Improving Teaching, Improving Learning, Improving as a Teacher
Mathematical Knowledge for Teaching as an Object of Learning

Jorryt van Bommel
Improving Teaching, Improving Learning, Improving as a Teacher - Mathematical Knowledge for Teaching as an Object of Learning

Jorryt van Bommel

DISSERTATION

Karlstad University Studies I 2012:31

ISSN 1403-8099


© The author

Distribution:
Karlstad University
Faculty of Technology and Science
Mathematics
SE-651 88 Karlstad, Sweden
+46 54 700 10 00

Print: Universitetstryckeriet, Karlstad 2012

WWW.KAU.SE
What you see and what you hear depends a great deal on where you are standing.
It also depends on what sort of person you are.

C.S. Lewis
Abstract

This thesis concerns teaching in mathematics teacher education and is based on the implementation of a learning study at teacher training. The overall purpose was to investigate in what way teacher training could facilitate and improve student teachers’ Mathematical Knowledge for Teaching (MKT). In the learning study design, MKT was conceptualized as an object of learning with a meta-character, which meant that it was applicable to and transferable between different content areas of mathematics. This made it possible to vary the mathematical content between lessons but to keep the object of learning constant. Four critical features of the object of learning were found, giving insight in some of the problems related to teacher education. Student teachers had to be able to formulate proper aims for a lesson and to give detailed descriptions of elements of MKT for coherence in their MKT to occur. A focus on student teachers’ role as mathematics teachers had to be established and finally, sufficient mathematical knowledge was found to be a prerequisite for their MKT to develop. The study shows that enactment of these critical features improved the teaching by the teacher educators, which in its turn improved the student teachers’ learning with regard to MKT. The study also indicates that the prescribed design is worth considering for future collaborative efforts of improving teaching where other objects of learning with a similar meta-character are involved.

Keywords: Mathematical Knowledge for Teaching, Object of Learning, Learning Study, variation theory, mathematics teacher education.
Acknowledgements

“What do you want to be when you grow up?” As people have asked me this question through the years, my reply has become more specific over time. At the age of four, having just started school in the Netherlands, I wanted to become a teacher. Some years later, when introduced to different subjects, I would answer that I wanted to become a mathematics teacher. After teacher training, I taught for some years and my great interest in teacher education made me become a teacher educator in mathematics. Further studies in mathematics education attracted me and I became a PhD student on her way towards becoming a Doctor of Mathematics Education. Now at the final stage of my doctoral studies I do not know what I want to become next. Mathematics Education fascinates me and it feels as if its future holds exciting paths to explore, new roads to pave and lots of doors to open.

I would like to thank Karlstad University and two of its departments in particular: the Department of Mathematics for their support and the confidence they have had in me these past years and also the Department of Educational Science for their support during the final year of my thesis. I especially would like to mention my supervisors: Bo and Ola, for giving me the freedom to find out what I wanted to do and for guiding me in the writing of this thesis. I also owe thanks to Anita and Birgitta for all the administrative work my courses abroad, conferences and study trips have caused them; to Åsa and Mirela for their moral support during breakfasts, lunches, fika breaks and dinners; to the members of the research group at our department and to SMEER, for giving me the opportunity to present and discuss my work at different stages. In specific, I would like to mention Yvonne for her valuable input and comments to my questions and doubts, for good company during conferences and study visits and for becoming my friend over these years.
Further, I am grateful to the NoGSME for providing good opportunities to build a network; to FLUM at Gothenburg University, who have invited me to present and discuss my work on several occasions; to Ference Marton and Mona Holmqvist who both contributed with valuable input and comments at the initial stage of my project; to Ulla Runesson who pointed out important issues at my 50% seminar; and to Andreas Ryve who carefully read my 90% version and provided highly valued response; and of course to the teacher educators and their student teachers who participated in and contributed enormously to the study. You have made the study both valuable and enjoyable and without you I would not have been able to do this!

Others who deserve a special mention are my brother Kjeld for his thorough language check of a previous version of the thesis; Maria R for her help with the translations of quotes; Jari for his help with statistics; Andreas, Ann, Annika, Karin, Maria B-H, Maria F, Nina, Olov, Sorina and Yvonne for reading and commenting on separate parts of the final draft; friends and family, both in Sweden and abroad, who have been waiting patiently and wondering when I would complete my PhD and who have boosted my energy during leisure time by just being there.

And finally Stefan, Noam and Esli, thank you for reminding me daily that work is only work and that there are other, more important things to life! Stefan, I am ever grateful for your tremendous support, choosing when to just listen and when to actually answer – you are just perfect for me!
Figure 1 Study placed in didactical triangle ................................................................. 5
Figure 2 In focus: Object of Learning and Learning Study .................................................. 9
Figure 3 Domains of Mathematical Knowledge for Teaching (Ball, Phelps & Thames, 2008, p.403) 14
Figure 4 Regular design of a learning study ................................................................ 25
Figure 5 The triad of the Object of Learning .................................................................. 36
Figure 6 How many triangles? ......................................................................................... 39
Figure 7 Rubin’s Vase by Edgar Rubin ............................................................................. 41
Figure 8 Shape ................................................................................................................ 42
Figure 9 Shape with contrast .......................................................................................... 42
Figure 10 In focus: Student teachers and Object of Learning as specifics for this study ....... 45
Figure 11 In focus: Design of Learning Study ................................................................ 48
Figure 12 Course Structure ............................................................................................. 49
Figure 13 Adjusted design learning study ....................................................................... 50
Figure 14 The five elements of MKT focused on in this study within the six domains of MKT ... 52
Figure 15 Three elements mentioned means three possible connections.
Four elements mentioned means six possible connections ...................................... 63
Figure 16 Visualisation of Table 4 .................................................................................. 65
Figure 17 In focus: Student teachers’ results concerning the Object of Learning ............... 73
Figure 18 Percentage of appearance of code 2 (element directly referred to) and code 5 (connection relevant) in test 1 and test 6 ........................................................................ 76
Figure 19 In focus: Analysis of seminars and tests with focus on the Object of Learning ........ 85
Figure 20 Mind map - adapted from Rysted & Trygg (2005, p 72) .................................... 103
Figure 21 Common Content Knowledge as a fundament for the MKT domains ............... 125
Figure 22 Design of learning study including mathematical topics addressed .................. 135
Figure 23 Suggested design for learning study with meta-Object of Learning .................. 142
Table 1 Three different data sets ........................................................................................................ 57
Table 2 Percentage of participants divided over age groups .............................................................. 57
Table 3 Codes used in analysis ............................................................................................................ 64
Table 4 Example of form for analysis of student teacher test with codes used ............................... 64
Table 5 Student teachers’ results at two previous courses in teacher education in percentage of total per group ........................................................................................................... 69
Table 6 Occurrence of codes 0, 1, 2 in percentage of total elements per test ................................. 74
Table 7 Occurrence of codes 3, 4, 5, 8 in percentage of total connections per test ....................... 75
Table 8 Occurrence of codes 3, 4, 5 in percentage of total possible connections per test ............... 75
Table 9 Occurrence of code 0 – Element not referred to – in frequency and percentage of total per test .................................................................................................................... 78
Table 10 Occurrence of code 5 – Connection relevant – in frequency and percentage of total per test ........................................................................................................................................ 79
Table 11 Occurrence of codes 3, 4, 5 in average score and as percentage of total possible connections per test ................................................................................................................. 80
Table 12 Occurrence of codes 3, 4, 5 concerning the element – Curricular documents – for Test 1 and Test 6, in percentage of total connections per test ......................................................... 95
Table 13 Occurrence of codes 0, 1, 2 for test 1-6, in percentage of total elements per test ............ 104
Table 14 Occurrence of codes 0, 1, 2 concerning the element – models for explanation – throughout test 1-6 in percentage of total per test ...................................................................................... 137
Table 15 Occurrence of codes 0, 1, 2 in lesson plan on Probability for Control group and Learning study group in percentage of total per group ........................................................................... 139
Table 16 Occurrence of codes 3, 4, 5 in lesson plan on Probability for Control group and Learning study group in percentage of total possible connections per group .................................. 139

Table 1

<table>
<thead>
<tr>
<th>Three different data sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Percentage of participants divided over age groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 3

<table>
<thead>
<tr>
<th>Codes used in analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>Example of form for analysis of student teacher test with codes used</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 5

<table>
<thead>
<tr>
<th>Student teachers’ results at two previous courses in teacher education in percentage of total per group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 6

<table>
<thead>
<tr>
<th>Occurrence of codes 0, 1, 2 in percentage of total elements per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 7

<table>
<thead>
<tr>
<th>Occurrence of codes 3, 4, 5, 8 in percentage of total connections per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 8

<table>
<thead>
<tr>
<th>Occurrence of codes 3, 4, 5 in percentage of total possible connections per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 9

<table>
<thead>
<tr>
<th>Occurrence of code 0 – Element not referred to – in frequency and percentage of total per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 10

<table>
<thead>
<tr>
<th>Occurrence of code 5 – Connection relevant – in frequency and percentage of total per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 11

<table>
<thead>
<tr>
<th>Occurrence of codes 3, 4, 5 in average score and as percentage of total possible connections per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 12

<table>
<thead>
<tr>
<th>Occurrence of codes 3, 4, 5 concerning the element – Curricular documents – for Test 1 and Test 6, in percentage of total connections per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 13

<table>
<thead>
<tr>
<th>Occurrence of codes 0, 1, 2 for test 1-6, in percentage of total elements per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 14

<table>
<thead>
<tr>
<th>Occurrence of codes 0, 1, 2 concerning the element – models for explanation – throughout test 1-6 in percentage of total per test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 15

<table>
<thead>
<tr>
<th>Occurrence of codes 0, 1, 2 in lesson plan on Probability for Control group and Learning study group in percentage of total per group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 16

<table>
<thead>
<tr>
<th>Occurrence of codes 3, 4, 5 in lesson plan on Probability for Control group and Learning study group in percentage of total possible connections per group</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>
Chapter 1 – Rational

Imagine you are teaching a lesson on number sense. You want pupils to choose appropriate strategies to solve problems like 28+6 or 55+9. What kind of approaches can you expect from your pupils? Are there any models or illustrations you would use or not use? How could you support the transition from counting towards a more proficient strategy? Describe your lesson plan...

1.1 Introduction

The above scenario is a typical scenario used within teacher education. Such a scenario serves to prepare for daily practice and illuminates the specific knowledge needed for teaching mathematics. The ultimate aim of the study reported in this thesis, was to improve student teachers’ knowledge of how to teach mathematics. Teacher educators conducted the study during a method course in mathematics teacher education. The findings outline consequences for teaching in mathematics teacher education, which is the main focus for this thesis.

The teacher educators were part of a research group and together they wanted to find ways of addressing knowledge and skills of how to teach mathematics, within a course and chose a so-called learning study (see for instance Holmqvist, 2006; Marton & Lo, 2007; Runesson, 2011) as a means to reach this goal. When planning the study, the research group had to reflect on the existing design of a learning study and, for practical reasons, had to look for alternative ways of conducting such a study. This resulted in an alternative design, the usefulness of which was reflected upon during and after the study.
1.2 Mathematics teacher education

Research on mathematics teacher education is of importance since the quality of teacher education influences the quality of future teachers, who in their turn influence pupils’ achievements. The quality of teaching is the most crucial school-related factor influencing pupils’ achievements (Darling-Hammond, 2000; Grossman, 2010; Nye, Konstantopoulos, & Hedges, 2004; Sanders, Saxton, & Horn, 1997). In combination with the fact that mathematics is considered to be fundamental to society, a consensus around what constitutes a good mathematics teacher education could perhaps be expected. However, that is not the case. As the different reports of the TEDS-M study¹ reveal, teacher education programs demonstrate major differences across the countries and within countries (Blömeke, Suhl, & Kaiser, 2011; Grønmo & Onstad, 2012; Schmidt, Cogan, & Haouang, 2011; Tatto et al., 2012). The diversity, lack of coherence and lack of established principles and standards for good teacher education in general and mathematics teacher education in particular, have been pointed out by Shulman (2005). Also in Sweden there are indications of similar diversity (Hejzlar, 2008; SOU 2008:109, 2008), which has led to an active stance of the Swedish government resulting in mathematics education as a national priority development. They have initiated several undertakings to implement their policy. One such undertaking, the so-called Lärarlyftet has given over 30 000 mathematics teachers a chance to take part in professional development courses, offered at request of the government by teacher education institutes. In 2008, the teacher education institutes in Sweden were audited and examined, resulting in the report En Hållbar Lärarutbildning (A Sustainable Teacher Education) (Hejzlar, 2008; SOU 2008:109, 2008). The reform of teacher education resulted in a compulsory renewed application for all universities and colleges offering teacher education, to retain or obtain the right to offer teacher education.

¹ Teacher Education and Development Study in Mathematics
² In Swedish: Innehållets pedagogik, Fackdidaktik (Marton, 1986)
Research on teacher education, and mathematics teacher education in particular, is of importance and has recently been summarized in a four volume international handbook on mathematics teacher education (T. Wood, 2008a, 2008b, 2008c, 2008d). The four volumes each address different aspects of mathematics teacher education; the \textit{what}, the \textit{how}, the \textit{who}, and teacher educators. The \textit{who} refers to the participants (of all kind, \textit{e.g.} student teachers, teachers, teacher educators) and their organizational context are considered and analysed (T. Wood, 2008a). The professionalization of teacher educators is explored by various examples in one of the volumes, and is considered to be an “important area for investigation and development” (T. Wood, 2008b, p. ix) especially as it has only recently been looked at as an aspect of mathematics teacher education.

The \textit{what} of teacher education (T. Wood, 2008d) is heavily dominated by trying to formulate the specific knowledge and competencies a mathematics teacher needs to have, in order to teach mathematics. Shulman’s classical article (1986) coined the notion Pedagogical Content Knowledge (PCK), which has been used and worked upon by several researchers both from a theoretical philosophical perspective and from a practical perspective (Borko & Putnam, 1996; Cochran, King, & De Ruiter, 1991; Graeber & Tirosh, 2008; Marks, 1990). Specific frameworks describing the nature and structure of the knowledge have been constructed; one such is called Mathematical Knowledge for Teaching (MKT) (Ball & Hill, 2008; Ball, Phelps, & Thames, 2008), a notion which is central for this thesis.

As the main question of this thesis is how the body of knowledge termed MKT by Ball and colleagues, or PCK by Shulman, can be taught in teacher education, the \textit{how} question is of most interest (T. Wood, 2008c). It seems however that literature on how to implement this knowledge as a substantial part of teacher education is far more sparse than literature on the character of the knowledge as such (Graeber & Tirosh, 2008; Jaworski, 2008). As indicated by Burkhart and Schoenfeld (2003), in their plea for more useful and influential educational
research, the idea of letting teachers read research and implement it in their classrooms is, for several reasons, not a very effective model of connecting research with practice. Having looked at several models, they end up recommending connecting research and practice by means of “design experiments [that] represent a significant attempt to conduct research in (experimental) practice and to contribute to both research and practice” (ibid., p. 4). Whereas Burkhart and Schoenfeld seem to focus on teachers at compulsory school, a similar line of reasoning should be valid for higher education. The practitioners in this study are the teacher educators and the research they participated in can be seen as a design experiment. It will be a matter of discussion to what degree the experiment is informed by and contributes to theory and to what extent it is informed by and contributes to practice. It is noteworthy that the object of inquiry itself, MKT, involves both theoretical knowledge and practical skills. Two levels appear, where the bow of MKT, actually means theoretical knowledge and practical skills of theoretical knowledge and practical skills. This twofold character of MKT, pending between theoretical knowledge and practical skills, between a description of a capability and the actual capability itself; means that a balance has to be found between knowing and doing.

1.3 Research questions

The main research question that will be addressed in this thesis is:

In what way would it be possible to address mathematical knowledge for teaching during a teacher-training course so that student teachers would more systematically consider elements of MKT in planning their future teaching?

If we look at the study through the perspective of the didactical triangle (Figure 1), three related questions arise. Concerning the student teachers (1), did their
learning improve? Concerning the teacher educators (2), did their teaching improve, and concerning the content that was taught – the so-called object of learning (3) – in what ways can this content be taught during a course in teacher education?

![Diagram of the didactical triangle](image)

Figure 1 Study placed in didactical triangle

The second goal of the study concerns the design of the experiment or intervention (4). Was the process of trying to improve the teaching a fruitful one? Could it be applied in other but similar situations? As the design is specific, these questions will be reflected on, in order to contribute to the development of theory and method of the teaching experiment.

1.4 Disposition

As MKT plays a prominent role in this study, research on MKT and its implications for mathematics teacher education will be considered. It will also be related to the issue of improving teaching in teacher education in general, in chapter 2.
Since the method chosen for this study heavily depends on a particular theoretical perspective on learning, namely variation theory, chapter 3 will describe and discuss this theory. Learning study and its background are explained in detail and specifics for this study are made explicit.

The fourth chapter clarifies considerations regarding the learning study and how it was designed for this project. Moreover, the role of MKT within the study is explained.

In chapter 5, participants and data are described together with the coding system used in the analysis of the learning study. Validity and reliability are addressed, as well as ethics.

The analysis and results are looked at in chapter 6, to answer the first set of research questions mentioned earlier (1-2-3 in Figure 1). Both quantitative and qualitative results of the study are presented. Student results’ as well as so-called critical features found during the analysis and conduction of the learning study are described.

In chapter 7, these critical features are discussed and put in relation to existing research and present trends in mathematics education.

The design for the study (4 in Figure 1) is reflected upon and its implications for further studies are looked at in the eight chapter.

Finally, in chapter 9, suggestions for future research on mathematics teacher education are given.
1.5 Lexicon

Teachers, student, pupils, tests, lesson, seminars… a lot of words have different meanings and over the past years, when communicating with others about this work, some key concepts came up often and were confusing to many. This lexicon only attempts to make clear what is meant in this thesis by: teacher, student, pupil etc.

In a learning study, teachers conduct a lesson with pupils, who are tested by means of some test exercises.
In this learning study, teacher educators conduct a seminar with student teachers, who are tested by the exercise of writing a lesson plan.

The conducted learning study lessons are called seminars in order to be able to distinguish between these seminars in the learning study and the fictive lessons referred to in the lesson plans. In the thesis the lesson plan exercises will be referred to as tests and not lesson plans, to avoid confusion with the plans of the seminars conducted in the learning study.

The student teachers will be called as such. In addition one has to keep in mind the fact that student teachers play a double role in this study. They are students when attending the seminars, but they have a role as a teacher when writing their tests (planning a lesson).

When pupils are mentioned, this will be in relation to the fictive pupils for the lesson plans written by the student teachers. Also, pupils will be used as the term for ‘students’ used by others, when referring to such studies.

The teacher educators are the ‘teachers’ conducting the learning study and will be referred to as teacher educators. In a learning study they are also part of the research team and are therefore referred to as researchers in such settings.
One of the five elements worked with in the learning study is called *preconceptions* and it describes conceptions needed to be able to work with an exercise, content, or material.

The following abbreviations are used in the thesis:

MKT – Mathematical Knowledge for Teaching

PCK – Pedagogical Content Knowledge

OoL – Object of Learning
Chapter 2 – Research review

This chapter serves to provide a background to the conceptual framework central for this study, MKT. This framework is of particular importance and its development over time and relation to earlier research will be described in the first section. A literature search with a very specific focus would include research on the teaching of MKT at pre-service mathematics teacher education. As could be suspected, such research is very scarce and for the purpose of this thesis the search was broadened to teaching and implementation of MKT and related frameworks in teacher education. The chapter also provides a background to the method used in the study. The concept of teaching experiment with learning study in particular will be discussed, to give an idea of what the study is about.

These two concepts, MKT as the Object of Learning and Learning Study, are central in this thesis and are therefore explored in this chapter (Figure 2).

![Diagram of Object of Learning and Learning Study](image)

**Figure 2 In focus: Object of Learning and Learning Study**

### 2.1 MKT and earlier research

In this section a historical perspective is given on the arising of the concept MKT. Research on MKT and its related frameworks has focused on different
aspects. Some researchers have tried to capture the essence of such frameworks; others have focused on the implications of such frameworks for pre-service and in-service training. In this section the historical overview is supported by some examples of the different types of research. Near the end of the section MKT as a framework is explained in more detail.

2.1.1 From Stoffdidaktik to PCK

Student teachers are instructed both in subject matter and pedagogy. These two fields could be seen as disjunctive domains, but over the past decades the interest in the intersection of these fields has grown. Already in the late 50’s in Germany, Klafki (1963) recognized that pure content knowledge and pure pedagogy is not enough for teachers and introduced the term *Stoffdidaktik* which can be translated as subject related pedagogics used in the Anglo-Saxon research community or subject-matter didactics as used in the European and American research community (see for instance Hopmann, 2000; Senn-Ferell, 2000). Later in the USA during the 80’s, Shulman coined the term Pedagogical Content knowledge (PCK) (1986): Pedagogical knowledge related to the subject. When Shulman introduced the term PCK he started by distinguishing three categories of content knowledge: PCK, subject matter knowledge and curricular knowledge. He then continued by describing PCK as knowledge that

… goes beyond the knowledge of the subject matter per se to the dimension of subject matter knowledge for teaching … the particular form of content knowledge that embodies the aspects of content most germane to its teachability. *(ibid. p. 9)*

By introducing the phrase PCK, teachers' knowledge of the content and its importance for teaching practices was redefined (De Corte, Greer, & Verschaffel, 1996; Graeber & Tirosh, 2008). The same year, Marton (1986)
wrote about “a pedagogy of content”, which according to Marton can be considered as comparable with the PCK concept. Marton’s concept “pedagogy of content” was exemplified to teachers in a book titled “Fackdidaktik”. Thompson expressed in this book that it is the relation between the pupil and the content which is of importance for didactics, not the pupil nor the content per se (J. Thompson, 1986). He implies three consequences for this way of looking at didactics. First, the focus is on learning and not on teaching, which gives the content a functional value, and thirdly knowledge as such is of interest and has to be defined epistemologically.

In the years that followed, Shulman’s notion of PCK has been built upon in framing, describing and positioning research on teacher education (Borko & Putnam, 1996; Cochran et al., 1991; Marks, 1990).

2.1.2 A focus on in-service training

Borko and Putman (1996) see learning from a cognitive science perspective, as a constructive process in which prior beliefs and knowledge influence the learning. Their review of research on learning to teach, concluded that it was crucial to use the experience teachers bring to in-service courses on PCK, for learning on PCK to take place. Marks (1990) also focused on teachers and conducted interviews with eight 5th grade teachers about their teaching of fractions. On the basis of these interviews, he proposed four components of PCK: Subject matter for instructional purposes, students’ understanding of the subject matter, media for instruction in the subject matter, and instructional processes for the subject matter. Marks explicitly included particular students’ understanding, to show that knowledge of student’ understanding is not pre-determined. Given that PCK is knowledge that teachers hold and use while teaching, some researchers have focused on teachers (Marks, 1990), whereas

---

2 In Swedish: Innehållets pedagogik, Fackdidaktik (Marton, 1986)
others have focused on implications for the learners and learning in general (Borko & Putnam, 1996). The distinction between learning and teaching might seem trivial but often indicates from which standpoint the theory has developed. For instance PCK was further developed and adjusted by Cochran et al. (1991) in order to fit a constructivist perspective on knowledge. Instead of Knowledge, Knowing was used and the abbreviation became PCKg: Pedagogical Content Knowing to emphasize both knowing and understanding as active processes. To look upon PCK as knowing requires that teachers “understand students’ learning and the environmental context in which teaching occurs” (ibid., p. 263). As they focus on the learning of PCK, they suggest a tentative model for implementation of PCK in teacher education. In this model professionalization of teacher educators through collegial planning and observation is central. A solid foundation of PCK throughout teacher education in different subjects sets the base for PCK development later. Finally, like Borko and Putman above, Cochran et al. (1991) conclude that PCK development goes beyond pre-service teacher education and should be integrated in in-service teacher training courses.

2.1.3 A focus on specific elements of importance for PCK

When Grossman (1990) conducted case studies with secondary English school teachers in order to capture relationships between the different components of PCK, she expanded the notion of PCK by including aspects of Shulman’s specific curricular knowledge. She saw that teachers had to be able to “connect their choice of instructional activities to their understanding of the underlying purpose for the teaching of English” (Grossman, 1990, p. 121), which Grossman saw as part of the curricular knowledge. Moreover, through a focus on the pedagogical understanding of the subject matter, Grossman could differentiate between the importance of knowledge on subject matter and teaching experience.
Grossman’s (1990) distinction between “the subject matter expert and the experienced teacher” (p. 9) challenged the assumption that anyone with solid subject matter knowledge can teach, but also stressed the importance of such knowledge.

The importance of the content knowledge was pointed out in Ma’s (1999) study as well, where she showed that teachers with profound understanding of fundamental mathematics knew how and when to use their relevant knowledge in classroom situations. Teachers without profound understanding of fundamental mathematics were not able to the same extend to make relevant didactical decisions when probed with typical classroom situations. Such knowledge implies an awareness of “conceptual structure and basic attitudes of mathematics inherent in elementary mathematics” (ibid., p. xxiv) and most importantly it means that one can teach this to students. Instead of a focus on the learning, Ma focused on the teaching, like Shulman did. Some years later, again with a focus on teaching, An, Kulman and Wu (2004) compared the PCK of mathematics teachers in the USA and China. In line with Ma’s notion of profound understanding, they introduced profound pedagogical content knowledge, consisting of both content knowledge, knowledge of teaching and curriculum knowledge. In their study they state that knowledge of teaching is the most important aspect, including in particular knowledge of students’ ways of thinking.

2.1.4 A mathematical framework of PCK

Specific for the subject mathematics, Ball, Hill and Bass (2005) have expanded on the notion of PCK and introduced the term Mathematical Knowledge for Teaching (MKT) - the mathematical knowledge needed to carry out the work of teaching mathematics. The focus on teaching shows in the description of their specialized content knowledge: Mathematical knowledge that is used in teaching
but not directly taught to students. When Ball and colleagues synthesized and developed Shulman’s notion of PCK, the focus was on getting insight in the mathematical work of a teacher. Studying a teacher in action through a mathematical framework would give such an insight. MKT as a special type of knowledge needed for teachers was defined. MKT first consisted of four domains, but later developed into a diagram, consisting of 6 domains (Figure 3) where the domains Horizon Content Knowledge and Knowledge of Content and Curriculum were added. Horizon Content Knowledge was added to capture the knowledge teachers need, to link topics within mathematics to other topics or to higher mathematics of the same topic. Knowledge of Content and Curriculum was included within the model as opposed to Shulman who initially, described it as a separate category.

Ball et al. (2008) situated PCK within their conception MKT – represented by the oval shape – and described the relationship between three of the domains as follows:

…recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an
unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS). (Ball et al., 2008, p. 401)

In the domain Knowledge of Content and Teaching, knowledge about mathematics and knowledge about teaching are combined. For instance, the knowledge of mathematics influences in what order a sequence of tasks is presented to pupils. The MKT model with its six domains was developed to be able to study the relationships between teacher content knowledge and pupils’ achievement. Furthermore it would serve to look at the effect of different approaches of teacher development on particular aspects of the domains. Results in their turn could inform teacher education and its supporting materials. To be able to study the relationships between teacher content knowledge and pupils results, researchers decided to measure the MKT of teachers by administering multiple-choice items and these scores were in their turn correlated to the scores on mathematical tests of the teachers’ pupils. The findings showed that MKT had a positive influence on pupils’ achievements in mathematics (Ball et al., 2005; Hill, Rowan, & Ball, 2005). By classifying teachers according to a score on the MKT measure test, a class taught by a teacher in the top quartile gained three weeks of teaching compared to a class taught by a teacher with average score on the MKT test. MKT knowledge thus makes a difference.

2.2 The enactment of MKT in teacher education

MKT has been researched at different levels of the educational system and different educational contexts. The previously described research concerned
teachers and their applied knowledge of MKT. In this section the focus will be on MKT in teacher education. In order to be able to include more relevant research, not only literature on MKT but also on PCK and other frameworks are included.

2.2.1 What kind of knowledge do future teachers need?

Numerous articles have been written about what type of mathematics is needed for future teachers (Ball, 2005; Clarke et al., 2004; Ma & Kessel, 2000; Manouchehri, 1995) and conclusions have been just as many. In this section some examples are given that try to describe what knowledge to address in teacher education.

In Denmark, the KOM project\(^3\) describing mathematics as a set of competencies, was followed up by some articles of which some had a focus on implications for teacher education (Højgaard, 2009; Niss, 2004). Besides stating that “A good mathematics teacher is one who can effectively foster the development of mathematical competencies with her/his students” (Niss, 2004, p. 188), other competencies a good mathematics teacher needs to possess, are described. The KOM project points out curriculum competency, teaching competency, uncovering of learning competency, assessment competency, collaboration competency and professional development competency. These competencies are referred to as didactical and pedagogical competencies, describing more than just subject related competencies.

In *Adding it Up* (National Research Council, 2001) recommendations for teacher education programs and professional development are given as a result of the description of mathematics in terms of five strands: Adaptive reasoning,  

---

\(^3\) KOM abbreviation in Danish – Kompetencer og matematiklæring - Mathematical Competencies and the learning of mathematics
strategic competence, conceptual understanding, productive disposition and procedural fluency. These strands require specific skills of mathematics teachers. For teachers to be able to teach mathematics envisioned in the report, they need to be able to develop “profound and useful mathematical knowledge” (p 428) which will guide them in their decisions on when to use such knowledge in discussions, modifying problems etc. Connections among mathematical ideas and their development have to be understood and they have to acquire knowledge of the curriculum. The report stresses the importance of professional development just like the KOM-project suggested through its professional development competency. More specific suggestions for teacher education programs are not given though.

Both PCK and MKT describe a need for knowledge beyond subject matter knowledge, which has been acknowledged in teacher educations. However, the TEDS-M reports, mentioned earlier in chapter 1, reveals huge differences in the way these types of mathematics are addressed. In an attempt to characterize the structure of teacher education programs, the courses in such programs were categorized: Liberal arts courses, courses in academic mathematics, courses that emphasize the mathematics content related to the school mathematics curriculum, courses on mathematics pedagogy, foundation courses and finally general pedagogy courses (Tatto et al., 2012). These categories can be considered as another description of the type of knowledge needed for mathematics teachers.

The MKT model described earlier, defined the knowledge needed for teaching mathematics in terms of six domains (Figure 3, page 14). Each of these domains is considered to be of crucial importance for teachers to be able to teach mathematics. Similarities between all the above-mentioned descriptions; KOM, Adding it Up, TEDS-M and MKT, are apparent. The importance of knowledge of the curriculum is for instance visible in each of the descriptions, curriculum competency, knowledge of curriculum, courses related to the school
curriculum and knowledge of content and curriculum. Also, knowledge of the subject matter is described as profound mathematical knowledge, academic mathematics and common content knowledge.

The categorization of the TEDS-M report gives an idea on what kind of knowledge is needed and offered. The TEDS-M report however also showed differences in the impact of the courses (Tatto et al., 2012). In the following section the knowledge that future teachers actually address during and after courses will be explored.

2.2.2 What kind of knowledge do future teachers address?

Within teacher education, student teachers address obtained knowledge at several instances both during course work and during their teaching practice in classes. Huckstep, Rowland and Thwaites (2003) have identified what kind of Mathematical Content Knowledge was addressed by student teachers in their teaching. Their focus was mostly on Shulman’s subject matter knowledge and PCK and resulted in four broad dimensions, known as the knowledge quartet: Foundation, transformation, connection and contingency. The knowledge quartet is useful when observing and describing a teaching situation, where one can focus on one of the four dimensions at a time. The first dimension, Foundation, includes beliefs, knowledge and understanding. From this dimension the other three dimension flow. Transformation shifts the attention to knowledge in action including both planning and teaching. The third dimension, Connection, concerns the coherence of the mathematical content in both planning and teaching. Finally, Contingency is concerned with classroom events that are impossible to plan for.

The link between planning and teaching is crucial within the knowledge quartet and in the dimension foundation, the student teachers’ existing knowledge is
merely described. How this existing knowledge, beliefs and understandings were obtained and what parts of the offered knowledge during teacher education were incorporated in planning and teaching were researched by Fennema and Romberg (1999). In *Mathematics classrooms that promote understanding*, Fennema and Romberg found from a variety of classroom observations, some common elements which seemed to be “critical to mathematical understanding” (*ibid.*, p. x). Amongst other things they found that student teachers come to understand the process of teaching-learning through engagement in the same mental activities as their pupils. By doing so, the mathematics they had to involve in was directly connected to the teaching-learning process.

The two examples above describe research that tried to look at teaching and relate the choices and decisions to student teachers’ knowledge and understanding. Bjorneby Häll (2006) concluded that beginning teachers’ practice was not an indicator of the MKT of those beginning teachers. The pressure to cover the course and to prepare pupils for the national tests, are in conflict with their view of mathematics.

Ball and Bass (2003) tried to characterize the task of a teachers’ work and looked at the work of beginning and experienced teachers. What PCK or, in their terms, MKT does a teacher demonstrate while teaching? Choosing tasks to assess pupils’ understanding, interpreting and evaluating pupils’ non-standard mathematical ideas and making and evaluating explanations were the three tasks that were illustrated in their work.

### 2.2.3 Models for developing PCK – MKT in teacher education

pre-service teachers and found that the development of PCK is not just a transformation of subject matter knowledge, as Shulman (1987) described it. She used Skemp’s theory of instrumental and relational understanding in order to clarify this process. Instrumental understanding refers to the what and how, and relational understanding refers to the why, the reasons for the what and how.

Kinach distinguishes between instrumental pedagogical content knowledge (PCKi) and relational pedagogical content knowledge (PCKr) and tried to stimulate the process of transformation from PCKi to PCKr during a pre-service teacher education course. The suggested cognitive strategy was centred around one aspect of PCK, instructional explanation, and consisted of several steps, amongst others, Identifying the student teachers’ PCK, Assessing the adequacy of explanations given by student teachers and Challenging the student teachers in their conceptions of good explanations.

Gess-Newsome (1999) proposed two models to implicate PCK in training courses, both for in-service and pre-service teachers: The integrative and transformative model. In the integrative model all parts of PCK are developed separately and integration takes place through the act of teaching. In the transformative model, new knowledge can be created by integrating experiences in such a way that connections can be made between mathematical and pedagogical understanding. “Teaching experience reinforces the development, selection, and the use of PCK” (ibid., p. 13). Thompson and Silverman build upon this model using Simon’s concept of key developmental understanding in combination with the transformative model of Gess-Newsome (Simon, 2006; P. W. Thompson & Silverman, 2008). They suggest teacher education to develop a practice that supports pre-service teachers’ ability to continually develop within their profession, concerning MKT.

Instead of in-service courses, Peterson & Williams (2008) tried to work with student teachers, in order to increase student teachers’ understanding of mathematics in and for teaching. They found that “student teaching can have a
profound effect on prospective teachers’ understanding of mathematics in and for teaching” (p. 459). In Malaysia Saad (2009) concluded that development of PCK should not be limited to pre-service education but should continue in various forms of in-service education. Ngoh (2009) continued the research and tried to find ways of working with PCK at in-service training of mathematics teachers and found that combining subject matter and pedagogy, was a fruitful way of implementing PCK, as opposed to teaching these as separate entities at pre-service training in Malaysia. Six forms of implementation of PCK at pre-service science teacher education were suggested by van den Berg (2009). Taking a graduate course, writing a thesis, team teaching, workshops, course development and coaching were the suggested forms, but these forms only state when PCK could be addressed after pre-service training. Moreover, they do not give any indication of how to address PCK. Sowder (2007) addressed the question of what can be learned from research on in-service training of mathematics teachers for the teaching of pre-service teachers. The development of a new frame of reference for teaching, improvement of mathematics teacher preparation and the mathematicians’ role in teacher preparation are mentioned as crucially different from in-service training courses.

If one sees PCK as an applied form of mathematics (Stylianides & Stylianides, 2009) it certainly has its place in teacher education and can and should be worked with as part of a pre-service course. Ball et al. (2008) suggest that it would be useful to study the effect of in-service teacher training (with its different approaches) on particular aspects of teachers’ MKT. In Norway, a research group used the so-called MKT measures to map the knowledge of Norwegian teachers in order to develop proper teaching materials for the professional development of teachers (Fauskanger, Mosvold, Bjuland, & Jakobsen, 2011; Mosvold, Fauskanger, Jakobsen, & Melhus, 2009).
2.2.4 Lack of research on teaching PCK

It appears that the research on MKT and PCK has either focused on the teachers in the classrooms and described their possession or lack of knowledge, or focused on justifying its importance. There is no clear definition of what PCK is and therefore new descriptions continue to arise. PCK has been adapted into different subjects each with their own characterizations. PCK is “unarticulated and tacit in nature” (De Jong, 2009, p. 1) and therefore difficult to investigate. The lack of a clear description might explain researchers’ focus on the need for a definition. However PCK might be too complex for researchers to reach a consensus definition and if so, such a definition is not something worth striving for. But if PCK is considered to be so important, different ways of teaching PCK should be developed instead. So far, little research has focused on the teaching of PCK in for instance teacher education. Teacher educators work implicitly or explicitly with PCK during teacher training courses, but in what way, and which ways are more efficient? If research states the lack of PCK amongst teachers, their teacher education might not have been successful in teaching PCK. How then should PCK be implemented during teacher training courses?

2.3 Design research and Learning Study

In this section design research, and learning study in specific, will be elaborated on.

2.3.1 Design research

Design research arose to overcome the gap between educational research and practice, by transforming educational research into practice. Cobb et al.
characterized design research as research that “develops theories about both the process of learning and the means that are designed to support that learning” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9). The goal is to determine how designed artefacts or interventions behave (Collins, 1991). It is not just about evaluating the artefact or intervention, but about answering the question why. Why is it effective or not and what can that say about the learning and the teaching? Such artefacts can be videos, assessments, teaching approaches, developed materials etc. Its iterative nature supports the innovative and developmental aspects of design. Over the past decade, design research has received increased attention in several parts of the world, for instance in the USA.

Design research is commonly viewed as an approach that makes use of existing traditional research findings in order to develop some type of ‘product’, in the case of teacher education, programs for professional development (T. Wood & Berry, 2003, p. 195).

In Sweden, design research has also gained ground within research on mathematics education and the last decade one example of such design research, learning study, has been developed in a collaboration between the University of Hong Kong and Gothenburg University (Marton & Pang, 2004). In the next section this concept of learning study is explored, and some examples of studies are given.

2.3.2 Learning Study

Before giving some explicit examples of research on and in learning study, an overview is given to describe where learning study comes from and what the specifics are.
The role of theory

Learning Study is inspired by the lesson study model used in Japan and nowadays widely adopted in, for instance, the USA (Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999). It differs from a lesson study on several key aspects. One such difference is the fact that a learning study leans on a theoretical ground both in planning and analysis of the lessons conducted (Pang & Marton, 2003) unlike lesson study where no such theoretical ground is manifested. A learning study's execution is flexible but must be associated with a theory (Holmqvist, Gustavsson, & Wernberg, 2008). Variation theory is the most commonly used theory within a learning study, although some have criticized the use of variation theory in Learning Study (Dahlin, 2007; Lenz Taguchi, 2010). It provides the teachers involved with a theory to support and strengthen their general pedagogical and subject related pedagogical skills and provides the researchers involved with tools for analysing the data obtained (Marton & Booth, 1997; Marton & Lo, 2007; Runesson, 1999). The theory is used in order to describe and give an account for differences in learning, it tries to depict the conditions necessary for learning to take place (Runesson, 2005).

From this perspective learning is seen as a change in understanding or experiencing a phenomenon through an awareness of the critical features of that particular phenomenon. In a classroom situation such a phenomenon could be the equal sign. By giving an account for the differences in understanding this phenomenon, (equal sign as a symbol, as an indicator for a operation to be carried out, but even as a double minus) teachers develop both their content knowledge and their pedagogical content knowledge concerning the equal sign, which offers them new ways of teaching the equal sign in order to change pupils' understanding of the equal sign – which makes it possible to learn. The implication for teaching is to make the critical features visible and learnable, the learner should get the possibility to discern and simultaneously focus on the critical features. These key concepts; discernment, simultaneity and critical features will be explained in more detail in section 3.5.
The cyclical design

The cycle of a learning study could be described as follows: A group of teachers first decides on an Object of Learning (OoL), for instance understanding the equal sign, of which they then try to identify critical features. Features critical for understanding this OoL. For the equal sign it is for instance important to identify the two features of the use of the equal sign ($\equiv$); the operational feature $4 + 3 = \ldots$ and the feature of balance which can also be described as an operation at both sides as in $4 + 3 = 2 + \ldots$. Another critical feature would be to understand when two things are not equal $4 + 3 \neq 6$. Studying relevant literature will indicate some common misconceptions, leading to a description of possible critical features. To confirm such possible critical features and to get insight in the understanding of the OoL for the specific group of pupils a pre-test is conducted and analysed, after which a lesson is planned and conducted, followed by a post-test. The post-test and the conducted lesson are analysed and a revised lesson is planned and conducted in a new class, with new pupils. This cycle is repeated several times and can be represented as depicted below (Figure 4).

Through a common lesson preparation and an analysis of pupils’ pre-, and post-test results with the help of variation theory, critical features will emerge and thus the teachers will develop their knowledge concerning their OoL –
understanding the equal sign. For example, the pre-test results of a learning study on use of equal sign showed that pupils only regarded the operational feature. When asked to solve problems using the feature of balance, pupils did not manage, which resulted in a formulation of a critical aspect to be addressed during the learning study lesson. In the post-test that followed after the lesson, it appeared that pupils now did consider both features. Such developed knowledge on the OoL, offers the teachers new ways of teaching the OoL in order to change pupils’ understanding of the OoL – which make it possible to learn.

The specific focus on the content is another crucial difference with the lesson study. A lesson topic as focused on in a lesson study is often more broad and often includes a focus on methods and organizational aspects used for teaching. For the teachers involved, variation theory strengthens and supports their didactical and pedagogical skills and for researchers involved, variation provides them with tools for analysing the data obtained (Marton & Booth, 1997; Runesson, 1999).

Specific characteristics

Pang, Linder and Fraser, (2006) state that a learning study basically consists of two aspects.

First, a learning study aims to build innovative learning environments and to conduct research into theoretically grounded innovations. Second, it aims to pool teachers’ valuable experiences into one, or a series of, research lessons to improve teaching and learning. (ibid., p 31)

They point out that the primary focus is not on the teaching methods but on the OoL. The stages a learning study has to go through, according to Pang and
his colleagues, are the following: 1) choosing an OoL; 2) ascertaining students’ existing understanding; 3) planning and implementing the lessons; and 4) evaluating and revising the lessons. These stages describe the process of a learning study. Runesson (2011) describes a learning study in more detail, as a systematic investigation of the conditions of learning and the meaning inherent in knowing. According to Runesson (ibid.) a learning study has the following characteristics:

1. A form of cooperation between researchers and a team of teachers,
2. The focus is on what pupils have to learn, not the lesson itself
3. Teachers own the research problem
4. The empirical study is carried out in a cyclic process
5. Systematic work:
   a. gather data of pupils understanding before and after
   b. videotape lessons
   c. analysis of pupils’ understanding and videotaped lesson
6. The main aim is to capture what is critical for learning and use that to improve teaching in such a way that pupils learn what they are intended to learn.

As learning study often is described as a hybrid between design research and lesson study, aspects of both are clearly visible in the characteristics above. Aspects (1), (4) and (5) concern the design and its process and have clear similarities with design research. But also an important difference with design research becomes clear. In a learning study, teachers’ ownership of both the OoL and of the teaching is central (Carlgren, 2012), aspect (3) above. Carlgren even states that

… involving teachers’ tacit knowledge in the development of designs, results will probably become more sustainable than what
is normally the case in educational research and developmental work (ibid., p.13).

Such a prominent role for the teachers is important in the lesson study tradition and learning study has drawn upon that. However, the focus on the content (2) and (6) is specific for learning study and neither related to lesson study nor to design research.

### 2.3.3 Examples

**Learning study as professional development**

For the teachers conducting a learning study, it is often seen as a form of professional development or school development (Holmqvist, 2006; Runesson, 2011). The Mathematics Initiative\(^4\) initiated by the Swedish Government 2009 has over a period of three years, supported teachers in their projects in the mathematics classroom, with 35 million Euro. Learning study and ICT\(^5\) were the two most common forms for projects (Swedish National Agency for Education, 2011). As learning study is widely adapted and enacted in several schools in Sweden, knowledge concerning the chosen OoL is obtained as well as another type of knowledge, gained by the teachers who participate in a learning study. One could argue that learning study only influences the lessons included in it, since the nature of the study is to focus on one single lesson with a single OoL. However, when teachers reflect on the learning study they participated in, they often state that it has influenced other lessons because they started thinking in another way, inspired by variation theory (Elliott & Yu, 2008; Gustavsson, 2008). Recently, research has started to look at such sustainability of a learning study (Carlsgren, 2011, 2012; Eklund, 2011; Runesson, 2011).

\(^4\) In Swedish: Matematikövergripningen
\(^5\) Information and Communication Technology
Research focused on variation theory and its importance

During the arising of learning study, the focus of research on learning study was to compare it with lesson study, and to see if a lesson planned with variation theory as a theoretical background gave better results than a similar lesson, planned without variation theory. Pang and Marton (2003) found that this was the case when they compared two lessons in economics. Over 70% of the students in the learning study group compared to fewer than 30% of the students in the lesson study group, developed a good grasp of the concept.

Other studies also showed the value of variation theory in planning and analysing teaching in a learning study (Holmqvist et al., 2008; Pang et al., 2006; Pang & Marton, 2005; Runesson & Kullberg, 2010).

Research focused on the outcome

When the importance and the effect of the theory was verified, the outcome of learning study came into focus (see for instance Holmqvist, 2006; Kullberg, 2004; Lo, Pakey, & Pang, 2006; Marton & Morris, 2001; Runesson, 2006, 2007). This outcome was often described in terms of the particular knowledge obtained concerning the OoL, that is, its critical features. Kullberg continued and researched the generalizability of the outcome, i.e. could the outcome of a learning study be used in following learning studies? She found that that was indeed the case and stated that the critical features were transferable (Kullberg, 2010, 2011). Her research suggested that the notion of critical features can be used by teachers to enhance student learning.

2.4 From background to theory

This chapter has served as a background to place the study presented in this thesis within the broad landscape of mathematics education research. The two concepts MKT and Learning study have placed the study in the area of both
design research and teacher education. In the next chapter the theoretical
ground for learning study, variation theory, will be described.
Chapter 3 – Variation theory

Two frameworks are central in this thesis: Variation theory as the supporting theory in the study conducted, and Mathematical Knowledge for Teaching (MKT) to frame the OoL used in this study. In this chapter variation theory and its role within learning study is clarified. The issue of why variation theory is suitable to illuminate the particular question driving a learning study is addressed: How is it possible for the learner to grasp a specific concept? Hereeto one needs to understand the theoretical setting (variation theory) in which the learning study is conducted. One way of doing this is by looking at the key concepts of variation theory to see whether these are closely related or could have been placed within other theories of learning and teaching mathematics. In the following sections an overview of the origins of variation theory is given, followed by an account of what learning is, as seen from the variation theory perspective. Subsequently some of the key concepts of variation theory will be explained, which in turn will be compared to seemingly similar concepts used in other theoretical perspectives, like the constructivist and the socio-cultural perspective. These theoretical findings will be summed up in the last section.

3.1 Phenomenography

The roots of variation theory lie in phenomenography, and variation theory could be seen as an expansion of phenomenography (Pang, 2003), a process started by the so called INOM group during the 1970’s. The INOM group wanted to focus on what pupils learn and how such learning is established. However, whereas explanations for misconceptions and other issues concerning pupils’ learning, were previously given from an outsiders perspective, phenomenography tried to explain these from an insiders

---

6 INOM: Inlärmning och Omvärldsuppfattning
perspective (Marton, 1986). The INOM group also did not want to quantify the outcome of learning: How much do pupils learn or not learn? They wanted to describe the learning from the subjective perspective of the pupil. This focus on the content to be learned, on the phenomenon as experienced or understood by the individual, was rather unique for that time.

Phenomenography explores the qualitatively different ways of experiencing or understanding a phenomenon and is a qualitative research methodology that originally emerged from empirical research rather than from a philosophical foundation (Åkerlind, 2005; Tan, 2009). Variation theory has been developed out of phenomenography and focuses on conceptions: The experience and understanding of a phenomenon. The choice of the phenomenon is usually taken to be a conception related to disciplinary knowledge, which leads to the next section where the epistemological standpoint is explained.

3.2 Non-Duality

The ontological and epistemological grounds for variation theory are interwoven. Since the object of research has the character of knowledge, the ontological standpoint represents the epistemological one (Uljens, 1993). The ontology of variation theory (taken from phenomenography) focuses on non-duality. There is only one world, which can be experienced and understood in different ways, depending on previous experiences, background etc. An individuals' understanding or experience, can only be put side by side and compared to other individuals' understanding or experience. In contrast to dualistic standpoint in which an individuals' understanding or experience can be compared to the world *per se*, as a result of the standpoint in which the individual and the world can be separated. The non-dualistic standpoint makes it impossible to separate the individual from the experience, or the experience from the experienced.
An example concerning notation in understanding the *equal sign* will be given to illustrate the difference between a dualistic and a non-dualistic standpoint and its consequences. Some pupils notate their calculations in the following manner: 

\[ 4 + 3 = 7 + 1 = 8 - 5 = 3. \]

It is only when pupils and the mathematical concept *equal sign* meet, that such an understanding becomes evident. Mathematics *per se* could explain why such notation is problematic, but does not lead to a formulation of possible critical features for the concept *equal sign*. A non-example in mathematics could be given and would address the non-equal part as in \[ 4 + 3 \neq 7 + 1. \] But such a non-example is not addressing the cumulative use of the equal sign. The number 7 in the expression above, could be replaced by another number and the mathematical value of the non-example would stay the same: \[ 4 + 3 \neq 8 + 1. \] However, to address the problem in notation, the 7 is of importance as it shows the calculation of \[ 4 + 3, \] and shows that pupils continue within the same calculation.

Such specific comprehension concerning the object of learning, cannot be found by looking at the content, but can be found when exploring pupils’ (mis)understandings. Pupils’ (mis)understandings lead in a non-dualistic view to a formulation of critical features concerning the OoL. In a dualistic view such (mis)understandings would lead to an explanation about learning and maybe, but not necessarily, lead to a formulation of consequences concerning the teaching of the object of learning. In the non-dualistic view the difference between teaching and learning is of secondary concern as the primary concern is the OoL.

If the individual cannot be separated from the experience, what implication does that have for a possible definition of learning? In the next section such definition will be given.
3.3 What is learning?

Instead of defining what knowledge is, variation theory focuses on the process of learning. Just like Piaget, variation theory does not call attention to how to take in knowledge from the world but emphasizes the question how we can develop knowledge about the world (Piaget, 1979). Gaining knowledge is important but it is the development of knowledge that is the focus of variation theory. Although similar, variation theory differs from Piaget and the constructivist point of view. The most crucial and obvious difference is probably the dualism in constructivism and the non-dualism of variation theory as clarified in the previous section. In a dualistic perspective, learning takes place internally through stimulus from the external environment whilst in a non-dualistic perspective, learning is seen as a relationship between people and environment (Merriam & Caffarella, 1991). Concepts are to be learned in relationship with the world and variation theory tries to answer the question of how to give all pupils/students access to knowledge.

Previous experiences are accounted for, with explicit focus on the content as previous experiences influence the current learning. Learning is seen as a process; “The process of becoming capable of doing something (“doing” in the wide sense) as a result of having had certain experiences (of doing something or of something happening)” (Marton & Tsui, 2004, p. 5). Bauersfeld (1995) touches upon this concept of previous experiences as well, when he describes learning “as knowledge dependent, people use learned knowledge to construct new knowledge” (p. 140). The importance of reflection is stressed when learning is seen as a successful organization of one's own experience. Glaserfeld (1987) expressed a view of learning as constructing a general representation from experience through reflection, whereas variation theory focuses on the awareness of critical features of the OoL. The ability to observe your own learning process (reflection) does not have a focus on the OoL but on learning in general. On the other hand, a focus on the content, on the subject is
characteristic for variation theory and puts the key element previous experience in a different perspective capturing other aspects than learned knowledge used by constructivists. Constructivists, like Bauersfeld (1995) and Glaserfeld (1987), see learning as adding or constructing new knowledge to the previous knowledge, whereas in variation theory, learning is seen as differentiation. Learning is seen as a change in understanding or experiencing a phenomenon through an awareness of the critical features (explained in section 3.5) of such a phenomenon. Within variation theory the how is most important and the process is analysed and described in detail and in content-related terms; i.e. the how is not separated from the what.

Now that learning has been defined from a variation theory point of view, the next section continues by looking at what is learned, the OoL.

3.4 Object of Learning

Learning is always the learning of something. Research on learning may focus on this something: A phenomenon of which one wants to know how it is learned. Within variation theory such a focus is called the Object of Learning (OoL) and it is defined as a capability. Any capability has a general and a specific aspect.

The general aspect has to do with the nature of the capability…[t]he specific aspect has to do with the thing or subject on which these capabilities are carried out (Marton & Tsui, 2004, p. 4).

The general and specific aspects are also referred to as the indirect and direct OoL, aiming at the acts and what is acted upon. As for the example previously used concerning the equal sign, the direct OoL would be to understand both the operational feature and the feature of balance, to understand that the equal sign
can be used in diverse statements. The indirect OoL would then describe what one has to be able to act upon; for instance they would have to be able to decide when to use or not to use the equal sign and write correct statements like \(4 + 3 = 7\) and \(5 + 2 = 4 + 3\).

Where some analyses within educational research gives information about the content, about teaching arrangements and about artefacts to use; variation theory only points at a specific aspect, the content. Variation theory is not aiming at, for instance, other teaching arrangements, although this can be a side effect of the analysis, neither does it directly suggest which artefact to use. This explicit focus on the content is necessary in order to deepen the (pedagogical) content knowledge.

Within a learning study the OoL, both direct and indirect, can be looked at, at three stages: Before the lesson, during the lesson and after the lesson. In the first stage, planning and describing the OoL can be called the intended OoL. While teaching the intended OoL becomes the enacted OoL and one can discover discrepancies between the intended and the enacted OoL. During a lesson the pupils will perceive different aspects of the OoL, which can be described as the lived OoL. (Figure 5).

![Figure 5 The triad of the Object of Learning](image-url)
Learning studies in mathematics have dealt with an OoL like adding and subtracting negative numbers, equal sign etc. Adding and subtracting negative numbers lies within the domain of negative numbers but addresses only a part of this concept, and a broad view on negative numbers and connections to other domains in mathematics might get lost with such narrow focus. The OoL chosen in most learning studies are often related to a narrow content. Dahlin (2007) criticizes the way variation theory is commonly used in learning studies and states that it supports a narrow view of the OoL. According to Dahlin, the philosophical aspect of the subject as such, for instance the essence of mathematics, gets lost with such narrow OoL. The narrowness of an OoL is not to be mistaken for a focus on the direct OoL instead of on the indirect OoL.

An example of a learning study in which the OoL in the school subject Swedish was problematized in the work of Mossberg Schüllerqvist and Olin-Scheller (2011). They distinguish between a main goal – an overarching OoL, and partite goals. In order to be able to work with the overarching OoL, one has to distinguish smaller partite goals, each to be considered as a subordinate OoL (ibid.). In another report, a learning study in the school subject History resulted in the distinction between the explicit OoL and a second, implicit OoL (Holmqvist, Björkman, & Ohlin, 2010). This second OoL concerned what was needed to pass the upcoming exam. These two studies show that not all OoL are of the same type. In the following section, the OoL for this study will be defined and related to such different types of OoL as described above.

### 3.4.1 A Meta-Object of Learning

As the main research question of the study was to find possible ways of addressing MKT during a teacher–training course so that student teachers more

---

7 Which in this case was knowledge of fiction
systematically would consider elements of MKT in planning their future teaching, the elements of MKT can be considered the OoL. In chapter 4 the specific elements and their interrelating character will be described, but at this stage, the OoL can be considered as an OoL with a meta-character. The meta-character is visible in both the complexity of the interrelating domains described in MKT, and in the complexity of each of the domains per se, as they are applicable in different topics in mathematics. To contrast the type of OoL described as narrow before, the OoL chosen in this learning study will be described as a meta-OoL. Meta refers to the multidimensionality of the OoL. Such an OoL is characterised by a capability that is applicable over a range of different subject contents. For example, the capability to reason mathematically is applicable in all areas of mathematics. The OoL consisting of a capability to reason mathematically would be a meta-OoL.

An OoL, both a narrow OoL and a meta-OoL, is experienced and understood differently by different persons. Variation theory is used in order to describe and give an account for such differences in learning. It tries to depict the conditions necessary for learning to take place (Runesson, 2005). Seeing the critical features is a necessary condition, and the next section will deal with the concept critical features.

3.5 Critical features

Each OoL has its own features, critical for understanding that OoL. The three concepts used for noticing critical features are discernment, simultaneity and variation, which will be explained later on in the next section. These three concepts are not new, but their uniqueness lies in the fact that they are combined together to describe learning. For now an example will illustrate these concepts, which in their turn will illustrate what critical features are about (Figure 6).
A well-known question to this picture is: How many triangles do you see? For untrained eyes it is difficult to discern all triangles, to be able to see the big triangle as a triangle as well. Others might not recognize the inner triangle as a triangle because the base is not situated at the bottom of the triangle. This might be because of a lack of variation in presentation when they learned the mathematical concept triangle. The persons who do recognize the inner triangle, as well as the other ones, show simultaneous awareness of (some) critical features of the mathematical concept triangle. A triangle is a closed figure with three sides, the placement of a triangle in space is irrelevant and does not affect if the figure is a triangle or not.

Critical features as conditions necessary for learning the OoL are found by analysing the different ways a phenomenon is portrayed by different persons. The analysis focuses in turn on the three key concepts mentioned above. These three concepts are now expanded on and compared to other concepts that have inspired and influenced them.

If one would ask two persons (a Chinese person and a Canadian person) to describe a person standing in front of them, they will probably describe different features of that person: Each feature just as important to them, but remarkably different. The colour of the hair will probably be described by the Canadian but not by the Chinese, as Chinese people generally have black hair.
For the Canadian, colour is a feature to pay attention to when describing a person. In China, the hair colour is not a feature to pay attention to when describing a person. So even if the hair colour of person in front of them would be different from their own, it would not necessarily become a point of attention. The Canadian person probably would not describe the width or length of the nose, something more likely to be described by a Chinese (Lillichöök, 2006). The way they can discern features and critical features is influenced by previous experiences. To be able to recognize the same person two weeks later demands simultaneous awareness. The memory of the person and the image of the actual person in front of them, have to work simultaneously in order to be able to recognize the person. To be able to point out the person in an out-dated picture, as well as the person’s parents and children on the picture, requires a capability to vary the aspects focused on earlier. This might seem as a farfetched example, but if some words are replaced the relevance within a classroom context might become more visible.

If one would ask two pupils (Ellen, a first grade pupil; and Nicole, a sixth grade pupil) to describe a black rectangle in front of them, they will probably describe different features of that figure: Each feature just as important to them, but remarkably different. The colour of the rectangle will probably be described by Ellen but not by Nicole. Ellen has been working with colours that week during class and therefore colour is a feature to pay attention to when describing the rectangle. Nicole has not been working with colours at all that week so even if the colour would be red, it would not necessarily become a point of attention. Ellen probably would not describe the precise width or length of the figure, something more likely to be described by Nicole, who has been working with geometrical shapes during class that week. The way they can discern features and critical features is influenced by previous experiences. To be able to recognize the same rectangle two weeks later demands simultaneous awareness. The memory of the rectangle and the image of the actual rectangle in front of them, have to work simultaneously in order to be able to recognize the rectangle. To be able
to point out the rectangle and other rectangles amongst other quadrilaterals requires a capability to *vary* the aspects focused on earlier.

In the above example, “vary the aspects” was mentioned, and variation theory states four patterns for such variation. For *separation*, a pupil needs to encounter each of the critical features one at a time, separate. For instance the critical feature that a rectangle has four angles of 90 degrees each. When all critical features have been addressed *fusion* follows, where the critical features are presented simultaneously in order to grasp the phenomenon. *Generalization* as a *pattern of variation* would make it possible to recognize other rectangles. *Contrasting*, another pattern of variation would help the pupil to distinguish a kite or a parallelogram from a rectangle.

Having illustrated and clarified the three terms discernment, simultaneity and variation, these can be connected to other terms used in other fields. The concept of discernment draws upon theories developed by Gibson, Rubin and Wertheimer, all psychologists (Holmqvist et al., 2008). The picture of the vase (Figure 7), for which Rubin is well known, clearly shows the connection with discernment.

Some people instantly see the vase, others the two faces. In order to be able to see the vase (or the faces) one has to focus on certain aspects of the vase (faces), for instance the colour. Michaels and Carello (1981) explain Gibson’s notion of indirect perception: The object to be *learned* is rich enough and contains all the information. The learner does not need to elaborate on it but
has to be able to attain the information from the object in order to obtain a rich perception of it. Attaining information is associated with both discernment and with the patterns of variation. Research within perceptual learning emphasizes the importance of distinction and separation, also linked to the patterns of variation. Garner (1974) talks about pattern discrimination and illustrates the important role contrasting cases have in noticing. When people are asked to describe a shape as in Figure 8 the description changes remarkably when the shape is contrasted with another shape as in Figure 9, in which certain aspects are varied while other aspects are kept invariant.

![Figure 8 Shape](image1)

![Figure 9 Shape with contrast](image2)

Describing the rectangle as in Figure 8 will differ when seen on its own or seen alongside another rectangle, as in Figure 9. More contrasts can be made so other features can become visible. Discrimination, variation and simultaneity are not new inventions. What is new is that they are put together and are used to describe how learning develops, connected to a specific OoL. This development of learning is elaborated on in the next section, where variation theory as a theory of learning is addressed.

### 3.6 Variation theory as a theory of learning

A theory of learning should according to Lave (1996) consist of three kind of stipulations: A telos (what learning is aiming at), a subject-world relation and a learning mechanism.
Previously, the subject-world relation of variation theory, has been stated as being non-dualistic (section 3.2). The learning mechanisms have been described in terms of discernment, simultaneity and variation. Finally, the telos, could be described as inclusiveness, increasing the differentiation of the whole. Emanuelsson (2001) continues and adds a fourth: Learning agent – who or what drives the learning towards the telos? He describes this for variation theory, as the relation of a person towards the world and towards others in this world, as the interplay between the outer and the inner. Both “the person” and the “others” have a relation to the world and to each other in this world. This interplay explains the distinction between the intended, enacted and lived OoL (as shown in Figure 5) since people that intend, enact or live this object most often are different from each other (e.g. Course plan writers, teachers, students). Furthermore, differentiation and inclusiveness take place within this interplay, resulting in learning. The learning of the specific aspect of the capability, the direct OoL, is depending on the interplay between the outer and inner as it is described as interaction between the subject and the capability.

3.7 The specific character of variation theory

In the above, key concepts used within variation theory have been put side by side with other concepts used in other theories of learning. Similarities have been explained and differences have been pointed out. The most important difference is the fact that concepts used in variation theory are defined in relation to the content to be learned and how it is learned. This focus on the content makes variation theory different from other theories of learning. A short summary will illustrate this difference in focus. The OoL in variation theory is content oriented. Critical features are described and defined with a focus solely on the OoL, which means that the critical features describe the necessary conditions that make it possible for the content to be learned.
(Marton & Pang, 2004). The critical features emerge out of discernment, simultaneous awareness, and variation in order to depict the differences in understanding a phenomenon.

By consciously varying certain critical aspects of the phenomenon in question while keeping other aspects invariant, a space of variation is created that can bring the learner's focal awareness to bear upon the critical aspects, which makes it possible for the learner to experience the object of learning (Pang & Marton, 2005, p. 164).

As stated, variation theory has its roots in phenomenography with its non-dualistic standpoint. From that point of view the critical features can only be found by including the subjective dimension. In a dualistic view, critical features will be searched for and found within the content as such. The found critical features will therefore not necessarily be the same. Variation theory provides a base for analysis of data where the content is in focus while learning is the aim of the research.
Chapter 4 – Methodology

In this chapter, the specifics for the study described in this thesis are clarified and compared to the regular design of a learning study. The aim of the chapter is to clarify in what way the study was conducted and to shed some light upon the character of the chosen OoL and its consequences for the study.

4.1 Specifics for this study

As Figure 10 shows, the focus for the specifics of this study is on both the OoL and on the group of student teachers. The learning study was conducted in one course dealing with only one group of student teachers, but where the OoL has a meta-character, as explained in section 3.4.1. Since the study focuses on the relations between different elements of MKT, mathematical topics (geometry, scale, algebra) can be used as a means by which MKT can be exposed. The idea is that student teachers that get exposed to such MKT on several occasions with different mathematical content would be able to transfer this knowledge to a new mathematical content. MKT is a competence, which exceeds the mathematical topic and at the same time depends on the topic. You cannot
teach MKT without relating it to a topic; neither can you develop an understanding of MKT only through a topic. If the example in the beginning of this thesis is looked at again, the meta-character of MKT becomes clear.

Imagine you are teaching a lesson on number sense. You want pupils to choose appropriate strategies to solve problems like 28+6 or 55+9. What kind of approaches can you expect from your pupils? Are there any models or illustrations you would use or not use? How could you support the transition from counting towards a more proficient strategy? Describe your lesson plan…

The questions asked here; "What kind of approaches can you expect from your pupils? Are there any models or illustrations you would use or not use?” can be asked for situations in different mathematical topics and their answers will be different for different topics. In the example the questions were related to arithmetic, but the questions are just as appropriate to for instance statistics.

Imagine you are teaching a lesson in statistics. You want your pupils to choose appropriate strategies to illustrate a set of data. What kind of approaches can you expect from your pupils? Are there any models or illustrations you would use or not use? How could you support the pupils in motivating their chosen strategy? Describe your lesson plan…

To answer such questions requires knowledge on how to teach mathematics, the answers differ between the different topics, but are supported by the knowledge within the same domains as described in the MKT model (Ball & Hill, 2008). The hypothesis is that if student teachers gain profound understanding in MKT, they would be able to transfer MKT taught in specific topics to other topics. Being aware of the on-going debate on the notion of transfer, Marton’s notion of transfer is used, as it is both suitable and consistent with the used theory of this thesis.
Discussions about transfer have mainly dealt with how people manage to do something in a situation thanks to having done something similar in a previous situation. From an educational point of view, however, it appears more fruitful to consider the case when the learner, having learned to do something in one situation, might be able to do something different in other situations, thanks to perceived differences (and similarities) between situations. (Marton, 2006, p. 499)

For the study reported here, this meant that the OoL was dealt with in three different seminars, through three different mathematical topics, but all along taught to one class. In a classical learning study though, one deals with different classes, (Holmqvist, 2006; Runesson, 2011; Runesson, Holmqvist, & Marton, 2003). This, together with the character of the chosen OoL, led to an alternative design of the learning study (van Bommel & Liljekvist, 2008a). Whereas one normally would alternate student groups to teach revised lessons such alternation would in this case mean that the study would have had to be conducted over a period of three terms, which could take up to three years. This would mean an insecurity considering the participating teachers. No assurance could be given that all teachers would continue working at the same university during these years and even if they would, it would not be certain that they would teach this specific course together. Another option would have been to cooperate with different universities. Besides the logistical problems, it proved difficult to find similar courses at different universities.

To develop an alternative design was not a goal of the study when it started but rather a result of the crucial differences in circumstances compared to other conducted learning studies (Holmqvist, 2006; Runesson et al., 2003). The next section will deal with the implications for the design of the study.
4.2 Implications for the design of the study

The two specifics described above (one student group, three topics) are interrelated and provide the possibility as well as the need for a new design. This section describes the implications for the design of the learning study as shown in Figure 11.

One cannot teach a certain seminar three times to a group: It would be unethical to do this and also it would not generate valuable results. However, since the mathematical content of the seminars differs, the seminars are not the same and therefore can be taught to the same group (van Bommel & Liljekvist, 2008b). So instead of dealing with three different classes, three different mathematical topics are dealt with, like geometry, statistics and algebra. Each and every topic contains its own models for explanation, materials for concretisation, connections to other topics and so on.

To understand the design of the study the structure of the course needs to be presented. The course is a 20-week fulltime course situated in the later part of the teacher-training program for future primary school teachers. About 50 student teachers take the course each term. The content is divided into several mathematical topics. In Figure 12 the vertical arrow represents the course and
the mathematical topics are listed in order of appearance within the course (problem solving, number sense etc.). All topics were addressed during the course through seminars, workshops and lectures. Three topics were chosen for the pre-tests (number sense, statistics and equations). That meant that the regular seminars on these topics were given to the student teachers and the student teachers were tested afterwards. Three other topics (rational numbers, algebra and area) functioned as mathematical content for learning study seminars and the post-tests (on the same content as in the preceding seminar) took place after these learning study seminars. At the end of the course a final test was given concerning the topic Probability. Within that topic student teachers had received regular seminars. The reason to have the last test was that a comparison would be possible between a control group and the group of student teachers involved in the learning study.

<table>
<thead>
<tr>
<th>Course week</th>
<th>Topic</th>
<th>Pre-test</th>
<th>Post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Problem Solving</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2+3</td>
<td><strong>Number Sense</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Number base systems</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Natural numbers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Whole numbers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td><strong>Rational numbers</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Irrational numbers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Powers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Coordinates</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Functions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Statistics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13+14</td>
<td><strong>Algebra</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>Equations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td><strong>Space Awareness</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>Proportions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>Scale</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>Probability</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure 12 Course Structure*
Changing the topic of the seminars also implicated a change of topic in both pre-, and post-tests as shown in Figure 12. This may seem a minor change but it has major implications: Other ways of comparison are now possible which enables validation of the results.

![Change of Topic](image)

*Figure 13 Adjusted design learning study*

This adjusted design (Figure 13) gives three dimensions of comparison, which now will be elaborated on. The first dimension of comparison is the regular one: Comparing pre-test results with post-test results in order to find critical features. This leads to the second dimension: Compare in order to judge the expected improvement made within a course. It is now possible to compare the different pre-test results with each other, and if the same is done for the post-test results these two comparisons can be evaluated and again compared. A third dimension of comparison opens up for comparing the short-, and long-term effect of a learning study on the teaching of the *teacher educators* involved.

One could argue that a learning study, only influences the lessons included in the learning study, since the nature of the study is to focus on one single lesson with a single OoL. However, when teachers reflect on the learning study they participated in, they often state that it has influenced other lessons because they started thinking in another way, inspired by variation theory (Elliott & Yu, 2008; Gustavsson, 2008). In order to be able to look at this effect two other teacher-training courses were involved: The courses given by the (same) teacher educators both prior to and after the course in which the learning study took
In these courses the student teachers were tested in a similar way, but only at the end of the teacher-training course. By doing so, it would be possible to compare the results of these three groups. Such comparison could give evidence for the effect of a learning study on future teaching of participating teachers. Some recent studies indicate such improvement (Nilsson, 2012, pp. 46-59) but this kind of research has only just started.

Summarized, the developed cycle takes into account the development by the learning study as well as the natural progression of the student teachers and the teaching of the teacher educators.

4.3 MKT in this learning study

The main research question was to find ways to address MKT during a teacher-training course so that student teachers more systematically would consider elements of MKT in planning their future teaching, as described in section 1.3. The teacher educators participating in the study were dissatisfied with some results of student teachers at the end of previous courses. They noticed that student teachers could formulate elements of MKT but could not put two such elements in relation to each other. The teacher educators did not only want to answer the question Ball et al. asked themselves: What kind of mathematical reasoning, insight, understanding and skill teaching mathematics demands (Ball et al., 2005), but also wanted to find a way to teach this knowledge. While planning the study several elements were considered to focus on during the study, amongst which representation forms, teaching aids, real life adjustments, applications in other subjects, pupils’ preconceptions, models for explanation, related (hands on) materials, suitable exercises and curricular documents. The teacher educators chose to focus on five of these elements: Pupils’

---

8 The data for the course after the learning study has been gathered but will not be looked at in this thesis.
9 This comparison, however, goes beyond the scope of this thesis and will not be developed here.
preconceptions, models for explanation, related (hands on) materials, suitable exercises and curricular documents. Once these elements were defined a suitable framework to relate these elements to, was searched for. From the frameworks described in chapter 2, MKT was the framework that seemed most coherent with the choice of the elements and the intentions of the study. PCK included four of the elements but not the element curricular documents, which is a separate category in Shulman’s description (1986). The knowledge quartet (Huckstep et al., 2003) as well as the competences described in the KOM-project (Højgaard, 2009; Niss, 2004) were focussing more on the actual teaching going on, which was not relevant in this study since student teachers would only plan lessons, not conduct them. Within the domains of MKT, all five elements could be placed (Figure 14). The element curricular documents could be linked to Knowledge of Curriculum in the MKT model. The main part of suitable exercises and preconceptions would be placed in the section Knowledge of Content and Teaching, whereas models for explanations and related (hands on) materials could be situated in Specialized Content Knowledge as well as in Knowledge of Content and Students but even in Knowledge of Content and Teaching.

![Figure 14](image)

Figure 14 The five elements of MKT focused on in this study within the six domains of MKT.
Good understanding of MKT would implicate *an awareness of interrelating elements of importance for teaching a particular topic in mathematics* (van Bommel, 2009, in press). For instance, when dealing with Cuisenaire rods, student teachers would start by saying that the material was appropriate to use, which implicates an understanding of one element of MKT: Related (hands on) materials. But when they continued and said that the material would be particularly suitable to work with within the number base system (fix the unit using the smallest cube of the Cuisenaire rods, thereafter calculate the other rods - whereas the finesse in Cuisenaire rods is that the unit is not fixed) they showed a lack of coherence between two elements of MKT. The chosen *model for explanation* and the chosen *hands on* material do not relate. A good understanding of MKT would show by either choosing a different material for this mathematical concept or at least problematize the choice of the related (hands on) material.

MKT is central for this learning study and is part of the defined OoL: Being capable of using (elements of) MKT in planning future teaching, by developing a structure between the elements. Observe that the direct OoL, the content, is MKT whereas the indirect OoL, the capability of using that content, focuses on using MKT in planning future teaching, through development of a structure of relations between the five elements of MKT.
Chapter 5 – Method

During a study a researcher makes decisions, poses questions and tries to be critical of one's own work. Such considerations are clarified in this chapter. Information concerning the participants is given and the different types of data gathered are described. The codes used in the data analysis will be described in detail and supported with examples. Ethics, validity and reliability are addressed at the end of this chapter.

5.1 Challenges

5.1.1 Challenges concerning the group

As explained in section 4.1 and 4.2, the design of the study changed partly due to the fact that there was only one group of student teachers available. A change of topic made it possible to teach all seminars to the same group. However, working in this way demanded more from the student teachers than normally would have been the case. Whereas pupils normally conduct two tests each; a pre-, and a post-test, each student teacher now conducted seven tests within this learning study. The teacher educators saw however a possibility of incorporating the tests within the exercises normally given in the course. By doing so, the learning study did not increase the workload for the student teachers, as the tests were now part of the course.

5.1.2 Challenges concerning the OoL

Working with a meta-OoL was challenging and demanded a lot with respect to the planning of the seminars. Both the OoL and the mathematical topic had to
be taken into account in the planning and execution of the seminars. This also made the analysis more complicated: In some cases changes could be suggested merely connected to the mathematical topic, whereas such changes were not possible to implement in another topic. However such instances gave insight in how to deal with the OoL even in other topics. For instance, during the analysis of the seminar on rational numbers, a suggestion was made concerning one of the exercises used during the seminar. It was suggested to use other numbers next time, so the value of the used additions would be the same. Such a change was not possible to implement in the following seminar, on space awareness, as the exercise would not be part of that lesson. However, the theoretical construct behind the suggestion, to keep certain things invariant, was considered during planning of the subsequent seminar.

5.1.3 Challenges concerning not participating teachers

Not all teachers involved in the course were participating in the learning study. One teacher stated very clearly that (s)he did not want to be involved. The lectures of this teacher were not looked at although in these lectures the same mathematical content was dealt with. In the section on validity (5.5.4) this issue will be further addressed.

5.2 Participants

5.2.1 Research team

The researcher team involved in this learning study consisted of five teacher educators and one researcher. Two of the teacher educators were only involved at the starting phase of the study, which included planning and preliminary deciding on the OoL. A third teacher educator participated in the beginning
and occasionally attended meetings during the learning study cycle. The two remaining teacher educators were the ones teaching the seminars and attended all meetings and sessions in which data was analysed. Both are senior teachers with several years of experience as primary and secondary school teachers but also as mathematics teacher educators. The teacher educators each got 30 hours working time to participate in the study which would cover their time for reading literature, time for analysis, observation and preparation. Teaching, regular preparation and examining the tests were already included in their normal working load.

5.2.2 Student Teachers

During the autumn of 2008 a pilot study was conducted and five student teachers out of a class with 53, volunteered to participate. From these five student teachers three test results were collected. At the end of their course all 53 student teachers handed in an examination exercise as part of their course, which also served as data for the study.

The actual learning study was conducted at a mathematics course in teacher education and student teachers’ results were analysed. In total 101 student teachers were involved, divided over two terms (Table 1). In the autumn of 2008 there were 53 student teachers that participated. During spring 2009, the term after, 48 student teachers were involved. The two groups are referred to in this article as control group (autumn 2008, the group taught a term before the learning study started) and learning study group (spring 2009, the group participating in the learning study). The learning study group contributed most of the data and consisted, as stated before, of 48 student teachers of which 39 participated in all tests. Some student teachers started the course half term, others quit before the end of the course. The data collected from these student teachers is not included in the analysis. Furthermore some student teachers did
not hand in one or several pre-, or post-tests in time and therefore in the analysis of the data, differences in the total numbers will occur.

**Table 1 Three different data sets**

<table>
<thead>
<tr>
<th>Time</th>
<th>Pilot</th>
<th>Control Group</th>
<th>Learning study Group</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Autumn 2008</td>
<td>Autumn 2008</td>
<td>Spring 2009</td>
</tr>
<tr>
<td>Number of Participating Student Teachers</td>
<td>5 voluntary student teachers (out of 53)</td>
<td>53 student teachers</td>
<td>48 student teachers</td>
</tr>
<tr>
<td>Number of Collected tests</td>
<td>3 per student teacher</td>
<td>1 per student teacher</td>
<td>7 per student teacher (individual differences)</td>
</tr>
</tbody>
</table>

Of the 48 student teachers in the learning study group, 10 were male (21%), which is relatively high compared to previous courses (<15%). Six student teachers (12%) were over thirty and 30 student teachers (64%) were 25 or younger (Table 2).

**Table 2 Percentage of participants divided over age groups**

<table>
<thead>
<tr>
<th>Age</th>
<th>Participants in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;25</td>
<td>64</td>
</tr>
<tr>
<td>25-30</td>
<td>24</td>
</tr>
<tr>
<td>&gt;30</td>
<td>12</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
</tr>
</tbody>
</table>

All student teachers had chosen a field of study when they entered teacher education. One of the fields had Mathematics as the main subject and student teachers that had chosen that field could not take the course in which this study was conducted. This meant that the student teachers in the study did not choose mathematics as their main subject. This could have implications for the level of their mathematical skills, which had to be taken into account during the analysis of the tests, since although the tests did not focus on the mathematics, the level of mathematics knowledge may have played a role.
The course was situated in the later parts of the teacher education program and student teachers had one (38%), two (28%) or three (34%) semesters left after taking the course.

5.3 Data

Data was collected at three different stages of the study. In the sections below these different stages preparation – seminars – tests, are presented and the process of transcription is also clarified.

5.3.1 Preparation

During the preparation of the seminars and analysis of the video of the seminars, notes were made and the sessions were audio taped. This resulted in 6 hours of audio data, which was transcribed and analysed using NVivo8. The prepared materials for the seminars were collected and consisted of (hand) written notes, PowerPoint slides and worksheets.

5.3.2 Seminars

The seminars prepared through the learning study were both observed by the team and recorded on video, resulting in handwritten notes of 3 parallel seminars of 2½ hours each and 15 hours of video data. The parallel seminars were held on the same day, one in the morning and one in the afternoon, separated by a one-hour break for the teacher educator. At two occasions a significant difference between the morning and afternoon seminar was noticed by one of the observers and communicated during the (afternoon) seminar, followed by a
complementation later in the seminar. One such difference concerned a reflection within the seminar on curricular goals. During the morning seminar the teacher educator related the curricular goals to grades 3, 5 and 9. During the afternoon seminar, the teacher educator mentioned only grade 3 and 9.

Not all data were transcribed: Of each parallel seminar only one was transcribed verbatim (7½ hour in total) and analysed using NVivo8. The parallel seminars were then looked at once more and compared in order to find any possible differences in how the OoL was dealt with; the team found no such differences.

5.3.3 Tests

Pre-, and post-tests from the teachers in the learning study group (6 per student teacher, around 300 in total) were collected as electronically (word documents and pdf). These tests were collected at different times during the course, alongside the learning study cycle. Additionally the student teachers in all three groups completed a handwritten test at the end of the course as part of an examination exercise (140 tests).

The test results had to be able to show the appearance and the coherence of the five elements of MKT focused on in this study: Pupils’ preconceptions, models for explanation, related (hands on) materials, suitable exercises, and curricular documents. The tests needed to give an indication of the different kinds of perceptions student teachers expressed in the tests. What did they manage and how did they develop the parts they knew already? The pilot study showed that an answer to these questions was possible to get through the chosen test form; writing a lesson plan. The form and content of the pre-, and post-test were as follows: Student teachers got the task (appendix 1) to write a lesson plan for a primary school lesson within a certain mathematical topic (e.g. number sense, rational numbers). The control test taken by both groups of student teachers at
the end of their course in mathematics for teacher education, also consisted of writing a lesson plan for a lesson, in this case for probability.

Different forms of tests were considered before deciding upon a lesson plan. Other forms were drawing Mind maps and Question and Answer forms. Both were judged to be suitable to show what student teachers knew about the separate elements, but not suitable for showing connections between the elements. Such connections could be asked for but the teachers were more interested in the ways the student teachers themselves would show the connections. In a lesson plan, the student teachers did not specifically have to be asked for the connections, these would emerge from the plan, and could then be judged as such. When learning about MKT one should be able to apply this knowledge for instance by making a preparation for instruction (An et al., 2004). Shulman (1987) even stated that he hoped a framework would be developed that would lead to an instrument which would test aspects of knowledge only for teachers. Such tests should only be possible to complete by (in-service) teachers. Writing a lesson plan was therefore deemed as a suitable exercise.

A practical advantage of taking lesson plans as pre-, and post-tests was that such tests could be implemented in the course without any major changes compared to how the course was taught previous years. In corresponding courses previous years, student teachers handed in a lesson series during the course, consisting of several lesson plans, linked together. The lesson plans they had to hand in during this learning study did not have such a link between them. It was calculated that it would take the same amount of time for the student teachers to hand in six separate lesson plans compared to handing in one lesson series.
5.3.4 Transcription

The transcription of the first seminar was done by the author and by an independent transcriptionist. These two transcripts were compared and judged as comparable, similar enough for the purpose of the research, after which only the independent transcriptionist transcribed the other two seminars.

In the video data both the teacher educator and the student teachers are present and in the scripts used for analysis the following denotations are used: Teacher educator is denoted with TE when referring to the seminars, whereas the student teachers are denoted with StA, StB etc. No connections can be made between the student teachers on video and student teachers’ tests. When referring to the tests, all student teachers have their own number: For instance St35T2 denotes student teacher 35 - test 2; Finally the researchers, including the teacher educators, participating during the preparation and analysis are denoted with RE in such sessions.

In some cases an explanation of the situation is given. Such comments within the transcription are written within closed brackets [ ]. *Italics* are used when someone reads from a text.

5.4 Data analysis

5.4.1 Analysis of the preparation

The transcriptions of the audio taped material i.e. preparation and analysis of seminars, have at some instances been used during the learning study to check if changes for the seminars were made according to the preparation. However, the transcripts were most valuable after the learning study and were used to recapture the process as a whole. As suggested in section 2.3.3 it would be
interesting to look at the short and long-term effect of a learning study on the participating teachers/teacher educators. A discourse analysis on the material could give insights on the way the teacher educators work with learning study and variation theory. However, as also stated in the same section 2.3.3 these questions go beyond the scope of this thesis.

5.4.2 Analysis of the seminars

During the learning study, the research team met for both planning and analysing the conducted seminars. The three seminars were analysed as follows: The videotaped seminar was distributed via DVD to the participating teacher educators to watch individually. Some days later the research team would gather and specific episodes chosen by each of the participants were looked at together. The reason to deal with the seminars in this way was that the seminars were long (2½ hours each) and it was difficult to find such a long period of time for all of the participants to gather and analyse. When picking out episodes the focus and attention was given to the OoL and related to student teachers’ results. Notes were made on how to change things in the next seminar.

5.4.3 Analysis of the tests

In total just over 400 tests were collected. These tests were analysed in both a quantitative and a qualitative way. The quantitative analysis was done after the learning study and served as a basis for describing the outcome of the learning study regarding student teachers’ results concerning the OoL. The analysis was done using the statistics application SPSS.

The qualitative analysis of the tests was done throughout the learning study. Every test was analysed regarding different items. First of all an inventory was made regarding the five elements of MKT considered in this study. In what way
were they referred to: Indirectly, directly or not at all. Depending on how many of the elements were referred to, a different amount of connections between these were possible, as illustrated in Figure 15. Each dot represents an element of MKT and each line marks a possible connection. Mentioning three MKT elements result in three possible connections, as shown in the figure to the left. Whereas mentioning a fourth element gives the opportunity to connect the elements in 6 ways, as shown in the figure to the right.

![Figure 15 Three elements mentioned means three possible connections. Four elements mentioned means six possible connections.](image)

The result of a student teachers’ test could contain all from one element resulting in no possible connections, up to ten connections when mentioning all five elements. So even though the OoL was only to focus on the connections, these results should be related to the number of elements mentioned.

When analysing the tests it became clear that elements and their connections were dealt with in different ways. Sometimes the lesson goals for instance were stated very clearly, in other cases these were more vague, only mentioned implicitly or lacking totally. To capture the complexity of the data, a coding scheme was made where each of the tests (circa 400) was coded concerning 15 items: Five elements and ten connections.

The codes used were 0, 1, 2 for the elements and 3, 4, 5 and 8 for their connections. Code 0 = not referred to; Code 1 = indirectly referred to; Code 2 =
directly referred to; Code 3 = irrelevant connection, Code 4 = vague connection, Code 5 = relevant connection and Code 8 = connection not possible (Table 3). In the next section examples of each code will be given.

Table 3 Codes used in analysis

<table>
<thead>
<tr>
<th>Code</th>
<th>Element</th>
<th>Connection</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Not referred to</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Indirectly referred to</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Directly referred to</td>
<td>Irrelevant</td>
</tr>
<tr>
<td>4</td>
<td>Vague</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Relevant</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Not possible</td>
<td></td>
</tr>
</tbody>
</table>

This codification resulted in a form for every student teacher for every test, of which a fictive example is given in Table 4.

Table 4 Example of form for analysis of student teacher test with codes used

<table>
<thead>
<tr>
<th>Student teacher X</th>
<th>Curricular documents</th>
<th>Preconceptions</th>
<th>Models for explanation</th>
<th>Related (Hands On) Materials</th>
<th>Exercises</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test #</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Curricular documents</td>
<td>0 (D)</td>
<td>- (F)</td>
<td>8 (E)</td>
<td>8 (E)</td>
<td>8 (E)</td>
</tr>
<tr>
<td>Preconceptions</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

| Models for explanation | 1 | 4(B) | 3 | |
| Related (Hands On) Materials | 2 (A) | 5(C) | |
| Exercises             | 1 | | | |

In the example above student teacher X refers directly to hands on materials (A) with a vague connection to the models for explanation (B) but with a relevant connection to the exercise referred to (C). Curricular documents are not referred to at all (D) and therefore ‘8’ (connection not possible) is marked in (E). After looking at the pilot study the research team decided that the
connection between Curricular documents and Preconceptions was difficult to state and the decision was made to omit this connection from the analysis, (F) in Table 4. The figure below shows a visualisation of Table 4.

![Figure 16 Visualisation of Table 4](image)

### 5.4.4 Examples of codes

To clarify the codes that were obtained during analysis, some examples are provided in this section. The examples are all deliberately taken from the same test (T 1 – first pre-test) and all focus on the same element (preconceptions) to make a comparison between the codes more easy.

*Element directly referred to (2)*

St7 T1 Preconceptions: Pupils already practiced classifying and sorting.

St27 T1 Preconceptions: Pupils do not need any specific preconceptions.

Student teacher 7 clearly states which preconceptions are considered necessary for the lesson. Student teacher 27 shows that preconceptions have been taken into consideration and the outcome was that no preconceptions were necessary.

It is important to realize here that whether this description is right or wrong is not relevant here. The analysis only states whether preconceptions are taken
into account in a clear way. Later, when looking at connections between, for instance, the exercise and preconceptions, it may become clear that some preconceptions indeed would have been relevant to describe and therefore a ‘3’, indicating irrelevant, will appear in the connection between these two.

*Element indirectly referred to (1)*

St19 T1  Pupils do not have deep preconceptions around number sense.
St3 T1  The lesson is conducted in whole class where all pupils have almost the same potential.

Student teacher 19 shows that preconceptions are considered, but the word “deep” makes it difficult to interpret and therefore it is classified as indirect. Student teacher 3 describes that all pupils have almost the same potential, which can be seen as an indication of preconceptions, but also here it is not stated clearly what the preconceptions are.

*Element not referred to (0) – Connection not possible (8)*

Not describing an element in the test (code 0) resulted automatically in coding it’s connections as ‘8’.

*Connection relevant (5)*

St42 T1  Pupils have learned all numbers up to ten. They can write the symbols and we have worked with number sense. Now we will strengthen their number sense and work practically with partitioning numbers.

(…)

Every pupil has a small bag with chestnuts in it (…) They work in pairs. First they decide how many chestnuts they will have in their bag (…) Now one pupil takes out an amount of chestnuts and the other pupils has to ‘guess’ how many chestnuts are left in the bag.

The example above shows a relevant connection between preconceptions and the exercise the pupils will engage in during the (planned) lesson. The preconceptions are both relevant and sufficient for the exercise.
I count on the fact that pupils earlier have calculated with plus and minus…

I write a number on the board, for example 351 … I tell the pupils that the 3 stands for three hundreds, the 5 for five tens and the 1 for one unit. So it actually says: $3 \cdot 100 + 5 \cdot 10 + 1 \cdot 1$ …

In the suggested exercise, pupils need to be able to add and multiply numbers, therefore the considered preconception “plus and minus” is not sufficient, although relevant.

Student teacher 27 in this example above clearly stated the preconceptions – no preconceptions are needed – but looking at the rest of the lesson plan the connection with the exercise for the pupils was lacking. Certain preconceptions would have been necessary for the pupils to make sense of the lesson. For pupils to write something in the clay that makes sense, meaning they do not just copy a symbol, it is of importance to master their own number system in order to “translate this to the Mayan symbols”.

The eight codes described above (Table 3) arose during the pilot study and were used in analysing all the tests. The need to check this codification, and also
the codes as such emerged. The following section will deal with these kinds of needs and other measures of reliability and validity.

5.5 Validity and reliability

5.5.1 Codification

To check whether the codification was valid, a reference group was used who also analysed some tests and coded the tests in their own way. The reference group consisted of doctoral students and researchers in the field. The codes they described were very similar to the codes defined above and therefore the codification was judged as valid. A second reference group was used in order to check whether the codes used were valid and clear enough. This group consisted doctoral students, and they received several tests but they had to use the predefined codes. The most difficult part for the reference group was to ignore the way the student teachers described the elements. If an element is referred to it should be marked as (2) regardless of whether it was a right or wrong description. The definition of code (2) was therefore made more precise. Other inconsistencies within the codes concerned the decision when to state a connection as vague and when a connection would be clear. After discussion, statements were unanimously coded the same way, but such discussions were not possible for all tests during the whole learning study. However, the study wanted to look at a possible improvement of student teachers, therefore for as long as the individual opinion would not differ over time this would not lead to inconsistency. To validate that tests were not judged differently at different stages of the learning study, an internal validation was conducted. On two occasions a re-analysis of a set of test was done, and no differences were found.
Two different student groups are taken into account in this learning study: The group of student teachers participating in the learning study, the learning study group; and the group of student teachers attending the course one semester prior to the learning study, the control group. However, no data was gathered to describe the level at the beginning of the course for each of these two groups. This problem was considered only afterwards and a solution was found by comparing the results for two compulsory courses attended by the student teachers earlier during their studies. The following results were obtained during these courses (Table 5).

<table>
<thead>
<tr>
<th>Table 5 Student teachers' results at two previous courses in teacher education in percentage of total per group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Course 1</td>
</tr>
<tr>
<td>Fail</td>
</tr>
<tr>
<td>Course 1</td>
</tr>
<tr>
<td>Course 2</td>
</tr>
</tbody>
</table>

Observe that the grades in this table are their final grades. It might have been the case that student teachers previously did not pass but with extra exams managed. Therefore the control group could have a better result, having had the opportunity to do more exams. However, since the mathematics course is in the later part of their studies, they will all have had a reasonable amount of opportunities for passing their exams. The table above shows a difference between the two groups concerning the two courses, but the difference was not statistically significant. A log-linear test showed that neither the two courses nor the two groups were dependent (p>0.60).
Writing a lesson plan

One aim of the pilot study was to test in what way the assignment write a lesson plan was understood. It also tested whether it was possible to measure connections between the five chosen elements of MKT. Additionally it showed whether or not all five elements were mentioned by the student teachers. Already during this pilot study it became clear that a connection between curricular documents and preconceptions was not relevant to judge and it was therefore left out in the analysis of further data, as mentioned before.

Using a similar format for all tests could lead to an improvement in results because student teachers had a better understanding of the test format – writing a lesson plan. In that case the results of the final test of both groups (control group and learning study group) should be better for the learning study group. However, as will be shown later in section 8.1.2, this was not the case and therefore such improvement of understanding the format did not seem to exist. As the results from the pilot study already indicated, writing a lesson plan was understood in a satisfactory way from the beginning.

One could discuss whether the student teachers answer what the teacher educators want to know. If that would be the case then student teachers have grasped the OOl and therefore this is not a problem. Nonetheless, this learning study does not test how these student teachers actually behave in teaching. Therefore there could be a great discrepancy between the lesson plan handed in by the student teachers and an actual lesson, for several reasons. The idea though is that a good lesson plan increases the chances of the lesson being good. Within teacher education the balance always has to be found between exercises suitable for teaching practices and actual practice. The choice not to look at student teachers practice was made for several reasons. Practical reasons: No teaching practice was included in the teaching course. Also, looking at 48 student
teachers conducting a lesson would have been difficult to incorporate as a test to analyse in a learning study. Furthermore as research has shown that the practice of beginning teachers not necessarily show their knowledge within MKT (Bjerneby Häll, 2006), such practice was considered as not suitable to observe in relation to the aim of this study. The research team discussed and stated that a good lesson plan would make it more likely for teachers to conduct a good lesson. If the student teachers would answer what they think the teacher educators wanted to hear, they at least understood the essence of the seminars. Teacher education is about bringing such essence across (Benken & Wilson, 1998; Blomhøj & Valero, 2007; Bramald, Hardman, & Leat, 1995; Gómez, 2009; Grevholm & Anthony, 2010; Grossman, 1990) and therefore answering what the teacher wants was not considered to be a problem for this study. On the contrary, such answers would give clear information about the lived OoL.

5.5.4 Non-participation

One teacher of the course, as mentioned in section 5.1.3, clearly stated (s)he did not want to be involved in the learning study. The lectures of this teacher were a part of the course and even touched upon the mathematical content dealt with in the learning study. The lectures of this teacher most likely did not influence the outcome of the learning study for several reasons. The critical features found were enacted in the learning study seminars but were not of the type to incorporate in the lectures focussing on the mathematics. But most of all, as the teacher educator took an active stance not to participate and all participants were aware of this: Nothing was directly communicated to or with this teacher educator concerning the learning study.
5.6 Ethics

Two groups of participants had to be considered in the learning study: The participating student teachers and the teacher educators. As the teacher educators themselves indicated that they wanted to conduct a learning study, they also agreed on the learning study to be research based. Before the start of the study they were informed on the way the research would be documented and analysed. The withdrawal of one of the teacher educators could have put pressure on the other teacher educators, making them feel obliged to participate in the study. In contrast, it could also have given the others an opportunity to express their doubts about the study and/or their participation.

The student teachers wrote the lesson plans as part of their course, which meant that their workload was not increased due to the learning study. To use the results of their lesson plans in the learning study was part of the professional development and course development of the teacher educators. However to use the results within the research was something they had agreed to. The student teachers got information (appendix 2) at the beginning of the course, both oral and in writing during the first lecture. They got a form to sign where the choice was given to disagree/agree on the results being used for research purposes (appendix 3). Furthermore permission to videotape the seminars was asked before each seminar.
Chapter 6 – Results

In the previous chapters the OoL has been considered, as well as the content and form of pre-, and post-test (writing of a lesson plan) used in this study. The qualitative analyses of the pre-, and post-tests, combined with the analysis of the seminars, helped the researchers to find critical features of the OoL. To be able to give a full account of the analysis, the quantitative results of the pre-, and post-tests will be presented first. These results were quantified after the study and give an idea of how the student teachers developed concerning the OoL. The three seminars are then presented, followed by a description of the evolution of the four critical features found.

6.1 Did the student teachers improve?

In this section the results obtained through a quantitative analysis of the pre-, and post-tests are presented. In these tests the student teachers’ understanding of the OoL are made explicit as shown in Figure 17. In section 5.4.4, examples
of each code have been given to clarify the quantitative analysis of the tests used in this study. The next section presents these results.

6.1.1 Elements and connections in total

Seven tests were given throughout the study. Six of these were done during the study and the seventh at the end of the course. The results of the first six tests gave the research team an insight in student teachers’ understanding concerning the OoL and helped in planning and analysing the seminars. The seventh test also informed on student teachers’ understanding of the OoL but was done to distinguish between the impact of the course and the impact of the seminars. The results of it will be dealt with in chapter 8. The results of the first six tests were analysed both qualitatively (on an individual level) and quantitatively (on a group level). On a group level the results indicate that the student teachers became more clear in their descriptions of individual elements (Table 6) The frequency for code 2 – element directly referred to, increased from only 46% to 73% at the end of the study. In addition there were changes in the other codes as well. The frequency for code 1 – element indirectly referred to, decreased from 17% in the first test to 8% in the last test.

<table>
<thead>
<tr>
<th>Test Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code 0 – Element not referred to</td>
<td>37</td>
<td>31</td>
<td>35</td>
<td>24</td>
<td>17</td>
<td>19</td>
</tr>
<tr>
<td>Code 1 – Element indirectly referred to</td>
<td>17</td>
<td>18</td>
<td>13</td>
<td>17</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>Code 2 – Element directly referred to</td>
<td>46</td>
<td>51</td>
<td>52</td>
<td>59</td>
<td>71</td>
<td>73</td>
</tr>
</tbody>
</table>

Although it was the connections between the elements, rather than the mentioning of the elements themselves that constituted the measurement of the OoL, Table 6 still forms an important background in the analysis. The table shows that in test 6, on average four out of five elements (81%) were directly or indirectly mentioned (code 1 and 2) which led to more possible connections.
Leaving out one element immediately resulted in decreasing the possible connections with 33-44%, depending on the element. Table 7 shows the distribution over the four types of connections for each test.

**Table 7 Occurrence of codes 3, 4, 5, 8 in percentage of total connections per test**

<table>
<thead>
<tr>
<th>Test Number</th>
<th>Code 3 – Connection Irrelevant</th>
<th>Code 4 – Connection Vague</th>
<th>Code 5 – Connection Relevant</th>
<th>Code 8 – Connection Not Possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>17</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>21</td>
<td>13</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>22</td>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>61</td>
<td>53</td>
<td>63</td>
<td>36</td>
</tr>
</tbody>
</table>

In Table 7 the change of code 8 – *connection not possible* is visible and decreases from 61% to 37%. In what follows, the percentage of each type of connection is measured in relation to number of possible connections for each student teacher and test (Table 8).

**Table 8 Occurrence of codes 3, 4, 5 in percentage of total possible connections per test**

<table>
<thead>
<tr>
<th>Test Number</th>
<th>Code 3 – Connection Irrelevant</th>
<th>Code 4 – Connection Vague</th>
<th>Code 5 – Connection Relevant</th>
<th>Code 8 – Connection Not Possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43</td>
<td>36</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>38</td>
<td>44</td>
<td>35</td>
<td>39</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>20</td>
<td>60</td>
<td>46</td>
</tr>
</tbody>
</table>

When measured like this, the percentage of code 3 – *connection irrelevant* decreased from 43% in test 1 to only 3% in test 6. As a result the percentage of code 5 - *connection relevant* increased from 19% in test 1 to 77% in test 6. Note that even in test 3 good results were obtained concerning the connections: 60% of the connections were relevant (code 5), which decreased to 46% in test 4. However, keeping in mind the results of Table 7, these were 60% of the possible connections, and as shown in Table 7, this changed remarkably after test 3.

In the analysis above, all elements and connections have been treated as a undifferentiated whole. The two figures below illustrate the change in results between test 1 and test 6, when looked at from the point of view of each
element and its connections. The numbers given are the percentage of code 2 – element directly referred to (written near the element itself) and code 5 – connection relevant (written near the line connecting two elements).

Figure 18 Percentage of appearance of code 2 (element directly referred to) and code 5 (connection relevant) in test 1 and test 6

Comparing the results of the two tests in Figure 18, an improvement is seen for all connections and for four of the elements. The only element that did not improve was the element models for explanation. This was surprising at first, especially since this element did score just over 56% in test 5. When re-analysing the tests it became clear why this element scored so low on test 6: 30% of the student teachers chose in their test to focus on prepositions for position10. For such a focus it appeared difficult to fully describe a model for explanation. An example of an activity is the following:

St31 T6 Pupils need around 5 big and small rectangles, circles, squares and triangles. Then the pupil has to make a figure by picking out some geometrical shapes and put them in front of

10 In Swedish: Lägesord
In this activity the pupils have to use prepositions for position in order to explain to another child what their figure looks like. This type of activity was not classified as containing models for explanation but comes close to such a classification. One could argue that such an exercise can be considered as a model for explanation as well. But to stay in line with classification decisions taken in the analysis of earlier tests, the decision was made not to classify this type of activity as models for explanation.

Nevertheless, the small increase when going from test 1 (45% give a clear description of models for explanation) to test 5 (56% give a clear description) is still a point of concern. The other element that increased, but still did not obtain a satisfactory level, is the element preconceptions. Taking a look at the connections made between the two elements models for explanation and preconception, and the other remaining elements, gives an indication of the way student teachers may or may not have changed their view of the elements models for explanation and preconception. One indicator is to look at code 5 – connection relevant as shown in Figure 18. All of the connections related to these two elements score higher in the last test. One could also look at code 3 – connection irrelevant and the data show that these decreased remarkably: Just under 1% (2 instances) in test 6 were irrelevant connections concerning either models for explanation or/and preconceptions, whereas up to 30% of these connections were irrelevant in the first test.

The analysis above focused on the elements of MKT and the connections between them. An analysis for each of the student teachers has also been conducted and will be dealt with in the next section.
6.1.2  Elements and connections per student teacher

In the previous section the results from the analysis of the tests have been described with focus on the elements and their connections and it showed whether the description of the amount of elements increased and whether the connections between the described elements became more relevant. For this study it was also interesting to look at the different codes per se and see what happened with these during the study. In this section interesting trends and results are presented.

The coding gives 7 codes to look at (see chapter 5). Codes 0, 1 and 2 serve to identify in what way the five elements are described. Codes 3, 4 and 5 serve to code the relevance of the connections, and finally code 8 makes it possible to relate the frequencies obtained for code 3, 4 and 5 to a proper total.

Recalling the OoL for this study is being capable of using elements of MKT in planning future teaching by developing a structure between the elements. Therefore it is first of all of interest to look at how the elements were present or not during the tests.

In Table 9, the number of times that code 0 – element not referred to was obtained by each student teacher in the first and last test are given.

<table>
<thead>
<tr>
<th>Code</th>
<th>Test 1</th>
<th>Test 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>6% (5)</td>
<td>32% (13)</td>
</tr>
<tr>
<td>1%</td>
<td>23% (11)</td>
<td>41% (17)</td>
</tr>
<tr>
<td>2%</td>
<td>53% (25)</td>
<td>27% (11)</td>
</tr>
<tr>
<td>3%</td>
<td>15% (7)</td>
<td>-</td>
</tr>
<tr>
<td>4%</td>
<td>2% (1)</td>
<td>-</td>
</tr>
<tr>
<td>5%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Total</td>
<td>100 (47)</td>
<td>100 (41)</td>
</tr>
</tbody>
</table>

Looking at the table, with individual results, it can be observed that code 0 was obtained 1 time for 23% of the student teachers in the first test. Code 0 was obtained 2 times for 53% of the student teachers at the first test and for 27% at the sixth test. In the first test, only 6% of the student teachers described all
elements (score 0 on code 0), whereas in the least test, 32% managed. From these data it can be concluded that more student teachers managed to mention more elements as the course proceeded. Where in the beginning some student teachers mention only one element, at the end all student teachers mention three elements. From this, a continuation can be made towards the connections between the five elements. Regarding the connections, code 3, 4 and 5 are of interest. Tables, similar to Table 9, can be made showing how many times student teachers obtained a certain code. Below, such a table is provided for code 5 – *connection relevant* is (Table 10).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Test 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>64%</td>
<td>28%</td>
<td>-</td>
<td>2%</td>
<td>4%</td>
<td>-</td>
<td>-</td>
<td>2%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(30)</td>
<td>(13)</td>
<td>(1)</td>
<td>(2)</td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Test 6</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>7%</td>
<td>7%</td>
<td>24%</td>
<td>7%</td>
<td>22%</td>
<td>7%</td>
<td>7%</td>
<td>2%</td>
<td>7%</td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td>(3)</td>
<td>(3)</td>
<td>(10)</td>
<td>(3)</td>
<td>(9)</td>
<td>(3)</td>
<td>(3)</td>
<td>(1)</td>
<td>(3)</td>
</tr>
</tbody>
</table>

The data expressed in this table shows that at the first test 64% of the student teachers did not manage to give any relevant connection between described elements. At test 6, only 5% failed to give any connections. Of course, the observed number of connections has to be looked at in relation to the number of possible connections. (Again, for each element that is omitted by the student teacher, several connections become impossible to describe). If the numbers in Table 10 are related to the amount of possible connections, the data shows that only 15% of the student teachers managed to describe half of their possible connections during the first test. At the time of the last test this increased to 88% of the student teachers.

In relation to the OoL it is also interesting to look at code 3 – *connection irrelevant*. An average of 1.6 was obtained on code 3 at the first test, which decreased to a value of 0.18 at test 6. These averages should be seen in relations to the number
of possible connections (depending on the number of mentioned elements) as shown in Table 11. In the first test on average 3.6 connections were possible, for 44% of these cases were coded as code 3. At the last test 5.66 of the connections were possible (on average), which makes the value 0.18 for code 3 comprising 3% of these cases. This is a clear improvement. Not only did the amount of connections increase (due to the fact that more elements were mentioned) but the quality of these connections were much higher, as code 3 – connection irrelevant only was coded 3% of the times at test 6, in contrast to 44% at test 1. The quality increased as shown in the averages for code 4 and 5. Code 4 – connection vague decreased from 39% of all possible connections to 20% of all possible connections. This leaves an increase for code 5 – connection relevant which increased from an average score of 0.6 in test 1 to an average of 4.33 in the last test. In relation to the amount of possible connections this leaves an increase for code 5; from only 17% of all possible connections in the first test to 77% of all possible ones in the last test.

Table 11 Occurrence of codes 3, 4, 5 in average score and as percentage of total possible connections per test

<table>
<thead>
<tr>
<th>Code Description</th>
<th>Test 1</th>
<th></th>
<th>Test 6</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>Percentage of possible connections</td>
<td>Average</td>
<td>Percentage of possible connections</td>
</tr>
<tr>
<td>Code 3 – Connection Irrelevant</td>
<td>1.6</td>
<td>44</td>
<td>0.18</td>
<td>3</td>
</tr>
<tr>
<td>Code 4 – Connection Vague</td>
<td>1.4</td>
<td>39</td>
<td>1.15</td>
<td>20</td>
</tr>
<tr>
<td>Code 5 – Connection Relevant</td>
<td>0.6</td>
<td>17</td>
<td>4.33</td>
<td>77</td>
</tr>
</tbody>
</table>

The results described above show that certain learning took place. Student teachers’ results improved both regarding the elements described and more importantly, as a focus for this study, regarding the connections between the elements. This quantitative analysis of the results of the pre-, and post-tests is now followed by a qualitative analysis of the teaching of the teacher educators, in which the identification and enactment of the critical features of the OoL are analysed and described. It is proposed that the identification and enactment of
these critical features contributed significantly to the improvement of the student teachers’ test results and for each critical feature relevant test results are presented explicit.

6.2 Three seminars

Prior to each seminar the student teachers were given a lecture on a purely mathematical topic. The seminars described below are those that were part of the study and focused on the enactment of this mathematical topic in the classroom. One could say that the lectures were mostly mathematical and the seminars mostly (mathematics) didactical. The mathematical topic of the lecture and the seminar were the same during one course week and changed for every course week, as explained in section 4.2, Figure 12. Each seminar was supported with a PowerPoint presentation. The description below, of the three seminars, merely serve to set the scene for the analysis.

Seminar 1 – Rational numbers

The teacher educator introduces today’s topic - rational numbers - through a short review of the mathematical content dealt with in the previous lecture, after which the applicable curricular goals are stated and discussed. Rational numbers written as fractions are discussed with a focus on the different types of fractions: a) as a number: 2/5 has a place on the number line; b) as part of a whole e.g. 2/5 of a cookie; c) as part of a total e.g. 2/5 of the pupils are boys; d) as a ratio e.g. 2 part syrup, 5 parts water. Different hands on materials available are shown and discussed in terms of when to use these, as well as in terms of what kind of understanding a specific material enhances or moderates. Models for explanations combined with some of the related (hands on) materials are put forward. Some example exercises are given to illustrate the different related (hands on) materials and models for explanations. The exercises deal with
addition, multiplication and division of two fractions. Limitations of specific hands on materials and models for explanation are taken into consideration when looking at the examples. The class is divided into groups of four, working for half an hour with Cuisenaire rods. The task given is to work through some exercises using the Cuisenaire rods in order to get acquainted with the hands on material. The teacher educator walks between the different groups and helps and supports the groups. Gathering the whole class again, the focus now shifts from rational numbers written as a fraction to rational numbers written as a decimal number. A whole group exercise (physical) and a game are given as examples for use in classrooms after which hands on materials related to decimal numbers are shown and discussed. Some misconceptions are considered (such as 2,12 > 2,9) and ways of dealing with these are shown. Example exercises on multiplication and division of two decimal numbers serve as an illustration for the models for explanations related to hands on materials.

The seminar is summarized by reflecting upon the goals described in the syllabus for the teacher-training course in relation to today's seminar.

Seminar 2 – Algebra

Today’s topic is introduced by asking the question what is algebra. The origin of the word Algebra is explained and its place within mathematics is clarified by giving some historical background. Arithmetic and algebra are put side by side to accentuate their relation. Algebra is discussed as the relation between equations, functions, patterns and generalizations. The applicable curricular goals (available for grade 3, 5 and 9) are presented where the discussion focused on how the goals change between the three grades. The question if preceding curricular goals can be considered as preconceptions for a subsequent curriculum is raised. The focus then shifts to the algebraic cycle as dealt with in the course literature. Three examples of exercises in algebra, taken from textbooks used in primary schools, are related to working with the algebraic cycle. Problems pupils might have with the exercises are explained within the
cycle and suggestions for helpful models for explanations are stated. Two algebraic games are explained and the class is divided into small groups of three or four. The task given is to reflect upon the games with respect to what kind of preconceptions a player needs, how the game could be developed, what the role of the teacher is, what kind of curricular goals are addressed, how to continue working in class after the game, and finally does the game support the step from concrete to abstract? The groups work for an hour on these tasks and the teacher assists them when possible. Gathering the whole class, the student teachers report their findings related to the two games. Finally the seminar is reflected upon, by relating the events to the goals of the teacher education course.

Seminar 3 – Space awareness

The seminar starts with the question on today’s topic what is space awareness? The focus is on space awareness for young pupils, age 6-8. What is meant by fundamental space awareness? The preconceptions pupils bring to school when they start are reflected on in relation to the concepts dealt with within space awareness. The curricular goals for grade 3 are considered and distinctions in the different formulations are discussed. For instance: Compare and sort length is balanced against Conceptual understanding of length. What do these two have in common and in what aspects do they differ from each other? An analytical scheme (Rystedt & Trygg, 2005) is given as an example to work more practically with the curricular goals as the scheme already divides the goals up into smaller goals. How pupils develop their space awareness is discussed and daily life experiences versus school experiences are considered. Finally, words and notions in geometry are reflected on. The group is divided into groups of three, each getting their own activity to work with. All activities have a mathematical content related to space awareness. The student teachers have to start with working through the activity, after which they have to reflect on the activity keeping the following questions in mind: What is the goal of this
activity? What kind of preconceptions do pupils need? Which words or concepts should be paid attention to? Are there different models for explanations applicable? Is there any bridge from concrete to abstract to work with? What kind of questions can I ask the pupils working on this activity? All these questions have to be taken into account and notes should be made in a predefined mind map. The teacher educator assists each group during their group work and supports them in their work. The seminar continues with a gathering of the whole group, where all groups present their activity and their comments made in the table. At the end of the seminar the goals of the teacher education course are reflected upon in relation to the activities and discussions of the day’s seminar. The mind maps are distributed through the course platform and available for all student teachers to download after the seminar.

These descriptions of the three seminars in combination with the quantitative analysis of the pre-, and post-tests given earlier make a description of the qualitative analysis possible. In the qualitative analysis carried out in the process of the study, four critical features were found and enacted during the seminars. In the sections that follow each critical feature will be portrayed by tracking its identification and enactment throughout the study.

6.3 Critical features

When conducting a learning study, the analysis serves to find critical features of the OoL. A critical feature starts as a possible critical feature and is reformulated throughout the study towards a final formulation as one of the critical features found in that particular study. Having found four critical features in this study does not mean that all critical features were found. Another analysis of the data might give more insights and could result in the formulation of other critical features. The four critical features described below were the ones found by this research team. Some critical features were “suspected” after analysis of the first
pre-test; others were detected further on in the study. For each critical feature the process of identification, enactment and finally validation will be clarified and the analysis supporting this process will be shown through explicit examples. Figure 19 shows in what way the identification, enactment and validation fit in the didactical triangle.

![Diagram showing the didactical triangle for identification, enactment, and validation]

**Figure 19 In focus: Analysis of seminars and tests with focus on the Object of Learning**

### 6.3.1 First critical feature – Lesson goals

**Identification**

The analysis of the first test (pre-test – number sense) showed that 37 out of 47 student teachers stated goals related to the curriculum, however, only 17 student teachers (36%) managed to refer directly to such goals, the other 20 did so in an indirect manner. In other cases the goals described were too broad, covering in most cases the whole curriculum. The curricular documents in Sweden describe more overarching curricular goals as well as syllabus-type of goals and a lot of the student teachers stated both these overarching and the whole syllabus as a goal to work with during one lesson.
The connection between *curricular documents* and the other elements were in most cases not satisfactory. Only 10% of the possible connections (7 out of 69 possible connections) were coded as a relevant connection.

Against the background of these test results, the team formulated a critical feature dealing with *curricular documents* and stated that the student teachers should learn to address these more specifically. Therefore it was decided to work with the curricular goals of compulsory school as well as with the goals of the teacher education course itself. The seminar started with a discussion around the applicable goals for compulsory school, related to the mathematical topic of the seminar. Below three quotes are given from the first seminar to show in what way these curricular goals were addressed during the seminar.

---

**TE1** Well, “the goals to strive towards"”, which should steer the teaching. That is the situation we have now, but there is a change and you know, that the goals to aim for surely in some way will be included into our “goals to attain” and they will form a joint steering document. Today it is a bit confusing. But here it says, anyway: [reading from the syllabus]: “Pupils should develop their number sense and spatial awareness and they should develop their ability to understand and use basic number concepts and calculations with real numbers”.

... 

**TE1** If you only focus on rational numbers, this is what it says, what goals, at the least, the pupils should reach by the end of year 3 in school: “They should be able to divide different wholes into different parts. And then they should be able to describe, compare and name the parts as simple fractions”.

... 

**TE1** And these goals to strive towards, is what should steer. But many many times those who steer the teaching […] are the goals to attain. The goals to strive towards, are actually the way to work that will give pleasure to the pupils.

(from transcription of first seminar)

---

The quotes above show three occasions in which the teacher educator addresses the goals during the seminar. Expressions like “goals to strive towards”, and “goals to attain” are mentioned in these quotes. In the analysis of
the seminar, the teacher educators realized that their way of addressing the goals to strive towards might be confusing. In the curriculum these goals exist on two levels or places. One level is the subject level, which the research team was aiming at. The other level is an overarching level where goals are described as for instance: “The school should strive to ensure that all pupils develop a sense of curiosity and the desire to learn” (Swedish National Agency for Education, 2006, p. 10). This specific goal is repeatedly addressed by the student teachers as one of the goals in their lesson plan. In the discussion during the analysis of the recordings of the first seminar, the research team realized that they had not been specific enough concerning which of these goals they referred to during the seminar. Therefore it might have been difficult for the student teachers to discern what the teacher educator was aiming at.

RE2 The goals to strive towards, they are the umbrella under which we are the whole time.
RE1 Actually, I don’t even think… I think I find it so obvious. So I probably don’t even say it, I mean when I say goals to strive towards, I mean the goals to strive towards in mathematics. Yes. But they don’t necessarily think of that.
RE3 No.
RE2 No. But when you mean, well, the goals to strive towards in mathematics, I do not think it is enough for them to refer to the goals to strive towards.
RE1 No, no.
RE2 Cause if you have a lesson when you talk about fractions, perhaps it is ‘part of a whole’ this lesson is about and that is not described anywhere in the goals to strive towards.

(from transcription analyses first seminar)

When the teacher educator listens again to the following “The goals to strive towards is actually the way to work that will give pleasure to the pupils” (from transcript first seminar above), the teacher educator reacts.

RE1 And what do I mean, by saying that?
RE3 I don’t know.
RE1 No, I probably thought more of methods, we can create methods in a different way than, I mean when you look at the goals to strive towards, you can’t just work with a book, that’s what I think.
RE2 No.
The discrepancy between the intended, enacted and lived OoL are revealed here. The intended OoL was to address the goals to strive towards for mathematics. During the seminar the teacher educator is not specific enough – “there are many implicit meanings when you say that” – and the expressions used can be placed into both mathematics and the more general goals. The enacted OoL therefore is dual. The results of the post-tests of the student teachers show that the lived OoL has been the general goals and not the mathematical goals. Just over 40% of the student teachers stated goals directly which in most cases meant that the whole syllabi was quoted as a goal for the lesson.

The goals for the teacher education course were reflected upon at the end of the seminar. A PowerPoint slide showed the course goals and student teachers were asked to state what they had been doing during the seminar, in relation to these goals. This resulted in brief statements where the student teachers explained what they had done, on request of the teacher educator:

TE1     Now, tell me what you have done today.
StA     Number one [referring to the first goal stated on the slide: Upon completion of the course, the student teachers should be able to demonstrate: solid subject-theoretical and pedagogical knowledge based on the mathematics taught in primary school].
StB     Yes.
TE1     Why?
StA     Cause we have worked with that which you can use in compulsory school.
TE1     Yes.
In the quotes above, the student teachers tend to give short answers – “Yes”, “Mm” – and the teacher educator tries to get more answers by asking questions “But we have also looked at how pupils think and how one can aim to correct such thinking”. When analysing this part of the seminar, it was concluded that the student teachers were not contributing to making connections; it was the teacher educator who did most of the work in this respect.

It seemed that the student teachers had a hard time to choose a proper goal for a lesson from the curricular documents they used. The documents were seen as a whole and treated as such, meaning that the student teachers were not capable of choosing one element and specify it to a goal for one lesson.

After the analysis of the first seminar and carrying out the first pre-, and post-test, the discovered critical feature was formulated more precisely:

To be able to give relevant connections between the five elements of MKT, a student teacher needs…

To be able to relate to curricular documents in a supportive way, seeing the curricular documents as documents to choose appropriate components from.
Enactment

For the second seminar it was decided to specify the goals more clearly by giving specific examples of goals within mathematics, excluding the general goals.

In the PowerPoint presentation that the teacher educator used in the seminar, the difference in formulation of goals for the current curriculum (Lpo94) and for the new planned curriculum were shown and discussed. By putting the two curricula side by side, a pattern of variation was created: Contrast.

The following was shown on the slides:

The school, in its teaching of mathematics, has to strive to ensure that pupils
• recognize the value of, and use, mathematical expressions
• develop their ability to understand, manage and use logical reasoning, draw conclusions and generalize well orally and in writing and explain the reasons for their thinking

The aim should also be that pupils develop number sense and spatial awareness and develop the ability to understand and use
• basic algebraic concepts, expressions, formulas, equations and inequalities
• properties of some functions and corresponding graphs

Goals that pupils should have attained at the end of the fifth school year are:
• to understand and use addition, subtraction, multiplication and division and to detect number patterns and determine unknown numbers in simple formulas

(Lpo94)

Goals that pupils should have attained at the end of the third school year are:
• having the ability to interpret information that includes mathematical concepts and symbols
• being able to describe addition, subtraction, multiplication, division and their relationships using concrete materials and pictures as well as using mathematical symbols

(Lpo11 – at time under construction)
These goals were referred for consideration, and so they were sent to schools and there was a huge team working with this. And then they came back, and then they were told that they were hard to interpret. There was a request that it should be a bit more organized, so they look entirely different and you can see that in your hand-outs, how they turned out later.

And these I have shown to you [teacher educator shows slides of Lpo94] I can show them again in a bit, but we had a thought – Why did it turn out like this? Among other things, these goals include the ability to describe patterns in simple sequences, that's what it says in the third goal [on the hand-outs]. And to handle mathematical similarities for whole numbers between zero and twenty. That is how explicitly it is précised and we see a difference, just by reading what it says right here [teacher educator shows slide with new curriculum].

That's how they changed it.

We see clearly as well, in the grade 3 goals, it is very clear for every topic what pupils have to do and what a pupil has to be able to do at minimum for every topic of mathematics. There it is very clear. This was the only thing standing in the 5th grades goal. And 5th grade will of course become a target of 6th grade later on. As a starting point, before secondary school.

“Understand and be able to use addition, subtraction, multiplication and division, and be able to detect number patterns and determine unknown numbers in simple formulas” [quote from hand-outs]. That was all there was. Then you see that it requires, in fact, an education to know what is behind every such word.

To use contrast by means of two relevant curricula made it possible for student teachers to reflect upon this critical feature related to the OoL. For the researcher team it supported their intentions. In the analysis of this part of the seminar the consensus was that curricular documents really had been addressed but that the way of addressing the documents throughout the seminar could be improved. The teacher educator concludes that there had not been a final comment during the seminar to tie it altogether.

But this is how I think, I don’t tie it up in the end. That is also one of the things that I always do otherwise, when I work with children or young people. I mean, I make a point of tying up the beginning of the lesson towards the end. I don’t do that here. I don’t mention it,
because from these curricular documents [Lpo94] I go to the goals of the syllabus [teacher education course]. I should link back to the curricular documents [Lpo94].

(from transcript analysis second seminar)

The results for the student teachers on the fourth test (post-test – algebra) showed that some improvements were made on how the curricular goals were addressed. Only 2% did not state any goals at all. For the first three tests, respectively 21%, 15% and 7% did not state any goals in their lesson plan. The connections between the described curricular documents and the other four elements of MKT did not improve between the third and fourth test. This was a point of concern, which was dealt with later when describing the second critical feature.

For the third seminar it was decided to work in the same way as in the second seminar, using the curricular goals for compulsory school. The teacher educator gave clear examples of mathematical goals to strive towards. During the seminar, they looked at a part of the curricular document in detail and contrasted one of the specified goals with a less precise goal. For the goals related to the course, the team decided to work in detail there as well. The student teachers were asked to write down what they had learned during the seminar, connected to the goals for the course. Three aspects had to be noted and connected to the goals for the course.

TE2  What did you learn about the activity? Write three aspects and refer to syllabi for this course [teacher education course]. That is our syllabus, but since we only have 20 minutes, and you are many groups, stick with three aspects, short and concisely.

(from transcript third seminar)

As the class had been divided into small groups of four working on different tasks, all groups could contribute to the discussion with their own comments related to their own specific task. At the end of the seminar, one group
managed very well in expressing in what way the exercise worked with related to the goals for the course.

StA  The first connection we make to this, is an “understanding of how mathematical knowledge is built on conceptual understanding and the role of language in development and the learning of mathematical terms” [quote from course goals]. The task we had was to describe different shapes, which we had built with blocks. And this is the thing, had we not had those mathematical terms, we could not have made it at all, and despite the fact that I have the terms, that’s how I feel anyway, it was difficult to use them in that situation. And there was a lot of, seeing things from a different direction and turning around. So I felt that I got to work with that today.

Then the next aspect is the “example of the content and methods that can stimulate creativity” [quote from course goals]. We built houses and got to discover; Well, how does it look? And in the mirror and - Was this really a reflection of it? And it felt real, since you got to build yourself and discover yourself.

And then we had examples of different explanatory models, discuss the pros and cons of these models and demonstrate an understanding of how the pupil can perceive different mathematical terms. There was most of the last part, demonstrate an understanding of how the pupil can perceive different mathematical terms. About taking it in and that the terms can come out weird and Sofia and Jenny did a task in which they would describe a figure they had built up with just words, then Sofia would try to build it. And there was a lot of: “horizontal” and “vertical” and “what was that and what did it look like in 3D?” And we really felt that, yes, but then you must go back and think, what is it like for someone who does not know what the terms mean?

(from transcript third seminar)

This group managed to explain in what way the goals related to the exercise and also managed to specify within a goal what part of the goal they found most relevant, again supported with an example. “There was most of the last part [referring to one part of one of the goals], demonstrate an understanding of how the student can perceive different mathematical concepts”.

Another group expressed their thoughts in the following way:

StB  We have seen in what way you can work with geometrical figures in a creative way and how you can get pupils to discover them themselves instead of giving it directly to them…

(from transcript third seminar)
The way student teacher B from this last group expressed their thoughts differs from the previously cited group (student teacher A). For student teachers that had not worked with this exercise the comments given only contain elements of the goal for the course. The teacher educator tries to open up by asking this group information on the exercise, if it would be useful in school, upon which the group only answers a short “Yes”.

TE2 Is this an exercise one could use in school?
StB Yes.

(from transcript third seminar)

Just prior to this the teacher educator had already asked the group to reveal the product of their work (a dog made of boxes) in order to give the rest of the class something to relate to. The group did not seem to be able to give relevant details on their task on their own.

Validation

The test results concerning curricular documents and their connections to the other elements of MKT changed as presented in Table 12. Again for these percentages the number of possible connections is seen as the total (100%). But the absolute values of these totals differ. In the first test 21% did not relate to the curriculum, and 43% related indirectly. At the sixth test none of the student teachers left out goals for the lesson and only one student teacher (2%) referred to the goals indirectly. Combined with the response on the other elements only 49% of the available connections were possible in the first test, changing to 77% in the sixth test.
As shown in Table 12, a clear improvement had been obtained throughout the study. The irrelevant connections decreased from 36% irrelevant in the first test to only 4% in the sixth test. Consequently the amount of vague and relevant connections increased, which finally resulted in just over 71% relevant connections in the sixth test compared to 9% in the first test.

In this section the first critical feature concerning lesson goals has been looked at and the evolvement over the study has been described. The study resulted in four critical features and the second critical feature will now be addressed.

6.3.2 Second critical feature – Detailed descriptions

Identification

One of the problems visible in the tests was the fact that student teachers addressed elements indirectly. Very seldom when an element was addressed indirectly, the connection between other elements was clear. Near the end of the study, the research team pointed out the importance of clear descriptions of the individual elements. By then the formulation of the critical feature was the following:

To be able to give relevant connections between the five elements of MKT, a student teacher needs...

to be able to describe each of the five elements accurately and in detail.
However, the formulation of this critical feature was developed during the study. During the first part of the cyclic process, the feature was not distinguished from the other elements but was very closely related to the first critical feature, dealing with the curriculum. Some indications of the existence of the critical feature were visible during the preparation of the first seminar. However, it was not until after the first seminar that the critical feature was depicted and worked with. Some indications found afterwards, when analysing the transcript of the preparation of the first seminar, are displayed in the quotes below.

RE1  They write: this is what one should be able to do; this is what you should know. Just like you said, they write ‘one should have preconceptions’ or ‘one should have good number sense’. Yes, and then you should write what that means.

RE4  We need to get the student teachers to understand or, to know how to express, that is, in a mathematical, or in some didactical language, in a professional language, express mathematically what is happening.

(from transcript preparation first seminar)

These transcripts show that the team was aware of the fact that a precise formulation of the elements was lacking, however, it was not depicted as a critical feature. Preparing the second seminar (on algebra), the suggestion to use a model to describe the cyclic process of algebra was made. Such a model would support student teachers in describing some of the five elements of MKT in more detail.

RE2  So, I have been thinking that I should use this cycle over and over again. Just to get the cycle into their head. It is not so important what is written in the cycle, as long as I know that I go from the concrete, I have to talk and then I go through drawing or constructing or something, to the abstract, and then I have to be able to describe in my own words.

(from transcript preparation second seminar)
The algebraic cycle was shown during the PowerPoint presentation in the second seminar. The teacher educator and the student teachers discussed the use of the algebraic cycle and in what way it could support the student teachers. Exercises worked with previously, during the seminar were briefly referred to again but now in relation to the algebraic cycle.

Throughout the whole second seminar the five elements of MKT were addressed more in detail. For curricular documents it meant a focus on the goals in mathematics. For the related (hands on) materials and exercises worked with during the seminar, more time was given beforehand to read through the instructions. Preconceptions and models for explanation were thoroughly addressed on several occasions. These changes made it possible for the student teachers to discern the five elements of MKT. One such example of preconceptions is given in the quote below, where student teachers try to formulate the necessary preconceptions needed for a specific exercise. The teacher educator tries to get a total view of preconceptions whereas the student teachers seem to focus on one preconception at a time. The focus on preconceptions however is clear for both the student teachers and the teacher educator.

StA \[ x \] equals \( a \) times 2. Or \( a \) equals \( x \) divided by 2.

TE1 What do you need to be able to do? What kind of preconceptions does such an exercise require?

StB The basic operations.

TE1 Basic operations, anything else, something you did just now?

StC Quotition division.

TE1 Quotition division, for example, you need to know.

StD These laws, like the opposite.

TE1 Yes, the inverse of multiplication, which is division.

StE Exactly.

TE1 Well, to be able to do that, those are preconceptions…

(from transcript second seminar)

97
The fourth test showed that the connections between the five elements of MKT did not improve remarkably from the third to the fourth test. During the analysis of the test this was pointed out and in the analysis of the second seminar the team looked for instances that could explain this result.

RE3  I was taking notes on what you were doing and that kind of stuff, when there was an exercise where you … let’s see, where you really explained why the exercise was good and why … yes, with thousands, for example.
RE1  Yes.
RE3  And then you say something like, well – what preconceptions do they need to have in this case?
RE1  Yes.
RE3  And they were to do mental calculations, counting numbers by sort, so you were talking a lot about prior knowledge. So I thought, they might bring up that exercise but …
RE1  No.
RE3  Well, some of them brought up this exercise and those who brought it up also described the right preconceptions.
RE1  Yes.
RE3  Then the other exercise we did, when they are lined up.
RE1  With decimal numbers?
RE3  Decimal numbers, there you did not talk that much about…
RE1  Preconceptions.
RE3  Hmm, preconceptions and it is clear that they place that exercise.
RE1  Totally wrong.
RE3  In completely wrong context really, so is a bit interesting.

(from transcript analysis second seminar)

In the series of quotes above, the team recognizes that student teachers only seem to be able to duplicate relevant connections when dealt with explicitly during the seminar. The exercise, which had been dealt with in detail in the seminar, led to relevant connections in the test of the student teachers that used that exercise. Another exercise, which had not been dealt with in the same detail, resulted in placing the exercise in the *wrong context*. The discrepancy in these results indicated a need of a more general approach.
Well, I have been thinking, when you talk, it is important to focus on preconceptions. Now it is a bit spread out. And that may be unclear, because if you put it in an exercise I think it is connected. But if you really want to focus on preconceptions, maybe we should have a separate part in the seminar when we discuss focus on preconceptions, what it means and also give some examples within a specific mathematical topic, I thought. And then tell them [student teachers] “Keep that thought in mind, everything about preconceptions, in every topic and exercise you do.” Cause I really don't do that now.

I think so too. And we talk about it too rarely here, because here we take it for granted that, here, they should have this preconceptions.

That is perhaps why we focus on it too rarely.

Yes. Because I think it was too spread out, I think I talked about it quite a lot. But maybe not well enough, what can I say? Maybe, from their point of view, it was presented without connection to anything really. That it's more a general discussion on preconceptions. But, I mean, I do not point out, well maybe at some occasions, but that you first during the seminar discuss preconceptions a bit and give an example, that they should have that thought in mind in every exercise.

And you can just stop at an exercise. ‘What preconceptions are at hand here?’

Yes, that is what I mean.

And then you don’t have to discuss it, but just…

Yes. Just think…

Yes, sure.

That's how we could do this.

The team expresses that the student teachers cannot discern preconceptions (as part of the OoL); “Maybe, from their view, it was presented without connection to anything really”. By focusing on preconceptions in a “separate part in the seminar” discernment would be easier for student teachers. Transferability would be suitable to use during work on exercises. As one teacher educator says, “keep that thought in mind” and later “have that thought in mind in every exercise”.

One of the researchers brings up an example of a student teacher’s work earlier in the course. The student teacher handed in an assignment in which she had

---

12 Another assignment within the course, not one of the tests included in this study
to describe an outdoor activity. The mathematics, however, was not visible. On a second try when she was asked to write into detail where the mathematics was, the whole assignment became more coherent.

RE2  She wanted to combine it [mathematics] with gymnastics. So they had to run an obstacle course. And she wrote that very clear and in detail, first they jump over a gymnastic horse, then they crawl under it; then they roll over on the carpet and climb up on the stall bars and then they come to a mathematical exercise.

RE1  But she did not explain what kind of math?

RE3  No.

RE1  No. But she got it in return, and then I got it back yesterday. She’d used the same gymnastic tools, but now she had described what kind of exercises she thought of. And those were really based on problem solving in groups, good exercises, I think. Now she had changed the goals, she kind of focused on problem solving now in stead, so that is a good thing.

RE3  Something happened there.

(from transcript analysis second seminar)

From this example the idea that elements described in detail would lead to relevant connections developed into the second critical feature. Thus, through working on detailed explanations on how to describe the elements, relevant connections would follow. For the third seminar it was decided that student teachers themselves would work on such detailed descriptions. The algebraic cycle used in the second seminar did support some student teachers but was not supportive enough. Another format to describe more parts of an exercise, or a topic in mathematics was used. A structured mind-map (Rystedt & Trygg, 2005) seemed suitable to open up for dimensions of variation, in which the mathematical content could vary.

Enactment

At the beginning of the third seminar the context of the mathematical topic – spatial awareness – was dealt with in detail. The teacher educators had worked on several relevant questions to get student teachers thinking about some of the
MKT-elements. During the beginning of the seminar some of these questions were raised by the teacher educator, as shown in the quote below.

TE2 What does spatial awareness mean? … Like Bengt Johansson and Göran Emanuelsson have written in 1992, this is what they think is incorporated in spatial awareness: “First of all, it means to have understanding and knowledge and skills which form the basis of being able to orientate in a space” [On PowerPoint slide]. And then one could wonder, what does it mean “knowledge and skills to be able to orientate in space?” That you have to figure out yourself, what that is. And when you have thought for a while you can ask yourself; what does this mean for me as a teacher and what does this mean for my pupils? To get such knowledge and skills, what does it mean, in practice? Of course, there are different aspects to consider. But,”to understand where in space, an object is situated in relation to its surrounding” [On PowerPoint slide], that is one thing, I will talk about this more later on. “Can describe how to move an object” [On PowerPoint slide], the word describe tells us we have to know quite a few words, because otherwise we do not have the possibility to describe…

(from transcript third seminar)

The questions raised by the teacher educator – What does it mean ‘knowledge and skills to be able to orient in space?’, What does this mean for my pupils? – were meant to start the student teachers to think into detail. Just some minutes later, when the specific goals to strive towards concerning spatial awareness were on the PowerPoint slide, the teacher educator continued by posing some more questions to the student teachers. They were, for instance, asked to think about the difference between compare and sort, again in order to start the detailed thinking.

TE2 “Compare and sort length, volume, mass, area and angles. Understanding the concept length” [On PowerPoint slide]. What is the difference between “compare and sort” and “understanding the concept”? Have you thought about if you have a lesson where we work with length, is the aim of the lesson to compare and sort length, or is it concept understanding? What is the difference?

StA To be able to explain.

StB They go hand in hand.

TE2 But they don’t necessarily go hand in hand. You could measure with a ruler, even if you really don’t have a clue, if it says one hundred, what it means, really.

StC Then I might use something else.
Then I might use something else, right.

I believe that the issue of terms is important to us pedagogues, well what terms do I need now, if we are to compare and sort length? What terms do I need, to be able to do this, so that the terms are introduced but we cannot take it for granted and I think it is important that we think about; what terms do we need? Which are important to know? Nothing is obvious.

(from transcript third seminar)

At several occasions during the seminar questions like what concepts do I need now?, were raised and the importance of detailed descriptions was in focus. In such a way the student teachers got a general idea how to work with it, but had to apply this idea to specific situations. The example from the second seminar shows that student teachers are only able to duplicate relevant connections when made explicit during the seminar, but not able to make relevant connections when these had not been dealt with during the seminar. This example stresses the importance of generalisation.

The mind-map (Rystedt & Trygg, 2005, p. 72) was introduced at the stage of the seminar when the student teachers had to pick an exercise to work with. During group work they had to work through the exercise and its instructions and fill in the mind-map for that specific exercise. The format of the mind-map imposed the idea of generalization of the five elements of MKT and their connections. At the same time it stimulated both separation and fusion of the critical features found. The mind-map (adapted and translated) looked as in Figure 20.
Working with the mind maps produced during the seminar seemed to open a dimension of variation. The mathematical content varied, but the way to address the elements of MKT was constant. Student teachers seemed to be able to generalize their knowledge concerning the five elements of MKT and use it in different situations.

**Validation**

In this section it is important to look at the change within the amount of relevant connections observed in the student teachers’ tests. But, as the critical feature focused on a detailed description of the five elements, it is also interesting to look at the change within that code.

A focus on the elements gives Table 13 in which a change can be observed from test 4, *i.e.* after the second seminar when looking at the percentage of elements not referred to. An increase in the percentage of elements directly referred to is also observable from that point (test 4).
Table 13 Occurrence of codes 0, 1, 2 for test 1 - 6, in percentage of total elements per test

<table>
<thead>
<tr>
<th>Code 0 – Element not referred to</th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
<th>Test 4</th>
<th>Test 5</th>
<th>Test 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code 1 – Element indirectly referred to</td>
<td>37</td>
<td>31</td>
<td>35</td>
<td>24</td>
<td>17</td>
<td>19</td>
</tr>
<tr>
<td>Code 2 – Element directly referred to</td>
<td>17</td>
<td>18</td>
<td>13</td>
<td>17</td>
<td>12</td>
<td>8</td>
</tr>
</tbody>
</table>

The critical feature suggested that if elements would be described more in detail, that relevant connections would be a consequence. However the team did notice that a structure, the mind-map, had to be offered to the student teachers in order to be able to generalize and work with this structure in different mathematical contexts.

Table 7 and Table 8, previously shown in section 6.1.1, show that the number of relevant connections starts to increase already in the third test, i.e. prior to the second seminar. Looking at these tests in more detail it became clear that most student teachers had used a context dealt with in one of the textbooks for the course. This problem was also observed after the second seminar. Student teachers apparently could reproduce obtained knowledge on the elements of MKT in situations that are known, but they could not apply knowledge on the elements of MKT into new contexts.

The use of the mind map, introduced in the third seminar, gave the student teachers a tool to use their knowledge and apply it in different settings. A detailed look at the type of exercises chosen in the last test showed that student teachers chose unfamiliar exercises, not dealt with within the seminar, and managed to refer directly to the elements and give relevant connections between the elements. Of course these results cannot be explained only by the way this second critical feature was addressed, since all four critical features together were aiming at improving the quality of the connections.

---

13 Due to rounding the total of some columns does not add up to 100%
6.3.3 Third critical feature – Mathematical knowledge

Identification

Mathematical knowledge was not tested *per se*, but analysis of the tests did show indications of the influence mathematical knowledge had on the way the connections were expressed. Several student teachers used the word *digit* when they meant *number*, which some people might argue, is not a point of concern. But when the lesson deals with number sense specifically it actually *is* crucial, as shown in the following quotes, taken from the first (pre-) test.

St5T1 This lesson plan is aimed at pupils who already have learned digits up to 1000, for instance grade 3. As an introduction to the lesson one could discuss what pupils know about digits and give different examples on numeral positions and hundreds, tenths and units…

St6T1 The teacher has made space in the classroom and has put down papers with the digits from 1 to 20…

St10T1 For instance, you can draw a number line with the digits 0 and 100.

All these examples show an incorrect use of terminology, which, as explained above, is crucial terminology in these lessons. The following example, taken from the first (pre-) test, shows that the student teacher does not master the mathematical content and therefore incorrect conclusions are drawn.

St10T1 Afterwards you, the teacher, can tell [the pupils] that 50 can be written as 0.50 in decimal form or as ½ as a fraction. [because 50 is half of 100]

One of the student teachers (St43T1) describes an exercise where all digits are represented and with these digits numbers can be made. However, according to this student teacher there are only nine different digits (zero was not accounted for).

Such examples give an indication that mathematical knowledge might influence the results of the tests. A third possible critical feature arose and concerned the
student teachers’ knowledge of mathematics and its relevance. Caution had to be exercised here when drawing conclusions: The student teachers that show a lack of mathematical knowledge might have equal mathematical knowledge compared to other student teachers. Mathematical mistakes only appear if the mathematical content of the lesson exceeds the mathematical knowledge of that student teacher. However, some student teachers might have chosen mathematical content that they can cope with and therefore these student teachers were not detected in these tests.

During the seminars more indications of the lack of mathematical knowledge were found. Some student teachers got stuck when explaining, or talking about, an exercise. Others did not manage to leave the mathematics and never came to discuss the educational value of the exercise: They got stuck in the mathematics.

In the quotes below some examples are given of a lack of mathematical knowledge and how this obstructs the didactical discussion.

TE1: What I talked about was this diagram [referring to a Venn diagram on PowerPoint Slide] with all the natural numbers. What are those? … What numbers are natural numbers?
StA: One to ten.
TE1: And?
StA: Positive.
TE1: Positive, anything else? What about zero, as positive? It's often forgotten, but it is there. But you said one to ten?
StA: Yes, because those are the numbers one learns first.
TE1: OK, that’s how you were thinking.

(from transcript first seminar)

This student teacher thinks of natural numbers as numbers appearing in a natural, everyday way. Building further within the number system to integers and rational numbers will not be the next logical step for this student teacher, because of a lack in mathematical knowledge. In some cases, however, the
mathematical knowledge might be good enough, but the terminology is lacking. When multiplying fractions, a student tries to explain how to multiply:

TE1 And here, what do we need to do?
StA Change them into the same unit.
TE1 You change them into the same unit – how?
StA Timesing[^14][gångrar]
StB Multiply.
TE1 And you do that in the numerator?
StC You do that both up and down.

(from transcript first seminar)

Numerator and denominator were not used as terms by the student teachers. In this quote one student teacher corrects another student teacher when using the expression timesing. On other occasions this does not happen and different forms of timesing are used [gångrar, gångréar].

Such ways of thinking led to some adjustments for the exercises used in the seminars. During the seminar the mathematics was not supposed to be in focus but rather the educational perspective on mathematics. If the mathematical knowledge influences the quality of the discussions and descriptions of this educational perspective, then one could choose to work on the mathematics with the student teachers. Another choice is to find exercises that are suitable to discuss from a mathematics educational point of view, but that are not demanding from the mathematical perspective. Both approaches were applied during the second seminar. This third critical feature was defined as:

```
To be able to give relevant connections between the five elements of MKT, a student teacher needs…
to understand the underlying mathematics.
```

[^14]: Unofficial verbal form of the noun times. In Swedish: From gånger (times) to gångrar

107
Understanding was not defined more specifically at this stage.

**Enactment**

The mathematical topic of the second seminar was algebra. During this seminar one specific problem was brought up to problematize the role of variables. For this problem the student teachers were first given time to work on the mathematics. Student teachers asked questions related to the mathematics, which were crucial for a fruitfully proceeding didactical discussion. In the following two excerpts such questions are stated. The student teacher in the first quote asks a question related to the commutative law of multiplication and needs to clarify if it is valid or not. In the second excerpt the group of student teachers work with the terminology and definition of the different number sets.

**StA** When one writes such expressions, does it matter where the letter is positioned? Because she writes \( r = \text{three times } n \) and I write \( n \times 3 \).

(from transcript second seminar)

**StB** A Rational number is like the square root of… fractions right?

**StC** No.

**StD** No they are irrational.

**StC** Irrational… Rational ones are like fractions, kind of.

**StB** Rational ones are fractions?

**StC** Yes.

**StB** Yeah, right and then irrational, the square root of…

**StD** Hmm.

**StB** That’s maybe what we have, the square root of and powers, in some way.

(from transcript second seminar)

When the mathematics was dealt with, a discussion about the problem in relation to mathematics education could start. The goal for the teacher was not to solve the problem in class but to talk about the problem from a teacher’s perspective. In some instances mathematical understanding supported the
discussion about the problem, whereas in other cases, the lack of mathematical understanding seemed to make it more difficult.

In the following two excerpts, two student teachers describe how one could make changes in a proposed game, Parcheesi\textsuperscript{15}, in order to elaborate the game.

StA And it is possible to develop it in many ways, with or without the rule of capturing the pawn. I mean you are the one making the rules. And you could also use more difficult numbers, so it gets more advanced…

(from transcript second seminar)

This student teacher first focuses on the game and changes rules in order to elaborate – \textit{with or without the rule of capturing the pawn}. Such a suggestion is not supporting an elaboration of the mathematics or of the mathematical understanding. When the mathematics is addressed, the suggestion is too vague and not specific enough. Another student teacher is more specific and expresses the same type of elaboration in the following way:

StB And then we talked about changing the rules in both games. Develop the games. And that you can, in the game of Parcheesi, perhaps even have them practice odd and even numbers, that the player has to get to an odd number, or an even.

(from transcript second seminar)

This student teacher exemplifies what type of numbers one could elaborate with. To be able to give such an example one needs good understanding of the mathematics used in this game. It is not said that the student teacher in the first example (StA above) has a lack of mathematical understanding. However, by choosing broad terms, like \textit{more difficult numbers}, the argument for elaboration of the mathematics used and learned in this game, becomes very thin.

During the third seminar even more time was given to each group to understand the mathematics of the problem student teachers had to work on. It

\textsuperscript{15} In Swedish: Fia med knuff
was decided to let each of the groups work on a specific problem. In this way only one problem would be dealt with per group. Each of the exercises had a sheet with instructions guiding the student teachers through the problem, starting with solving (or exploring) the problem and continuing with questions focusing on the problem from a teacher's perspective. They had to problematize the problem, discuss positive and negative parts of the problem, had to look at a broader context, connect it to both mathematics and the curriculum etc. At the end of the seminar, one of the student teachers said the following:

StA Had we not had the mathematical terms we would not have made it at all and despite the fact that I have the terms, as I feel, it was hard to use them in that [right] context.

(from transcript third seminar)

This student teacher states that having mathematical notions is not enough; one has to be able to use them as well. At this point, the teacher educators talked again about the third critical feature, where understanding the mathematics was stated. The student teacher who said “I have these terms” but at the same time is not capable of using the notions in context, might actually not have the terms.

Validation

As stated before, mathematical knowledge was not tested *per se* in the tests. Indications of the importance of mathematical knowledge were found, but no evidence can be given. The only conclusion given at this stage is that in those cases where student teachers described the mathematical content in detail, relevant connections are indeed found. However relevant connections were also found in cases were the mathematical content had not been described at all. In contrast, in those cases were the mathematical content had been described improperly or incorrect no relevant connections were found.
The development over the six tests in the study did not show any clear changes amongst student teachers in the use of their mathematical knowledge. Since every test deals with a different topic in mathematics such a change would not be likely to be depicted.

After the final analysis of the tests, the third critical feature was looked at again in a minor meeting, with only two members of the research team present. At this meeting it was pointed out that the critical feature as formulated until then had focused on the student teachers’ mathematical knowledge. It might however be that the student teachers’ needed to be aware of the importance of proper articulation of mathematical terms and notions during discussion and planning. Using proper mathematical language allows one to express ideas and connections. Such focus had not been made explicit during the seminars and had not been possible to discern for the student teachers.

### 6.3.4 Fourth critical feature – Shift in perspectives

**Identification**

The lack of relevant connections between the five elements was a point of concern for the teacher educators and the starting point of the study. Previously the teacher educators had not been able to find specific reasons for this lack of relevance, but they did have some thoughts about the causes. They believed there to be a relation between the involvement of the student teachers in an exercise or problem and their ability to take a teacher’s perspective on the problem. This ability seemed crucial for being able to connect the five elements of MKT and was therefore defined and tested as a critical feature. This critical feature was indicated at the beginning of the study and an early stage definition of this critical feature was: To be able to give relevant connections between the five elements of MKT, a student teacher needs to look at the elements from a teacher’s perspective. As the research team believed there to be a relation
between the involvement and the ability to take a teacher’s perspective, increasing possibilities for involvement were searched for. Giving student teachers the opportunity to actively engage in exercises by performing the exercises themselves seemed a suitable approach. Such an approach however is not possible for all existing exercises and the aim of a teacher-training course is to give student teachers a tool that allows them to view upon teaching aids (exercises, support materials, computer programs etc.) from a teacher’s perspective.

From the first critical feature – Lesson goals (section 6.3.1)– closely related to curricular goals, it was already found that student teachers often focused on all the goals at once. Even when the teacher educator concentrated on talking about the goals to strive towards within mathematics, the student teachers listened and applied the comments on the more overarching curricular goals, written for all subjects in school. This observation led to the idea that student teachers needed to shift from seeing themselves as (general) teachers to looking at themselves as mathematics teachers. However, an additional shift in attitude seemed necessary. Student teachers needed to change the way they saw their role within mathematics. As the course was a course in mathematics education, the mathematics was not the goal of the course. However, for a lot of student teachers their insecurity with respect to their own mathematical knowledge and understanding dominated and they saw themselves as mathematics learners instead of mathematics teachers. The ability to involve in exercises as mathematics teachers would make it possible to take a (mathematics) teacher’s perspective during discussions. After analysis of the first seminar, the critical feature was defined as:

To be able to give relevant connections between the five elements of MKT, a student teacher needs… to look at the elements from a mathematics teacher’s perspective.
Enactment

During the preparation of the first seminar, the team talked about multiplication and the chocolate bar model, which a lot of student teachers used in their tests. One of the team members explained the problem with the use of the model as follows:

RE2 OK, this is where they show them the chocolate bar to make it very clear, very clear for the pupils. And still they don’t get it.
RE1 No.
RE2 I mean, because they still believe it’s enough just to bring out the material and play around with it.

(from transcript preparation first seminar)

The concern the teacher educator expresses involves the shift from being teacher oriented to being mathematics teacher oriented. Showing a chocolate bar was often in the tests connected to pleasure and fun in class. But the mathematical goals are not instantly obtained just by bringing a chocolate bar. In the following excerpt, the shift from mathematics learners to mathematics teachers, is pointed out.

The team discussed in what way they themselves have used a specific booklet in previous course and what problems they encountered. The first pages of the booklet contained exercises that focused on simple mathematics when using a specific material – Cuisenaire rods – and the discussion in the excerpt below concerns whether or not these first exercises can be left out. One of the teacher educators had tried just that in a previous course but without success. The team concludes that the beginning exercises are crucial for further work and support a profound understanding of the material and the mathematics.

RE4 Well, it is my experience from previous years, that they [student teachers] need to work with those pages. Unfortunately I should say. I don’t think it is possible to start …
RE1 So they might not need it then, or?
RE4 I don’t know, I mean…
If we presume that they come to understand what methods to use for younger children at this stage, to have a solid ground, then they'll have to do this. Because otherwise they don't think it is necessary, that you can just start wherever. But you can't.

Yes, because I tried to, it was Mary who did this before, and then when I took over I altered it, and I made the mistake to take away from these first ones [exercises]. It did not work.

No.

Because they don’t know the material. And they are too poor at their own…, no not too poor, I mean they have too few representation models for fractions in their heads. So it was not possible to discuss with them. That was my experience. That is probably why these particular exercises are in the booklet. It's not a coincidence.

No. It is not a coincidence that I chose them. Really, but it is possible to do them, but you’ll have to be efficient and work hard.

Yes, you’d better make sure that they don’t drag their feet, but that they understand that “this is the amount of time we will have for this” and, maybe, push them a bit. I don’t know. But I think it is important, if you are to get a discussion with quality, that they have had their own experience.

But you cannot just let them work, because they do have to discuss afterwards.

Yes, that's important, you have to bring it [the discussion] up.

In the above excerpt the team sees the importance of getting acquainted with unknown material like Cuisenaire rods and letting student teachers get their own experience. At this stage, the team did not articulate the importance of distinguishing between the two roles the student teachers have when working with materials. In a later stage of the study the situation was recognized as dealing with shift, where student teachers work with the material as learners of mathematics and have to switch to look at the material from a mathematics teacher point of view.

During the first seminar, the instructions to the student teachers was as follows:

So your task will be to get acquainted with these [Cuisenaire rods] and we will get back to them when we work with percentages.

In this instruction it is not apparent for the student teachers from what point of view they have to “get acquainted with the rods”. In the two groups who were
recorded while working on this exercise two different approaches were visible. One group mostly worked with the material as learners of mathematics whereas the other group managed to work both as mathematics learners and as mathematics teachers (van Bommel & Liljekvist, 2009). Later in the seminar other exercises were worked on where the student teachers had to participate as pupils. The subsequent discussion focused on (mathematical) didactical issues. Since the team had not yet made explicit the importance of clarifying the shifts described above, the student teachers were not given the opportunity to discern this shift explicitly.

One part of the seminar dealt with the different support materials a teacher can use when working with rational numbers. These materials were mostly shown and put on tables for student teachers to look at by themselves. Student teachers engaged in the material as learners of mathematics. A shift towards teachers of mathematics was not made nor was it supported by the teacher educator. In the analysis of the seminar this lack of shift was not noticed, but some changes for the second seminar were made which afterwards could be explained in terms of the shifts described above. The teacher educator clearly wanted to focus more on the role of mathematics teacher while getting student teachers involved in the exercises. Suggestions for possible questions for the second seminar focus on the content – what pupils need to know.

RE1 And I was thinking, my plan is that they play both [games]. Starting with this, they get to think for a minute – what do their pupils need to know? What do you need to know to be able to play this game? And what could be a problem? What could be misunderstood, what do I as a teacher need to discuss with the pupils?

(from transcript preparation second seminar)

In the same meeting, the point already made prior to the first seminar, that the timeslot is important, was taken up again. The team talked about how much time the different parts of working on the exercises had to take and concluded
that it was *the discussion afterwards which was the most important*. In such a discussion the shift could be made transparent.

RE1 Yes, well let’s say ten minutes.
RE2 Yes, at most, since it is the discussion about it, which is the most important.  
(from transcript preparation second seminar)

During the second seminar it becomes clear that the student teachers were asked to take the role as a mathematics teacher in their discussions and reflections, as shown in the following quote:

TE1 Well, here are some questions I would like you to address: Look at preconceptions, what is needed? Can the game be used to give pupils such conceptions? How can you develop these games? How can you use them in your teaching? And what is your role as a teacher when the pupils play such a game? What role and what aim? What connection can you see with the steering documents? Think about what parts of the steering documents they need to know … Think about that. You don’t have to go into details, but these are didactical questions.  
(from transcript second seminar)

While the intended way of communicating related to this critical feature was to stress the importance of taking a mathematics teachers role in the reflections, the enacted way of communicating only addresses a part of this: “How can you develop these games? How can you use them in your teaching?” When student teachers later in the seminar expressed their thoughts concerning the exercise, it became clear that only one of the above-described shifts had been made clear to the student teachers: The shift from learner to teacher. The shift from *teacher to teacher in mathematics* was not made explicit and therefore was not possible to discern. Therefore the lived OoL, for some, was only to reflect as a teacher. Some student teachers expressed their thoughts on *the role of the teacher* as follows:

StA You role as a teacher? Well, to explain the game.
StB In a better way than the written rules.
StA Yes, and making up your own rules is perhaps not a good idea for small children.
The student teachers quoted above, point out important elements of the role of a teacher, but the elements are not specific for the role of a mathematics teacher – “Everybody should get the chance to think” or “Everyone is taking part”. Such a role should be taken at all times, regardless what subject a game is placed in.

When the same group of student teachers argues what part of the curriculum is dealt with in this game, they relate some goals of the curriculum to the game:

- Connection to the steering documents.
- An interest in mathematics...
- Yes.
- Confidence in one’s own thinking. Yes, actually you will be corrected quite quickly if you do it wrong.
- Use mathematics on different occasions.

The goals addressed – interest in mathematics, confidence in one’s own thinking and use mathematics on different occasions – are not related to any content in mathematics. Later the group continues and goes through the list of goals to attain (in mathematics) and says that all goals are applicable, if you add things to the game. The game deals with algebra, equations etc. and a motivation for connecting the game to a goal related to geometry is that one could calculate the area of the board [board of the game]. In order to connect statistics to this game pupils could make diagrams on how often a player gets to
certain fields on the board. Such solutions are given to be able to connect the
game to whatever goal:

Yes, well it is possible to get everything in. Just set up the game according to the goal you
want to attain by playing the game

(from transcript second seminar)

This way of reasoning touches upon the problem addressed in the first critical feature as well, but from a different angle. The first critical feature put the curriculum in focus and tried to address the problem student teachers had when choosing appropriate goals for their lesson plans described in the tests. In this case the game/exercise should be the starting point and student teachers have to be able to focus on the game and see its potential. Adding a new activity (like calculating the area of the board) in order to be able to connect the game to another goal means that the game is not in focus but rather the goals are.

In the third seminar the teacher educators gave student teachers some instructions for discussion, which would help the student teachers to make the two shifts. The instructions to each exercise (each group worked on a specific exercise) started by working through the problem. The exercises were of the kind that the student teachers would not encounter difficulties with at this stage. After working through the problem specific questions were asked with a focus on mathematics teaching to make discernment of mathematical content possible for pupils. So instead of: “In what way could we develop the game?” the question was: “What kind of questions can we ask pupils to stimulate progression within the same mathematical content?”. Before the student teachers started working with the exercises the teacher educator gave very strict instructions on how to work on the exercises, and said the following:

Work carefully through the entire task. It can be that, in many of the tasks it says: “1. Do this. 2. Do that 3. Do this. And I want you to work through all of them. If you work with these in class, they can be developed all the time, but let's stick to these ones [the
The teacher educator continued with more instructions for reflection on the exercises and as mentioned earlier the student teachers had to fill in a mind-map capturing these reflections. In this instruction the teacher educator managed to clarify the difference between working as a learner of mathematics with the exercise and working as a teacher in mathematics with the exercise. Simultaneously, the shift from a (general) teacher towards a mathematics teacher was made, by placing the questions specifically within a mathematical context – what is the aim of this activity. This shift was also made possible through a clear focus on the steering documents for mathematics as addressed in the first critical feature, and a focus on the mathematics as expressed in the third critical feature.

Validation

In previous sections the improved results of the connections between the five elements of MKT have already been shown. The shift itself was not possible to detect in the lesson plans, but only in student interaction as shown in the excerpts above. The teacher educators wanted to offer a tool that allows looking at teaching aids from a mathematics teacher’s perspective. The tool chosen in the second seminar (the algebraic cycle) was connected to the specific mathematical content whereas the tool in the third seminar (mind-map) was not content related. It seemed that this last tool offered, suited the student teachers and the OoL of the teacher educators. In the last seminar, student teachers worked with only one exercise, but in the post-test several topics were addressed, not related to the exercises worked with in the seminar. It seemed that the student teachers had used the tool in another setting and thus used its transferability.
6.4 Conclusion

In this chapter the results of the student teachers on the tests have been displayed. These results have been related to the critical features found during the learning study. These critical features give an answer to the main question addressed in this thesis:

In what way would it be possible to address MKT during a teacher-training course so that student teachers would more systematically consider elements of MKT in planning their future teaching?

When addressing MKT during a teacher-training course, these critical features should be possible to discern for the student teachers, which can be established through the use of the patterns of variation. The three related questions mentioned in section 1.3 concerned the student teachers’ learning, the teaching of the teacher educators and the OoL. The results presented in this chapter showed that the student teachers’ learning did improve. Their test results showed that the connections between the five chosen elements of MKT, became more relevant and also more frequent. Through an enactment of the critical features in the seminars, the intended, enacted and lived OoL seemed to come closer to each other. The enactment of the critical features made it possible for the student teachers to become aware of the critical features concerning the OoL of the learning study. Such awareness makes a change in understanding and experience more likely is a necessary for learning to occur according to variation theory (Marton & Booth, 1997; Runesson, 2006). The study does not claim to have found all possible critical features concerning the OoL, but the critical features which were found and enacted during the study, seem to have caused a change in understanding which had not been the case in previous groups. In the next chapter, the critical features will be reflected upon within the light of previous research findings. As Kullberg (2010) showed,
critical features found can be used in other situations (e.g. classes, lessons) and 
the consequences of such generalisation are considered.
Chapter 7 – Reflections on critical features

The critical features found during the learning study and described in the previous chapter, are reflected upon and put in a broader context using results of other research in the field of mathematics education.

7.1 Lesson goals

During the study it was observed that student teachers had difficulties in describing their goal(s) for lessons. Student teachers’ way of treating the existing curricular documents resulted in lesson plans where they stated the whole curriculum as their goal for a lesson and a clear focus within the lesson plan was therefore difficult to find. Niss (2004) also pointed out the importance of knowledge on curriculum as one of the competencies needed for mathematics teachers. To be able to compose and evaluate a curriculum in relation to teachers’ own teaching is essential according to Niss (ibid). Grossman (1990) included the curricular knowledge in PCK, as she felt the need to emphasize the importance of the curricular knowledge when differentiating between knowledge on subject matter and teaching experience. A similar need for including curricular knowledge appeared during the development of the MKT framework. The framework first did not include knowledge of content and curriculum as one of the domains, but became later one of the hypothesised domains (Ball et al., 2008). The critical feature encompassed that student teachers had to learn to work with the curricular documents, that is, to choose appropriate components from them, applicable for one particular lesson, rather than treating these documents as a whole.

Since 2011 new curricular documents have been implemented in Swedish schools. The new curriculum is much more detailed in several ways. The
content is described more in detail, with specified goals for specific topics. In addition, curricular documents have been made for three stages of the education, (year 3, 6 and 9), instead of the previous two stages (year 5 and 9). This means that the goals described, and the content dealt with, is specified for only three years of education at a time, whereas before, the documents were valid for 4 or 5 years. Smaller units in both time and content may help future teachers in planning the lesson, as the study showed that student teachers had to learn to focus on the subject as well as within the subject, since there seemed to be a desire to work with the whole curriculum during one lesson.

7.2 Detailed descriptions

For the critical feature lesson goals described above, student teachers had difficulties with being explicit and in detail. The second critical feature described in section 6.3.2 also concerned such explicitness. Detailed descriptions of elements resulted immediately in better connections. Is it then not possible to give a detailed description of the elements but still no relevant connections? The data showed no such instances. A detailed description requires insights in what to include and exclude in such a description. Once a student teacher knows how to describe an element in detail, the connections are clear, as they are part of the detail-ness of the description.

Kinach (2002) suggested a cognitive strategy to guide PCK development in pre-service courses. Adequacy of explanations was seen by Kinach as an indicator of PCK and student teachers’ conception of ‘good’ explanations were challenged to develop their PCK. Adequacy and ‘good’ explanations could be related to the critical feature detailed description. Such adequacy, or detailed description, are difficult to find in larger studies on teachers’ knowledge, as the tests used to capture teachers’ scores on MKT, use multiple-choice questions (Ball & Hill, 2008). The answers to such questions do not give any information
concerning a teacher’s capability to describe in detail. The tests can show if two
domains are related or not, but the question raised in this critical feature is, if
there might be an aspect in common for both domains which is actually the
reason for correlation. The critical feature suggests that the use of detailed
descriptions of the different elements resulted in the fact that connections were
established.

7.3 Mathematical knowledge

The importance of content knowledge for mathematics teachers has been
pointed out over the past years (Grønmo & Onstad, 2012; Sullivan, 2008; Tattoo
et al., 2012). The longitudinal study conducted by Begle in the sixties in the US
pointed out that there was no relation between the level of mathematics that
the teachers had studied and their pupils’ results (Askew, 2008). However, it was
claimed that a certain amount of mathematical knowledge was required. The
MKT model, as used in this thesis, also points out the difference between
(pure) content knowledge and knowledge of content and teaching. The
difference between these types of knowledge is emphasized and although no
clear boundaries can be given at all times, the focus is on describing different
types of knowledge needed to teach. Only a limited number of studies have
been investigating in what way these types of knowledge are interconnected.
The Michigan team working with MKT, has constructed large tests where all
items can be linked to one of the domains as described in their model (Ball &
Hill, 2008). In order to be able to look at the domains, it is important to first
look at them one at a time. However, as one of the critical features showed,
these domains may be interrelated and further research may have to move
towards a description of in what way the domains depend on each other. The
critical feature mathematics found in this study suggests that the domains of
MKT partly depend on the domain Common Content Knowledge. The study
revealed that in order to be able to discuss on a didactical level, the
mathematical knowledge (common content knowledge) has to be adequate. Common content knowledge *per se* will not lead to good knowledge of teaching, but has to be considered as a requirement. In the knowledge quartet (Huckstep et al., 2003), this requirement is described as well. Foundation, as their first unit, includes understanding and from that unit the other three flow. Also Gess-Newsome (1999) and Kinach (2002) stressed the interrelation between mathematical and pedagogical understanding. For Kinach it resulted in a focus on the use of relational questions to stimulate the transformation process to connect subject matter knowledge to PCK (section 2.2.3) whereas Gess-Newsome talks about a transformative model. The importance of mathematics, expressed either by seeing it as a foundation, or as interrelating with pedagogical understanding, has implications for the MKT model, as shown in Figure 21, where Common Content Knowledge is seen as the fundament and the other domains building on that.

![Figure 21 Common Content Knowledge as a fundament for the MKT domains](image)

Some of the domains might be interrelated but, as mentioned, further research is needed to find out more about such interrelation. Common content
knowledge somehow sets the boundaries for the growth of the other domains. If the amount of MKT-knowledge would be represented by the size of the egg shaped figure, it would depend on the size of the common content knowledge. Stylianides and Stylianides (2009) described MKT as an applied form of mathematics and their idea could be developed further: Mathematics is a pre-requisite for MKT (and thus an applied form of mathematics).

In applying lesson study, Perry and Lewis (2009) showed that teachers who themselves engaged in a mathematical content were able to see some misconceptions of pupils related to that mathematical content. Such engagement in the subject was also pointed out by Nilsson when she studied science pre-service teachers and their development of PCK during a pre-service course in teacher education (2008). In her conclusion she writes

... for student teachers to be able to build on, and challenge, students’ conceptions of science, it is necessary to obtain deeper understanding of the subject content and pedagogy within a context... (p. 1297).

Nilsson (ibid.) also mentions the importance of student teachers participating in activities for the development of their PCK. In her study the experience of doing science was important for the development of student teachers’ PCK. The importance of further research is pointed out as Nilsson states that the research conducted on student teachers’ learning seldom influences teacher education practice. Ryve et al. (Ryve, Nilsson, & Mason, 2011) studied how Mathematics for Teaching was established in teacher training courses and conclude that it was the use of variation and dimensions of variation that established Mathematics for Teaching. They suggest the possibility for variation to open up the ambiguity of the OoL.
7.4 Shift in perspectives

The lack of focus on subject and content is closely connected to the critical feature describing the shift in perspectives the student teachers have to make. Student teachers needed to shift from the perspective of a teacher to the perspective of a mathematics teacher. Such shifts are not unknown and have been described in different ways. In a study on the potential of lesson study for US teachers, Fernandez et al. (2003) described that teachers have to be able to develop lenses in order to benefit from the didactical discussions in a lesson study. The lenses described are the researcher lens, the curricular developer lens and the student lens. Even though the lenses are not the same as the shifts in perspectives described earlier, there are similarities. The shifts the student teachers need to make are of the same kind, they have to leave their habitual perspective and look at both mathematics, as well as mathematics teaching from a different point of view. Anticipating pupils’ thinking and understanding (student lens) is included in the shift towards mathematics teacher. In the UK, Wood (2000) characterized the development of student teachers during their teacher education as a shift from a more simple to a more complex conception of teaching. In Japan, Sato (plenary talk at the WALS conference 2011, Tokyo) also addressed these different roles or shifts when describing in what way novice teachers can engage in professional development through lesson study. They go through four stages in their discourse where they start with questions on how to teach mathematics. At the second stage the question is narrowed: How to teach mathematics to children. At the third stage the question is specified as follows: How to teach children doing mathematics in a collaborative context. Finally, at stage four, the teachers’ own learning is taken into account: How to learn to teach children doing mathematics in a collaborative context. These stages are similar to the shifts described above. Teaching is central at first, then a shift towards mathematics is made, and then more detailed questions concerning the pupils who have to learn mathematics can be raised. The critical feature shift in perspectives shows that the
central question for student teachers is not *Learning to learn how to teach mathematics*, which is an issue to consider in teacher education.

During most teacher education courses teacher educators try to address a vast amount of subject content knowledge during a course of limited time. Such constrains imply that student teachers need to learn how to learn, as teacher education courses cannot cover all. Developing tools to help student teachers to learn how they can learn will improve teacher education. Connecting *learning to teach* to a framework gives the student teachers the opportunity to put new topics and situations in a familiar context. Earlier the idea of transfer was mentioned as an important issue for the chosen OoL. Taking MKT as the OoL makes transfer between topics in mathematics possible. Both frameworks used in this study (MKT and variation theory) were only used for analysis and planning by the research team. The teacher educators decided not to use the MKT-framework explicitly during the seminars. As the strength of such a framework became visible to the teacher educators during the learning study, MKT as a framework has been introduced explicitly in proceeding courses. As for the other framework, variation theory, the teacher educators have continued using some aspects of it in their teaching. Interest in using learning study at preservice training has grown and at several places in Sweden (Gävle, Göteborg, Karlstad, Kristianstad, Skövde) and in Hong Kong it has been implemented. Up till now this has mostly been done through their final examination work (see for instance Bengtsson & Vennerlund, 2009; Dolk, 2011; Frisk, 2010; Mårtensson, 2010), connected to variation theory or a learning study. In addition some courses have tested learning study as a model within the course (Davies & Dun nell, 2008; Ko & Marsh, 2009).

Again, both frameworks give student teachers support in developing their learning. Whereas variation theory, often through a learning study, can be implemented at all subjects it can support both the development as a teacher as well as the development within the subject. MKT on the other hand focuses
only on mathematics as the subject and therefore the subject is more explicitly in focus. All frameworks have their limitations and for the course given an appropriate framework can be chosen to meet the goals of the course, and to support the shift in perspectives.

7.5 Future teacher education

The four critical features are directly linked to the OoL, and since the essence of the OoL has been linked to the MKT framework, it is interesting to look at the relation between the critical features and the MKT framework. As for the critical feature dealing with mathematical knowledge, the link has already been described (section 7.3) and mathematical knowledge was suggested to serve as a fundament, for the other domains to build on. The critical feature that dealt with the detailed descriptions also has its implications for how to understand or define the MKT framework. As the study suggested, the elements chosen were interrelated and the connections were of interest. In the framework however, the domains are described as distinctive domains and it is not evident that there might be an overlap between domains. However, this critical feature showed that detailed descriptions of the specific elements did lead to relevance between all described elements, which would indicate that the domains are not to be seen as disjoint. The critical feature on lesson goals (section 7.1) is closely related to the domain Knowledge of Content and Curriculum in the MKT framework. However, this relation is not of the same type as the ones described above, where the critical features give renewed insights in the framework. Lesson goals show in what way the student teachers meet the framework. If a student teacher’s attitude towards the curriculum is to adopt it instead of to adapt it for a specific lesson, the domain Knowledge of Content and Curriculum gets another meaning. The curricular documents are then seen as a domain on its own, which does not have to be incorporated with the other domains. The last critical feature, shift in perspectives, illustrates that the MKT framework can be
perceived and treated in different ways. When the framework, or parts of it, is the OoL; what becomes the lived OoL depends on if student teachers have the opportunity to make a shift in perspectives or not.

In this chapter some reflections were given on each of the critical features found during the learning study, by relating the critical features to results of other research. These reflections were made to show possible consequences for future teacher education courses in mathematics. Critical features can be used in other situations (Kullberg, 2010) and a more broad generalisation would even mean that the critical features found would be applicable for in-service courses in mathematics teacher education. Such generalisation would be desired as it is suggested that MKT should be developed throughout a teachers’ profession (Ngoh, 2009; Saad, 2009; Simon, 2006; P. W. Thompson & Silverman, 2008).
Chapter 8 – Reflections on design

Two goals were stated in chapter 1. The first goal related to the OoL of the study, where the research team wanted to know in what way it would be possible to address MKT during a teacher training course. In the previous chapters (chapter 6 and 7), this goal and its related questions have been answered and reflected upon. The second goal of the study concerned the design of the study. The alternative design that was developed has to be evaluated to see whether the developed design would work out during this study and hold for further studies. In this chapter reflections upon the design are made. The first sections relate the study to the existing design of a learning study and clarify if this study can be called a learning study. In the final section, a more broad reflection is made to see in what way the study contributes to the existing field of design research.

8.1 The study as a learning study

The teacher educators where inspired by Learning Study and wanted to conduct such a study. However, both the meta-character of the OoL, and the practical implications of having only one group of student teachers, made it necessary to develop an alternative design (chapter 4). A reflection on the design raises a question: Does the design of the conducted study retain the characteristics needed for it to qualify as a learning study? To be able to answer that question, the study as such had to be considered as the unit of analysis. This section serves to make clear which parts of the study have been focused on and in what way they have been analysed.

Summarizing from chapter 2, the following can be considered basic tenets of a learning study (Runesson, 2011):
1. A form of cooperation between researchers and a team of teachers (teacher educators),
2. The focus is on what pupils (student teachers) have to learn, not the lesson (seminar) itself
3. Teachers (teacher educators) own the research problem
4. The empirical study is carried out in a cyclic process
5. Systematic work:
   a. gather data of pupils (student teachers) understanding before and after
   b. videotape lessons (seminars)
   c. analysis of pupils’ (student teachers’) understanding and videotaped lesson (seminars)
6. The main aim is to capture what is critical for learning and use that to improve teaching in such a way that pupils (student teachers) learn what they are intended to learn.

Each of these characteristics has to be considered in the subsequent evaluation of the design of this study. Since the interest lies in comparing this study’s particular design to what can be considered a classical learning study-design, the design, process and results have to be related to the characteristics 1 to 6 from the list above. In what follows, the characteristics are divided into two sections: The participants and how they participated (section 8.1.1) and the procedure (section 8.1.2). This means that in the list above, characteristic 1-3 are dealt with in section 8.1.1 whereas section 8.1.2 deals with the characteristics 4 and 5. The last characteristic, 6, has been dealt with in the previous chapter, where the critical features found have been described as well as the results for the student teachers during the learning study.

Since the cyclic aspect of the design is central for a learning study, the design as such is considered, as well as its implementation. The qualitative analysis based
on variation theory, as well as the OoL for this learning study, will be looked at once again in order to examine their suitability. The three seminars will be looked at to see if the OoL stays the same when the mathematical content of the seminars is changed. Finally the role of the participating teacher educators has to be considered.

8.1.1 Reflections on participants’ roles

The initial idea for conducting a learning study came from the teacher educators. They came to hear about the concept and wanted to conduct such a study. As the mathematics is secondary in the course they were teaching, the possible OoL’s suggested by the teacher educators were all didactical in character. The teacher educators expressed their discontent with student teachers results and, together with the other members of the team, an OoL was found describing what they wanted to work with. The OoL originated from the teacher educators and was kept under their control during the learning study. At a later stage the researchers got involved and related the formulated OoL to existing frameworks. From the considered theoretical frameworks in chapter 2, Ball’s framework on MKT (Ball et al., 2008) was chosen to describe the OoL in a more theoretical setting. The framework was communicated to the teacher educators but was not used explicitly during their seminars.

Cooperation between researcher and teacher educators is central in a learning study. Some measures were taken in order to let practicalities facilitate the cooperation to be productive and constructive. In this study the teacher educators and the researchers met at scheduled times, convenient for the teacher educators who had to fit it into their existing schedule. One of the researchers took care of arranging the meetings, including the bookings of rooms as well as equipment and such. The teacher educators were compensated with 30 hours extra from their department for taking part in the research and
development work related to the course. The tests could have resulted in an extra workload for both the teacher educators and the student teachers. In order to prevent this, when planning the learning study, ways of incorporating the tests in the course were searched for and one of the previous assignments was replaced. In this way the student teachers would not have to devote any extra time to write the tests and as for the teacher educators they would analyse the tests instead of correcting the regular assignment. The analysis of the tests was mainly done by one of the teacher educators, together with one of the researchers; however, in the analysis of the seminars all teacher educators were involved.

One of the points addressed by Runesson (2011) is focus on the learning of the pupils (student teachers in this study). Since the initial thought of the teacher educators was to improve their teaching in order to obtain more satisfying results in student teachers’ learning, the focus was on the learning of the student teachers. Analysis of the tests was first done through codification of the elements and the connections. Thereafter the analysis continued by trying to find explanations for the results in the way the OoL was dealt with during the corresponding seminar. Through a detailed analysis of what was said, the enacted OoL was connected to the results, i.e. the lived OoL. In this analysis the learning of the student teachers was central. The research team often found different explanations for the discrepancy between the intended, enacted and lived OoL. This resulted in the four critical features found. These critical features were formulated in such a way that the learning of the student teachers was in focus. What the critical feature would imply for teaching was the next part to develop. The critical feature describing the importance of sufficient mathematical knowledge resulted in a change in the type of instructions for some exercises. Instructions would for instance incorporate time to work with the mathematics if necessary. But, also the type of exercises that were chosen had changed to the extent that the mathematics was on a non-problematic level for the student teachers in order to be able to focus on the didactical discussion.
In the above the role of, and relation between, the different participants have been described and reflected upon, the conclusion being that the characteristics 1-3 are fulfilled in this study. In the following section the characteristics concerning the procedure are considered.

8.1.2 Reflection on the procedure

The cyclic process of the learning study was maintained. In Figure 22, the different stages are shown and the different mathematical topics have been added to the design. Since the class is the same throughout the whole study, each student teacher wrote six tests during the learning study and one delayed post-test (not depicted in Figure 22).

It is obvious that this is a design that incorporates iterative improvement, i.e. the cyclic process (characteristic 2). Such cyclic process makes it possible for analysis and changes to build on previous results. In this learning study all the tests could be compared to each other, and a progression over time was possible, because they all dealt with the same group of student teachers. The first pre-test is actually the only true pre-test, all other tests can be seen as both pre-, post-, and delayed post-tests. For this study each test was valuable as they
allowed a comparison to be made between the levels of the student teachers at various points of the study, showing whether there was a progression in-between two learning study seminars. In addition, the tests provided information on the knowledge of student teachers concerning the OoL, right before the seminar.

The cyclic process incorporated systematic work both for gathering data and for analysis. The tests were collected through an Internet-based platform\textsuperscript{16}. The choice for the type of test has been explained in section 5.3.3. The seminars were videotaped and transcribed as described in section 5.3.4.

At this point the change of topic has to be problematized, and then two questions arise. First of all, change of topic includes a change of topic in the tests, which could lead to incommensurability of the tests. The question is whether the results of a test on a specific topic, for instance statistics, will give any generalizable information concerning the OoL. The analysis shows that such generalizability was possible and that it could give information on possible critical features, regardless of the mathematical topic.

The critical features found could be implemented in other topics and implementation resulted in better results. It could of course be the case that other critical features simply were not detected because of the narrow focus on the five elements of MKT. However, such focus is specific for a learning study and not a point of discussion here. The question then is if the OoL can be seen as independent from the mathematical topic, or if the OoL changes when changing the topic. In this study mathematics has been treated as one dimension of variation of MKT as explained in sections 4.1 and 6.3.3, therefore the OoL is independent of the choice of mathematical topics.

\textsuperscript{16} Its learning
Changing the topic leads to a second question: Do two different topics offer the same opportunities to show connections between the five elements of MKT? Even though the test results were generalizable, a reflection has to be made upon the fact that the different topics used in the tests may have led to unequal possibilities in showing the five components of MKT. The analysis showed a major difference in the development of models for explanation compared to the other elements, as mentioned in section 6.1.1. Table 14 shows how this element was referred to in the different tests. It shows that student teachers in the first test manage to refer directly to an element in almost 45% of the cases. This gradually changes to just over 56% in test 5, but drops to 20% in the sixth test. The type of activities chosen by many of the student teachers in the sixth test, had a focus on prepositions for position and a model of explanation appeared difficult to obtain for such an activity, as explained previously (section 6.1.1). Those kinds of activities were coded as code 0 – element not referred to, and therefore that code suddenly increased to 60%.

<table>
<thead>
<tr>
<th>Codes for Models for Explanation</th>
<th>Test 1 Number Sense</th>
<th>Test 2 Rational Numbers</th>
<th>Test 3 Statistics</th>
<th>Test 4 Algebra</th>
<th>Test 5 Equations</th>
<th>Test 6 Space Awareness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Code 0 – Element not referred to</td>
<td>29,8</td>
<td>34,0</td>
<td>51,1</td>
<td>27,3</td>
<td>17,1</td>
<td>60,0</td>
</tr>
<tr>
<td>Code 1 – Element indirectly referred to</td>
<td>25,5</td>
<td>21,3</td>
<td>24,4</td>
<td>38,6</td>
<td>26,8</td>
<td>20,0</td>
</tr>
<tr>
<td>Code 2 – Element directly referred to</td>
<td>44,7</td>
<td>44,7</td>
<td>24,4</td>
<td>34,1</td>
<td>56,1</td>
<td>20,0</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Figure 18
This difference in results, specifically the difference concerning the last test, may indicate that the tests do not offer an equal possibility for all five elements to be addressed. Test 3 also points out this difference and it was only with this specific component — models for explanation — that these differences occurred. The actual research question, however, focused on the connections between the elements referred to. In the presentation of the quantitative results, the possible connections were considered as the total for each individual student teacher, and therefore these results did not depend on such (un)equalities between tests.

Reflecting upon the design, a third aspect has to be taken into account. Merely taking a course can be seen as a type of intervention, and student teachers that take a course are therefore expected to obtain better results at the end of such a course than at the beginning of the course. Regular lectures and seminars can be seen as intervention, something that is valid for all learning studies. The design presented here tries to tackle this problem in two ways. First of all, a control group was tested in order to distinguish the results obtained after regular intervention (regular lectures and seminars) from results after an intervention by the learning study. In a previous course, when the teacher educators had not yet started with the learning study, the student teachers were tested at the end of the course by writing a lesson plan on the topic of Probability. The learning study group also wrote such a lesson plan at the end of their course. These data show that the learning study group obtained better results compared to the control group. (For a comparison of the starting level of the two groups see section 5.5.2, Table 5). Table 15 shows in what way the components were described in this test (lesson plan in Probability) for these two groups, expressed in the codes: Element not referred to, element indirectly referred to, and element directly referred to. The student teachers in the learning study group more often managed to directly refer to one of the five elements. In the control group 50% of the elements were directly referred to, compared to 66% in the learning study group.
Table 15 Occurrence of codes 0, 1, 2 in lesson plan on Probability for Control group and Learning study group in percentage of total per group18.

<table>
<thead>
<tr>
<th>Code 0 – Element not referred to</th>
<th>Control group</th>
<th>Learning study group</th>
</tr>
</thead>
<tbody>
<tr>
<td>38</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>Code 1 – Element indirectly referred to</td>
<td>13</td>
<td>10</td>
</tr>
<tr>
<td>Code 2 – Element directly referred to</td>
<td>50</td>
<td>66</td>
</tr>
</tbody>
</table>

In Table 16, the codification of the possible connections is given and the results clearly show that the learning study group obtained better results. The percentages for vague connections are almost the same, and the percentages for relevant connection are doubled from 24% in the control group to 48% in the learning study group. It was code 3 – connections irrelevant which initiated the teacher educators to conduct a learning study and the data in Table 16 show clearly that this type of connection diminished drastically; from 22% in the control group to only 1% (2 instances) in the learning study group.

Table 16 Occurrence of codes 3, 4, 5 in lesson plan on Probability for Control group and Learning study group in percentage of total possible connections per group

<table>
<thead>
<tr>
<th>Code 3 – Connection irrelevant</th>
<th>Control group</th>
<th>Learning study Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Code 4 – Connection vague</td>
<td>54</td>
<td>51</td>
</tr>
<tr>
<td>Code 5 – Connection relevant</td>
<td>24</td>
<td>48</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

By analysing the progression that the student teachers made between the different tests, the hypothesis that student teachers would be able to transfer knowledge concerning MKT obtained within a specific topic to another topic, was tested. If transfer would not take place, the results of all pre-tests would have been rather similar. However a change is observable in results as shown in chapter 4. This (positive) change could be explained by the natural progression

18 Due to rounding, the total of the columns does not add up to 100%
as described above, however the change was much larger. This provides a strong indication that the hypothesis could indeed be valid: Student teachers are able to transfer knowledge concerning MKT between two topics in mathematics.

8.1.3 Implications of new design

According to the characteristics stated at the beginning of this chapter the study conducted can be considered a learning study. The design of the learning study reported here makes it possible to conduct such a study when dealing with a meta-OoL and only one group of student teachers. In previous studies, some problems have been described when conducting a learning study. As mentioned before in chapter 2, the complexity of an OoL in the subject Swedish has been problematized. Mossberg Schüllerqvist and Olin-Scheller expressed that within the school subject Swedish an OoL is often so complex that the teachers themselves need to work a great deal on their own understanding of the OoL (2011). The development of the OoL in such settings was central and the main focus during the study. Their distinction between the Overarching OoL and Subordinate OoL indicate a need for development within the OoL. The design suggested in this thesis would support such development and would be more compliant to the need expressed. Moreover, the meta-character of the OoL described in this thesis would make a distinction between the overarching and subordinate OoL possible within a learning study. Also mentioned in chapter 3, the distinction between the OoL and a second, implicit OoL was observed in a learning study in the school subject history (Holmqvist et al., 2010). The authors concluded that an effective learning strategy was difficult to accomplish because of the twofold OoL. Pupils were focussing on both understanding the OoL as well as on trying to find out what was needed to pass the exam. Long-term outcomes showed a decrease in the learning of isolated facts whereas there was an increase in comprehension-based knowledge. Long-term learning is
supported by *generative learning*, which is described as learning more about the OoL after the learning situation in new settings. For the OoL in history, such settings outside school are not likely to occur spontaneously (Holmqvist et al., 2010). Therefore, the design suggested in this thesis would support the need for generative learning in order to work with the long-term learning outcomes. Besides the design, the use of meta-OoL could give insight in such generative learning.

Another type of problem with Learning Study concerns the practicalities. As Learning Study is gaining popularity amongst teachers, and more schools and their teachers want to conduct a learning study, or want to continue working with learning study, practical problems occur. Some of the schools do not have parallel classes as is required, when conducting a learning study. There have been many solutions to this problem: Cooperation between schools, choosing an OoL that is not bound to a specific age group, or splitting the class into two groups. With the design described in this thesis one could work with a meta-OoL within one class. Naturally, a team of teachers is still required, however, these teachers can teach at different levels.

Evaluation of the design has led to some suggestions for improvement of the tests. In order to diminish the amount of pre-, and post-tests one could consider the post-tests as pre-tests for the upcoming lesson. This would mean that there is only one true pre-test, which takes place before the study, one true post-test, at the end of the study, and one test in between every learning study lesson as shown in Figure 23. If one conducts a cycle with three lessons this would mean four tests in total instead of the six tests normally needed.
Another opportunity this design offers is the possibility to follow individual pupils throughout all tests. The individual pupil’s development over time can be analysed, which makes it possible to work with a meta-OoL. If one is aiming at generative learning, another adjustment could be to let pupils reflect on previous answers in their tests and to let them add comments or change answers. In such a way, the change in understanding would become clear and comparable between pupils as well as between test results of the individual pupils.

8.2 Design research in mathematics teacher education

The study presented is unique, for several reasons. One of the reasons have been dealt with in the previous chapters: A design for the study was developed to be able to address a meta-OoL and simultaneously, to conduct the study in one group. Another reason to call the study unique is due to the fact that it was conducted in teacher education, with teacher educators. The gap between educational research and practice has been problematized and design research was one of the solutions (Cobb et al., 2003). The practice aimed at in this gap is often the practice of teachers at primary or secondary school. The gap is seldom spoken of as a gap between teaching in teacher education and research on teacher education. This could be explained by the non-existence of such a gap, i.e. research on teacher education gives clear indications and is easy to implement for teacher educators in their practice. However, as others have
found, research on teacher education focuses on the implication or the outcome of teacher education or its organizational structure, but seldom on the teaching in teacher education (see for instance De Jong, 2009; Grevholm, 2010; Gronmo & Onstad, 2012; Nilsson, 2008; Tatto et al., 2012).

The study presented here gives an opportunity to open up for such studies in teacher education. The suggested design makes it possible to conduct studies within one single course in cooperation with other teacher educators. Teacher educators could in such settings make use of existing research findings and develop their own practice, both as professional development and as part of an outside research project.
Chapter 9 – Final remarks

The study described and reflected upon in this thesis has given some insight in mathematics teacher education practice. In this final chapter possible directions for further research are stated. Finally the title of this thesis is clarified in relation to the study described.

9.1 Further study

The study clearly stated four critical features to consider when teaching the chosen OoL. These critical features gave insight in some of the problems concerning mathematics teacher education. One of the critical features concerned the mathematical knowledge of the student teachers and it would be interesting to see if this critical feature is just as prominent for other groups in teacher education. Is the connection between mathematical knowledge and the other domains of MKT as suggested in section 7.3, just as evident in a group of student teachers studying to become secondary school teachers? Moreover, the suggested relation between mathematical knowledge and the other domains should be researched more specifically and therefore other groups might be interesting to look at.

Although research on mathematics teacher education has increased enormously (Adler, Ball, Krainer, Lin, & Novotna, 2005; Björkqvist, 2003; Grevholm, 2010; Jaworski, 2008), further research on mathematics teacher education needs to focus on “[r]esearch based design of the pre-service mathematics teacher education, its content and form; Development of research based competence development programs for teachers” (Grevholm & Anthony, 2010, p. 360). Since the content of a teacher education course in mathematics is in some way related to MKT or other similar frameworks (PCK for instance), such a focus
can be obtained by researching in what way MKT can be taught. For this study the focus has only been on five elements of MKT and other relations between the domains of MKT may be found with other foci. A description of such relations would enhance the quality of the teaching of MKT, and thus would be profitable for mathematics teacher education. Such relations can be obtained by conducting new learning studies, or through other design experiments. However it is important to focus on obtaining both mathematical and pedagogical knowledge for teaching MKT.

Finally it would be interesting to see if teaching MKT differs for the different levels of education. Do the four critical features hold in other situations, as suggested by Kullberg (2010)? Other situations can be defined as other courses in teacher education, as suggested above, but could also be defined as different forms of teacher educations, for instance in other countries. In what way are the critical features found applicable in such settings? Commonalities and differences might give insight in understanding how MKT is established and what consequences such establishment has for teaching.

9.2 Improving Teaching, Improving Learning, Improving as a Teacher…

Imagine you are teaching a lesson on number sense. You want pupils to choose appropriate strategies to solve problems like 28+6 or 55+9. What kind of approaches can you expect from your pupils? Are there any models or illustrations you would use? How could you support the transition from counting towards a more proficient strategy? Describe your lesson plan…

Several approaches can be expected.

\[
\begin{align*}
28 + 1 + 1 + 1 + 1 + 1 + 1 \\
28 + 2 + 4
\end{align*}
\]
Each of the approaches show in what way the pupil thinks and a teacher has to decide if and how materials or models of explanations can support that thinking. Ten base block material are available to support the tens transition. The concept of ‘ten friends’ would support the partitioning of the number 6 in 2 and 4, to nicely add from 28 to 30, to 34. Counting doubles helps some pupils to see the connection between 6 + 8 and 7 + 7. Simultaneously a teacher has to think of the consequences of the strategy for doubles, and has to follow up with that specific pupil in what way 55 + 9 will be calculated. If the ten base blocks are used, a teacher has to think of the impact this can have on the understanding of the number base system. A teacher also has to look at pupils who show advanced strategies and decide in what way these pupils can be stimulated within the same mathematical concept (addition) but with challenges at their level.

In the above example the complexity of teaching mathematics is illustrated. To learn to teach this mathematics is, if possible, even more complex, as expressed by the term “meta-OoL”. In the end, the aim is to improve pupils’ learning of mathematics. For that, teachers have to teach in a suitable way. MKT supports such teaching and should be incorporated in mathematics teacher education courses. To support the learning of MKT, the teaching in teacher education had to improve, which was the main aim for this study.

As stated in the first chapter, the quality of teaching is considered the most crucial school-related factor influencing pupils’ achievements (Darling-Hammond, 2000; Grossman, 2010; Nye et al., 2004; Sanders et al., 1997). To improve this quality of teaching, a learning study was conducted in teacher education and the results of the learning study have been described in this thesis. The aim of the learning study was to improve teacher educators’
teaching, leading to an improvement in student teachers' learning. These improvements led to a professional improvement for the teachers involved in the study, both for the teacher educators and for the student teachers.

Improving Teaching, Improving Learning, Improving as a Teacher.
References


Elliott, J., & Yu, C. (2008). *Learning studies as an educational change strategy in Hong Kong: An independent evaluation of the "variation for the improvement of teaching and learning" (VITAL) project.* Hong Kong, China: Hong Kong Institute of Education.


van Bommel, J., & Liljekvist, Y. (2008b). Testing the same group again and again: An alternative design for a learning study. Poster presented at the WALS 08, Hong Kong, China.


van den Berg, E. (2009). The PCK of laboratory teaching: Turning manipulation of equipment into manipulation fo ideas. In O. De Jong & L. Halim (Eds.), Teachers’ professional knowledge in science and mathematics
education: Views from Malaysia and abroad (pp. 85-110). Bangi, Malaysia: University Kebangsaan.


Hej!

Den här veckan handlar undervisningen om rationella tal. Ni kommer att få se att rationella tal redan används av de yngsta elever och att de spelar en väsentlig roll i hela grundskolans (och gymnasiums) matematik.


När du planera lektionen får du gärna tänka på följande:
- Koppling till styrdokument
- Förkunskaper/förförståelse
- Förklaringsmodeller
- Lämpliga övningar (läromedel/arbetshasd)
- Laborativt material

Det är inte meningen att ni uppfinner den perfekta lektionen där alla elever lär sig allt. Meningen är att ni planerar en helt vanlig, enkel och bra lektion. Syftet med lektionsplaneringen är att vi ska kunna se hur mycket av lärarnas mål med veckans undervisning som har kommit fram. Genom en noggrant analys av både lektionsplaneringarna och lärarnas undervisning kan vi då genomföra vår Learning Study, med mål att förbättra undervisningen i X.

Lektionsplaneringen lämnas in senast fredagen den 13 februari på its i mappen ’lektionsplaneringar’ – 13/2 Rationella Tal. Som sagt tidigare kodas alla planeringar om, så att ingen utomstående kan härleda vem som har skrivit de olika planeringarna. Planeringen ska ta ungefär 1 timme och ni väljer själva om ni vill skriva med papper och penna och scanna det eller skriva direkt på datorn i t.ex Word.

Lycka till!

Kontakta mig gärna om det är något du undrar över,

/Jorryt

Jorryt van Bommel  
054 700 2315  
jorryt.vanbommel@kau.se
Appendix 2 – Information to student teachers (1/2)

Forskning på kursen X


Målet med forskningen är att förbättra undervisningen genom en så kallad 'Learning Study'. Jag har bifogat information om vad en Learning Study är och om du vill läsa mer kan du titta på: http://www.ipd.gu.se/forskning/forskningsprojekt/learningstudy/.

För dig som läser X under våren 2009 betyder det att du kommer att planera sex lektioner (vecka 5, 7, 11, 14, 15, 17). Att planera en lektion kan ta olika lång tid, men vi räknar med ungefär en timme per lektion. Lektionerna är fiktiva och ska riktas mot elev i grundskolan i det området i matematik som ni behandlar just då i kursen (t ex taluppfattning, rationella tal). Syftet är att genom era lektionsplaneringar kunna se vad ni har lärt er och vad ni behöver ändra på när ni undervisar er – vilka kunskaper har kommit fram och vilka kunskaper har inte kommit fram? Resultatet ska ligga till grund för att utveckla och förbättra undervisningen på X. Det är alltså lärarna i kursen som genomför själva Learning Study. Genom videospelningar och genom att låta er skriva lektionsplaneringar som kontroll, hoppas vi kunna utveckla undervisningen i kursen.


Jag ser fram emot att få fortsätta med denna Learning Study kommande termin med er!

Vid frågor eller övrig information, vänligen kontakta mig.

Hälsningar,

Jorryt van Bommel
054 700 2315
jorrytvombmel@kau.se
Appendix 2 – Information to student teachers (2/2)

Learning Study

- Hur kan man på bästa sätt undervisa om något som är svårt?
- Vad är det i undervisningen som gör skillnad om elever (studenter) lär sig eller inte?
- Vad innebär det att kunna det som vi vill att eleverna (studenterna) skall lär sig?

Dessa är några frågor som lärare arbetar med i en Learning study.

I en Learning study är det elevernas lärande som står i fokus. De lärare som deltar i en Learning study arbetar systematiskt med att understödja och utveckla sin undervisning så att förbättra elevernas lärande. Ett antal Learning studies som har genomförts i Sverige och utomlands visar att ett sällskar arbete ger resultat. Elevernas lärande förbättras med i väsentlig grad. Speciellt är det de elever som har störst problem som utvecklas i en Learning study.

Det som eleverna skall lära och hur de förstår detta står i fokus. Att sätta fokus på elevernas lärande innebär i detta fall att det är en viss förmåga, ett visst kunnande eller en viss färdighet som uppmärksammas.

En undervisningssituation planeras gemensamt i ett arbetslag där alla deltagare undervisar i samma ämne, till exempel i matematik, men i olika elevergrupper. Vad eleverna kan om det aktuella området innan undervisningen undersöks. Den kunskap som lärarna får om detta utgör grunden för resonemang om hur undervisningen skall läggas upp. Det är alltså sådant som eleverna har svårighet med som bildar utgångspunkt i en Learning study.

En av lärarna genomför den planerade lektionen med en elevgrupp och lektionen videofilmas.

Efter lektionen tar man reda på om eleverna har lärt sig det man ville och man studerar den inspelade undervisningen för att komma underförd om möjligheterna att lära sig var de alltför bättre och vad man eventuellt behöver ändra på.

Därefter genomförs samma lektion men med de ändringar som gruppen har kommit överens om i en ny elevgrupp. Så fortsätter proceduren tills alla lärare i gruppen har genomfört lektionen.

En Learning study kan sammanfattas i följande fem punkter:

- Målet är att förbättra elevernas lärande
- Fokus på de förmågor, det kunnande som eleverna skall utveckla
- Lagarbetes
- Pedagogiskt utveckling direkt i klassrummet
- Den är teoretiskt grundad. Det innebär att ett specifikt teoretiskt perspektiv på lärande, vissa teoretiska begrepp osv. används som redskap för att skapa möjligheter för eleverna att utveckla de önskade förmågorna

Källa: http://www.ipd.gu.se/forskning/forskningsprojekt/learningstudy/
Hej!


Jag (fyll i ditt namn)……………………………………………………………………………………………………………………………………………….. klass ……………
☐ medger
☐ medger ej

att jag bli filmad på tisdag den 10 februari på Xs lektion

Vid frågor eller övrig information, vänligen kontakta mig.

Hälsningar,

Jorryt van Bommel
054 700 2315
jorryt.vanbommel@kau.se
Bilaga: Sammanfattning Forskningsäthiska principer

Regel 1
Forskaren skall informera uppgiftslämnare och undersökningsdeltagare om deras uppgift i projektet och vilka villkor som gäller för deras deltagande. De skall därvid upplysas om att deltagandet är frivilligt och om att de har rätt att avbryta sin medverkan. Informationen skall omfatta alla de inslag i den aktuella undersökningen som rimligen kan tänkas påverka deras villighet att delta.

Regel 2
Forskaren skall inhämta uppgiftslämnarens och undersökningsdeltagarens samtycke. I vissa fall bör samtycke dessutom inhämtas från föräldrar/vårdnadshavare (t.ex. om de undersökta är under 15 år och undersökningen är av etiskt känslig karaktär).

Regel 3
De som medverkar i en undersökning skall ha rätt att självständigt bestämma om, hur länge och på vilka villkor de skall delta. De skall kunna avbryta sin medverkan utan att detta medför negativa följder för dem.

Regel 4
I sin beslut om delta eller avbryta sin medverkan får inte undersökningsdeltagarna utsättas för otillbörlig påtryckning eller påverkan. Beroendeförsäkranden bör heller inte förelägga mellan forskaren och rättintäkta undersökningsdeltagare eller uppgiftslämnare.

Regel 5
All personal i forskningsprojekt som omfattar användning av etiskt känsliga uppgifter om enskilda, identifierbara personer bör underteckna en förbindelse om tystnadsplikt beträffande sådana uppgifter.

Regel 6
Alla uppgifter om identifierbara personer skall antecknas, lagras och avrapporteras på ett sådant sätt att enskilda människor ej kan identifieras av utomstående. I synnerhet gäller detta uppgifter som kan uppfattas vara etiskt känsliga. Denna innebär att det skall vara praktiskt omöjligt för utomstående att komma åt uppgifterna.

Regel 7
Uppgifter om enskilda, insamlade för forskningsändamål, får inte användas eller utlånas för kommersiellt bruk eller andra icke-vetenskapliga syften.

Regel 8
Personuppgifter insamlade för forskningsändamål får inte användas för beslut eller åtgärder som direkt påverkar den enskilde (vård, tvångsintagning, etc.) utom efter särskilt medgivande av den berörda.

Improving Teaching, Improving Learning, Improving as a Teacher

This thesis concerns teaching in mathematics teacher education and is based on the implementation of a learning study at teacher training. The overall purpose was to investigate in what way teacher training could facilitate and improve student teachers’ Mathematical Knowledge for Teaching (MKT). In the learning study design, MKT was conceptualized as an object of learning with a meta-character, which meant that it was applicable to and transferable between different content areas of mathematics. This made it possible to vary the mathematical content between lessons but to keep the object of learning constant. Four critical features of the object of learning were found, giving insight in some of the problems related to teacher education. Student teachers had to be able to formulate proper aims for a lesson and to give detailed descriptions of elements of MKT for coherence in their MKT to occur. A focus on student teachers’ role as mathematics teachers had to be established and finally, sufficient mathematical knowledge was found to be a prerequisite for their MKT to develop. The study shows that enactment of these critical features improved the teaching by the teacher educators, which in its turn improved the student teachers’ learning with regard to MKT. The study also indicates that the prescribed design is worth considering for future collaborative efforts of improving teaching where other objects of learning with a similar meta-character are involved.