s = s_+^\alpha = (s_1, ..., s_{n_\alpha^+}) \text{ and } R_-^\alpha s = s_-^\alpha = (s_{n_\alpha^+ + 1}, ..., s_{n_\alpha})$$

for $s^{\alpha} = (s_1, ..., s_{n_\alpha})$.

We will below assume that $n_B^- = n_B^+$, and that the symmetry relation (10) is fulfilled. Furthermore, we assume that

$$k_B^- = p_B.$$

Then

$$k_B^+ = l_B = 0, m_B^+ = n_B^+ \text{ and } D_B^- = D_B^+.$$

At $x = 0$ we assume the boundary conditions

$$f_+^A(0) = h_0 \text{ and } f_+^B(0) = Cf_-^B(0) \tag{14}$$

where C is the $n_B^+ \times n_B^+$ matrix, with the elements

$$c_{ij} = \frac{\xi_j^{B,1}\sqrt{M_{n_B^+ + j}^B}M_{0i}^B}{\left\langle D_B^- M_{0-}^B, 1\right\rangle \sqrt{M_i^B}}.$$

and

$$h_0 = \frac{1}{\sqrt{M_+^A}}(a_0 - M_+^A) \in \mathbb{R}^{n_A^+},$$

where $M_0^B = K_0^B e^{c_0 m_B \left|\xi^B\right|^2}$, with $K_0^B > 0$, and $a_0 \in \mathbb{R}^{n_A^+}$. This corresponds to the boundary conditions

$$\begin{cases} F_+^A(0) = a_0 \\ F_+^B(0) = C_0 F_-^B(0) \end{cases}$$

where C_0 is the $n_B^+ \times n_B^+$ matrix, with the elements

$$c_{0ij} = \frac{\xi_j^{B,1} M_{0i}^B}{\left\langle D_B^- M_{0-}^B, 1\right\rangle}$$

(the discrete version of the diffusive boundary conditions, cf. [10], [2], or [4]), before the expansion (6).

We consider the case of condensation, i.e. we assume that $b < 0$, where b is the first component of $\mathbf{b}$ in Eq.(4). For the Boltzmann equation there is a critical number $b_- < 0$ (where $-b_-$ is the speed of sound) [9], such that

$$\begin{cases} k_A^+ = 1 \text{ and } l_A = 0 \text{ if } b_- < b < 0 \\ k_A^+ = 0 \text{ and } l_A = 1 \text{ if } b = b_- \\ k_A^+ = l_A = 0 \text{ if } b < b_- \end{cases}. \tag{15}$$

We assume that we have a DVM with a critical number $b_- < 0$, such that Eq.(15) is fulfilled. In fact, this number can be explicitly calculated for a plane axially symmetric 12-velocity model (assuming that the solution is symmetric with respect to the x-axis) see Section 7 below.

Furthermore, for $b_- < b < 0$ we will assume that

$$\begin{aligned} & R_+^A\sqrt{M^A} \notin R_+^A U_A^+, \\ & \text{with } U_A^+ = \operatorname{span}(u : L_{AA}u = \lambda D_A u,\ \lambda > 0) = \operatorname{span}(u_1^A, ..., u_{m_A^+}^A), \end{aligned} \quad (16)$$

or, equivalently,

$$\dim(R_+^A \widetilde{U}_A^+) = m_A^+ + 1 = n_A^+, \text{ with } \widetilde{U}_A^+ = \operatorname{span}(u_1^A, ..., u_{n_A^+}^A, \sqrt{M^A}).$$

In this case, we can assume that $y_{p_A}^A = \sqrt{M^A}$ without loss of generality, since $l_A = 0$.

Remark 1 In fact, we could instead of $\sqrt{M^A}$ take any vector $y \in N(L_{AA})$, such that

$$R_+^A y \notin R_+^A U_A^+ \text{ and } \left\langle L_{BA} f^B, y \right\rangle = \left\langle S_{BA}(f^B, f^A), y \right\rangle = 0,$$

as $y_{p_A}^A$.

We introduce the operator $\mathcal{C} : \mathbb{R}^{n_B} \to \mathbb{R}^{n_B^+}$, given by

$$\mathcal{C} = R_+^B - C R_-^B.$$

We will assume that the set

$$U_B^+ = \operatorname{span}(\mathcal{C}u : L_{AB}u = \lambda D_B u, \lambda > 0) = \operatorname{span}(\mathcal{C}u_1^B, ..., \mathcal{C}u_{n_B^+}^B)$$

has non-zero codimension, i.e.

$$\dim U_B^+ < n_B^+, \quad (17)$$

but, also that the set

$$\widetilde{U}_B^+ = \operatorname{span}(\mathcal{C}u_1^B, ..., \mathcal{C}u_{n_B^+}^B, \mathcal{C}\sqrt{M^B})$$

has codimension 0, i.e.

$$\dim \widetilde{U}_B^+ = n_B^+. \quad (18)$$

Therefore, the set U_B^+ has codimension 1, i.e.

$$\dim U_B^+ = n_B^+ - 1.$$

We can without loss of generality assume that

$$\mathcal{C}u_{n_B^+}^B \in \operatorname{span}(\mathcal{C}u_1^B, ..., \mathcal{C}u_{n_B^+-1}^B).$$

If the set U_B^+ would have had codimension 0, i.e. if $\dim U_B^+ = n_B^+$, then the only possibility would have been $f^B(x) = 0$.

We fix ϵ to be

$$\epsilon = \min\left\{|h_0|, 1\right\},$$

and the total mass of the gas B to be m_B^{tot}, i.e.

$$\epsilon m_B \sum_{i=1}^{n_B} \int_0^\infty \sqrt{M_i^B} f_i^B(x)\, dx = m_B^{tot}, \tag{19}$$

for a given positive constant m_B^{tot}. Clearly, the case $m_B^{tot} = 0$, corresponds to the case of single species considered in [4].

We now state our main result.

Theorem 2 *Assume that we have a DVM with a critical number $b_- < 0$, such that Eq.* (15) *is fulfilled, let conditions* (17) *and* (18)*, for $b_- < b < 0$ also condition* (16)*, be fulfilled, and suppose that $\langle h_0, h_0 \rangle_{D_A^+}$ is sufficiently small and that m_B^{tot} is sufficiently small relatively $|h_0|$. Then with*

$$k_A^+ + l_A = \begin{cases} 1 \text{ if } b_- \le b < 0 \\ 0 \text{ if } b < b_- \end{cases}$$

conditions on h_0, the system (12) *with the boundary conditions* (13),(14) *under the condition* (19)*, has a locally unique solution.*

Theorem 2 is proved in Section 5.

Remark 2 If

$$\frac{M_0^B}{\sqrt{M^B}} \in U_B^+$$

then condition (17) is fulfilled, since

$$\mathcal{C} \frac{M_0^B}{\sqrt{M^B}} = 0.$$

Half-space problems for the Boltzmann equation are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers [12],[13]. Half-space problems provide the boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary. Theorem 2 tells us that the number of parameters to be specified in the boundary conditions depends on whether the condensing vapor flow is subsonic or supersonic. This behavior has earlier been found numerically in [16] and [17] as the vapor molecules leaving the condensed phase are distributed according to the Maxwellian at the condensed phase. We can't be sure that there is any Maxwellian at rest close enough to the Maxwellian at infinity, but if this is the case, our results are still valid. To our knowledge, this is the first rigorous analytical result of this kind and no corresponding results exist for the full Boltzmann equation.

5 Proof of the main result

We add (cf. Refs. [19] and [4]) a damping term

$$-\gamma(\Psi_A, \Psi_B) = -\gamma(D_A P_A^+ f^A, D_B P_B^+ f^B),$$

to the right-hand side of the system (12) and obtain

$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA} f^A = -\epsilon L_{BA} f^B + S_{AA}(f^A, f^A) + \epsilon S_{BA}(f^B, f^A) - \gamma \Psi_A \\ D_B \dfrac{df^B}{dx} + L_{AB} f^B = \epsilon S_{BB}(f^B, f^B) + S_{AB}(f^A, f^B) - \gamma \Psi_B \end{cases}, \tag{20}$$

where $\gamma > 0$ and $\Psi_\alpha = D_\alpha P_\alpha^+ f^\alpha$, with

$$P_A^+ f^A = \begin{cases} \dfrac{\left\langle f^A(x), y_{p_A}^A \right\rangle_{D_A}}{\left\langle y_{p_A}^A, y_{p_A}^A \right\rangle_{D_A}} y_{p_A}^A \text{ if } b_- < b < 0 \\ \left\langle f^A(x), z_1^A \right\rangle_{D_A} w_1^A \text{ if } b = b_- \\ 0 \text{ if } b < b_- \end{cases}, \text{ and}$$

$$P_B^+ f^B = \frac{\left\langle f^B(x), \sqrt{M^B} \right\rangle_{D_B}}{\left\langle \sqrt{M^B}, \sqrt{M^B} \right\rangle_{D_B}} \sqrt{M^B}.$$

We can, without loss of generality, assume that

$$y_{p_B}^B = \sqrt{M^B},$$

since $l_B = 0$.

First we consider the corresponding linearized inhomogeneous system

$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA} f^A = g_A - \gamma D_A P_A^+ f^A \\ D_B \dfrac{df^B}{dx} + L_{AB} f^B = g_B - \gamma D_B P_B^+ f^B \end{cases}, \tag{21}$$

where $g_\alpha = g_\alpha(x) : \mathbb{R}_+ \to \mathbb{R}^{n_\alpha}$ are given functions such that

$$\left\langle g_\alpha(x), \sqrt{M^\alpha} \right\rangle = 0. \tag{22}$$

Below, we will consider the case

$$b_- < b < 0.$$

The system (21) with the boundary conditions (13) has (under the assumption that all necessary integrals exist) the general solution, using the notations in Eqs.(7)-(9),

$$\begin{cases} f^A(x) = \sum\limits_{r=1}^{q_A} \beta_r^A(x)\, u_r^A + \sum\limits_{i=1}^{p_A} \mu_i^A(x)\, y_i^A \\ f^B(x) = \sum\limits_{r=1}^{q_B} \beta_r^B(x)\, u_r^B + \sum\limits_{i=1}^{p_B} \mu_i^B(x)\, y_i^B \end{cases}, \tag{23}$$

where $q_\alpha = n_\alpha - p_\alpha$ and

$$\begin{cases} \mu_i^\alpha(x) = -\int\limits_x^\infty \widetilde{\mu}_i^\alpha(\tau)\, d\tau \text{ and }, \ i = 1, ..., p_\alpha - 1, \text{ and } \mu_{p_\alpha}^\alpha(x) = \mu_{p_\alpha}^\alpha(0)\, e^{-\gamma x} \\ \beta_r^\alpha(x) = \beta_r^\alpha(0)\, e^{-\lambda_r^\alpha x} + \int\limits_0^x e^{(\tau-x)\lambda_r^\alpha} \widetilde{\beta}_r^\alpha(\tau)\, d\tau, \ r = 1, ..., m_\alpha^+, \\ \beta_r^\alpha(x) = -\int\limits_x^\infty e^{(\tau-x)\lambda_r^\alpha} \widetilde{\beta}_r^\alpha(\tau)\, d\tau, \ r = m_\alpha^+ + 1, ..., q_\alpha, \end{cases} \tag{24}$$

with

$$\widetilde{\mu}_i^\alpha(x) = \langle g_\alpha(x), y_i^\alpha \rangle \text{ and } \widetilde{\beta}_r^\alpha(x) = \frac{\langle g_\alpha(x), u_r^\alpha \rangle}{\lambda_r^\alpha}. \tag{25}$$

By the boundary conditions (14), we obtain the systems

$$\begin{aligned} &\sum_{r=1}^{m_A^+} \beta_r^A(0)\, R_+^A u_r^A + \mu_{p_A}^A(0)\, R_+^A y_{p_A}^A \\ &= h_0 + \sum_{i=1}^{p_A-1} \int\limits_0^\infty \widetilde{\mu}_i^A(\tau)\, d\tau\, R_+^A y_i^A + \sum_{r=m_A^+ + 1}^{q_A} \int\limits_0^\infty e^{\tau\lambda_r^A} \widetilde{\beta}_r^A(\tau)\, d\tau\, R_+^A u_r^A, \\ &\text{and } \sum_{r=1}^{n_B^+ - 1} \beta_r^B(0)\, \mathcal{C} u_r^B + \mu_{p_B}^B(0)\, \mathcal{C} y_{p_B}^B = \beta_{n_B^+}^B(0)\, \mathcal{C} u_{n_B^+}^B \\ &+ \sum_{r=n_B^+ + 1}^{q_B} \int\limits_0^\infty e^{\tau\lambda_r^B} \widetilde{\beta}_r^B(\tau)\, d\tau\, \mathcal{C} u_r^B + \sum_{i=1}^{p_B-1} \int\limits_0^\infty e^{\tau\lambda_r^B} \widetilde{\mu}_i^B(\tau)\, d\tau\, \mathcal{C} y_i^B, \\ &\text{with } \mathcal{C} = R_+^B - C R_-^B. \end{aligned} \tag{26}$$

For $a_0 = M_+^A$, we have the trivial solution $f^A = 0$. Therefore, we consider only non-zero h_0,

$$h_0 = \frac{1}{\sqrt{M_+^A}} (a_0 - M_+^A) \neq 0,$$

below. The system (26) has (under the assumption that all necessary integrals exist) a solution, with a free parameter $\vartheta = \beta^{B}_{n_B^+}(0)$, if we assume that conditions (16), (17), and (18) are fulfilled.

Theorem 3 *Assume that conditions* (16)*,* (17)*,* (18)*, and* (22) *are fulfilled and that all necessary integrals exist. Then the system* (21) *with the boundary conditions* (13),(14)*, has a solution, with a free parameter*

$$\vartheta = \beta^{B}_{n_B^+}(0),$$

given by Eqs.(23)-(26)*.*

Note that ϑ will be determined by condition (19).

We fix a number σ, such that

$$0 < 2\sigma \leq \min\{|\lambda^{\alpha}| \neq 0;\ \det(\lambda^{\alpha} D_{\alpha} - L_{A\alpha}) = 0\} \text{ and } 2\sigma \leq \gamma$$

and introduce the norm (cf. [11] and [4])

$$|h|_{\sigma} = \sup_{x \geq 0} e^{\sigma x} |h(x)|,$$

the Banach space

$$\mathcal{X} = \left\{ h \in \mathcal{B}^0[0,\infty) \mid |h|_{\sigma} < \infty \right\}$$

and its closed convex subset

$$\mathcal{S}_R = \left\{ h \in \mathcal{B}^0[0,\infty) \mid |h|_{\sigma} \leq R\,|h_0| \right\},$$

where R is a, so far, undetermined positive constant.

We assume that conditions (16), (17), and (18) are fulfilled and introduce the operator $\Theta(f) = (\Theta_A(f), \Theta_B(f))$ on $\mathcal{X}$, defined by

$$\begin{cases} \Theta_A(f) = \sum\limits_{r=1}^{q_A} \beta_r^A(f(x))\, u_r^A + \sum\limits_{i=1}^{p_A} \mu_i^A(f(x))\, y_i^A \\ \Theta_B(f) = \sum\limits_{r=1}^{q_B} \beta_r^B(f(x))\, u_r^B + \sum\limits_{i=1}^{p_B} \mu_i^B(f(x))\, y_i^B \end{cases},$$

where $q_{\alpha} = n_{\alpha} - p_{\alpha}$ and

$$\begin{cases} \mu_i^{\alpha}(f(x)) = -\int\limits_x^{\infty} \widetilde{\mu}_i^{\alpha}(f(\tau))\, d\tau \text{ and }, \ i = 1, ..., p_{\alpha} - 1, \\ \mu_{p_{\alpha}}^{\alpha}(f(x)) = \mu_{p_{\alpha}}^{\alpha}(f(0))\, e^{-\gamma x} \\ \beta_r^{\alpha}(f(x)) = \beta_r^{\alpha}(f(0))\, e^{-\lambda_r^{\alpha} x} + \int\limits_0^x e^{(\tau - x)\lambda_r^{\alpha}} \widetilde{\beta}_r^{\alpha}(f(\tau))\, d\tau, \ r = 1, ..., m_{\alpha}^+, \\ \beta_r^{\alpha}(f(x)) = -\int\limits_x^{\infty} e^{(\tau - x)\lambda_r^{\alpha}} \widetilde{\beta}_r^{\alpha}(f(\tau))\, d\tau, \ r = m_{\alpha}^+ + 1, ..., q_{\alpha}, \end{cases}$$

with $\beta_1^\alpha(f(0)),...,\beta_{m_\alpha^+}^\alpha(f(0))$, and $\mu_k^A(f(0))$ given by the systems

$$\sum_{r=1}^{m_A^+}\beta_r^A(f(0))\,R_+^A u_r^A+\mu_{p_A}^A(f(0))\,R_+^A y_{p_A}^A=h_0$$
$$+\sum_{i=1}^{k_A-1}\int_0^\infty \widetilde{\mu}_i^A(f(\tau))\,d\tau\,R_+^A y_i^A+\sum_{r=m^++1}^{q_A}\int_0^\infty e^{\tau\lambda_r^A}\widetilde{\beta}_r^A(f(\tau))\,d\tau\,R_+^A u_r^A,$$
$$\sum_{r=1}^{n_B^+-1}\beta_r^B(f(0))\,\mathcal{C}u_r^B+\mu_{p_B}^B(f(0))\,\mathcal{C}y_{p_B}^B=\vartheta\mathcal{C}u_{n_B^+}^B$$
$$+\sum_{r=n_B^++1}^{q_B}\int_0^\infty e^{\tau\lambda_r^B}\widetilde{\beta}_r^B(f(\tau))\,d\tau\,\mathcal{C}u_r^B+\sum_{i=1}^{p_B-1}\int_0^\infty e^{\tau\lambda_r^B}\widetilde{\mu}_i^B(f(\tau))\,d\tau\,\mathcal{C}y_i^B,$$
$$\text{and } \beta_{n_B^+}^B(f(0))=\vartheta,\ \vartheta\in\mathbb{R},$$

$\mathcal{C}=R_+^B-CR_-^B$, and

$$\begin{cases}\widetilde{\mu}_i^A(f)=\left\langle -\epsilon L_{BA}f^B+S_{AA}\left(f^A,f^A\right)+\epsilon S_{BA}\left(f^B,f^A\right),y_i^A\right\rangle\\ \widetilde{\beta}_r^A(f)=\left\langle -\epsilon L_{BA}f^B+S_{AA}\left(f^A,f^A\right)+\epsilon S_{BA}\left(f^B,f^A\right),u_r^A\right\rangle\\ \widetilde{\beta}_r^B(f)=\left\langle \epsilon S_{BB}\left(f^B,f^B\right)+S_{AB}\left(f^A,f^B\right),u_r^B\right\rangle\end{cases}.$$

Lemma 1 *Let $f,h\in\mathcal{X}$, assume that conditions* (16), (17), *and* (18) *are fulfilled, and fix a positive constant $K_\vartheta>0$. Then there is a positive constant K (independent of f and h), such that*

$$|\Theta(0)|_\sigma\le K\,|h_0|\,, \tag{27}$$
$$|\Theta(f)-\Theta(h)|_\sigma\le K(|f|_\sigma+|h|_\sigma+|h_0|)\,|f-h|_\sigma\,, \tag{28}$$

for all ϑ, such that $|\vartheta|\le K_\vartheta\,|h_0|$.

Proof The proof can be carried out in a similar way to the proof of the corresponding lemma in the case of single-component gases (Lemma 5.2 in [4]), noting that

$$\begin{aligned}\left|S_{\alpha\beta}(f^\alpha,f^\beta)-S_{\alpha\beta}(h^\alpha,h^\beta)\right|_{2\sigma}&=\left|S_{\alpha\beta}(f^\alpha-h^\alpha,f^\beta)+S_{\alpha\beta}(h^\alpha,f^\beta-h^\beta)\right|_{2\sigma}\\&\le K_1^{\alpha\beta}(\left|f^\beta\right|_\sigma|f^\alpha-h^\alpha|_\sigma+|h^\alpha|_\sigma\left|f^\beta-h^\beta\right|_\sigma)\end{aligned}$$

and

$$\left|L_{BA}(f^B-h^B)\right|_\sigma\le K_2\left|f^B-h^B\right|_\sigma.$$

□

Theorem 4 *Let conditions* (16), (17), *and* (18) *be fulfilled and fix a positive constant $K_\vartheta>0$. Then there is a positive number δ_0, such that if*

$$|h_0|\le\delta_0,$$

then the system (20) *with the boundary conditions* (13),(14), *has a unique solution $f=f(x)$ in $\mathcal{S}_R$ for a suitable chosen R, for all ϑ, such that*

$$|\vartheta|\le K_\vartheta\,|h_0|\,.$$

Proof By estimates (27) and (28), there is a positive number K such that

$$|\Theta(f)|_{\sigma} = |\Theta(f) - \Theta(0) + \Theta(0)|_{\sigma} \leq K(|h_0| + |f|_{\sigma}^2 + |h_0|\,|f|_{\sigma}) \tag{29}$$

if $f \in \mathcal{X}$.

Let

$$R = 2K \text{ and } \delta_0 = \frac{1}{R^2 + R}.$$

By estimates (28) and (29)

$$|\Theta(f)|_{\sigma} \leq (\frac{1}{2} + \frac{R^2 + R}{2}\,|h_0|)R\,|h_0| \leq R\,|h_0|$$

and

$$|\Theta(f) - \Theta(h)|_{\sigma} \leq (2R+1)K\,|h_0|\,|f-h|_{\sigma} \leq \varrho\,|f-h|_{\sigma}\,,$$
$$\text{with } \varrho = \frac{2R^2 + R}{2}\delta_0 < 1,$$

if $f, h \in \mathcal{S}_R$ and $|h_0| \leq \delta_0$.

The theorem follows by the contraction mapping theorem. □

Theorem 5 *The solution of Theorem 4 is a solution of the problem* (12),(13),(14) *if and only if*

$$P_A^+ f^A(0) = 0.$$

Proof The relations

$$\mu_{p_\alpha}^{\alpha}(f(x)) = \mu_{p_\alpha}^{\alpha}(f(0))e^{-\gamma x},$$

are fulfilled if $f(x)$ is a solution of Theorem 4. Hence,

$$P_\alpha^+ f^\alpha(0) = 0 \text{ if and only if } P_\alpha^+ f^\alpha(x) \equiv 0.$$

But,

$$P_B^+ f^B(0) = \frac{\left\langle f^B(0), \sqrt{M^B}\right\rangle_{D_B}}{\left\langle \sqrt{M^B}, \sqrt{M^B}\right\rangle_{D_B}}\sqrt{M^B}$$

and

$$\begin{aligned}
&\left\langle f^B(0), \sqrt{M^B}\right\rangle_{D_B} \\
&= \left\langle CR_-^B f^B(0), R_+^B\sqrt{M^B}\right\rangle_{D_B^+} - \left\langle R_-^B f^B(0), R_-^B\sqrt{M^B}\right\rangle_{D_B^-} \\
&= \sum_{i,j=1}^{n_B^+} \xi_i^{B,1} M_{0i}^B \frac{\xi_j^{B,1}\sqrt{M_{n_B^+ + j}^B}}{\langle D_B^- M_{0-}^B, 1\rangle} f_{n_B^+ + j}^B(0) - \sum_{j=1}^{n_B^-} \xi_j^{B,1}\sqrt{M_{n_B^+ + j}^B} f_{n_B^+ + j}^B(0) = 0,
\end{aligned}$$

since $D_B^- = D_B^+$, $n_B^- = n_B^+$, $M_{0-}^B = M_{0+}^B$ etc.. Hence,

$$P_B^+ f^B(x) \equiv 0 \text{ and } P_A^+ f^A(0) = 0 \text{ if and only if } P_A^+ f^A(x) \equiv 0.$$

□

We remind that, since $b_- < b < 0$,

$$\Psi_A = P_A^+ f^A = \frac{\left\langle f^A(x), y_{p_A}^A \right\rangle_{D_A}}{\left\langle y_{p_A}^A, y_{p_A}^A \right\rangle_{D_A}} y_k^A$$

and denote by $\mathbb{I}^\gamma$ the linear solution operator

$$\mathbb{I}^\gamma(h_0, \vartheta) = f^A(0),$$

where $f(x)$ is given by

$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA} f^A + \gamma \Psi_A = -\epsilon L_{BA} f^B \\ D_B \dfrac{df^B}{dx} + L_{AB} f^B = 0 \\ R_+^A f^A(0) = h_0 \text{ and } \mathcal{C} f^B(0) = 0 \\ f = \left(f^A, f^B\right) \to 0, \text{ as } x \to \infty \end{cases}.$$

Similarly, we denote by $\mathcal{I}^\gamma$ the nonlinear solution operator

$$\mathcal{I}^\gamma(h_0, \vartheta) = f^A(0),$$

where $f(x)$ is given by

$$\begin{cases} D_A \dfrac{df^A}{dx} + L_{AA} f^A + \gamma \Psi_A = S_{AA}\left(f^A, f^A\right) + \epsilon S_{BA}\left(f^B, f^A\right) - \epsilon L_{BA} f^B \\ D_B \dfrac{df^B}{dx} + L_{AB} f^B = \epsilon S_{BB}\left(f^B, f^B\right) + S_{AB}\left(f^A, f^B\right) \\ R_+^A f^A(0) = h_0 \text{ and } \mathcal{C} f^B(0) = 0 \\ f = \left(f^A, f^B\right) \to 0, \text{ as } x \to \infty \end{cases}.$$

By Theorem 5, the solution of Theorem 4 is a solution of the problem (12),(13),(14) if and only if

$$P_A^+ \mathcal{I}^\gamma(h_0, \vartheta) = 0.$$

Let

$$r_1 = \frac{r_1'}{\sqrt{\left\langle r_1', r_1' \right\rangle_{D_A^+}}}, \text{ with } r_1' = R_+^A y_{k_A}^A - \sum_{r=1}^{m_A^+} \frac{\left\langle R_+^A y_{k_A}^A, R_+^A u_r^A \right\rangle_{D_A^+}}{\left\langle R_+^A u_r^A, R_+^A u_r^A \right\rangle_{D_A^+}} R_+^A u_r \neq 0.$$

Then

$$P_A^+ \mathbb{I}^\gamma(h_0, 0) = 0 \Leftrightarrow h_0 \in \mathcal{R}^{\perp_{D_A^+}}, \text{ where}$$
$$\mathcal{R}^{\perp_{D_A^+}} = \left\{ u \in \mathbb{R}^{n_A^+} \;\middle|\; \langle u, r_1 \rangle_{D_A^+} = 0 \right\},$$

and

$$\mathcal{I}^\gamma(h_0, \vartheta) = \widetilde{\mathcal{I}}^\gamma(a_1, h_1, \vartheta), \text{ where}$$
$$h_0 = a_1 r_1 + h_1, \text{ with } h_1 \in \mathcal{R}^{\perp_{D_A^+}} \text{ and } a_1 = \langle h_0, r_1 \rangle_{D_A^+}.$$

Lemma 2 *Suppose that*

$$P_A^+ \mathcal{I}^\gamma(h_0, \vartheta) = 0.$$

Then h_0 *is a function of* h_1 *and* ϑ*, if* $\langle h_0, h_0 \rangle_{D_A^+}$ *and*

$$\frac{m_B^{tot}}{\epsilon} = m_B \sum_{i=1}^{n_B} \int_0^\infty \sqrt{M^B} f_i^B(x)\, dx$$

are sufficiently small.

Proof It is obvious that $\mathcal{I}^\gamma(0,0) = 0$ and that we for the Fréchet derivative of $\mathcal{I}^\gamma(\varepsilon h_0, 0)$ have

$$\left.\frac{d}{d\varepsilon} \mathcal{I}^\gamma(\varepsilon h_0, 0)\right|_{\varepsilon=0} = \mathbb{I}^\gamma(h_0, 0).$$

Then

$$\begin{aligned}
\left.\frac{\partial}{\partial a_1} \left\langle \widetilde{\mathcal{I}}^\gamma(a_1, h_1, \vartheta), y_{k_A}^A \right\rangle_{D_A}\right|_{(0,\dots,0)} &= \left.\frac{d}{d\varepsilon} \left\langle \mathcal{I}^\gamma(\varepsilon r_1, 0), y_{k_A}^A \right\rangle_{D_A}\right|_{\varepsilon=0} \\
&= \left\langle \mathbb{I}^\gamma(r_1, 0), y_{k_A}^A \right\rangle_{D_A} \neq 0.
\end{aligned}$$

By the implicit function theorem,

$$\left\langle \widetilde{\mathcal{I}}^\gamma(a_1, h_1, \vartheta), y_{k_A}^A \right\rangle_{D_A} = 0$$

defines

$$a_1 = a_1(h_1, \vartheta)$$

if $\langle h_0, h_0 \rangle_{D_A^+}$ and ϑ are sufficiently small. Clearly,

$$K_m \vartheta \leq \frac{m_B^{tot}}{\epsilon},$$

for some positive constant K_m. Hence, ϑ is sufficiently small if $\dfrac{m_B^{tot}}{\epsilon}$ is sufficiently small. □

The cases $b < b_-$ and $b = b_-$ can be treated in a similar way. In the case $b < b_-$, we are done by Theorem 4. In the case $b = b_-$ we can modify the proof inspired by the proof for single species in a recent unpublished work by the author. In this work the result in [4] is improved, by getting rid of some quite restrictive conditions in the degenerate cases (as $l > 0$), essentially built on the use of a different damping term in the proof than in [4].

6 Exact solution for a reduced six+four-velocity model

We now consider the case when the vapor, gas A, is modeled by a six-velocity model with velocities

$$(\pm 1, 0) \text{ and } (\pm 1, \pm 1),$$

and the non-condensable gas B is modelled by the classical Broadwell model [8] in plane with velocities

$$(\pm m, \pm m), \text{ where } m = \frac{m_A}{m_B}.$$

Note that for the Broadwell model we have only two linearly independent collision invariants, as the mass vector and the energy vector are linearly dependent, even if mass, momentum, and energy all are preserved. For a flow axially symmetric around the x-axis we obtain the reduced system

$$\begin{cases} \dfrac{dF_1^A}{dx} = \sigma_1 q_1 + \sigma_2 q_2 \\ \dfrac{dF_2^A}{dx} = -\sigma_1 q_1 + \sigma_3 q_3 \\ -\dfrac{dF_3^A}{dx} = -\sigma_1 q_1 - \sigma_2 q_2 \\ -\dfrac{dF_4^A}{dx} = \sigma_1 q_1 - \sigma_3 q_3 \\ m\dfrac{dF_1^B}{dx} = \sigma_2 q_2 + \sigma_3 q_3 \\ -m\dfrac{dF_2^B}{dx} = -\sigma_2 q_2 - \sigma_3 q_3 \end{cases},$$

where

$$q_1 = F_2^A F_3^A - F_1^A F_4^A, q_2 = F_3^A F_1^B - F_1^A F_2^B, q_3 = F_4^A F_1^B - F_2^A F_2^B,$$
$$F_1^A(x) = F^A(x, (1,0)), F_2^A(x) = F^A(x, (1,1)) = F^A(x, (1,-1)),$$
$$F_3^A(x) = F^A(x, (-1,0)), F_4^A(x) = F^A(x, (-1,1)) = F^A(x, (-1,-1)),$$
$$F_1^B(x) = F^B(x, (m,m)) = F^B(x, (m,-m)), \text{ and}$$
$$F_2^B(x) = F^B(x, (-m,m)) = F^B(x, (-m,-m)),$$

or equivalently

$$\begin{cases} D_A \dfrac{dF^A}{dx} = Q^{AA}(F^A, F^A) + Q^{BA}(F^B, F^A) \\ D_B \dfrac{dF^B}{dx} = Q^{AB}(F^A, F^B) + Q^{BB}(F^B, F^B) \end{cases},$$

where

$$D_A = \text{ diag}(1,1,-1,-1),\, D_B = \text{ diag}(m,-m),\, F^A = \left(F_1^A, F_2^A, F_3^A, F_4^A\right),$$
$$F^B = \left(F_1^B, F_2^B\right),\, Q^{AA}(F^A, F^A) = \sigma_1 q_1 (1,-1,-1,1),$$
$$Q^{BA}(F^B, F^A) = \sigma_2 q_2 (1,0,-1,0) + \sigma_3 q_3 (0,1,0,-1),$$
$$Q^{AB}(F^A, F^B) = (\sigma_2 q_2 + \sigma_3 q_3)(1,-1), \text{ and } Q^{BB}(F^B, F^B) = 0.$$

The set of collision invariants are generated by the collision invariants

$$\begin{cases} \phi_0 = (1,1,1,1,0,0) \\ \phi_1 = (0,0,0,0,1,1) \\ \phi_2 = (1,1,-1,-1,1,-1) \\ \phi_3 = (1,2,1,2,2m,2m) \end{cases},$$

and the Maxwellians are of the form

$$M = \left(M^A, M^B\right), \text{ with } M^A = s^A(1,q,p,pq) \text{ and } M^B = s^B(1,p), \qquad (30)$$

where $p, q, s^A, s^B > 0$.

We consider the case when $p > 1$ (corresponding to condensation) and assume the boundary conditions

$$\begin{cases} \begin{pmatrix} F_1^A(0) \\ F_2^A(0) \end{pmatrix} = s_0^A \begin{pmatrix} 1 \\ q_0 \end{pmatrix} \\ F_1^B(0) = F_2^B(0) \end{cases},$$

at the condensed phase, and

$$F^A \to M^A = s^A(1,q,p,pq), \text{ and } F^B \to 0 \text{ as } x \to \infty,$$

at the far end. We note that for this simplified model diffuse and specular reflection coincide for gas B.

We denote

$$F = \left(M^A, 0\right) + \sqrt{M} f,$$

for the Maxwellian of the form (30), such that $F^A \to M^A$ and $s^B = 1$. We let $\epsilon = 1$ here, cf. Eq.(6). We obtain the system (note that $S_{BB}(f^B, f^B) = 0$)

$$\begin{cases} \dfrac{df^A}{dx} + D_A^{-1} L_{AA} f^A = D_A^{-1}(S_{AA}(f^A, f^A) + S_{BA}(f^B, f^A) - L_{BA} f^B) \\ \dfrac{df^B}{dx} + D_B^{-1} L_{AB} f^B = D_B^{-1} S_{AB}(f^A, f^B) \end{cases}, \qquad (31)$$

where

$$D_A^{-1}L_{AA} = s^A\sigma_1 \begin{pmatrix} -pq & p\sqrt{q} & \sqrt{p}q & -\sqrt{pq} \\ p\sqrt{q} & -p & -\sqrt{pq} & \sqrt{p} \\ -\sqrt{p}q & \sqrt{pq} & q & -\sqrt{q} \\ \sqrt{pq} & -\sqrt{p} & -\sqrt{q} & 1 \end{pmatrix},$$

$$D_A^{-1}L_{BA} = \sqrt{s^A} \begin{pmatrix} \sigma_2 p & -\sigma_2\sqrt{p} \\ \sigma_3 p\sqrt{b} & -\sigma_3\sqrt{pq} \\ \sigma_2\sqrt{p} & -\sigma_2 \\ \sigma_3\sqrt{pq} & -\sigma_3\sqrt{q} \end{pmatrix},$$

$$D_B^{-1}L_{AB} = \frac{s^A}{m}(\sigma_2+\sigma_3 b) \begin{pmatrix} p & -\sqrt{p} \\ \sqrt{p} & -1 \end{pmatrix},$$

$$D_A^{-1}S_{AA}(f^A,f^A) = \sigma_1 q_1\sqrt{s^A}(\sqrt{pq},-\sqrt{p},\sqrt{q},-1),$$

$$D_A^{-1}S_{BA}(f^B,f^A) = \sigma_2 q_2(\sqrt{p},0,1,0)+\sigma_3 q_3(0,\sqrt{p},0,1), \text{ and}$$

$$D_B^{-1}S_{AB}(f^A,f^B) = \frac{\sqrt{s^A}}{m}(\sigma_2 q_2+\sigma_3 q_3\sqrt{q})(\sqrt{p},1).$$

The new boundary conditions are

$$\begin{cases} \begin{pmatrix} f_1^A(0) \\ f_2^A(0) \end{pmatrix} = \dfrac{1}{\sqrt{s^A q}} \begin{pmatrix} \sqrt{q}(s_0^A - s^A) \\ s_0^A q_0 - s^A q \end{pmatrix} \\ f_1^B(0) = \sqrt{p} f_2^B(0) \end{cases}, \tag{32}$$

at the condensed phase, and

$$f^A \to 0 \text{ and } f^B \to 0 \text{ as } x \to \infty,$$

at the infinity.

The linearized collision operators L_{AA} and L_{AB} are symmetric and semi-positive and have the null-spaces

$$N(L_{AA}) = \text{span}(y_1^A, y_2^A, y_3^A) \text{ and } N(L_{AB}) = \text{span}(y^B) \text{ with } y^B = (1,\sqrt{p}),$$
$$y_1^A = (\sqrt{p},\sqrt{pq},1,\sqrt{q}), y_2^A = (1,0,\sqrt{p},0), \text{ and } y_3^A = (0,1,0,\sqrt{p}).$$

The non-zero eigenvalues of $D_A^{-1}L_{AA}$ and $D_B^{-1}L_{AB}$ are (remind that $p>1$)

$$\lambda^A = s^A\sigma_1(1+q)(p-1) > 0 \text{ and } \lambda^B = \frac{s^A}{m}(\sigma_2+\sigma_3 q)(p-1) > 0,$$

respectively, with corresponding eigenvectors

$$u^A = (\sqrt{pq},-\sqrt{p},\sqrt{q},-1) \text{ and } u^B = (\sqrt{p},1).$$

We decompose

$$\begin{cases} f^A = \mu_1^A y_1^A + \mu_2^A y_2^A + \mu_3^A y_3^A + \beta^A u^A \\ f^B = \mu^B y^B + \beta^B u^B \end{cases}.$$

By Eq.(31)

$$\frac{d\mu^B}{dx} = \frac{d\mu_2^A}{dx} = \frac{d\mu_3^A}{dx} = 0,$$

and, since $\mu^B \to 0$, $\mu_2^A \to 0$, and $\mu_3^A \to 0$ as $x \to \infty$,

$$\mu^B = \mu_2^A = \mu_3^A = 0.$$

Then

$$S_{AA}(f^A, f^A) = S_{BA}(f^B, f^A) = S_{AB}(f^A, f^B) = 0.$$

Hence, we have a linear system

$$\begin{cases} \dfrac{df^A}{dx} + D_A^{-1} L_{AA} f^A = -D_A^{-1} L_{BA} f^B \\ \dfrac{df^B}{dx} + D_B^{-1} L_{AB} f^B = 0 \end{cases},$$

or, equivalently,

$$\begin{cases} \dfrac{d\beta^A}{dx} + \lambda^A \beta^A = -\beta^B \sqrt{q}(\sigma_2 - \sigma_3) \\ \dfrac{d\mu_1^A}{dx} = -\beta^B \dfrac{p-1}{1+q}(\sigma_2 + \sigma_3 q) \\ \dfrac{d\beta^B}{dx} + \lambda^B \beta^B = 0 \end{cases}. \tag{33}$$

Solving system (33), with the boundary conditions $\beta^B \to 0$, $\beta^A \to 0$, and $\mu_1^A \to 0$ as $x \to \infty$, we obtain

$$\begin{cases} \beta^A = \beta_0^B \sqrt{q} \dfrac{\sigma_2 - \sigma_3}{\lambda^B - \lambda^A} e^{-\lambda^B x} + k e^{-\lambda^A x} \\ \mu_1^A = \beta_0^B \dfrac{m}{s^A (1+q)} e^{-\lambda^B x} \\ \beta^B = \beta_0^B e^{-\lambda^B x} \end{cases}, \text{ with } \beta_0^B = \beta^B(0) \text{ and } k \text{ constant.}$$

If we fix the total amount of gas B to be m_B^{tot}, then

$$m_B^{tot} = 2 m_B \beta_0^B \sqrt{p} \int_0^\infty e^{-\lambda^B x}\, dx = 2 m_B \beta_0^B \frac{\sqrt{p}}{\lambda^B},$$

and, hence,

$$\beta_0^B = \frac{\lambda^B m_B^{tot}}{2 m_B \sqrt{p}} = \frac{s^A m_B^{tot}}{2 m_A \sqrt{p}} (\sigma_2 + \sigma_3 q)(p-1).$$

By the boundary conditions (32) at the condensed phase,

$$\frac{1}{\sqrt{s^A q}} \begin{pmatrix} \sqrt{q}(s_0^A - s^A) \\ s_0^A q_0 - s^A q \end{pmatrix}$$
$$= \sqrt{p}(\beta_0^B \sqrt{q} \frac{\sigma_2 - \sigma_3}{\lambda^B - \lambda^A} + k) \begin{pmatrix} \sqrt{q} \\ -1 \end{pmatrix} + \sqrt{p} \beta_0^B \frac{m}{s^A (1+q)} \begin{pmatrix} 1 \\ \sqrt{q} \end{pmatrix}.$$

Then
$$k = \frac{s_0^A - s^A}{\sqrt{s^A pq}} + \beta_0^B \sqrt{q} \frac{\sigma_2 - \sigma_3}{\lambda^A - \lambda^B} - \frac{m\beta_0^B}{s^A \sqrt{q}\,(1+q)},$$
and we obtain the solvability condition

$$s_0^A(1+q_0) = s^A(1+q) + \frac{m_B^{tot}\sqrt{s^A}}{2m_B}(\sigma_2 + \sigma_3 q)(p-1).$$

All our assumptions in Section 4 are fulfilled for this reduced model, if we allow $b_- = -\infty$ in Eq.(15), even if it is a drawback that our model is simplified and that all the non-linear terms disappear. We have that

$$\dim(R_+^B - CR_-^B)U_B^+ = 0 = n_B^+ - 1,$$

and, especially,
$$\frac{M_0^B}{\sqrt{M^B}} = \frac{s_0^B}{\sqrt{p}}(\sqrt{p}, 1) \in \operatorname{span}(u^B).$$

Furthermore,

$$R_+^A\sqrt{M^A} = \sqrt{s^A}(1, \sqrt{q}) \notin \operatorname{span}(\sqrt{q}, -1) = \operatorname{span}(u^A) \text{ and}$$
$$N(L_{AB}) = \operatorname{span}(\sqrt{M^B}).$$

We don't need any smallness assumptions on the total amount of the gas B, or on the closeness of the far Maxwellian and the Maxwellian at the wall for the gas A, due to the disappearance of the quadratic terms. We note that, in agreement with our main result in Section 4, we have exactly

$$k_A^+ + l = k_A^+ = 1$$

solvability condition.

7 Twelve+six-velocity model

We now consider the case when the vapor, gas A, is modelled by a twelve velocity model with velocities

$$(\pm 1, \pm 1), (\pm 1, \pm 3) \text{ and } (\pm 3, \pm 1),$$

and the non-condensable gas B is modelled by a six-velocity model with velocities
$$(\pm m, 0) \text{ and } (\pm m, \pm m), \text{ where } m = \frac{m_A}{m_B}.$$

We assume that we have a flow axially symmetric around the x-axis and obtain the reduced system

$$D\frac{dF}{dx} = Q(F, F),$$

where

$$D=\begin{pmatrix} D_A & 0 \\ 0 & D_B \end{pmatrix}, \text{ with}$$

$$D_A=(1,1,3,-1,-1,-3) \text{ and } D_B=(m,m,-m,-m)\,,$$
$$F=\left(F_1^A,...,F_6^A,F_7^B,...,F_{10}^B\right),$$
$$F_1^A(x)=F^A(x,(1,1))=F^A(x,(1,-1)),$$
$$F_2^A(x)=F^A(x,(1,3))=F^A(x,(1,-3)),$$
$$F_3^A(x)=F^A(x,(3,1))=F^A(x,(3,-1)),$$
$$F_4^A(x)=F^A(x,(-1,1))=F^A(x,(-1,-1)),$$
$$F_5^A(x)=F^A(x,(-1,3))=F^A(x,(-1,-3)),$$
$$F_6^A(x)=F^A(x,(-3,1))=F^A(x,(-3,-1)),$$
$$F_1^B(x)=F^B(x,(m,0)),F_2^B(x)=F^B(x,(m,m))=F^B(x,(m,-m)),$$
$$F_3^B(x)=F^B(x,(-m,0)),F_4^B(x)=F^B(x,(-m,m))=^B F(x,(-m,-m)),$$

and

$$\begin{aligned}&Q(f,f)\\&=(\sigma_1q_1+\sigma_2q_2+\sigma_3q_3+\sigma_6q_6+\sigma_7q_7,\sigma_1q_1-\sigma_2q_2+\sigma_4q_4+\sigma_8q_8+\sigma_9q_9,\\&-\sigma_1q_1-\sigma_4q_4,-\sigma_1q_1-\sigma_2q_2-\sigma_3q_3-\sigma_6q_6-\sigma_7q_7,\\&\sigma_2q_2-\sigma_3q_3+\sigma_4q_4-\sigma_8q_8-\sigma_9q_9,\sigma_3q_3-\sigma_4q_4,\sigma_5q_5-\sigma_6q_6-\sigma_8q_8,\\&-\sigma_5q_5-\sigma_7q_7-\sigma_9q_9,-\sigma_5q_5+\sigma_6q_6+\sigma_8q_8,\sigma_5q_5+\sigma_7q_7+\sigma_9q_9),\end{aligned}$$

with

$$q_1=F_3^AF_4^A-F_1^AF_2^A,q_2=F_2^AF_4^A-F_1^AF_5^A,q_3=F_4^AF_5^A-F_1^AF_6^A,$$
$$q_4=F_3^AF_6^A-F_2^AF_5^A,q_5=F_2^BF_3^B-F_1^BF_4^B,q_6=F_4^AF_1^B-F_1^AF_3^B,$$
$$q_7=F_4^AF_2^B-F_1^AF_4^B,q_8=F_5^AF_1^B-F_2^AF_3^B,q_9=F_5^AF_2^B-F_2^AF_4^B,$$
$$\text{and } \sigma_1,...,\sigma_9\geq 0.$$

The Maxwellians are of the form

$$\begin{aligned}&M=\left(M^A,M^B\right), \text{ with } M^A=s^A(p^2,p^2q^8,q^8,p^4q^8,p^4q^8,p^6q^8),\\&M^B=s^B(1,q^m,p^2,p^2q^m), \text{ and } p,q,s^A,s^B>0.\end{aligned} \tag{34}$$

We consider the case when $p>1$ (corresponding to condensation) and assume the boundary conditions

$$\left\{\begin{array}{l}\begin{pmatrix}F_1^A(0)\\F_2^A(0)\\F_3^A(0)\end{pmatrix}=\begin{pmatrix}a_{0,1}\\a_{0,2}\\a_{0,3}\end{pmatrix}\\ \begin{pmatrix}F_1^B(0)\\F_2^B(0)\end{pmatrix}=\dfrac{1}{1+q_0^m}\begin{pmatrix}1 & 1\\ q_0^m & q_0^m\end{pmatrix}\begin{pmatrix}F_3^B(0)\\F_4^B(0)\end{pmatrix}\end{array}\right.,$$

at the condensed phase, and

$$F^A \to M^A = s^A(p^2, p^2q^8, q^8, p^4q^8, p^4q^8, p^6q^8) \text{ and } F^B \to 0 \text{ as } x \to \infty,$$

at infinity.

We denote

$$F = \left(M^A, 0\right) + \sqrt{M} f,$$

for the Maxwellian of the form (34), such that $F^A \to M^A$ and $s^B = 1$. Then we obtain a system

$$\begin{cases} \dfrac{df^A}{dx} + D_A^{-1} L_{AA} f^A = D_A^{-1}(S_{AA}(f^A, f^A) + \epsilon S_{BA}(f^B, f^A) - \epsilon L_{BA} f^B) \\ \dfrac{df^B}{dx} + D_B^{-1} L_{AB} f^B = \epsilon D_B^{-1} S_{BB}(f^B, f^B) + D_B^{-1} S_{AB}(f^A, f^B) \end{cases},$$

where, in particular,

$$\begin{aligned} & D_B^{-1} L_{AB} \\ & = \frac{s^A p^2}{m} \begin{pmatrix} p^2(\sigma_6 + q^8\sigma_8) & 0 & -p(\sigma_6 + q^8\sigma_8) & 0 \\ 0 & p^2(\sigma_7 + q^8\sigma_9) & 0 & -p(\sigma_7 + q^8\sigma_9) \\ p(\sigma_6 + q^8\sigma_8) & 0 & -(\sigma_6 + q^8\sigma_8) & 0 \\ 0 & p(\sigma_7 + q^8\sigma_9) & 0 & -(\sigma_7 + q^8\sigma_9) \end{pmatrix}. \end{aligned}$$

The new boundary conditions are

$$\begin{cases} \begin{pmatrix} f_1^A(0) \\ f_2^A(0) \\ f_3^A(0) \end{pmatrix} = \dfrac{1}{pq^4\sqrt{s^A}} \begin{pmatrix} q^4(a_{0,1} - s^A p^2) \\ a_{0,2} - s^A p^2 q^8 \\ p(a_{0,3} - s^A q^8) \end{pmatrix} \\ \begin{pmatrix} f_1^B(0) \\ f_2^B(0) \end{pmatrix} = \dfrac{p}{(1 + q_0^m) q^{m/2}} \begin{pmatrix} q^{m/2} & q^m \\ 1 & q^{m/2} \end{pmatrix} \begin{pmatrix} f_3^B(0) \\ f_4^B(0) \end{pmatrix} \end{cases},$$

at the condensed phase, and

$$f^A \to 0 \text{ and } f^B \to 0 \text{ as } x \to \infty,$$

at the far end.

The linearized collision operator L_{AB} has the null-space

$$N(L_{AB}) = \text{span}(y_1^B, y_2^B) \text{ with } y_1^B = (1, 0, p, 0) \text{ and } y_2^B = (0, 1, 0, p),$$

and the non-zero eigenvalues of $D_B^{-1} L_{AB}$ are (remind that $p > 1$)

$$\lambda_1^B = \frac{s^A p^2}{m}(\sigma_6 + q^8\sigma_8)(p^2 - 1) > 0 \text{ and } \lambda_2^B = \frac{s^A p^2}{m}(\sigma_7 + q^8\sigma_9)(p^2 - 1) > 0,$$

with corresponding eigenvectors

$$u_1^B = (p, 0, 1, 0) \text{ and } u_2^B = (0, p, 0, 1)\ .$$

Then

$$(R_+^B - CR_-^B)u_1^B = \frac{pq_0^m}{(1+q_0^m)q^{m/2}}(q^{m/2}, -1) = -\frac{q_0^m}{q^{m/2}}(R_+^B - CR_-^B)u_2^B,$$

and, therefore,

$$\dim(R_+^B - CR_-^B)\mathrm{span}(u_1^B, u_2^B) \ = 1 = n_B^+ - 1.$$

Especially,

$$\frac{M_0^B}{\sqrt{M^B}} = \frac{s_0^B}{pq^{m/2}}(pq^{m/2}, pq_0^m, q^{m/2}, q_0^m) \in \mathrm{span}(u_1^B, u_2^B).$$

Furthermore,

$$\begin{aligned}(R_+^B - CR_-^B)\sqrt{M^B} = (1 - \frac{p^2(1+q^m)}{1+q_0^m}, q^{m/2} - \frac{p^2(1+q^m)q_0^m}{(1+q_0^m)q^{m/2}}) \\ \notin (R_+^B - CR_-^B)\mathrm{span}(u_1^B, u_2^B),\end{aligned}$$

and, hence,

$$\dim(R_+^B - CR_-^B)\mathrm{span}(u_1^B, u_2^B, \sqrt{M^B}) \ = 2 = n_B^+.$$

Let us now consider the twelve-velocity plane DVM

$$(\pm1, \pm1)\,,\ (\pm3, \pm1)\ \text{ and }\ (\pm1, \pm3)\,.$$

The Maxwellians are of the form

$$M^A = s^A(p^2, p^2q^8, q^8, p^4q^8, p^4q^8, p^6q^8),$$

where $q = e^{-c}$, $p = e^{-b}$ and $s^A = e^a e^{3b-2c}$, a, b and c are constant, and the null-space of L_{AA} is

$$N(L_{AA}) = \mathrm{span}\,(\phi_1, \phi_2, \phi_3)\,,$$

where

$$\begin{cases} \phi_1^A = \sqrt{M^A}(1,1,1,1,1,1) = \sqrt{s^A}(p, pq^4, q^4, p^2, p^2q^4, p^3q^4) \\ \phi_2^A = \sqrt{M^A}(1,1,3,-1,-1,-3) = \sqrt{s^A}\left(p, pq^4, 3q^4, -p^2, -p^2q^4, -3p^3q^4\right) \\ \phi_3^A = \sqrt{M^A}(2,10,10,2,10,10) = 2\sqrt{s^A}\left(p, 5pq^4, 5q^4, p^2, 5p^2q^4, 5p^3q^4\right) \end{cases}.$$

If we denote

$$\begin{cases} \varphi_1 = (1,0,0,p,0,0) \\ \varphi_2 = (q^{-4}, -1, 0, 0, -2p, -3p^2) \\ \varphi_3 = \left(0, p, 1, 0, p^2, p^3\right) \end{cases},$$

then

$$\mathrm{span}\,(\varphi_1, \varphi_2, \varphi_3) = \mathrm{span}\left(\phi_1^A, \phi_2^A, \phi_3^A\right),$$

and,

$$K_A = (\langle \varphi_i, \varphi_j \rangle_{D_A}) = \begin{pmatrix} 1-p^2 & q^{-4} & 0 \\ q^{-4} & q^{-8}+1-4p^2-27p^4 & -p+2p^3+9p^5 \\ 0 & -p+2p^3+9p^5 & 3+p^2-p^4-3p^6 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \det K_A &= 3(1-p^2)(1-(4+q^{-8})p^2 - \frac{2}{3}(41+2q^{-8})p^4 - (4+q^{-8})p^6 + p^8) \\ &= 3(1-p^2)(1-2Sp^2+p^4)(1+2\widetilde{S}p^2+p^4), \end{aligned}$$

where

$$S = \frac{1}{4q^8}\left(1+\frac{4}{q^8}+\sqrt{1+\frac{40}{3q^8}(1+\frac{10}{q^8})}\right) > 1 \text{ and}$$
$$\widetilde{S} = \frac{1}{4q^8}\left(-1-\frac{4}{q^8}+\sqrt{1+\frac{40}{3q^8}(1+\frac{10}{q^8})}\right) > 0.$$

Therefore,

$$\det K_A = 0 \Leftrightarrow p = 1 \vee p = \sqrt{S \pm \sqrt{S^2-1}} \Leftrightarrow b = 0 \vee b = -\frac{\ln(S \pm \sqrt{S^2-1})}{2}.$$

But,

$$S - \sqrt{S^2-1} = \frac{1}{S+\sqrt{S^2-1}} < 1,$$

and hence,

$$\det K_A = 0 \Leftrightarrow b = 0 \vee b = b_\pm = \pm\frac{\ln(S+\sqrt{S^2-1})}{2}.$$

Furthermore,

$$\det K_A = -P(p-1)(p-\sqrt{S+\sqrt{S^2-1}})(p-\sqrt{S-\sqrt{S^2-1}}), \text{ where}$$
$$P = 3(p+\sqrt{S+\sqrt{S^2-1}})(p+\sqrt{S-\sqrt{S^2-1}})(1+p)(1+2\widetilde{S}p^2+p^4) > 0.$$

Hence, if we consider different values of negative b, $b < 0$, (i.e. $p > 1$) we obtain

$$\begin{aligned} b < b_- &: \det K_A < 0 \Rightarrow k_A^+ = 0 \vee k_A^+ = 2 \\ b = b_- &: \det K_A = 0 \Rightarrow k_A^+ = 0 \vee k_A^+ = 1 \vee k_A^+ = 2 \\ b_- < b < 0 &: \det K_A > 0 \Rightarrow k_A^+ = 1 \vee k_A^+ = 3. \end{aligned}$$

By the method of Routh-Hurwitz we obtain the following values for k_A^+, k_A^-, and l_A, depending on the value of b, $b < 0$;

$$\begin{aligned} b < b_- &: k_A^+ = l = 0, k_A^- = 3 \\ b = b_- &: k_A^+ = 0, k_A^- = 2, l_A = 1 \\ b_- < b < 0 &: k_A^+ = 1, k_A^- = 2, l_A = 0. \end{aligned}$$

Remark 3 One can, by similar arguments and the Routh-Hurwitz method obtain the following values for k_A^+, k_A^-, and l_A, for $b \geq 0$ (i.e. $p \leq 1$);

$$b = 0 : k_A^+ = 0, k_A^- = 1, l_A = 2$$
$$0 < b < b_+ : k_A^+ = 2, k_A^- = 1, l_A = 0$$
$$b = b_+ : k_A^+ = 2, k_A^- = 0, l_A = 1$$
$$b > b_+ : k_A^+ = 3, k_A^- = l_A = 0.$$

The only condition of our necessary conditions we haven't proved for this model is condition (16) when $b_- < b < 0$. This we have to leave open for now. However, it seems most likely, especially for b sufficiently close to 0, that also condition (16) is fulfilled for this model.

Acknowledgements This work was initiated during a stay at Kyoto University as JSPS Postdoctoral Fellow, with a grant from the Japan Society for the Promotion of Science (No. PE 09549). The author wants to thank Professor Kazuo Aoki for the nice hospitality and for proposing this work.

References

1. Bardos, C., Golse, F., Sone, Y.: Half-space problems for the Boltzmann equation: A survey. J. Stat. Phys. **124**, 275–300 (2006)
2. Bernhoff, N.: On half-space problems for the linearized discrete Boltzmann equation. Riv. Mat. Univ. Parma **9**, 73–124 (2008)
3. Bernhoff, N.: On half-space problems for the discrete Boltzmann equation. Nuovo Cim. C **33**, 47–54 (2010)
4. Bernhoff, N.: On half-space problems for the weakly non-linear discrete Boltzmann equation. Kinet. Relat. Models **3**, 195–222 (2010)
5. Bernhoff, N.: Boundary layers and shock profiles for the discrete Boltzmann equation for mixtures. Kinet. Relat. Models **5**, 1–19 (2012)
6. Bobylev, A.V., Bernhoff, N.: Discrete velocity models and dynamical systems. In: N. Bellomo, R. Gatignol (eds.) Lecture Notes on the Discretization of the Boltzmann Equation, pp. 203–222. World Scientific (2003)
7. Bobylev, A.V., Vinerean, M.C.: Construction of discrete kinetic models with given invariants. J. Stat. Phys. **132**, 153–170 (2008)
8. Broadwell, J.E.: Shock structure in a simple discrete velocity gas. Phys. Fluids **7**, 1243–1247 (1964)
9. Coron, F., Golse, F., Sulem, C.: A classification of well-posed kinetic layer problems. Comm. Pure Appl. Math. **41**, 409–435 (1988)
10. Gatignol, R.: Kinetic theory boundary conditions for discrete velocity gases. Phys. Fluids **20**, 2022–2030 (1977)
11. Kawashima, S., Nishibata, S.: Stationary waves for the discrete Boltzmann equation in the half space with reflective boundaries. Comm. Math. Phys. **211**, 183–206 (2000)
12. Sone, Y.: Kinetic Theory and Fluid Dynamics. Birkhauser (2002)
13. Sone, Y.: Molecular Gas Dynamics. Birkhauser (2007)
14. Sotirov, A., Yu, S.H.: On the solution of a Boltzmann system for gas mixtures. Arch. Rational Mech. Anal. **195**, 675–700 (2010)
15. Taguchi, S., Aoki, K., Latocha, V.: Vapor flows along a plane condensed phase with weak condensation in the presence of a noncondensable gas. J. Stat. Phys. **124**, 321–369 (2006)
16. Taguchi, S., Aoki, K., Takata, S.: Vapor flows condensing at incidence onto a plane condensed phase in the presence of a noncondensable gas. I. Subsonic condensation. Phys. Fluids **15**, 689–705 (2003)

17. Taguchi, S., Aoki, K., Takata, S.: Vapor flows condensing at incidence onto a plane condensed phase in the presence of a noncondensable gas. II. Supersonic condensation. Phys. Fluids **16**, 79–92 (2004)
18. Takata, S., Golse, F.: Half-space problem of the nonlinear Boltzmann equation for weak evaporation and condensation of a binary mixture of vapors. Eur. J. Mech. B/Fluids. **26**, 105–131 (2007)
19. Ukai, S., Yang, T., Yu, S.H.: Nonlinear boundary layers of the Boltzmann equation: I. Existence. Comm. Math. Phys. **236**, 373–393 (2003)
20. Yang, X.: The solutions for the boundary layer problem of Boltzmann equation in a half-space. J. Stat. Phys. **143**, 168–196 (2011)