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Staggered Ladder Spectra

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Abstract

We discuss aspects of a quantum mechanical system which has a staggered ladder spectrum. The raising and lowering operators of this spectrum are presented and calculated. The system we consider arises from a rewriting of a Fokker-Planck equation, which consists in constructing a generalized form of an Ornstein-Uhlenbeck system. From this Fokker-Planck equation we define a Hamiltonian operator, for which we study the eigenvalue problem. As we solve this eigenvalue problem, even and odd eigenfunctions are obtained with respective even and odd eigenvalues, which for given parity are equidistant, but for different parities are staggered.
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1 Introduction

In the first quarter of the twentieth century, the formulation of quantum mechanics was developed by W. Heisenberg, E. Schrödinger, P.A.M. Dirac, among others [5]. In 1925 Dirac derived the transition from classical mechanics of Poisson brackets to quantum mechanics as

$$[\hat{A}, \hat{B}]_{cl} \rightarrow \frac{[\hat{A}, \hat{B}]_{qm}}{i\hbar},$$

(1.1)

for operators $\hat{A}$ and $\hat{B}$. This gave rise to the construction of the quantum mechanical commutator algebra. Related to this, ladder operators were introduced, probably by Dirac. In an article by L.S. Ornstein and G.E. Uhlenbeck [8], published in 1930, an application of the statistical motion of quantum mechanical particles was given, taking the equation of motion as

$$\dot{p} = -\gamma p + f(t),$$

(1.2)

for the momentum $p$, damping coefficient $\gamma$ and a rapidly fluctuating force $f(t)$. The description of the processes made in their paper “On the Theory of the Brownian Motion” has later come to be referred to as Ornstein-Uhlenbeck processes. In recent years (2006-2007), articles have been published by Arvedson et al. [1, 6, 9, 10] approaching a generalization of the Ornstein-Uhlenbeck processes for the equation of motion

$$\dot{p} = -\gamma p + f(x,t),$$

(1.3)

which eigenvalue problem for certain conditions gives rise to eigenvalues constituting staggered ladder spectra.

In this bachelor-level thesis, we construct a staggered ladder spectrum and calculate transitions between eigenvalues of the same parity (even-even or odd-odd) by commutator algebra.

We first consider the one-dimensional quantum harmonic oscillator, for which an energy spectrum with equidistant energy levels is obtained. Raising and lowering operators for this case is presented. This spectrum with operators is illustrated in Figure 1. Introducing the statistics of Brownian motion, we consider an ensemble of particles for the Ornstein-Uhlenbeck process in its regular form (1.2). In this system, the probability distributions for momentum and position are calculated. Generalizing the Ornstein-Uhlenbeck process as (1.3), the Fokker-Planck equation is introduced, providing a description of the process. First in its general form, then derived into the so called generic case (for which we have a diffusion constant $D(p) \sim |z|^{-1}$), with a dimensionless form of the Fokker-Planck operator $\hat{F}$, which is given in Hermitian form. This equation is expressed in terms of the momentum probability density $P$. Taking $P_0 \propto e^{-|z|^3/3}$, we define the Hamiltonian operator $\hat{H}$ for our generalized system. By solving the eigenvalue problem for this Hamiltonian, we obtain eigenvalues for even and odd parts of the respective eigenfunctions. Their respective eigenvalues constitute a staggered ladder spectrum, which is illustrated in Figure 2. An exact solution is calculated to the Fokker-Planck equation, using the propagator for the obtained Hamiltonian. Raising and
lowering operators are introduced and their acting in the spectrum is calculated by commutator algebra.

Section 2 consists of the basic algebraic properties of quantum mechanics given in standard books on quantum mechanics, e.g. [2, 3, 5]. Section 3 introduces non-standard quantum mechanics of some special systems presented in e.g. [1, 6, 7, 8, 9, 10]. In Section 4 a special solution is presented to one of these systems. This is described in [1, 6, 9]. Conclusions and recommended further studies are given in Section 5. Finally, commutator algebraic calculations are presented in Appendix A and some additional calculations are given in Appendix B.

In this bachelor-level thesis, I have not added any new information to the field of research. My contribution in this paper has been to fill in calculations and performing derivations, verifying already existing results. The main goal has been to perform the derivations and calculations validating the staggered ladder spectra obtained in [1], for the generic case. I also mention here that some special polynomials and functions will be used in deriving some results. These results do not require any explicit calculations of these polynomials/functions, so they have not been described in detail. These are: the Hermite polynomials, the associated Laguerre polynomials and the modified Bessel functions.

I would like to thank my supervisor Jürgen Fuchs for interesting discussions throughout the making of this paper.
2 Algebraic properties of quantum mechanics

2.1 The quantum harmonic oscillator

Here, the one-dimensional quantum harmonic oscillator will be considered, together with solutions of the form of hypergeometric functions, which are expressed in series. This section will mainly follow [3], with some parts taken from [2]. Consider the one-dimensional harmonic oscillator, calling the coordinate $x$. For a mass point with mass $m$ moving in a central force potential described by Hooke’s law $F = -kx^2$, where $k = m\omega^2$ is the constant of the force and $\omega$ is the oscillation frequency, the potential is given by

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2. \tag{2.1}$$

The total energy can be described by the Hamilton operator, $H = T + V$, with the kinetic energy $T$ given by $T = \frac{\hat{p}^2}{2m}$ for the quantum mechanical momentum operator $\hat{p} = -i\hbar\frac{d}{dx}$. Thus, we get the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2, \tag{2.2}$$

which allows us to express the Schrödinger eigenvalue equation as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x), \tag{2.3}$$

where $\psi(x)$ is the wave eigenfunction for the mass point and $E$ is the total energy. To solve this differential equation, we begin by putting all terms on the same side of the equality:

$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + E\psi(x) - \frac{1}{2}m\omega^2x^2\psi(x) = 0. \tag{2.4}$$

Then, multiplying by $\frac{2m}{\hbar^2}$ gives the second order derivative-term by itself,

$$\frac{d^2\psi(x)}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m\omega^2x^2}{\hbar^2}\right)\psi(x) = 0. \tag{2.5}$$

Now, we introduce the notations

$$\alpha^2 = \frac{2mE}{\hbar^2} \tag{2.6}$$

and

$$\beta = \frac{m\omega}{\hbar}. \tag{2.7}$$

Then the Schrödinger equation becomes

$$\frac{d^2\psi(x)}{dx^2} + (\alpha^2 - \beta^2x^2)\psi(x) = 0. \tag{2.8}$$
Continue by making the transformation \( z = \beta x^2 \). Then we get
\[
\frac{d\psi(z)}{dz} = \frac{d\psi(z)}{dz} \frac{dz}{dx} = \frac{d\psi(z)}{dz} 2\beta x
\] (2.9)
and
\[
\frac{d}{dx} \left( \frac{d\psi(z)}{dz} 2\beta x \right) = 2\beta \frac{d\psi(z)}{dz} + (2\beta x)^2 \frac{d^2 \psi(z)}{dz^2} = 2\beta \frac{d\psi(z)}{dz} + 4\beta x \frac{d^2 \psi(z)}{dz^2}. \tag{2.10}
\]
Now, (2.8) can be expressed as
\[
4\beta z \frac{d^2 \psi(z)}{dz^2} + 2\beta \frac{d\psi(z)}{dz} + (\alpha^2 - \beta z) \psi(z) = 0. \tag{2.11}
\]
Dividing by \( 4\beta \) gives
\[
z \frac{d^2 \psi(z)}{dz^2} + \frac{1}{2} \frac{d\psi(z)}{dz} - \left( c + \frac{1}{4} \right) \psi(z) = 0, \quad c = -\frac{\alpha^2}{4\beta}. \tag{2.12}
\]
For \( z \to \pm \infty \), we get terms in the differential equation which tend to infinity. We take care of these asymptotic parts, by making the ansatz
\[
\psi(z) = e^{-z/2} \phi(z), \tag{2.13}
\]
leading to
\[
\frac{d\psi(z)}{dz} = \left[ -\frac{1}{2} \phi(z) + \frac{d\phi(z)}{dz} \right] e^{-z/2} \tag{2.14}
\]
and
\[
\frac{d^2 \psi(z)}{dz^2} = \left[ \frac{1}{4} \phi(z) - \frac{d\phi(z)}{dz} + \frac{d^2 \phi(z)}{dz^2} \right] e^{-z/2}. \tag{2.15}
\]
Putting this into (2.12) and dividing the resulting equation by the exponential factor \( e^{-z/2} \), one then gets
\[
z \frac{d^2 \phi(z)}{dz^2} + \left( \frac{1}{2} - z \right) \frac{d\phi(z)}{dz} - \left( c + \frac{1}{4} \right) \phi(z) = 0. \tag{2.16}
\]
The general form of this equation is expressed as
\[
z \frac{d^2 \phi(z)}{dz^2} + (b - z) \frac{d\phi(z)}{dz} - a \phi(z) = 0, \tag{2.17}
\]
with constants \( a, b \in \mathbb{R} \) and can be recognized as Kummer’s hypergeometric differential equation (by e.g. [3] and [4]).

Now, the general solution to (2.17) is given by
\[
\phi(z) = A \cdot \text{}_{1}F_{1}(a; b; z) + Bz^{1-b} \cdot \text{}_{1}F_{1}(a - b + 1; 2 - b; z), \tag{2.18}
\]
with constants \( A, B \in \mathbb{C} \). The the first term on the right hand side of the equality corresponds to the even part of the solution and the second term stands for the odd part of the solution. We have
\[
b = \frac{1}{2} \tag{2.19}
\]
in (2.17) (by (2.14)), that results in the solution
\[
\phi(z) = A \cdot {}_1F_1 \left( a; \frac{1}{2}; z \right) + B z^{1/2} \cdot {}_1F_1 \left( a + \frac{1}{2}; \frac{3}{2}; z \right),
\] (2.20)
where the so called Kummer’s hypergeometric function \( {}_1F_1 \) is expressed as
\[
{}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} = 1 + \frac{a z}{b 1!} + \frac{a(a+1) z^2}{b(b+1) 2!} + \ldots .
\] (2.21)
In this summation, we have the Pochhammer symbol
\[
(a)_n = a(a+1) \ldots (a+n-1) = \frac{(a+n-1)!}{(a-1)!},
\] (2.22)
for any parameter \( a \).

Remark 2.1 Throughout the text, Kummer’s hypergeometric function will be expressed as \( {}_1F_1 \), or in terms of the Gamma function [11]. In this case we have the relations
\[
{}_1F_1(a; b; z) \propto \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b}
\] (2.23)
and also
\[
{}_1F_1(a; b; z) \propto \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a},
\] (2.24)
where the Gamma function can be defined (cf. [4]) by
\[
\Gamma(z) := \int_{0}^{\infty} e^{-t} t^{z-1} dt, \quad \Re(z) > 0.
\] (2.25)
Inserting \( z = \frac{1}{2} \) into this definition, the Gamma function in this case becomes the integral due to Euler (derived from the well-known Gaussian integral \( \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \), by the substitution \( x = \sqrt{t} \)),
\[
\Gamma \left( \frac{1}{2} \right) = \int_{0}^{\infty} e^{-t} t^{-1/2} dt = \sqrt{\pi}.
\] (2.26)
The Pochhammer symbol can also be expressed as
\[
(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)},
\] (2.27)
since
\[
\Gamma(n) = (n-1)!.
\] (2.28)
The energy spectrum can be obtained by analyzing the following two cases for (2.20) (in the variable \( \beta z^2 \) obtained in our transformation):

**Case I** (even solutions): \( B = 0 \) and \( a = -n, \ n \in \mathbb{Z}_{\geq 0} \).
The eigenfunctions are given by
\[
\psi_n(z) = N_n e^{-\beta z^2/2} \cdot {}_1F_1 \left( -n; \frac{1}{2}; \beta z^2 \right),
\] (2.29)
with normalization constants (given by [3])
\[ N_n = \sqrt{\frac{\beta^2}{\sqrt{\pi} 2^n n!}} \] (2.30)

and the energy eigenvalues are obtained from \( a \) in (2.17), which by (2.16) is equal to \( c + \frac{1}{4} \), which by (2.12) is equal to \( -\frac{n^2}{4} + \frac{1}{4} \) and finally by (2.6) and (2.7) we obtain:
\[ a = -\frac{1}{2} \frac{E}{\hbar \omega} + \frac{1}{4}. \] (2.31)

We also have \( a = -n \), so
\[ -n = -\frac{1}{2} \frac{E_n}{\hbar \omega} + \frac{1}{4}, \] (2.32)
giving the energy eigenvalues
\[ E_n = \left( 2n + 1 \right) \frac{1}{2} \hbar \omega. \] (2.33)

**Case II** (odd solutions): \( A = 0 \) and \( a + \frac{1}{2} = -n \), \( n \in \mathbb{Z}_{\geq 0} \).
The eigenfunctions are given by
\[ \psi_n(z) = N_n e^{-\frac{(\beta z^2)}{2}} \cdot \mathbf{j}_n \left( -n, -\frac{3}{2}; \beta z^2 \right), \] (2.34)
with the energy eigenvalues
\[ E_n = \left( 2n + 1 \right) + \frac{1}{2} \hbar \omega. \] (2.35)

Summarizing the two cases, the entire discrete energy spectrum is given by
\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega, \quad n \in \mathbb{Z}_{\geq 0}. \] (2.36)

The energy eigenvalues (2.36) are bound states with evenly spaced energy levels, that are non-degenerate. The ground state energy \( E_0 = \frac{1}{2} \hbar \omega \) is called the zero-point energy, which is the lowest possible energy state. The energy spectrum we obtained (from our Hamiltonian (2.2)) is illustrated in Figure 1 (where the operators \( \hat{\alpha}^\dagger \) and \( \hat{\alpha} \) will be explained in Section 2.3).

We introduce Hermite polynomials \( H_n \) for the even and the odd cases respectively:
\[ \begin{align*}
H_{2n}(\sqrt{\beta} z) &= (-1)^n \frac{(2n)!}{n!} \cdot \mathbf{j}_n \left( -n, -\frac{3}{2}; \beta z^2 \right) ; \\
H_{2n-1}(\sqrt{\beta} z) &= (-1)^n \frac{2(2n+1)!}{n!} \sqrt{\beta} z \cdot \mathbf{j}_n \left( -n, -\frac{3}{2}; \beta z^2 \right) .
\end{align*} \] (2.37)

In terms of these relations, the wave function is expressed as
\[ \psi_n(z) = N_n e^{-\frac{(\beta z^2)}{2}} H_n (\sqrt{\beta} z). \] (2.38)
2.2 Properties of Hermite polynomials

Here we will present some results regarding Hermite polynomials. For detailed derivations cf. [3].

We shall consider the normalized eigenfunctions of the harmonic oscillator. The condition fulfilled for the eigenfunctions to be normalized is given by

\[ \int |\psi(z)|^2 \, dz = 1. \]  

(2.39)

The normalized eigenfunctions are expressed as

\[ \psi_n(z) = \sqrt{\frac{\beta^2}{\sqrt{\pi}2^n n!}} \, e^{-\beta z^2/2} H_n(\sqrt{\beta} z), \]  

(2.40)

in accordance with (2.38). Some relations can then be derived for the Hermite polynomials. Under the condition that

\[ \alpha^2 = 2 \beta \left(n + \frac{1}{2}\right), \]  

(2.41)

the Schrödinger equation (2.8) is fulfilled for functions \( \psi(z) = e^{-\beta z^2/2} H_n(\sqrt{\beta} z) \). Make this substitution for the wave functions, and then make the substitution \( \xi = \sqrt{\beta} z \). These derivations are presented in [3], so we skip to the results directly (since they are needed when introducing raising and lowering operators). One uses the relation of Hermite polynomials

\[ \xi H_n(\xi) = n H_{n-1}(\xi) + \frac{1}{2} H_{n+1}(\xi), \]  

(2.42)

to get the relation for wave functions,

\[ \xi \psi_n(\xi) = \sqrt{\frac{n}{2}} \psi_{n-1} + \sqrt{\frac{n+1}{2}} \psi_{n+1}. \]  

(2.43)
By using another relation of the Hermite polynomials,
\[
\frac{\partial H_n(\xi)}{\partial \xi} = 2n H_{n-1}(\xi),
\] (2.44)
one gets for the wave functions that
\[
\frac{\partial}{\partial \xi} \psi_n(\xi) = \sqrt{\frac{\beta}{\pi 2^n n!}} e^{-\xi^2/2} H_n(\xi) \left( \frac{\partial H_n(\xi)}{\partial \xi} - \xi \right)
\] (2.45)
\[
= \sqrt{\frac{\beta}{\pi 2^n n!}} e^{-\xi^2/2} H_n(\xi)(2n H_{n-1}(\xi) - \xi).
\]

Expressing the above equation in wave functions, one also gets
\[
\frac{\partial}{\partial \xi} \psi_n(\xi) = 2 \sqrt{\frac{n}{2}} \psi_{n-1}(\xi) - \xi \psi_n(\xi)
\] (2.46)
and finally, by inserting (2.43) into (2.46), one gets the expression
\[
\frac{\partial}{\partial \xi} \psi_n(\xi) = \sqrt{\frac{n}{2}} \psi_{n-1}(\xi) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(\xi).
\] (2.47)

### 2.3 Raising and lowering operators

The material here comes from [3].

Inserting \(\psi_{n-1}\) from (2.43) into (2.47) gives us
\[
\frac{\partial}{\partial \xi} \psi_n = \xi \psi_n - \sqrt{2(n+1)} \psi_{n+1}.
\] (2.48)

This can be put in a nicer form by collecting \(\psi_n\)–terms and dividing both sides by \(\sqrt{2}\):
\[
\frac{1}{\sqrt{2}} \left( \xi - \frac{\partial}{\partial \xi} \right) \psi_n = \sqrt{n+1} \psi_{n+1}.
\] (2.49)

Now, inserting \(\psi_{n+1}\) from (2.43) into (2.47) yields
\[
\frac{\partial}{\partial \xi} \psi_n = \sqrt{\frac{n}{2}} \psi_{n-1} + \sqrt{\frac{n}{2}} \psi_{n-1} - \xi \psi_n,
\] (2.50)
where we again collect \(\psi_n\)–terms and divide both sides by \(\sqrt{2}\):
\[
\frac{1}{\sqrt{2}} \left( \xi + \frac{\xi}{\partial \xi} \right) \psi_n = \sqrt{n} \psi_{n-1}.
\] (2.51)

Here we define the “raising” and “lowering” operators respectively:
\[
\hat{\alpha}^\dagger := \frac{1}{\sqrt{2}} \left( \xi - \frac{\partial}{\partial \xi} \right)
\] (2.52)
and
\[ \hat{\alpha} := \frac{1}{\sqrt{2}} \left( \xi + \frac{\partial}{\partial \xi} \right), \quad (2.53) \]
which implies that
\[ \hat{\alpha}^\dagger \psi_n = \sqrt{n + 1} \psi_{n+1}, \quad (2.54) \]
and
\[ \hat{\alpha} \psi_n = \sqrt{n} \psi_{n-1}. \quad (2.55) \]

By (2.54) and (2.55), we can make each \( \psi_n \) an eigenfunction of the operator product \( \hat{\alpha}^\dagger \hat{\alpha} \) as
\[ \hat{\alpha}^\dagger \hat{\alpha} \psi_n = \sqrt{n} \hat{\alpha}^\dagger \psi_n = n \psi_n. \quad (2.56) \]
Thus, we can define a number operator \( \hat{N} \),
\[ \hat{N} := \hat{\alpha}^\dagger \hat{\alpha}, \quad (2.57) \]
where
\[ \hat{N} \psi_n = n \psi_n. \quad (2.58) \]
So, the operator \( \hat{N} \) has eigenvalues \( n \) and eigenfunctions \( \psi_n, \ n \in \mathbb{Z}_{\geq 0} \). For an illustration of how the raising and lowering operators act, cf. Figure 1.

### 2.4 The bracket notation

Here we will follow [2].

Consider a system in one dimension, say in \( z \)-direction. Let \( \psi_1(z) \) and \( \psi_2(z) \) be two square integrable functions. Then the scalar product is by definition:
\[ \langle \psi_1 | \psi_2 \rangle \equiv \int \psi_1^* (z) \psi_2 (z) \, dz, \quad (2.59) \]
where \( * \) denotes complex conjugation. The following algebraic relations are satisfied by the scalar product, with \( \psi_3(z) \) a square integrable function and \( c \in \mathbb{C} \) an arbitrary number:
\[ \langle \psi_1 | \psi_2 \rangle = \langle \psi_2 | \psi_1 \rangle^*; \quad (2.60) \]
\[ \langle \psi_1 | c \psi_2 \rangle = c \langle \psi_1 | \psi_2 \rangle; \quad (2.61) \]
\[ \langle c \psi_1 | \psi_2 \rangle = c^* \langle \psi_1 | \psi_2 \rangle; \quad (2.62) \]
\[ \langle \psi_3 | \psi_1 + \psi_2 \rangle = \langle \psi_3 | \psi_1 \rangle + \langle \psi_3 | \psi_2 \rangle. \quad (2.63) \]

With respect to this scalar product, we express the relation between orthogonal eigenfunctions by Kronecker’s delta symbol, for \( n \in \mathbb{Z}_{\geq 0} \):
\[ \langle \psi_i | \psi_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j, \end{cases} \quad (2.64) \]
3 Ornstein-Uhlenbeck processes and the Fokker-Planck equation

3.1 Statistics of Brownian motion

The material here comes from [7] and [8].

The standard notation of angular brackets will be adapted for average values throughout the text. Consider a force $f(t)$ which is randomly fluctuating. The main principle of Brownian motion is that the average value of this force becomes zero, i.e.

$$\langle f(t) \rangle = 0. \quad (3.1)$$

We introduce the definition of variance, or diffusion of our force [7]:

$$\sigma^2 := \langle (f(t) - \langle f(t) \rangle)^2 \rangle, \quad (3.2)$$

which becomes, for systems fulfilling the condition (3.1),

$$\sigma^2 = \langle f^2(t) \rangle. \quad (3.3)$$

I.e. for such systems, the variance becomes the average of the square. We will only consider systems for which (3.1) holds, so we will simply denote the variance as the average of the square throughout the text. This is also referred to as the distribution of the force.

Proceeding with the statistics, we now introduce the so-called correlation functions.

3.2 Correlation functions

The notations that will be introduced here follow [1].

Consider the function $f(t)$ at an initial time $t_1 = t$ and at a final time $t_2 = t'$. We are interested in their correlation. Assuming that $f(t)$ satisfy the stochastic statistics (3.1), we require that it also satisfies

$$\langle f(t)f(t') \rangle = C(t - t'), \quad (3.4)$$

where the two-force correlation function $C$ is a function which depends only on the difference between the functions that are correlated (in this case it can be referred to as the time correlation function, since it only depends on time).

We will now introduce a coefficient, called the correlation length which will be denoted by $\tau$. This factor is a measure of the difference in the range of fluctuations between correlations. A typical behaviour of $C(t - t')$ for an Ornstein-Uhlenbeck process (which we will soon describe) is

$$C(t - t') \sim e^{-|t-t'|/\tau}, \quad (3.5)$$

where $|t - t'| < \tau$ (one can say that the correlation length is exponentially small). In the case of (3.9), it is sensible to require that $\tau$ is $< \frac{1}{\gamma}$. 

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The Brownian motion we will consider throughout the text will be confined to one spatial dimension. We will call this coordinate $x$. For the randomly fluctuating force depending on position $x$, as well as on time $t$, we get the correlation of the force $f(x,t)$ with $f(x',t')$ expressed as

$$\langle f(x,t)f(x',t') \rangle = C(x - x', t - t'). \tag{3.6}$$

### 3.3 Ornstein-Uhlenbeck processes

We will now consider stochastic processes satisfying Brownian motion. The material here follows from [7] and [8].

Consider an ensemble of heavy particles moving in a viscous medium (a fluid or a gas) consisting of light molecules. While the particles are moving in the medium, they are randomly subjected to forces from the molecules, such that the path of movement of each particle becomes completely random. The forces of which each of the heavy particles are subjected can be described by two parts. One damping force, which depends on the velocity of the particle, as well as its mass. That is, the momentum $p = mv$, as well as a coefficient of the damping, which we will denote by $\gamma$, gives the first part of the two forces,

$$F_1 = -p\gamma. \tag{3.7}$$

The second part is a rapidly fluctuating force, which we shall denote by $f(t)$, for the time variable $t$. This is called a random function and since its evolution is time-dependent, it is referred to as a stochastic process. We have

$$F_2 = f(t). \tag{3.8}$$

The equation of motion for each particle in the ensemble is given by Newton’s second law, which takes the form

$$\dot{p} = F_1 + F_2 = -p\gamma + f(t), \tag{3.9}$$

where $\dot{p}$ is the time-derivative of the momentum.

Ornstein-Uhlenbeck processes are stationary stochastic processes that follow the statistics of the Langevin approach [7]:

**The Langevin approach**

We will now analyze the momentum of Brownian particles. For short time intervals $|t - t'| < \tau$, the equation of motion (3.9) was integrated by Langevin, resulting in the momentum equation

$$p(t) = p(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t dt' e^{\gamma t'} f(t'), \tag{3.10}$$

where $p(0)$ is determined by an initial condition. We will assume each particle in the ensemble to be initially at rest, i.e. $p(0) = 0$. The average value of the momentum for the above equation, keeping (3.1) in mind, becomes

$$\langle p(t) \rangle = e^{-\gamma t}\int_0^t dt' e^{\gamma t'} \langle f(t') \rangle = 0. \tag{3.11}$$
The distribution of the momentum is given by

\[ \langle p^2(t) \rangle = e^{-2\gamma t} \int_0^t dt' \int_0^{t'} dt'' e^{\gamma(t'+t'')} \langle f(t')f(t'') \rangle. \]  
(3.12)

Using the notation (3.4) and making the substitution \( u = t' + t'' \) and \( v = t' - t'' \), we get the distribution as

\[ \langle p^2(t) \rangle = \frac{1}{2} e^{-2\gamma t} \int_0^{2t} du e^{\gamma u} \int_{-t}^t dv C(v) = \frac{1}{2\gamma} e^{-2\gamma t} (e^{2\gamma t} - 1) \int_{-t}^t dv C(v). \]  
(3.13)

The second integral has values (significantly) different from zero only for short time intervals (since as we stated earlier, the correlation length is exponentially small), so there will be no loss of generality to extend the integration limits to infinity. I.e. the process is stationary. At equilibrium, we get \( t' = 0 \) and \( t'' = t \) such that

\[ \langle p^2(t) \rangle = \frac{D_0}{\gamma} (1 - e^{-2\gamma t}), \]  
(3.14)

where the (time) diffusion constant \( D_0 \) is given by

\[ D_0 = \frac{1}{2} \int_{-\infty}^{\infty} dt \langle f(t)f(0) \rangle = \frac{1}{2} \int_{-\infty}^{\infty} dt C(t). \]  
(3.15)

The distribution (3.14) is called a momentum probability distribution (or probability density).

Proceeding after the Langevin approach, we require an equation describing the position \( x \) of the particle. Similar to the momentum approach, by integrating (3.10) (and dividing both sides by \( m \)), the equation

\[ x(t) = \frac{1}{m} \int_0^t dt' e^{-\gamma t'} \int_0^{t'} dt'' e^{-\gamma t''} f(t''), \]  
(3.16)

was obtained and integrated by parts by Ornstein and Uhlenbeck [8] resulting in:

\[ x(t) = \frac{1}{m\gamma} \left( \int_0^t dt' f(t') - e^{-\gamma t} \int_0^t dt' e^{\gamma t'} f(t') \right), \]  
(3.17)

where the boundary condition has been chosen as \( x(0) = 0 \), which is valid, since we can always choose our coordinate system arbitrarily without the loss of generality. The property (3.1) holds and the distribution of the position (i.e. its probability distribution) becomes

\[ \langle x^2(t) \rangle = \frac{1}{m^2\gamma^2} \int_0^t dt' \int_0^{t'} dt'' \langle f(t')f(t'') \rangle - \frac{2e^{-\gamma t}}{m^2\gamma^2} \int_0^t dt' \int_0^{t'} dt'' e^{\gamma t'} \langle f(t')f(t'') \rangle \\
+ \frac{e^{-2\gamma t}}{m^2\gamma^2} \int_0^t dt' \int_0^{t'} dt'' e^{-\gamma(t'+t'')} \langle f(t')f(t'') \rangle. \]  
(3.18)
We again use the notation (3.4) and we make the following substitutions: in the first and the third term, \( u = t' + t'' \) and \( v = t' - t'' \) and in the second term, \( u' = t' \) and \( v = t' - t'' \). Then,

\[
\langle x^2(t) \rangle = \frac{1}{2m^2\gamma^2} \int_0^{2t} du \int_{-t}^t dv C(v) - \frac{2e^{-\gamma t}}{m^2\gamma^2} \int_0^t du' e^{\gamma u'} \int_{-t}^t dv C(v) \\
+ \frac{e^{-2\gamma t}}{2m^2\gamma^2} \int_0^{2t} du e^{\gamma u} \int_{-t}^t dv C(v).
\]

By the same reasonings as before, we can extend the integration limits of the second integral in each term to infinity. At the same time evaluating the first integral of each term, we get

\[
\langle x^2(t) \rangle = \frac{1}{2m^2\gamma^2} \left( 2t \int_{-\infty}^\infty dv C(v) - \frac{4e^{-\gamma t}}{\gamma} (e^{\gamma t} - 1) \int_{-\infty}^\infty dv C(v) \right) \\
+ \frac{e^{-2\gamma t}}{\gamma} (e^{2\gamma t} - 1) \int_{-\infty}^\infty dv C(v) = D_x \left( 2t + \frac{(4 - e^{-\gamma t}) e^{-\gamma t} - 3}{\gamma} \right),
\]

with the space diffusion constant

\[
D_x = \frac{D_0}{m^2\gamma^2}.
\]

We will now consider a generalization of the standard Ornstein-Uhlenbeck process. The generalization consists in extending the force \( f(t) \) to be dependent on position as well. Then, the equation of motion (3.9) is generalized to:

\[
\dot{p} = -\gamma p + f(x, t),
\]

where \( f(x, t) \) follow the statistics of (3.1) and (3.6). For the study of this generalized equation of motion, we introduce the Fokker-Planck equation.

### 3.4 The Fokker-Planck equation

Here the material in [1] and [9] will be used.

The Fokker-Planck equation provides an expression for the generalized momentum probability density \( P(p, t) \), dependent of the momentum \( p \) and the time \( t \). It is obtained by first approximating the equation of motion (3.22) as the correlation length approaches zero. Arvedson et al. [1, 9] showed that this could be done by a Langevin equation:

\[
dp = -\gamma p \, dt + dw,
\]

where \( dw \) is assumed to satisfy the statistics \( \langle dw^2 \rangle = 2D_0 \, dt \) (and as usual \( \langle dw \rangle = 0 \)), where the diffusion constant \( D_0 \) is given by (3.15).

Introducing the general form of the diffusion constant, dependent of the momentum,

\[
D(p) = \frac{1}{2} \int_{-\infty}^\infty dt \, C(pt/m, t),
\]

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we can express the general form of the Fokker-Planck equation [9] of the momentum probability density $P(p,t)$ as

$$\frac{\partial P(p,t)}{\partial t} = -\frac{\partial}{\partial p}(v(p)P(p,t)) + \frac{\partial^2}{\partial p^2}(D(p)P(p,t)).$$  \hspace{1cm} (3.25)

In terms of the statistics of the Langevin equation (3.23), we can express the functions

$$v(p) = \frac{\langle dp \rangle}{dt},$$ \hspace{1cm} (3.26)

which corresponds to a force, and

$$D(p) = \frac{\langle dp^2 \rangle}{2dt},$$ \hspace{1cm} (3.27)

corresponding to diffusion. Evaluating the distribution $\langle dp^2 \rangle$, we get

$$\langle dp^2 \rangle = \gamma^2 p^2 dt^2 - 2\gamma p dt \langle dw \rangle + \langle dw^2 \rangle.$$ \hspace{1cm} (3.28)

It is argued in Arvedson et al. [1, 9] that, considering the equation of motion (3.23) as the forces depend principally on the small changes in position rather than the small changes in time, the force $dw$ can be approximated as an impulse $\delta w$. The variance of the impulse was obtained as

$$\langle \delta w^2 \rangle = 2D(p) \delta t,$$ \hspace{1cm} (3.29)

for a small time interval $\delta t$. The approximation also implies that the $dt$-term in (3.23) becomes very small. Thus, for the variance (3.28), the $dt^2$-term can be neglected. The diffusion constant, using this approximation, becomes

$$D(p) = \frac{\langle dw^2 \rangle}{2dt} = \frac{2D(p)dt}{2dt} = D(p),$$ \hspace{1cm} (3.30)

with $D(p)$ given by (3.24).

For the evaluation of $\langle dp \rangle = -\gamma p dt + \langle dw \rangle$, the approximation resulted in the average of the impulse as

$$\langle \delta w \rangle = \delta t \frac{d}{dp} D(p),$$ \hspace{1cm} (3.31)

which used in (3.26) becomes

$$v(p) = -\gamma p + \frac{d}{dp} D(p).$$ \hspace{1cm} (3.32)

Now, by (3.24) and (3.32), the Fokker-Planck equation (3.25) can be expressed as

$$\frac{\partial P(p,t)}{\partial t} = \frac{\partial}{\partial p} \left( \gamma p + D(p) \frac{\partial}{\partial p} \right) P(p,t).$$ \hspace{1cm} (3.33)

We will now consider the so called generic case, for which $D(p) = D_0 \frac{1}{|p|}$ (such that $D(p) \sim \frac{1}{|p|}$). For this generic case, the Fokker-Planck equation becomes

$$\frac{\partial P(p,t)}{\partial t} = \frac{\partial}{\partial p} \left( \gamma p + D_0 \frac{1}{|p|} \frac{\partial}{\partial p} \right) P(p,t).$$ \hspace{1cm} (3.34)
Introducing the dimensionless variable

\[ t' = \gamma t, \quad (3.35) \]

we get

\[ \frac{\partial P(p, t')}{\partial t'} = \frac{\partial P(p, t')}{\partial t} = \frac{\partial P(p, t')}{\partial t'} \gamma. \quad (3.36) \]

This alters (3.34) slightly into

\[ \frac{\partial P(p, t')}{\partial t'} = \frac{\partial}{\partial p} \left( p + \frac{D_0}{\gamma |p|} \frac{\partial}{\partial p} \right) P(p, t'). \quad (3.37) \]

Then, introducing the dimensionless variable (suggested in [9])

\[ z = \left( \frac{\gamma D_0}{1} \right)^{1/3} p, \quad (3.38) \]

we get

\[ \frac{\partial P(z, t')}{\partial z} = \frac{\partial P(z, t')}{\partial z} = \frac{\partial P(z, t')}{\partial z} \left( \frac{\gamma D_0}{1} \right)^{1/3} \quad (3.39) \]

and

\[ \frac{\partial}{\partial p} \left[ \frac{\partial P(z, t')}{\partial z} \left( \frac{\gamma D_0}{1} \right)^{1/3} \right] = 0 + \left( \frac{\gamma D_0}{1} \right)^{2/3} \frac{\partial^2 P(z, t')}{\partial z^2}. \quad (3.40) \]

Inserting this into (3.37), the Fokker-Planck equation becomes

\[ \frac{\partial P(z, t')}{\partial t'} = \frac{\partial}{\partial z} \left[ \left( \frac{\gamma D_0}{1} \right)^{1/3} p + \left( \frac{\gamma D_0}{1} \right)^{2/3} \frac{D_0}{\gamma |p|} \frac{\partial}{\partial z} \right] P(z, t') \]

\[ = \frac{\partial}{\partial z} \left( z + \frac{1}{|z|} \frac{\partial}{\partial z} \right) P(z, t'), \quad (3.41) \]

which we denote in terms of the so called Fokker-Planck operator \( \hat{F} \) as

\[ \frac{\partial P(z, t')}{\partial t'} = \frac{\partial}{\partial z} \left( z + \frac{1}{|z|} \frac{\partial}{\partial z} \right) P(z, t') \equiv \hat{F} P(z, t'). \quad (3.42) \]

We define the stationary state \( P_0(z) := P(z, 0) \) and use the proportionality relation \( P_0(z) \propto e^{-|z|^3/3} \), also suggested in [9]. Then we get

\[ \frac{\partial}{\partial z} \left( z + \frac{1}{|z|} \frac{\partial}{\partial z} \right) e^{-|z|^3/3} = \frac{\partial}{\partial z} z e^{-|z|^3/3} + \frac{\partial}{\partial z} \frac{1}{|z|} \frac{\partial}{\partial z} e^{-|z|^3/3} \]

\[ = e^{-|z|^3/3} - z^2 e^{-|z|^3/3} - \frac{1}{z^2} \partial e^{-|z|^3/3} + \frac{1}{|z|} \partial^2 e^{-|z|^3/3} \]

\[ = 2e^{-|z|^3/3} - |z|^2 e^{-|z|^3/3} - \frac{1}{z^2} \partial e^{-|z|^3/3} + \frac{1}{|z|} \partial^2 e^{-|z|^3/3} \]

\[ = 2e^{-|z|^3/3} - |z|^2 e^{-|z|^3/3} - 2e^{-|z|^3/3} + |z|^3 e^{-|z|^3/3} = 0, \]
i.e. $\hat{F} P_0 = 0$ and the Fokker-Planck equation can thus be put in the Hermitian form

$$P_0^{-1/2} \hat{F} P_0^{1/2} = \frac{1}{2} - \frac{|z|^3}{4} + \frac{\partial}{\partial z} \left( \frac{1}{|z|} \frac{\partial}{\partial z} \right).$$

(3.44)

We will refer to this as the Hamiltonian operator for our generalized Ornstein-Uhlenbeck system. We define our dimensionless Hamiltonian as

$$\hat{H} := \frac{\partial}{\partial z} \left( \frac{1}{|z|} \frac{\partial}{\partial z} \right) + \frac{1}{2} - \frac{|z|^3}{4}.$$  

(3.45)
4 Staggered ladder spectra

4.1 Eigenvalues and eigenfunctions

The material here will follow the line in [6].

For our Hamiltonian (3.45), we can set up the eigenvalue problem, from which we can determine the spectrum of energy eigenfunctions. For our generic case, the eigenvalue problem becomes

\[
\frac{d}{dz} \frac{1}{|z|} \frac{d\psi(z)}{dz} + \left( \frac{1}{2} - \frac{|z|^3}{4} \right) \psi(z) = \lambda \psi(z), \tag{4.1}
\]

where we assume that \( \psi(z) \propto e^{-|z|^3/6} \), as in the derivation of our Fokker-Planck Hamiltonian. Making the substitution \( x = |z|^3 \), we get

\[
\frac{d\psi(x)}{dx} = \frac{d\psi(x)}{dz} \frac{1}{3z|z|} \tag{4.2}
\]

and

\[
\frac{d}{dz} \frac{1}{|z|} \frac{d\psi(z)}{dz} = -\frac{1}{|z|} \frac{d\psi(z)}{dz} + \frac{1}{|z|} \frac{d^2\psi(z)}{dz^2} = -\frac{1}{|z|} \frac{d\psi(x)}{dx} \frac{d}{dx} \left( \frac{d\psi(x)}{dx} 3z|z| \right) \tag{4.3}
\]

\[
= -3 \frac{d\psi(x)}{dx} + 6 \frac{d\psi(x)}{dx} + (3z|z|)^2 \frac{1}{|z|} \frac{d^2\psi(x)}{dx^2} = 3 \frac{d\psi(x)}{dx} + 9x \frac{d^2\psi(x)}{dx^2}.
\]

We get the equation (4.1) expressed in functions \( \psi(x) \) as (placing all terms on the same side)

\[
9x \frac{d^2\psi}{dx^2} + 3 \frac{d\psi}{dx} - \left( \frac{x}{4} + \lambda - \frac{1}{2} \right) \psi = 0. \tag{4.4}
\]

Dividing this equation by \( 9x \), the behaviour as \( x \to \infty \) becomes

\[
\frac{d^2\psi}{dx^2} - \frac{1}{36} \psi \approx 0. \tag{4.5}
\]

I.e. the wavefunction that follows the asymptotic behaviour we have considered is indeed \( \psi(x) \propto e^{-x/6} \). To cut off this asymptotic behaviour, we make the transformation \( \psi(x) = \phi(x)e^{-x/6} \), giving us

\[
\frac{d\psi}{dx} = \left( -\frac{1}{6} \phi + \frac{d\phi}{dx} \right) e^{-x/6} \tag{4.6}
\]

and

\[
\frac{d^2\psi}{dx^2} = \left( \frac{1}{36} \phi - \frac{1}{3} \frac{d\phi}{dx} + \frac{d^2\phi}{dx^2} \right) e^{-x/6}. \tag{4.7}
\]

Inserting these relations into (4.4) and dividing by \( e^{-x/6} \), we get

\[
9x \left( \frac{1}{36} \phi - \frac{1}{3} \frac{d\phi}{dx} + \frac{d^2\phi}{dx^2} \right) + 3 \left( -\frac{1}{6} \phi + \frac{d\phi}{dx} \right) - \left( \frac{x}{4} + \lambda - \frac{1}{2} \right) \phi = 0 \tag{4.8}
\]

\[
= \frac{1}{4} x \phi - 3x \frac{d\phi}{dx} + 9x \frac{d^2\phi}{dx^2} - \frac{1}{2} \phi + 3 \frac{d\phi}{dx} - \left( \frac{x}{4} + \lambda - \frac{1}{2} \right) \phi = 0.
\]
Dividing this by 9 gives us the equation on the form, which we can recognize from (2.17) (p. 7 in this paper),

\[ \frac{d^2\phi}{dx^2} + \left( \frac{1}{3} - \frac{1}{3}x \right) \frac{d\phi}{dx} + \frac{\lambda}{9} \phi = 0. \]  

(4.9)

This is Kummer’s hypergeometric differential equation in the variable \( \frac{x}{3} \). The general solutions follow (2.18),

\[ \phi(x) = A \cdot \, _1F_1 \left( \frac{\lambda}{3}; \frac{1}{3}; \frac{x}{3} \right) + B x^{2/3} \cdot \, _1F_1 \left( \frac{2}{3} + \frac{\lambda}{3}; \frac{5}{3}; \frac{x}{3} \right). \]  

(4.10)

The wave eigenfunctions are given as the even and odd solutions. I.e. we have respectively, recalling that \( N^\pm \) are normalization constants (writing the variable \( x \) as \(|x|\), since it has only positive values, due to our transformation \( x = |z|^3 \)),

\[ \psi^+(x) = N^+ \cdot \, _1F_1 \left( \frac{\lambda}{3}; \frac{1}{3}; \frac{|x|}{3} \right) e^{-|x|/6} \]  

(4.11)

and

\[ \psi^-(x) = N^- |x|^{2/3} \cdot \, _1F_1 \left( \frac{2}{3} + \frac{\lambda}{3}; \frac{5}{3}; \frac{|x|}{3} \right) e^{-|x|/6}. \]  

(4.12)

Analyzing the behaviour of Kummer’s hypergeometric function \( \, _1F_1 \) as \( x \to \infty \), we can only get that \( \psi \to 0 \) if \( \, _1F_1 \) is a polynomial, resulting in eigenvalues equal to zero, or a negative integer. This can be seen by analyzing (2.21) (p. 8 in this paper). As this power series is expanded, we require that (using the Pochhammer symbol)

\[ \left( \frac{\lambda}{3} \right)_m = 0, \]  

(4.13)

respectively

\[ \left( \frac{2}{3} + \frac{\lambda}{3} \right)_m = 0, \]  

(4.14)

for some \( m \). This is obtained for our parameters (4.11) and (4.12) if they are negative integers. Then every other term in the expansion becomes negative. The dominant term in (4.11) and (4.12) then becomes \( e^{-|x|/6} \), which decreases exponentially towards zero as \( x \to \pm \infty \). Thus, the even eigenvalues are given by

\[ \frac{\lambda^+_n}{3} = -n, \]  

(4.15)

i.e.

\[ \lambda^+_n = -3n, \quad n \in \mathbb{Z}_{\geq 0} \]  

(4.16)

and the odd eigenfunctions are given by

\[ \frac{2}{3} + \frac{\lambda^-_n}{3} = -n, \]  

(4.17)

i.e.

\[ \lambda^-_n = -3n - 2, \quad n \in \mathbb{Z}_{\geq 0}. \]  

(4.18)
The eigenvalues for the even and odd cases are staggered with respect to each other. I.e. they are only equidistant with respect to theirselves. The even eigenvalues are equidistant with respect to each other, but not with respect to the odd eigenvalues, and vice versa. The energies here are non-positive, but the system we consider is not an ordinary quantum mechanical system, but a system with diffusion. It is a consequence of this that the energy tends towards minus infinity. Regarding Figure 2, the operators $\hat{A}^{++}$ and $\hat{A}^{--}$ will be explained in Section 4.3.

![Diagram](image)

Figure 2: The staggered ladder spectrum of the Hamiltonian (3.45) in the generalized Ornstein-Uhlenbeck system. The ladders are equidistant for the same parity, but staggered with respect to different parities (distance between even-odd states differs from the distance between odd-even states).

The eigenfunctions (4.11) and (4.12) can be expressed in terms of associated Laguerre polynomials. For convenience in the next section of propagators, we adapt the earlier notation of $|z|^3 = x$. We get (with relations and normalization constants from [9])

$$
\psi^+_n(z) = N^+_n {}_1F_1 \left( -n; \frac{1}{3}; \frac{|z|^3}{3} \right) e^{-|z|^3/6},
$$

(4.19)

with the normalization constants

$$
N^+_n = \frac{3^{1/3}}{\Gamma(1/3)} \sqrt{ \frac{1}{2} \frac{\Gamma(n + 1/3)}{\Gamma(n + 1)} }
$$

(4.20)

and the hypergeometric function can be expressed as

$$
{}_1F_1 \left( -n; \frac{1}{3}; \frac{|z|^3}{3} \right) = \frac{\Gamma(1/3)\Gamma(n + 1)}{\Gamma(n + 1/3)} L_n^{-2/3} \left( \frac{|z|^3}{3} \right).
$$

(4.21)

Then, the normalized even eigenfunctions become

$$
\psi^+_n(z) = 3^{1/3} \sqrt{ \frac{1}{2} \frac{\Gamma(n + 1)}{\Gamma(n + 1/3)} } L_n^{-2/3} \left( \frac{|z|^3}{3} \right) e^{-|z|^3/6}.
$$

(4.22)
Similarly, the normalized odd eigenfunctions become

$$\psi_n^-(z) = 3^{-1/3} \sqrt{\frac{1}{2} \frac{\Gamma(n + 1)}{\Gamma(n + 5/3)}} \frac{|z|^3}{3} e^{-|z|^3/6}. \quad (4.23)$$
4.2 Propagators

The material here comes from [1], [5] and [10].

Propagators are used to represent momentum transfers between functions separated by a short correlation length (i.e. correlation functions are determined by propagators). Consider a particle at an initial position \( y \) and at a final position \( z \). The wave function at a final position can be described by a function including both initial and final positions, as well as an initial wave function. It is expressed as

\[
\psi(z, t) = \int_{-\infty}^{\infty} dy \, K(y, t; z, t') \psi(y, t), \tag{4.24}
\]

where \( t \) is the time at an initial position and \( t' \) is the time at a final position. In this expression we have a kernel \( K \) within the integral. This is called the wave propagator and it gives the probability density for the particle to go from a position \( y \) to a position \( z \) in some time \( t' - t \). This is written as

\[
K(y, t; z, t') = \sum_{n=0}^{\infty} a_n(y) \phi_n(z) e^{\lambda_n(t' - t)}, \tag{4.25}
\]

for eigenvalues \( \lambda_n \) and eigenfunctions \( \phi_n \). The \( a_n \)'s are determined by the boundary condition \( K(y, 0; z, 0) = \delta(z - y) \) for the Dirac delta function [4]:

\[
\delta(z - y) = \begin{cases} 
0 & \text{if } z \neq y; \\
\infty & \text{if } z = y, 
\end{cases} \tag{4.26}
\]

which also satisfies the property

\[
\int_{-\infty}^{\infty} \delta(z) \, dz = 1. \tag{4.27}
\]

There is no loss in generality to put an initial time as \( t = 0 \), which we shall do for notational simplicity. We will then drop this variable in the notation of the propagator. Consider now the Hamiltonian (3.45). The propagator for this operator is given by

\[
K(y, z; t') = \sum_{n=0}^{\infty} P_n^{-1/2}(y) \psi_n(y) P_n^{1/2}(z) \psi_n(z) e^{\lambda_n t'}, \tag{4.28}
\]

with \( \sigma = \pm \), representing the even respectively odd eigenvalues and eigenfunctions, so we need to sum over both odd and even cases as well. Here, \( P_n^{-1/2}(y) \psi_n(y) \) act as the term determined by the boundary condition, i.e. in comparison with (4.25), we have \( a_n(y) = P_n^{-1/2}(y) \psi_n(y) \). By [10], \( P_n^{1/2}(y) = \psi_n^+(y) = e^{-|y|^3/6} \) and \( P_n^{1/2}(y) = \psi_n^-(y) = y |y| e^{-|y|^3/6} \) (and the same for the variable \( z \)). So we split up the summation into odd and even parts, dividing everything by \( \psi_0^-(y) \psi_0^+(y) \),

\[
K(y, z; t') = \frac{\psi_0^+(z)}{\psi_0^-(y)} \left( \sum_{n=0}^{\infty} \psi_n^+(y) \psi_n^+(z) e^{\lambda_n t'} + \sum_{n=0}^{\infty} \psi_n^-(y) \psi_n^-(z) e^{\lambda_n t'} \right). \tag{4.29}
\]
By (4.16), (4.18), (4.22) and (4.23), we get the propagator as
\[
K(y, z; t') = \frac{e^{-|y|^3/3}}{e^{-|y|^3/6}} \left[ \sum_{n=0}^{\infty} 3^{2/3} 2 \Gamma(n + 1) / 2 \Gamma(n + 1/3) L_n^{-2/3} \left( \frac{|y|^3}{3} \right) L_n^{-2/3} \left( \frac{|z|^3}{3} \right) e^{-3nt'} \right. \\
+ \left. \sum_{n=0}^{\infty} \frac{3^{-2/3} \Gamma(n + 1) y |y| |z| L_n^{-2/3} \left( \frac{|y|^3}{3} \right) L_n^{-2/3} \left( \frac{|z|^3}{3} \right) e^{(-3n-2)t'} \right] \\
= \frac{e^{-|y|^3/3}}{2} \left[ 3^{2/3} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + 1/3)} L_n^{-2/3} \left( \frac{|y|^3}{3} \right) L_n^{-2/3} \left( \frac{|z|^3}{3} \right) e^{-3nt'} \right. \\
+ \left. \frac{y |y| |z| e^{-2t'}}{3^{2/3}} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + 5/3)} L_n^{-2/3} \left( \frac{|y|^3}{3} \right) L_n^{-2/3} \left( \frac{|z|^3}{3} \right) e^{-3nt'} \right] .
\]

According to [10], the following formula provides an exact result for the summation:
\[
\sum_{n=0}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + \beta + 1)} L_n^\beta(y) L_n^\beta(z) e^n = \frac{(yz\varepsilon)^{-\beta/2}}{1 - \varepsilon} \exp \left[ -\epsilon \left( \frac{y + z}{1 - \varepsilon} \right) I_\beta \left( \frac{2 \sqrt{yz\varepsilon}}{1 - \varepsilon} \right) \right] .
\]

for $|\varepsilon| < 1$ and where $I_\beta$ are modified Bessel functions. We take $\varepsilon = e^{-3t'}$ ($t' > 0$) and use this formula for $\beta = -\frac{2}{3}$ and $\beta = \frac{2}{3}$. Keeping in mind that our variables are $\frac{|y|^3}{3}$ and $\frac{|z|^3}{3}$ respectively $L_n^\beta$, the formula we will use becomes explicitly:
\[
\sum_{n=0}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + \beta + 1)} L_n^{2/3}(y) L_n^{2/3}(z) e^n = \frac{(|y|^3 |z|^3 e^{-3t'})^{1+1/3}}{(3^{1/3}(1 - e^{-3t'}))} \times \exp \left( -e^{-3t'} \frac{|y|^3 + |z|^3}{3(1 - e^{-3t'})} \right) I_{2/3} \left( \frac{2(|y|^3 |z|^3 e^{-3t'})^{1/2}}{3(1 - e^{-3t'})} \right) .
\]

Inserting this into (4.30), we get
\[
K(y, z; t') = \frac{e^{-|y|^3/3}}{2} \left[ 3^{2/3} \frac{|y|^3 |z|^3 e^{-3t'})^{1/3}}{3^{2/3}(1 - e^{-3t'})} \exp \left( -e^{-3t'} \frac{|y|^3 + |z|^3}{3(1 - e^{-3t'})} \right) \right. \\
\times \left. L_{-2/3} \left( \frac{2(|y|^3 |z|^3 e^{-3t'})^{1/2}}{3(1 - e^{-3t'})} \right) + \frac{y |y| |z| e^{-2t'}}{3^{2/3}} \exp \left( -e^{-3t'} \frac{|y|^3 + |z|^3}{3(1 - e^{-3t'})} \right) I_{2/3} \left( \frac{2(|y|^3 |z|^3 e^{-3t'})^{1/2}}{3(1 - e^{-3t'})} \right) \right] \\
= \frac{e^{-t'}}{2(1 - e^{-3t'})} \exp \left( -\frac{|y|^3 - |y|^3 e^{-3t'}}{3(1 - e^{-3t'})} \right) \left[ |y| |z| I_{-2/3} \left( \frac{2(|y|^3 |z|^3 e^{-3t'})^{1/2}}{3(1 - e^{-3t'})} \right) \right. \\
\times \left. + yz I_{2/3} \left( \frac{2(|y|^3/2 |z|^3/2 e^{-3t'})^{1/2}}{3(1 - e^{-3t'})} \right) \right] .
\]

By this result, we can express the momentum probability density, using (4.24).
Choosing $t = 0$ as an initial condition, we get
\[
P(z, t') = \int_{-\infty}^{\infty} dy K(y, z; t') P(y, 0) .
\]
The initial condition for \( P(y, t') \) is given by: \( P(y, 0) = \delta(y) \), for Dirac’s delta function (cf. (4.26) and (4.27)). The nice consequence of this condition is that we get
\[
P(z, t') = K(0, z; t').
\]
For the modified Bessel functions, the following relation is presented in [10]:
\[
I_y \sim \frac{1}{\Gamma(\beta + 1)} \left( \frac{1}{2} y \right)^\beta,
\]
as \( y \to 0 \). Specifically for our case (4.33), we need the explicit relations
\[
I_{\pm 2/3} \left( \frac{2|y|^{3/2}|z|^{3/2}e^{-3t'/2}}{3(1 - e^{-3t'})} \right) \sim \frac{1}{\Gamma(\beta + 1)2^{2/3}} \left( \frac{2|y|^{3/2}|z|^{3/2}e^{-3t'/2}}{3(1 - e^{-3t'})} \right)^{2/3}.
\]
Now, taking the limit as \( y \to 0 \) in (4.33), we get
\[
K(0, z; t') = \lim_{y \to 0} \frac{e^{-t'}}{2(1 - e^{-3t'})} \exp \left( \frac{-|z|^3 - |y|^3e^{-3t'}}{3(1 - e^{-3t'})} \right) \times \left| \frac{|y||z|I_{-2/3} \left( \frac{2|y|^{3/2}|z|^{3/2}e^{-3t'/2}}{3(1 - e^{-3t'})} \right)}{\Gamma(1/3)} + yzI_{2/3} \left( \frac{2|y|^{3/2}|z|^{3/2}e^{-3t'/2}}{3(1 - e^{-3t'})} \right) \right|
\]
\[
= \lim_{y \to 0} \frac{e^{-t'}}{2(1 - e^{-3t'})} \exp \left( \frac{-|z|^3 - |y|^3e^{-3t'}}{3(1 - e^{-3t'})} \right) \left| \frac{|y||z|e^{-t'}}{\Gamma(1/3)} \right|^{2/3}
\]
\[
= \frac{1}{2\Gamma(1/3)} \left| \frac{|y||z|e^{-t'}}{\Gamma(1/3)} \right|^{1/3}
\]
i.e.
\[
P(z, t') = \frac{3}{2\Gamma(1/3)} \frac{1}{[3(1 - e^{-3t'})]^{1/3}} \exp \left( \frac{-|z|^3}{3(1 - e^{-3t'})} \right).
\]
From [11], we get the relation:
\[
\Gamma(n + 1/3) = \frac{1 \cdot 4 \cdot 7 \cdots (3n - 2)}{3^n} \Gamma(1/3) = \frac{(3n - 2)!!}{3^n} \Gamma(1/3),
\]
so for \( n = 1 \) we have that
\[
\Gamma(4/3) = \frac{\Gamma(1/3)}{3}.
\]
Changing back from our dimensionless variables, we insert (3.35) and (3.38) into the function (4.39) to get the momentum probability density dependent of the momentum and the time of our generic case,
\[
P(p, t) = \frac{1}{\Gamma(4/3)} \frac{\gamma^{1/3}}{[3D_0(1 - e^{-3\gamma t})]^{1/3}} \exp \left( \frac{-\gamma|p|^3}{3D_0(1 - e^{-3\gamma t})} \right),
\]
4.3 Commutators

Here we will construct raising and lowering operators that act within our staggered ladder spectra. Operators will act upon each other by taking the commutator of the operators considered. This is denoted by square brackets. We will present results, which will be motivated. Calculations are postponed to Appendix A, together with algebraic properties of the commutators. For notational simplicity, we introduce the following notations and definitions (similar notations are used in e.g. [1, 9]):

\[ \partial_z = \frac{d}{dz} \]
\[ \hat{a}^\pm := \partial_z \pm \frac{1}{2}z|z| \] (4.43)

and

\[
\begin{cases}
\hat{A}^{++} := \hat{a}^+|z|^{-1}\hat{a}^+; \\
\hat{A}^{--} := \hat{a}^-|z|^{-1}\hat{a}^-; \\
\hat{A}^{+} := \hat{a}^-|z|^{-1}\hat{a}^+; \\
\hat{A}^{--} := \hat{a}^+|z|^{-1}\hat{a}^-.
\end{cases} \] (4.44)

One can note that (4.44) are similar to the raising and lowering operators (2.52) and (2.53) (pp. 11-12 in this paper), but they are indeed not the same. In (4.45) the operators \( \hat{A}^{++} \) and \( \hat{A}^{--} \) gives rise to second order in derivation and they act as the raising respectively lowering operators in our generalized Ornstein-Uhlenbeck system. The operator \( \hat{A}^{--} \) can be put in another form,

\[
\hat{A}^{--} = (\partial_z - \frac{1}{2}z|z|)|z|^{-1}(\partial_z + \frac{1}{2}z|z|) = \partial_z|z|^{-1}\partial_z + \frac{1}{2}\partial_zz - \frac{1}{2}\partial_z - \frac{1}{2}|z|^3 \]
\[
= \partial_z|z|^{-1}\partial_z + \frac{1}{2} - \frac{1}{4}|z|^3,
\] (4.45)

which we recognize as our Fokker-Planck Hamiltonian (3.45) (the operator \( \hat{A}^{--} \) can be interpreted as a Hamiltonian operator acting in a different system).

In these abbreviations, one can calculate some interesting properties [1]:

\[ [\hat{A}^{+}, \hat{A}^{\pm}] = \pm 3\hat{A}^{\pm}; \]
\[ [\hat{A}^{++}, \hat{A}^{\pm}] = \pm 3(\hat{A}^{++} + \hat{A}^{--}); \]
\[ \hat{A}^{--} - \hat{A}^{++} = \hat{I}; \]
\[ \hat{A}^{--}\hat{A}^{++} = 2\hat{A}^{--} + (\hat{A}^{++})^2. \] (4.46)

(for details on the calculations, cf. Section I). The interesting property (4.47) is interpreted as by letting the operator \( \hat{A}^{--} \) act on \( \hat{A}^{++} \) respectively on \( \hat{A}^{--} \). Then one gets a resulting eigenfunction which has eigenvalues raised respectively lowered by three (for an illustration of this, cf. Figure 2). A consequence of this is, by letting \( \hat{A}^{++} \) operate on the ground state wavefunctions \( \psi_0^+(z) \) and \( \psi_0^-(z) \), we get a result which is identically equal to zero:

\[ \hat{A}^{++}\psi_0^+ \equiv 0, \] (4.47)
respectively

\[ \hat{A}^{++} \psi_0^- \equiv 0 \quad (4.52) \]

(which is calculated in Section II). This is interpreted as by raising the highest eigenvalue of our Hamiltonian, we get identically zero, since there are no eigenvalue states over the highest value (this is analogous to lowering the lowest energy state in an ordinary quantum mechanical system).
5 Conclusion

The Fokker-Planck equation was derived for the generalized Ornstein-Uhlenbeck process. The generic case was considered, for which the diffusion constant was put as $D(p) = D_0 |z|^{-1}$, with $D_0$ given by (3.15). Introducing dimensionless variables and deriving a Hermitian form of this equation, we could define a dimensionless Hamiltonian operator (3.45). Studying the eigenvalue problem for this operator, we obtained solutions expressed in even respectively odd eigenfunctions. The respective even and odd eigenvalues became (for $Z \geq 0$):

$$\begin{align*}
\lambda^+_n &= -3n; \\
\lambda^-_n &= -3n - 2.
\end{align*}$$

(5.1)

First, we note that the eigenvalues are negative (or zero). This might be interpreted as "negative energies", but one should not be put any particular weight to this, since the system we consider is not a fundamental quantum mechanical system (it is a system experiencing diffusion). Second, we note that each of the even eigenvalues are equidistant with respect to each other, with the intermediate spacing of three times the eigenvalue. This is equal to the intermediate spacing of the odd eigenvalues with respect to each other (the entire spectrum is obtained by letting the raising and lowering operators $\hat{A}^{++}$ respectively $\hat{A}^{--}$ repeatedly act on the Hamiltonian). Third, we note that the spacing between the even and the odd eigenvalues is staggered. This might have something to do with that our raising and lowering operators are expressed in terms of nonlinear second-order derivatives, instead of linear first-order derivatives as in the case of the harmonic oscillator treated in the beginning of this paper. According to [1], the staggered spectrum might be a consequence of the singularity at $z = 0$ of the Hamiltonian.

Proceeding from this paper, one can confirm the results of the staggered ladder spectra for the general case. As described by Arvedson et al. in [9], one would consider the constant of the diffusion as $D(p) \sim |z|^\zeta$, $Z \leq -2$ (since we have already considered the case $\zeta = -1$ in this paper). The Hamiltonian would become

$$\hat{H} = \frac{\partial}{\partial z} \frac{1}{|z|^\zeta} \frac{\partial}{\partial z} + \frac{1}{2} \frac{|z|^{2+\zeta}}{4}.$$  

(5.2)

The eigenvalues one would have to confirm are

$$\begin{align*}
\lambda^+_n &= -(2 + \zeta)n; \\
\lambda^-_n &= -(2 + \zeta)n - 1 - \zeta. 
\end{align*}$$

(5.3)

The eigenfunctions would be considered as $\psi^+_0(z) \propto e^{-|z|^{1+2/(4+2\zeta)}}$ and $\psi^-_0(z) \propto z|z|^{\zeta} e^{-|z|^{1+2/(4+2\zeta)}}$. The operators

$$\begin{align*}
\hat{A}^{\pm\pm} &= \hat{a}^\pm |z|^{-\zeta} \hat{a}^\pm; \\
\hat{A}^{\pm\mp} &= \hat{a}^\pm |z|^{-\zeta} \hat{a}^{\mp},
\end{align*}$$

(5.4)

would be considered for $\hat{a}^\pm = \partial_z \pm \frac{1}{2} |z|^\zeta$, which gives rise to the commutator

$$[\hat{A}^{--}, \hat{A}^{\pm\pm}] = \pm (2 + \zeta) \hat{A}^{\pm\pm}.$$  

(5.5)
I Appendix A

Commutators

Here, material from [2] will be used to calculate some commutator algebraic results presented in [1].

Consider two operators $\hat{A}$ and $\hat{B}$. The commutator is defined as

$$[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A}. \quad (I.1)$$

For a complete set of functions $\psi_n$, the operators are called compatible if and only if they have simultaneous eigenfunctions $\psi_n$:

$$\hat{A}\psi_n = a_n\psi_n; \quad (I.2)$$

$$\hat{B}\psi_n = b_n\psi_n, \quad (I.3)$$

with eigenvalues $a_n$ and $b_n$. Then we also have

$$\hat{A}\hat{B}\psi_n = a_nb_n\psi_n = b_na_n\psi_n = \hat{B}\hat{A}\psi_n. \quad (I.4)$$

Thus, if $\psi_n$ is complete, then $\hat{A}$ and $\hat{B}$ commute, i.e.

$$[\hat{A}, \hat{B}] = 0. \quad (I.5)$$

Commutator algebraic relations

Consider five operators $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ and $\hat{E}$. Their algebraic properties follow from the definition of commutators:

$$[\hat{A}, \hat{A}] = 0; \quad (I.6)$$

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]; \quad (I.7)$$

$$[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]; \quad (I.8)$$

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]; \quad (I.9)$$

$$[\hat{A}\hat{B}\hat{C}, \hat{D}\hat{E}] = \hat{A}\hat{B}[\hat{C}, \hat{D}\hat{E}] + \hat{A}[\hat{B}, \hat{D}\hat{E}]\hat{C} + [\hat{A}, \hat{D}\hat{E}]\hat{B}\hat{C} \quad (I.10)$$

$$= \hat{A}\hat{B}\hat{D}[\hat{C}, \hat{E}] + \hat{A}\hat{B}\hat{C}, \hat{D}]\hat{E} + \hat{A}\hat{D}[\hat{B}, \hat{E}]\hat{C}$$

$$+ \hat{A}[\hat{B}, \hat{D}]\hat{E}\hat{C} + \hat{D}[\hat{A}, \hat{E}]\hat{B}\hat{C} + [\hat{A}, \hat{D}]\hat{E}\hat{B}\hat{C};$$

$$\hat{A}\hat{B} = [\hat{A}, \hat{B}] + \hat{B}\hat{A}. \quad (I.11)$$

Calculations.

Now, some calculations will follow from the notations and definitions given in (4.43), (4.44) and (4.45). First, some simple calculations will be made which will be used in the latter, more laborious calculations.
Example I.1

\[ \hat{a}^{-} - \hat{a}^{+} = (\partial_{z} - \frac{1}{2}z|z|) - (\partial_{z} + \frac{1}{2}z|z|) = -z|z|. \]  

(I.12)

Example I.2

\[
\begin{align*}
[\hat{a}^{\pm}, \hat{a}^+] &= (\partial_{z} \pm \frac{1}{2}z|z|)(\partial_{z} \mp \frac{1}{2}z|z|) - (\partial_{z} \pm \frac{1}{2}z|z|)(\partial_{z} \mp \frac{1}{2}z|z|) \\
&= \partial_{z}^2 \pm \frac{1}{2}z|z|\partial_{z}\partial_{\bar{z}}z|z| \mp \frac{1}{2}z^2|z|^2 - \partial_{z}^2 \pm \frac{1}{2}z|z|\partial_{z}\partial_{\bar{z}}z|z| \mp \frac{1}{2}z^2|z|^2 \\
&= \pm z|z|\partial_{z} \mp \partial_{\bar{z}}z|z| = \mp 2|z|.
\end{align*}
\]  

Example I.3

\[
[\hat{a}^{\pm}, |z|^{-1}] = (\partial_{z} \pm \frac{1}{2}z|z|)|z|^{-1} - |z|^{-1}(\partial_{z} \pm \frac{1}{2}z|z|) = \partial_{z}|z|^{-1} \pm \frac{1}{2}z - |z|^{-1}\partial_{z} \mp \frac{1}{2}z \\
= -(z|z|)^{-1}.
\]  

(I.14)

Example I.4

\[
[\hat{a}^{\pm}, z^{-3}] = (\partial_{z} \pm \frac{1}{2}z|z|)z^{-3} - z^{-3}(\partial_{z} \pm \frac{1}{2}z|z|) = -3z^{-4} \pm \frac{1}{2}|z|^{-1} - z^{-4}\partial_{z} \mp \frac{1}{2}|z|^{-1} \\
= -3z^{-4}.
\]  

(I.15)

Now, the more laborious calculations:

Example I.5

\[
[\hat{\Lambda}^{\pm}, \hat{\Lambda}^{++}] = [\hat{a}^{-}z^{-1}\hat{a}^{+}, \hat{a}^{+}z^{-1}\hat{a}^{+}] = \hat{a}^{-}z^{-1}[\hat{a}^{+}, \hat{a}^{+}]z^{-1}\hat{a}^{+} \\
= \hat{a}^{-}z^{-1}\hat{a}^{+}[\hat{a}^{-}z^{-1}\hat{a}^{+}]z^{-1}\hat{a}^{+} + [\hat{a}^{-}, \hat{a}^{+}]z^{-1}\hat{a}^{+}]z^{-1}\hat{a}^{+} \\
= \hat{a}^{-}z^{-1}\hat{a}^{+}[\hat{a}^{-}z^{-1}\hat{a}^{+}]z^{-1}\hat{a}^{+} + \hat{a}^{-}\hat{a}^{+}z^{-1}\hat{a}^{+} + \hat{a}^{-}\hat{a}^{+}z^{-1}\hat{a}^{+} + [\hat{a}^{-}, \hat{a}^{+}]z^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} \\
= \hat{a}^{-}z^{-1}\hat{a}^{+}(-z|z|)^{-1}\hat{a}^{+} + \hat{a}^{-}\hat{a}^{+}z^{-1}(z|z|)^{-1}\hat{a}^{+} \\
= \hat{a}^{-}z^{-1}\hat{a}^{+}(-z|z|)^{-1}\hat{a}^{+} + \hat{a}^{-}\hat{a}^{+}z^{-1}(z|z|)^{-1}\hat{a}^{+} \\
+ \hat{a}^{-}(-z|z|)^{-1}\hat{a}^{+} + 4\hat{a}^{+}z^{-1}\hat{a}^{+} - \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} \\
= -\hat{a}^{-}(-z|z|)^{-1}\hat{a}^{+} + 4\hat{a}^{+}z^{-1}\hat{a}^{+} - \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} \\
= -\hat{a}^{-}(-z|z|)^{-1}\hat{a}^{+} - \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+} - 3\hat{a}^{-}z^{-4}\hat{a}^{+} + 2\hat{a}^{-}z^{-3}\hat{a}^{+} \\
+ 4\hat{a}^{+}z^{-1}\hat{a}^{+} - \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} = -\hat{a}^{-}(-3\hat{a}^{+}, z^{-3})z^{-1}\hat{a}^{+} \\
+ \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} + \hat{a}^{-}\hat{a}^{+}z^{-1}\hat{a}^{+} + \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} \\
= -2\hat{a}^{-}z^{-4}\hat{a}^{+} + \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} + 2\hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} \\
- \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} = \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} + 4\hat{a}^{+}z^{-1}\hat{a}^{+} \\
- \hat{a}^{-}(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} = (\hat{a}^{-} - \hat{a}^{+})(z|z|)^{-1}\hat{a}^{+}z^{-1}\hat{a}^{+} \\
+ 4\hat{a}^{+}z^{-1}\hat{a}^{+} = 3\hat{a}^{+}z^{-1}\hat{a}^{+} = 3\hat{A}^{++}.
\]
With completely similar calculations one shows that

\[ [\hat{A}^-, \hat{A}^-] = -3\hat{A}^-. \]  

(I.17)

Example I.6

\[ [\hat{A}^{--}, \hat{A}^{++}] = [\hat{a}^-|z|^{-1}\hat{a}^-, \hat{a}^+|z|^{-1}\hat{a}^+] = \hat{a}^-|z|^{-1}\hat{a}^-, \hat{a}^+|z|^{-1}\hat{a}^+ \]  

(I.18)

\[ + \hat{a}^-\hat{a}^+|z|^{-1}\hat{a}^+\hat{a}^- + |\hat{a}^-\hat{a}^+|z|^{-1}\hat{a}^+|z|^{-1}\hat{a}^- \]

\[ = \hat{a}^-|z|^{-1}\hat{a}^+|z|^{-1}[\hat{a}^-\hat{a}^+, \hat{a}^+|z|^{-1}\hat{a}^+] \]

\[ + \hat{a}^-|z|^{-1}[\hat{a}^-\hat{a}^+]|z|^{-1}\hat{a}^+ + \hat{a}^-\hat{a}^+|z|^{-1}|z|^{-1}\hat{a}^+\hat{a}^- \]

\[ = 2\hat{a}^-|z|^{-1}\hat{a}^+ - \hat{a}^-|z|^{-1}\hat{a}^+(z|z|)^{-1}\hat{a}^+ + 2\hat{a}^-|z|^{-1}\hat{a}^+ + \hat{a}^-\hat{a}^+z^{-3} \]

\[ + \hat{a}^-z^{-3}\hat{a}^+ - 2\hat{a}^-|z|^{-1}\hat{a}^+(z|z|)^{-1}\hat{a}^+ + \hat{a}^-\hat{a}^+|z|^{-1}\hat{a}^+ + 2\hat{a}^-|z|^{-1}\hat{a}^- \]

\[ = 4\hat{a}^-|z|^{-1}\hat{a}^+ + 4\hat{a}^+|z|^{-1}\hat{a}^+ - \hat{a}^-|z|^{-1}\hat{a}^+(z|z|)^{-1}\hat{a}^+ \]

\[ + \hat{a}^-[\hat{a}^+, |z|^{-1}|z|^{-1}\hat{a}^+ + \hat{a}^-|z|^{-1}\hat{a}^+(z|z|)^{-1}\hat{a}^- \]

\[ + \hat{a}^+(z|z|)^{-1}[|z|^{-1}\hat{a}^+] + \hat{a}^- (z|z|)^{-1}\hat{a}^+|z|^{-1}\hat{a}^- \]

\[ = 3\hat{A}^- + 3\hat{A}^-. \]

With completely similar calculations one also shows that

\[ [\hat{A}^{++}, \hat{A}^-] = -3(\hat{A}^- + \hat{A}^+). \]  

(I.19)

Example I.7

\[ \hat{A}^{--} - \hat{A}^{++} = \hat{a}^-|z|^{-1}\hat{a}^- - \hat{a}^+|z|^{-1}\hat{a}^+ \]  

(I.20)

\[ = [\hat{a}^-|z|^{-1}\hat{a}^+ + \hat{a}^-|z|^{-1}\hat{a}^- - [\hat{a}^-|z|^{-1}\hat{a}^+ + \hat{a}^-|z|^{-1}\hat{a}^- \]

\[ = -(z|z|)^{-1}\hat{a}^+ + |z|^{-1}\hat{a}^+ + (z|z|)^{-1}\hat{a}^- - |z|^{-1}\hat{a}^- \]

\[ = (\hat{a}^+ - \hat{a}^-)(z|z|)^{-1} + 2 = -1 + 2 = \hat{I}. \]

Example I.8

\[ \hat{A}^{--}\hat{A}^{++} = \hat{a}^-|z|^{-1}\hat{a}^-|z|^{-1}\hat{a}^+ \]  

(I.21)

\[ = \hat{a}^-|z|^{-1}[\hat{a}^-, \hat{a}^+]|z|^{-1}\hat{a}^+ + \hat{a}^-|z|^{-1}\hat{a}^+|z|^{-1}\hat{a}^- \]

\[ = 2\hat{a}^-|z|^{-1}\hat{a}^+ + (\hat{a}^-|z|^{-1}\hat{a}^+)^2 = 2\hat{A}^- + (\hat{A}^-)^2 \]  

(I.22)
II Appendix B

Here we calculate the ground state wavefunctions \( \psi_0^+ (z) \) and \( \psi_0^- (z) \), which are subjected to the raising operator \( \hat{A}^+ \) defined in (4.45). We approximate the wavefunctions as follows:

\[
\psi_0^+ = e^{-|z|^3/6},
\]

respectively

\[
\psi_0^- = z|z|e^{-|z|^3/6}.
\]

For the operator, we have

\[
\hat{A}^+ = \left( \partial_z + \frac{1}{2}z|z| \right) |z|^{-1} \left( \partial_z + \frac{1}{2}z|z| \right) = \partial_z |z|^{-1} \partial_z + \frac{1}{2}z \partial_z + \frac{1}{2} \partial_z z + \frac{1}{4} |z|^3 \quad (II.3)
\]

\[
= -(z|z|)^{-1} \partial_z + |z|^{-1} \partial_z^2 + \frac{1}{2}z \partial_z + \frac{1}{2} \partial_z z + \frac{1}{4} |z|^3.
\]

We also need the first and second order derivate of the wavefunctions. Let’s first consider the wavefunction \( \psi_0^+ \). The derivatives in this case becomes:

\[
\partial_z \psi_0^+ = -\frac{1}{2} z |z| e^{-|z|^3/6} = -\frac{1}{2} z |z| \psi_0^+
\]

and

\[
\partial_z^2 \psi_0^+ = -|z| e^{-|z|^3/6} + \frac{1}{4} z^4 e^{-|z|^3/6} = \left( -|z| + \frac{1}{4} z^4 \right) \psi_0^+.
\]

Now we can calculate:

Example II.1

\[
\hat{A}^+ \psi_0^+ = -(z|z|)^{-1} \partial_z e^{-|z|^3/6} + |z|^{-1} \partial_z^2 e^{-|z|^3/6} + \frac{1}{2} z \partial_z e^{-|z|^3/6} + \frac{1}{2} \partial_z z e^{-|z|^3/6} \quad (II.6)
\]

\[
+ \frac{1}{4} z^3 e^{-|z|^3/6} = \left( \frac{1}{2} - 1 + \frac{1}{3} |z|^3 - \frac{1}{4} |z|^3 + \frac{1}{2} - \frac{1}{4} |z|^3 + \frac{1}{4} |z|^3 \right) \psi_0^+ = 0.
\]

For the wavefunction \( \psi_0^- \) we get the following derivatives:

\[
\partial_z \psi_0^- = 2 |z| e^{-|z|^3/6} - \frac{1}{2} z^4 e^{-|z|^3/6}
\]

and

\[
\partial_z^2 \psi_0^- = 2 \frac{z}{|z|} e^{-|z|^3/6} - z^2 e^{-|z|^3/6} - 2 z^3 e^{-|z|^3/6} + \frac{1}{4} z |z|^5 \quad (II.8)
\]

\[
= 2 \frac{z}{|z|} e^{-|z|^3/6} - 3 z^3 e^{-|z|^3/6} + \frac{1}{4} z |z|^5.
\]

For which we can now calculate:

Example II.2

\[
\hat{A}^+ \psi_0^- = -(z|z|)^{-1} \partial_z z |z| e^{-|z|^3/6} + |z|^{-1} \partial_z^2 z |z| e^{-|z|^3/6} + \frac{1}{2} z \partial_z z |z| e^{-|z|^3/6} \quad (II.9)
\]

\[
+ \frac{1}{2} \partial_z |z|^3 e^{-|z|^3/6} + \frac{1}{2} z |z|^4 e^{-|z|^3/6} = -2 z^{-1} e^{-|z|^3/6} + \frac{1}{4} z |z| e^{-|z|^3/6}
\]

\[
+ 2 z^{-1} e^{-|z|^3/6} - 3 z |z| e^{-|z|^3/6} + \frac{1}{4} z^5 e^{-|z|^3/6} + |z| z e^{-|z|^3/6} - \frac{1}{4} z^5 e^{-|z|^3/6}
\]

\[
+ \frac{3}{2} z |z| e^{-|z|^3/6} - \frac{1}{4} z |z|^4 e^{-|z|^3/6} + \frac{1}{2} z |z|^4 e^{-|z|^3/6} = \left( -2 + 2 \right) z^{-1}
\]

\[
+ \left( \frac{1}{2} - 3 + \frac{3}{2} \right) z |z| + \left( \frac{1}{4} - \frac{1}{4} \right) z^5 + \left( -\frac{1}{4} + \frac{1}{4} \right) z |z|^4 e^{-|z|^3/6} = 0.
\]
References


