Albrecht Wurtz

Conformal Field Theory and D-branes
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Albrecht Wurtz. *Conformal Field Theory and D-branes*

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The main topic of this doctoral thesis is D-branes in string theory, expressed in the language of conformal field theory. The purpose of string theory is to describe the elementary particles and the fundamental interactions of nature, including gravitation as a quantum theory. String theory has not yet reached the status to make falsifiable predictions, thus it is not certain that string theory has any direct relevance to physics. On the other hand, string theory related research has led to progress in mathematics.

We begin with a short introduction to conformal field theory and some of its applications to string theory. We also introduce vertex algebras and discuss their relevance to conformal field theory. Some classes of conformal field theories are introduced, and we discuss the relevant vertex algebras, as well as their interpretation in terms of string theory.

In string theory, a D-brane specifies where the endpoint of the string lives. Many aspects of string theory can be described in terms of a conformal field theory, which is a field theory that lives on a two-dimensional space. The conformal field theory counterpart of a D-brane is a boundary state, which in some cases has a natural interpretation as constraining the string end point. The main focus of this thesis is on the interpretation of boundary states in terms of D-branes in curved target spaces.
This thesis is based on the following articles, published in international research journals:

P. Bordalo and A. Wurtz, *D-branes in lens spaces*,
Physics Letters B 568 (2003) 270-280,

J. Fuchs and A. Wurtz, *On the geometry of coset branes*,
Nuclear Physics B 724 (2005) 503-528,

A. Wurtz, *D-branes in the diagonal SU(2) coset*,
Journal of high energy physics 0601 (2006) 154,

A. Wurtz, *Finite level geometry of fractional branes*,

The following text contains (most of) the results of these articles, together with additional background and motivations.
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Introduction

In particle physics, there is a so-called standard model which describes elementary particles, as well as interactions between them. The model has been tested with a very high precision in various collision experiments; the agreement with experiments is excellent. Furthermore, the Maxwell equations and other well known low-energy phenomena of matter are incorporated in the standard model. Thus, it is fair to say that we have a good understanding of the physical properties of what we believe to be the fundamental constituents of matter. On the other hand, the standard model does not include gravity. We do have a good theory for gravity, which is Einstein's theory of general relativity. Unfortunately, it is not clear how to fit these theories together: the standard model is a quantum theory, while Einstein's theory of general relativity is classical (a quantum theory, unlike a classical theory, describes particles as having both wave- and particle properties). Questions that can arise are for example:

- General relativity predicts gravitational radiation. Is this radiation quantized?
- General relativity predicts that a sufficiently dense cluster of matter will collapse to a black hole, with all mass concentrated at the singularity. Is this consistent with Heisenberg uncertainty?
- In general relativity, black holes have the property that particles can fall into them, but not exit. Is this consistent with the quantum mechanical postulate that information cannot be destroyed (time evolution is unitary)?
- Are black holes consistent with thermodynamics, where entropy increases with time?

Physicists have different opinions to whether these questions need to be answered, can be answered, or if they even have been answered. Whatever opinion one might have, these questions serve to illustrate why some physicists seek to formulate a theory that contains both gravity and the standard model. Attempts to quantize
the gravitational field in similar ways as other force fields are quantized, have failed. If one quantizes the metric tensor field which appears in general relativity, the gravitons will self-interact and the theory gives divergent results (in a way which cannot be cured with renormalization).

In most of pre-string theory physics, one treats elementary particles as point-like objects. Perhaps, this is just an idealization which proved to be convenient until one tried to quantize gravity. One attempt to quantize gravity starts from postulating that the fundamental constituents of matter are not point-like, but string-like. This is the basic idea behind string theory.

As a particle moves forward in time, it sweeps out a world-line in space-time. As the string moves forward in time, it sweeps out a world sheet. Many aspects of string theory can be formulated in terms of a certain field theory on the world sheet. One is led to impose certain restrictions (conformal invariance) on this field theory. Then, string theory implies gravity in the sense that the metric of the target space (our spacetime) is curved according to Einstein's equations. Thus, gravity is not put in by hand, rather, it is in some sense explained (or predicted) by the theory. In fact, string theory was first invented as a theory of strong nuclear interactions, and failed because its predictions did not fit with experiments. Among other things, the theory predicted massless spin-2 particles, which one later learned to interpret as gravitons. Further, string theory predicts the existence of matter particles, and other interactions (apart from gravity).

Other predictions of (naive) string theory are that spacetime has 26 dimensions, and that there are tachyons (particles with $m^2 < 0$). One way to get rid of the tachyons is a procedure which involves the introduction of super-symmetry. Accordingly, the number of space-time dimensions is reduced to 10. One can further reduce these to four by postulating for example that 6 dimensions are rolled up to tiny circles, too small to have been observed yet. (The introduction of extra dimensions has been considered by physicists before the invention of string theory. In the 1920s, Kaluza and Klein discovered that Einstein gravity in five dimensions can reproduce Einstein gravity and Maxwell electromagnetism in four dimensions.)

Another way to explain why we do not see 10 dimensions is to postulate that open string end points are constrained to certain subsets of the target space, called D-branes. For example, if electrons, quarks and photons are open strings, one could imagine a 4-dimensional D-brane on which we are stuck. In a string model, one can specify both a geometry of target space, and a collection of D-branes. These
It turns out that there are many possibilities to construct string models that in some sense contain the standard model (and quantum gravity). The problem is that there are too many ways to do this, and it is difficult to find common features in the vast landscape of such models. Thus, string theory is not yet on a level where it can be falsified by experiment. Therefore, we need to better understand the theoretical foundations of string theory, hoping to exclude some of the models. The ultimate dream would be to prove that there exists exactly one string model which contains the standard model, and in addition some new particles, that are within reach of experiments. To reach closer to this goal, we need to better understand fundamental aspects of string theory. For instance, we would like to see how different formulations of string theory are related to each other. Therefore, it would be interesting to understand better how the notion of boundary states is related to the concept of D-branes (the relation turns out to be rather indirect).

The relation between the matter content of string models on one hand, and target space geometry on the other hand, turns out to be quite complicated. Different target space geometries can correspond to the same CFT on the world-sheet, and the same spectrum of string excitations.

Therefore, the interpretation of a certain string model as living in a certain target space, is rather indirect. Similar problems appear with the D-brane content in various models. The original description of a D-brane is as a subset of target space, to which the string endpoints are constrained. In an algebraic formulation of conformal field theory (which is used to describe scattering amplitudes in string theory), the basic objects are the correlation functions, and we impose that they solve certain differential equations. In such a formulation, one can introduce D-branes in terms of boundary states, which can (indirectly) be interpreted as having a certain shape. In some models, this shape is concentrated on the same subsets of target space at which the boundary conditions fixes the end-points of the strings. In some other models, however, different descriptions of D-branes do not seem to agree. For example, in WZW models where the target space is a compact Lie group of finite volume, the string endpoints do not seem to be well localized in the boundary state description. This indicates that the description of string theory in terms of path integrals is not quite equivalent to the description of string theory in terms of conformal blocks in conformal field theory.
The first chapter of this thesis is about basic concepts of conformal field theory, and provides background information for the later chapters. Vertex algebras are introduced in section 1.3, and some of the statements about vertex algebras appear here for (probably) the first time. It is shown that if the state space is a Hilbert space, then all fields in the theory commute. This is not entirely unexpected, as the state spaces considered in the literature do not have a Hilbert space structure. This is clear from the fact that the (topological) dual space contains states with infinite norm, namely the so-called Ishibashi blocks, which we shall discuss in the third chapter.

The second chapter of this thesis is a exposition of some CFT models and provides some background for the papers [1], [3], [4] and [5], as well as a few of the results in those papers (mostly in sections 2.4 and 2.5). In particular, the discussion in section 2.4.1 appeared in a slightly shorter form in [3], parts of it also in [2]. Section 2.3.6 contains a proof that was omitted in [5]. Most of the models discussed are related to curved target spaces, which complicates the formulation of string theory. It is mainly in such models that we shall discuss D-branes and boundary states, which is done in the third chapter, which contains (most of) the results of papers [1], [3], [4] and [5], with additional motivations (at some places).
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1 Conformal Field Theory

This chapter aims at explaining what a Conformal Field Theory (CFT) is. The main interest here is towards applications in string theory, but it is worth to mention that there are other applications. CFT is used in statistical mechanics, where critical systems and phase transitions can be described in terms of CFT (mainly in terms of minimal models, which are introduced later).

1.1 CFT in String Theory

In the 20'th century, much progress in theoretical physics has been made by describing physics in terms of action functionals. For instance, Newtons equations of motion can be derived from the assumption that the system evolves along the path that minimizes a certain quantity, called the action, which is an integral of the Lagrangian. For Newtons equations of motion, one uses as a Lagrangian the difference between kinetic energy and potential. Also in field theory, the Lagrangian is an important quantity. One can derive Einsteins field equations in vacuum from the Einstein-Hilbert action, which minimizes the (integrated) Ricci curvature scalar. The usual starting point of string theory is to introduce the so-called Nambu-Goto action, which minimizes the area swept out by the string. Another action which is often used is the so-called Polyakov action. This action gives the same equations of motion as the Nambu-Goto action, but in other ways it is quite different, as we shall see.

A classical theory which is formulated in terms of a Lagrangian can be quantized in the Feynman path integral formulation. Roughly, this means that one lets the system evolve along all paths, where each path is weighted with a factor which depends on the Planck constant $h$ and the action. This dependence is such that in the classical limit $h \to 0$, only the path with minimal classical action contributes. Thus, in the limit $h \to 0$, the classical field equations (or equations of motion) are restored. In general, two theories with equivalent equations of motion (minima of
the action) may be rather different as quantum theories, if for example the actions away from the minimum are different.

1.1.1 Polyakov action

The Nambu-Goto Lagrange density is the (generalized) area of the world-sheet swept out by the string, expressed as the square root of the determinant of the induced metric.

\[ S_{NG} = \int d\sigma d\tau \sqrt{\left| \det \partial_\alpha X^\mu \partial_\beta X_\mu \right|}, \tag{1.1.1} \]

where it is assumed that target space admits global coordinates \( X_\mu(\sigma, \tau) \). The matrix \( \partial_\alpha X^\mu \partial_\beta X_\mu \) is the embedding metric on the world sheet. In euclidean spacetimes, the determinant of the embedding metric gives the area. The Nambu-Goto action is difficult to quantize, and is rarely used. Instead, it is customary to consider the Polyakov\(^1\) action

\[ S_P = \int \Sigma \sqrt{\det h} \ h^{ab} \partial_a X_\mu(\sigma, \tau) \partial_b X^\mu(\sigma, \tau) \, d\sigma \, d\tau. \tag{1.1.2} \]

Here, \( h_{ab} \) is a world-sheet tensor field whose classical equations of motion are

\[ h_{ab} = \partial_a X^\mu \partial_b X_\mu. \tag{1.1.3} \]

Substituting (1.1.3) into (1.1.2), one obtains (1.1.1), hence the Nambu-Goto action and the Polyakov action are classically equivalent. We shall stick to the Polyakov action for yet a while. Many properties of string theory can be postulated with the motivation that they hold for this action. A nice feature of this particular action is that symmetries of the Polyakov action leads string theory to (in a certain sense) predict gravity, cf. [22, 57, 72].

If we picture a string moving in time, it sweeps out a 2-dimensional world sheet. At a given time \( t_0 \), the world sheet becomes again a string. If we consider a target space which has no time direction, there is a priori no reason to expect to find a world sheet. Likewise, if target space is a product \( M \times E \) where \( M \) contains the time direction and \( E \) contains only space like dimensions, we would not expect to find a world sheet in \( E \). Fortunately, with the Polyakov action, this works in a different way than one might expect. If the fields \( X^\mu \) are embedding coordinates on a product space \( M \times E \), we can write

\[ S_P(M \times E) = S_P(M) + S_P(E). \tag{1.1.4} \]

\(^1\)This action was first written down by Brink, Di Vecchia and Howe [16]. It is named after Polyakov, who made important contributions to its interpretation [74], see also [75].
The action $S_P(E)$ makes sense because we can choose (complex) coordinates on the world sheet such that $h^{ab}$ is Euclidean. This splitting is not possible for the Nambu-Goto action, due to the square root in the integrand. Such a splitting allows us to consider string theory on $E$ independently of string theory on $M$.

String theory predicts that the space in which we live has more than the observed three space-like dimensions. To explain why we do not see more, it is necessary to compactify these extra dimensions, so that only three space dimensions are large enough to be (directly) observed. The bulk of this thesis is concerned mainly with applications of CFT to string theory with target spaces that are compact and euclidean, that is, the focus is on strings that propagate in the extra dimensions.

1.1.2 Genus expansion

In a Feynman path integral quantization, one considers the time-evolution of a system along all possible paths, each path weighted with a factor $e^{iS/\hbar}$. A nice illustration is the double-slit experiment, where the electron wave function propagates through both slits. One observes an interference between the contributions of the two paths which are most likely in the classical theory. In the classical limit $\hbar \to 0$, the path which minimizes the action $S$ gives the main contribution, and the classical equations of motion are restored. In a path integral formulation of a field theory, one can calculate scattering amplitudes as a superposition of elementary scattering processes with an increasing number of internal loops.

![Figure 1.1: Expansion of a scattering amplitude in a quantum field theory.](image)

A string scattering amplitude is defined in terms of a series of world sheets. If we consider closed, oriented string theory, we sum over (all) closed oriented world-sheets, where the action in each term is exponentiated. We shall impose conformal invariance, which allows us to deform an infinite cylinder into the plane with a point removed. Similarly, we can deform all (from $t = \pm \infty$ incoming) string states into
a compact region, and then we remove a point from the world sheet. At these re-
moved, or special points, we insert certain vertex operators that describe the in- and
out coming states of the scattering process.

To calculate a string scattering amplitude, one takes the sum over all diagrams.
This means for closed oriented string theory a sum over all genera (the number of
handles). At each genus, one performs an integration over all inequivalent world
sheet metrics, as well as an integration over all insertion locations \( z_i \). In the sequel,
we shall discuss what it is that we integrate.

### 1.1.3 World sheet coordinates

The world sheet is taken to be a manifold, which implies that it admits local \( \mathbb{R}^2 \)
coordinates. In terms of light-cone coordinates \( x_\pm = \sigma \pm \tau \), the equations of motion
for the embedding field are

\[
\partial_- \partial_+ X(x_+, x_-) = 0. \tag{1.1.5}
\]

We shall think of \( \sigma \) and \( \tau \) to be complex-valued coordinates. Thus we extend \( X \)
(locally) to a field on \( \mathbb{C}^2 \) and introduce,

\[
z = \sigma + i\tau
\]
\[\bar{z} = \sigma - i\tau, \tag{1.1.6}\]

which are independent complex variables. If \( \sigma, \tau \) where real, we would have \( z = \bar{z}^* \).
The equations of motion for \( X(z, \bar{z}) \) are

\[
\partial \bar{\partial} X(z, \bar{z}) = 0. \tag{1.1.7}
\]

If we assume that the fields \( X \) have continuous derivatives, we can exchange the
order of differentiation. Thus, \( \partial X(z, \bar{z}) \) is a holomorphic field and can be written
\( \partial X(z, \bar{z}) = \partial X(z) \). Likewise, \( \bar{\partial} X(z, \bar{z}) \) does not depend on \( z \) and can be written as
depending only on \( \bar{z} \). These fields, and not the fields \( X(\sigma, \tau) \) without derivatives,
are those that make sense in the quantum theory. For more general string models,
we require that an analogous splitting can be done, so that we may consider fields
depending on \( z \) and fields depending on \( \bar{z} \) separately.

### 1.1.4 Weyl invariance

Since the Polyakov density is a world-sheet scalar, the action is trivially invariant
under all coordinate transformations. (Since (1.1.3) is a classical equation of motion,
we may not use it in the quantum theory. We may interpret $h_{ab}$ as the world sheet metric, but not as the embedding metric.) We can choose world sheet coordinates (gauge fix) so that the world sheet metric is simply $\eta_{ab}$ or $\delta_{ab}$. The difference is irrelevant since we consider fields on $\mathbb{C}^2$. The gauge fixed Polyakov action is still invariant under Weyl transformations, which means that the theory is invariant under

$$h_{ab} \mapsto \Omega(\sigma, \tau) h_{ab}.$$  

(1.1.8)

This transformation corresponds to a deformation of the world sheet. Tensors such as the metric transform covariantly under coordinate transformations. If we consider those coordinate transformations that would induce a Weyl shift on the metric, but refrain from doing the shift, we get an equivalent description of Weyl invariance. The coordinate transformations we are interested in thus include rotations and translations, but also dilations (scalings). All coordinate transformations that preserve angles will be allowed. Such coordinate transformations are analytic functions of $z = \sigma + i\tau$ or analytic functions of $\bar{z} = \sigma - i\tau$, cf. [77].

### 1.2 Conformal in- and covariance

The Polyakov action can be interpreted as describing a field theory on the world sheet. This theory is conformal, which means that it is invariant under Weyl transformations (1.1.8). Accordingly, we postulate that string theory in general is described as a conformal field theory on the world sheet. There are aspects of string theory that have little to do with any field theory on the world sheet, but these matters will not be discussed here. Conformal invariance allows us to deform incoming cylinders (incoming closed string states in a string scattering diagram) to punctures on a compact world sheet. Further, conformal invariance implies that the field theory on the world sheet is invariant under local analytic and anti-analytic coordinate transformations. It turns out that it is convenient to relax this condition to begin with. In the end, one can put theories together in such a way that full conformal invariance is restored. Thus, we shall in the intermediate steps consider theories that are perhaps better called conformal co-variant, in order to avoid confusion with conformal in-variant theories. Anyhow, the subject we discuss is called conformal field theory, or CFT for short. We shall assume that the CFT lives on a two-dimensional space, as is the case for the applications we are interested in.
1.2.1 Virasoro algebra

Local analytic coordinate transformations of the fields are generated by

\[ l_n = -z^{n+1} \partial_z ; \quad e^{\varepsilon l_n} z = z - \varepsilon z^{n+1}, \]  \hspace{1cm} (1.2.1)

to first order in \( \varepsilon \), for all \( n \in \mathbb{Z} \). These transformations act on the fields \( \partial X(z) \). There are corresponding transformations \( \bar{l}_n \) that act on the variable \( \bar{z} \), which are considered separately.

The generators \( l_n \) span the Witt algebra, whose basis \( \{ l_n \}_{n=-\infty}^{\infty} \) obeys the commutation relations

\[ [l_n, l_m] = (n-m)l_{n+m}. \]  \hspace{1cm} (1.2.2)

These generators commute with their anti-analytic counterparts \( \bar{l}_n \). Naively, we would like the state space in a conformal field theory to build up a unitary highest weight representation of these two algebras. Thus, we consider them one at a time. Unfortunately, the Witt algebra possesses no (nontrivial) unitary irreducible highest weight representations. Thus, all unitary (reducible) representations are tensor products of the trivial representation. Unitarity is needed because when we quantize the theory, we want the symmetry transformations to preserve the (positive) norm on the state space. Further, we want the spectrum of \( l_0 \)-eigenvalues to be bounded from below (\( l_0 \) generates scalings, which on a cylinder are time translations, so this is our hamiltonian). If one adds to the algebra a central term \( C \) that commutes with all other generators, but appears on the right hand side of some commutators, one obtains the Virasoro algebra with commutators

\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12} n(n^2-1)\delta_{n,-m}, \]  \hspace{1cm} (1.2.3)

which does admit nontrivial unitary highest weight representations. The commutator of two coordinate transformations does not have a central term. Thus, as long as \( C \neq 0 \) the action of the Virasoro generators \( L_n \) on the fields can not be interpreted as generating conformal coordinate transformations. A realistic string theory is constructed out of CFT models so that the resulting central charge vanishes, see [22], lecture 7. It is then not a unitary theory; the states with negative norm are called ghosts. One can show that these ghosts don’t interact with the other fields, and thus we may ignore them here.

We introduce a second copy of the Virasoro algebra with basis \( \bar{L}_n \) and analogous Lie product as above. One must not confuse the generators \( \bar{L}_n \) with the hermitian
conjugate of the \textit{Vir} generators

\[ L_n^\dagger = L_{-n}. \tag{1.2.4} \]

Since \( C \) commutes with all other generators, the expectation value of \( C \), called central charge, will be a constant \( c \in \mathbb{R} \) in each irreducible representation. This is analogous to the angular momentum squared, which is a constant in each \( \mathfrak{sl}(2) \) representation with some fixed spin \( j \).

The space spanned by the generators \( \{ L_{\pm 1}, L_0 \} \) is closed under commutation. Thus, these three generators span a finite dimensional subalgebra \( \mathfrak{sl}(2, \mathbb{C}) \) of the Virasoro algebra. As these three generators are not affected by the central term, they can be interpreted in terms of conformal transformations even in the quantum theory. They generate what is called the \textbf{conformal group} \( SL(2, \mathbb{C}) \). The corresponding coordinate transformations are the rotations, scalings and translations.

\subsection*{1.2.2 Primary fields}

We consider tensor fields on \( \mathbb{C} \), which implies that all tensor indices take only one value. Recall that a covariant tensor of rank \( r \) transforms under an infinitesimal coordinate transformation \( z \mapsto z' \) as

\[ A'(z') = \left( \frac{\partial z'}{\partial z} \right)^r A(z). \tag{1.2.5} \]

More generally, field with conformal weight \( h \) (not necessarily integer) transforms under an analytic coordinate transformation \( z \mapsto z + \varepsilon z^{n+1} \) as

\[ \phi'(z + \varepsilon z^{n+1}) = \left[ \partial (z + \varepsilon z^{n+1}) \right]^h \phi(z + \varepsilon z^{n+1}) \]

\[ = \phi(z) + \varepsilon h(n+1) z^n \phi(z) + \varepsilon z^{n+1} \partial \phi(z), \tag{1.2.6} \]

with the notation \( \partial \equiv \partial_z \equiv \frac{\partial}{\partial z} \). In the quantum theory, we define a Virasoro (or conformal) transformation generated by \( L_n \) as

\[ [L_n, \phi(z)] = h(n+1) z^n \phi(z) + z^{n+1} \partial \phi(z). \tag{1.2.7} \]

The CFT on the world sheet should have a state space that contains a vacuum state. We would like this state to be conformal invariant, and thus it should be annihilated by the \( L_n \). This is, however, inconsistent with \( C \neq 0 \). Thus, we impose in the spirit of Gupta and Bleuler (who performed a similar analysis in QED), that the vacuum expectation values of all \( L_n \) vanish. Further, we impose that the vacuum is invariant under the conformal group. These requirements can be fulfilled if we impose

\[ L_n |0\rangle = 0 \quad \text{for} \quad -1 \leq n. \tag{1.2.8} \]
The other states in the irreducible representation are created by acting on the vacuum with those operators that do not annihilate it. Compare this with the harmonic oscillator where the Fock space of states is built by acting with creation operators $a^\dagger$ on the ground state. Acting on the vacuum with a field at the origin $\phi(0)$ gives new highest weight states

$$\lim_{z \to 0} \phi(z)|0\rangle = |h\rangle \text{ with } L_0|h\rangle = h|h\rangle,$$

that are annihilated by all $L_n$ with $n \geq 1$. Again one can act with lowering modes, as we did with the vacuum, to obtain more states in the state space. One requires existence of a positive definite scalar product $(|\rangle, |\rangle) \to \mathbb{C}$ on this space. The action $(|a\rangle, \cdot) : |b\rangle \mapsto \mathbb{C}$ might just as well be described by the multiplication with some dual vector $|a\rangle^\dagger := \langle a|$ by the prescription $\langle a|b\rangle := (|a\rangle, |b\rangle)$. Hence, if there is a scalar product, it can be defined by assigning dual vectors $\langle a| = \langle 0|a^\dagger$ to vectors $a|0\rangle$ or equivalently, by defining conjugate operators $a^\dagger$. So let us try to define a nice (positive definite) scalar product by assigning

$$L_n^\dagger = L_{-n} \text{ and } \phi^\dagger = \phi.$$

With $\langle 0|0\rangle = 1$, this fixes all scalar products. The condition $\phi^\dagger = \phi$ is to insure reality of expectation values. Now if this scalar product is positive definite and $L_{-2}|0\rangle \neq 0$, we have the following restriction on the central charge;

$$\frac{c}{2} = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2L_{-2}|0\rangle > 0.$$

This proves in particular that if $c = 0$, the generator $L_{-2}$ annihilates the vacuum. Similarly, one can prove that all $L_n$ annihilate the vacuum. Thus, the Witt algebra (which is the Virasoro algebra with $C = 0$) does not have nontrivial irreducible unitary highest weight representations.

### 1.2.3 Ward identities

From (1.2.7) it follows, by taking the Hermitian conjugate, that

$$\langle 0|L_n^\dagger = 0 \text{ for } -1 \leq n.$$

This implies for $i = -1, 0, 1$,

$$0 = \langle 0|L_i\phi(z_1)\cdots\phi(z_n)|0\rangle$$

$$= \sum_j \langle 0|\phi(z_1)\cdots[z^{i+1}\partial\phi(z_j) + h(i + 1)z^i\phi(z_j)]\cdots\phi(z_n)|0\rangle.$$
by commuting $L_i$ through to the right, where it again kills the vacuum. The equation (1.2.13) is our first example of a Ward identity. Such identities play a key role in CFT, as they in many cases allow correlation functions to be calculated without using path integrals - and without doing a perturbation expansion.

Consider in particular the case of the two-point function with fields $\phi_1, \phi_2$ that have conformal weights $h_1, h_2$ respectively. The three differential equations (1.2.13) with $i = -1, 0, 1$ require $h_1 = h_2 = h$ for the correlator to be non-vanishing, which then is proportional to $(z_1 - z_2)^{-2h}$. Performing the same analysis with the barred algebra we find that the two point function on the complex plane takes the form

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2) \rangle = \frac{K}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}. \quad (1.2.14)$$

If $K \neq 0$, its value can be changed by scaling the fields. Similarly, the three symmetries (1.2.13) also fix the form of the three point function.

For $i < -1$, the state $\langle 0 | L_i^\dagger | 0 \rangle$ is (in general) not zero. Then, the Ward identities read

$$\langle 0 | L_i \phi(z_1) \cdots \phi(z_n) | 0 \rangle = \sum_j \langle 0 | \phi(z_1) \cdots [z^{i+1} \partial \phi(z_j) + h(i + 1) z^i \phi(z_j)] \cdots \phi(z_n) | 0 \rangle. \quad (1.2.15)$$

In many CFT's of interest with $1 \leq c$, there is a symmetry algebra which is larger than the Virasoro algebra (a so-called chiral algebra). We shall require that the vacuum is annihilated by certain generators of the chiral algebra. When the action of the chiral generators on the primary fields is known, one can derive analogous chiral Ward identities. We postpone the details of this discussion to later parts of this thesis, as they are highly model dependent.

### 1.2.4 Null vectors

For $0 \leq c < 1$, there exist unitary highest weight representations of the Virasoro algebra only for a discrete set of central charges $c$ [34]. In particular, the unitary CFT with $c = 0$ contains only one state with non vanishing norm, as follows from (1.2.11) and similar formulae (cf. [79]). Given a central charge, there is also a restriction from unitarity on the allowed highest weights of the Virasoro generator $L_0$, hence we have a discrete set of allowed representations in this range of $c$. These are the so-called minimal models, which appear in condensed matter physics as describing systems at critical temperatures, where scale invariance occurs. These models have
the nice feature that there appears to be infinitely many states with vanishing norm. 
This leads to an infinity of differential equations of correlation functions, obtained 
in a similar way as (1.2.13). It turns out that indeed there exist correlators that 
solve all these constraints.

The Ising model is the minimal model with smallest non-vanishing central charge, 
\( c = \frac{1}{2} \). One can easily check that there is a state with vanishing norm

\[
|\text{null}_2\rangle := \left( L_{-2} - \frac{3}{2(2h + 1)} L_{-1}^2 \right) |h\rangle,
\]

(1.2.16)

for \( h \neq 0 \). This state is also a highest weight state; \( L_n|\text{null}_2\rangle = 0 \) for \( n \geq -1 \). In 
order that \( |h\rangle \) generates an irreducible module, and that we have a positive definite 
scalar product with \( \langle v|v\rangle = 0 \) iff \( v = 0 \), we take the quotient by setting \( |\text{null}_2\rangle = 0 \). It follows that

\[
0 = \langle h'|\phi_1 \cdots \phi_n|\text{null}_2\rangle,
\]

(1.2.17)

which gives us a differential equation similar to (1.2.13) by commuting the Virasoro 
generators through to the left where they annihilate the highest weight state \( \langle h'| \). 
Since \( v + 0 = v \) we regard the new vector space as a set of equivalence classes in the 
old space with \( v \sim v + \text{null}_2 \). With this and analogous knowledge, we can compute 
all correlators for the minimal models, see [24], chapters 7 and 8.

In some CFT’s with \( 1 \leq c \), when the chiral algebra is constructed from a finite di-

mensional simple Lie algebra (these are the WZW models which shall be introduced 
detail later), there are other null-vector equations, called Knizhnik-Zamolodchikov 
equations, cf. [39], chapter 3.

### 1.3 Vertex Algebras

We would like to multiply fields on the world sheet. A vertex algebra (VA) is a 
mathematical structure that is believed to encode the properties of such an oper-
ator product. We begin with some (heuristic) motivations for the vertex algebra 
axioms.

In quantum field theory, it is customary to require that fields (corresponding 
to observable quantities) at space-like separations commute, so that no interac-
tions that violate causality can occur. We would like to allow for an expansion

\[
a(z) = \sum_n a(n) z^{-n-1}
\]

of the fields. One would perhaps want the product of fields \( a(z) \) and \( b(w) \) to be analytic in \( z - w \). However, equation (1.2.13) then implies \( h = 0 \).
For more general fields, with \( h \neq 0 \), we allow singularities of the product \( a(z)b(w) \) at coinciding points \( z - w = 0 \). There it has at most a finite order pole, so that we can do a contour integration.

By causality, we would like the commutators \([a(z), b(w)]\) to vanish at space-like separations, when \( z - w \) is interpreted as a space-time coordinate in \( \mathbb{R}^2 \). But an analytic function with more than an isolated set of zeroes in \( \mathbb{C} \) vanishes everywhere. Thus, the expansion of the commutator vanishes everywhere where it is analytic, that is, everywhere except at \( z = w \). We can see that if we would require the commutator to be meromorphic and causal, all fields would in fact commute. Hence the commutator needs to have some kind of essential singularity, and we need it to have support at \( z = w \). (It might seem that we have a contradiction, the operator product expansion is meromorphic but the commutator (which is a difference between two meromorphic objects) is not. We shall resolve this ambiguity by introducing certain ordering prescriptions, see (1.3.11.).)

We introduce so-called formal variables. Our goal is to calculate contour integrals of correlation functions of the form \( \langle ...a(z)... \rangle \). Then, \( z \) will have the interpretation of being a complex number. Outside the correlation however, \( z \) will not have such an interpretation, and we think of it as a formal variable.

### 1.3.1 Formal power series

Suppose \( R(z, w) \) is a rational function of complex variables, with poles at \( z = w \). We want to interpret it as a power series in formal variables, which can be done as follows. We need to decide whether to make the power series expansion at \( |z| > |w| \) or at \( |w| > |z| \). Denote by \( i_{z,w}R(z, w) \), or \( i_zR(z, w) \), the formal power series obtained from expanding the function \( R(z, w) \) in the domain \( |z| > |w| \). For example, when \( j \geq 0 \),

\[
\frac{1}{(z - w)^{j+1}} = \sum_{m \geq 0} \binom{m}{j} z^{-m-1} w^{m-j}.
\]

(1.3.1)

The quantity inside the brackets is the binomial coefficient, which can be written as

\[
\binom{m}{j} = \frac{m(m-1)(m-2)\cdots(m-j+2)(m-j+1)}{j!}
\]

(1.3.2)

for positive \( j \) (when \( m = 0 \) and \( j \geq 1 \), (1.3.2) vanishes). When \( m = j = 0 \), the binomial coefficient is 1. The definition (1.3.2) of the binomial coefficient can be
used also for $m < 1$, and we have
\[ i_{w,z} \frac{1}{(z - w)^{j+1}} = - \sum_{m < 0} \binom{m}{j} z^{-m-1} w^{m-j}. \] (1.3.3)

One can show that
\[ (z - w) i_{z,w} \frac{1}{(z - w)^{j+1}} = i_{z,w} \frac{1}{(z - w)^{j+1}}, \] (1.3.4)
which is not unexpected, but still nontrivial, when treating $z, w$ as formal variables and using the above definitions. For $j \geq 0$, the difference between these expressions is given by
\[ i_{z,w} \frac{1}{(z - w)^{j+1}} - i_{w,z} \frac{1}{(z - w)^{j+1}} = \sum_{m \in \mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j}. \] (1.3.5)

For $j < 0$, the two expansions $i_{z,w} (z - w)^{-j-1}$ and $i_{w,z} (z - w)^{-j-1}$ are equal. The right hand side of (1.3.5) with $j = 0$ coincides with the formal delta distribution, which is
\[ \delta(z, w) = \frac{1}{w} \sum_{n \in \mathbb{Z}} \left( \frac{z}{w} \right)^n. \] (1.3.6)

One can show that $\delta(z, w) = \delta(w, z)$, cf. [63], Proposition 2.1. The formal delta distribution has the following property, which justifies its name: Let $f(z) = \sum f(j) z^j$ be a formal power series. Then,
\[ \text{Res}_z [f(z) \delta(z, w)] = \sum_{j, n \in \mathbb{Z}} f(j) j^{j+n} \frac{1}{w^{n+1}} = f(w), \] (1.3.7)
where $\text{Res}_z$ denotes the $z^{-1}$ coefficient. In the literature, the delta distribution $\delta(z, w)$ is also denoted $\frac{1}{w} \delta \left( \frac{z}{w} \right)$, or $\delta(z - w)$. Note that if we would substitute numbers for the formal variables, $\delta(0, -1) \neq \delta(2, 1)$, so one should be careful with interpreting the formal delta distribution as ‘depending on $(z - w)$’.

From (1.3.7), it follows that $(z - w) \delta(z, w)$ integrated together with $f(z)$ gives zero, and therefore
\[ (z - w) \delta(z, w) = 0, \] (1.3.8)
in the sense of distributions (which means that the equation holds under the integral sign, when integrated with nonsingular objects in $z - w$). The derivative of the formal power series $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$ is
\[ \partial_z a(z) = \sum_{n \in \mathbb{Z}} (-n - 1) a(n) z^{-n-2}. \] (1.3.9)
The residue is
\[ \text{Res}_z \{ z^n a(z) \} = a_n. \]  
(1.3.10)

One can check that the derivatives of the delta distribution are given by the right hand side of (1.3.5),
\begin{align*}
&i_{z,w} \frac{1}{(z-w)^{j+1}} - i_{w,z} \frac{1}{(z-w)^{j+1}} = \frac{1}{j!} \partial_w^j \delta(z, w) \\
&= \sum_{m \in \mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j}.
\end{align*}
(1.3.11)

The derivative of the delta distribution has the property
\[ \text{Res}_z \left[ a(z) \partial_z \delta(z, w) \right] = -\text{Res}_z \left[ (\partial_z a(z)) \delta(z, w) \right]. \]  
(1.3.12)

### 1.3.2 The fields

The operator product expansion (OPE) of \( a(z)b(w) \) should have a finite order pole in \( z - w \). For a heuristic motivation of the vertex algebra axiom (1.3.23), let us assume that we can write the most singular term of \( a(z)b(w) \) as \( i_{z,w}(z-w)^{-j-1} \). Then the most singular term of the opposite expansion \( a(w)b(z) \) has the form \( i_{w,z}(z-w)^{-j-1} \). Thus, the commutator \([a(z), b(w)]\) has the form of (1.3.5), and by (1.3.11) its worst singularity is a finite order derivative of a formal delta distribution. Such derivatives are annihilated by \((z-w)^{j+1} \); 
\[ (z-w)^{j+1} \partial^j \delta(z, w) = 0, \]
(1.3.13)
as can be seen from (1.3.11). In general, for each pair of fields we require that there exists a number \( N_{ab} > 0 \) such that 
\[ (z-w)^n [a(z), b(w)] = 0 \quad \text{for} \quad n \geq N_{ab}. \]  
(1.3.14)

With this condition, the fields \( a(z) \) and \( b(w) \) are said to be local with respect to each other. As is clear from previous discussions, locality does not imply \([a(z), b(w)]=0 \). As a counterexample consider \( x \delta(x) = 0 \), but we have \( \delta(x) \neq 0 \). One can prove that the only counterexamples are delta distributions and derivatives thereof (see [63], Corollary 2.2). Thus, we can expand
\[ [a(z), b(w)] = \sum_n c_n(w) \partial^n \delta(z, w) \]  
(1.3.15)
as a finite sum, with some coefficient fields \( c_n(w) \).
It is often assumed (but in the vertex algebra literature not always stated explicitly) that the set of fields $\mathcal{F}$ is complete in the sense that all products and derivatives of fields are again fields in $\mathcal{F}$. Sometimes (when one is given a set of fields and wishes to check if it satisfies all axioms), it is convenient to have a small set of fields, check the axioms and then enlarge it. Dong’s lemma (cf. [63], Lemma 3.2) insures that if we have a set of fields with the locality property, and add to it all derivatives and products, the enlarged set of fields is still local. It is not too difficult [6] to convince oneself of the fact that other properties (see Axioms I-V as given below) go through as well.

1.3.3 The state space

In the state space, we would like to have a ground state, or a vacuum. This is a state which does not contain any particles. Further, we would like the state space to be fully reducible. We consider each irreducible sector at a time, and postulate that there is a unique vacuum $|0\rangle \in V$. We would like this vacuum to be translational invariant, hence there is a translation operator $T$ with the property that

$$T|0\rangle = 0.$$

(1.3.16)

To motivate its name, this operator should also generate infinitesimal translations on the fields,

$$[T, a(z)] = \partial_z a(z).$$

(1.3.17)

We shall see later on that

$$e^{\varepsilon T} b(0)|0\rangle = b(z)|0\rangle,$$

(1.3.18)

in analogy with the momentum operator in quantum mechanics. The Lorentz group is generated by $T$ together with the set of rotations, which in the present situation are described by multiplication of complex numbers of unit modulus.

The space of fields $\mathcal{F}$ should be large enough to generate all states. This means

$$V = \text{Span} \left\{ a_{(n_1)}^{j_1} \cdots a_{(n_k)}^{j_k} |0\rangle \right\}.$$

(1.3.19)

Every vector is a finite linear combination of basis vectors, which in turn are finitely generated by the modes of the fields.
The fields \( a(z) \in \mathcal{F} \) are supposed to create the states in the state space. We require that there exists some \( Z \in \mathbb{C} \) such that \( a(Z)|0\rangle \) is non-singular. Due to translational invariance, we might as well let this \( Z \) be the origin, and require
\[
\lim_{z \to 0} a(z)|0\rangle \in V. \tag{1.3.20}
\]
With the convention that fields are expanded \( a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \), we require \( a(n)|0\rangle = 0 \) for \( n \geq 0 \).

### 1.3.4 Definition of a vertex algebra

We summarize the above properties in the following definition. A **vertex algebra** (VA) consists of the following data:

- \( V \) (state space),
- \( |0\rangle \in V \) (vacuum, or ground state),
- \( \mathcal{F} = \{ a^j(z) \} \) (collection of fields, with an OPE)
- \( T \) (translation operator on \( V \) and \( \mathcal{F} \)).

These data are subject to the following five **axioms**:

**Translation invariance of the vacuum:**
\[
\text{Axiom I)} \quad T|0\rangle = 0. \tag{1.3.21}
\]

**Translation covariance of the fields:**
\[
\text{Axiom II)} \quad [T, a^j(z)] = \partial_z a^j(z). \tag{1.3.22}
\]
Note that \( T \) acts on both \( V \) and \( \mathcal{F} \).

**Locality:** The set \( \mathcal{F} \) is equipped with a bracket, which has the following property. For all pairs of fields \( a^i(z), a^j(z) \in \mathcal{F} \), there exists an \( N_{ij} \in \mathbb{Z} \) such that
\[
\text{Axiom III)} \quad (z - w)^{N_{ij}}[a^i(z), a^j(w)] = 0. \tag{1.3.23}
\]
This means that commutators between fields at different points vanish. At coincident points, it turns out that they are finite order derivatives of delta functions.
Completeness:

Axiom IV \( ) \quad V = \text{Span} \left\{ a_{(n_1)}^{j_1} \cdots a_{(n_k)}^{j_k}|0\rangle \right\}, \quad (1.3.24)\)
every vector is a finite linear combination of basis vectors, which in turn are finitely generated by the modes of the fields.

Existence of states:

Axiom V \( ) \quad a_{(n)}|0\rangle = 0 \quad \text{for} \quad n \geq 0. \quad (1.3.25)\)

This ends the definition of a vertex algebra.

In the literature, one can find different definitions of VA’s. In [23] and [6], a vertex algebra is defined almost as here, with the only (essential\(^2\)) difference that Axiom V is replaced with the following axiom, called the **quantum field property**: For any \( v \in V \), and any \( a(z) \in \mathcal{F} \), \( a(z)v \) is a \( V \)-valued Laurent series in \( z \). In [23] and [6] it is shown that the quantum field property (together with the other axioms) implies (1.3.25), our Axiom V. The proof of the converse implication is presented below. It is similar to the proof of Theorem 1 in [6], and Dong’s lemma (cf. [63], Lemma 3.2). Along the way, we shall reveal other interesting (well known) properties of the OPE that are implied by the VA axioms.

### 1.3.5 The quantum field property

The commutator of fields at \( z \) and \( w \) vanishes almost everywhere, in fact, it has support at \( z = w \). Therefore, it must be a linear combination of delta functions and derivatives thereof, cf. [63], Corollary 2.2. We have

\[
(z - w)\partial^{j+1}_w \delta(z, w) = \partial^j_w \delta(z, w)
\]

for \( j \geq 0 \), and else the right hand side vanishes, cf. [63] Proposition 2.1. Hence, locality implies that there may only be finitely many derivatives of the delta function appearing in the expansion of the commutator

\[
[a(z), b(w)] = a(z)b(w) - b(w)a(z) = \sum_{j=0}^{N_{ab}} c^j(w) \partial^j_w \delta(z, w) \frac{1}{j!}.
\]

We write \( c(0) \) whenever it makes sense to replace the formal variable in \( c(z) \) with a complex number \( z \) and taking the limit \( z \to 0 \). We shall now see what state these

\(^2\)It is not difficult to generalize all this to the super-symmetric case: replace all brackets with super-brackets and add some signs when needed. We refrain from doing this here, as we shall not discuss super-symmetry in this thesis.
fields create when acting on the vacuum at \( w = 0 \). First multiply by \((z - w)^j\) to single out the term

\[
c^j(w) = \text{Res}_z \left\{ (z - w)^j [a(z)b(w) - b(w)a(z)] \right\}
\]

\[
= \sum_{k=0}^j \binom{j}{k} \sum_{m \in \mathbb{Z}} (a_{(k)} b_{(m)} w^{j-k-m-1} - b_{(m)} w^{j-k-m-1} a_{(k)}) .
\]

Now, since \( k \geq 0 \), (1.3.25) guarantees that the second term on the RHS annihilates the vacuum. We proceed to calculate the state

\[
c^j(0)|0\rangle = \sum_{k=0}^j \binom{j}{k} \sum_{m < 0} a_{(k)} b_{(m)} w^{j-k-m-1}|0\rangle \bigg|_{w=0}.
\]

The term in this sum with the lowest power of \( w \) is the one with \( m = -1 \) and \( j = k \), which is proportional to \( w^0 \). All other terms are higher power in \( w \) and vanish in the limit. This gives

\[
c^j(0)|0\rangle = a_{(j)} b_{(-1)}|0\rangle = a_{(j)} b(0)|0\rangle.
\]

Translational covariance of the fields implies

\[
[T, a_{(N)}] = -Na_{(N-1)}.
\]

Translational invariance of the vacuum implies

\[
[T, a_{(N)}]|0\rangle = Ta_{(N)}|0\rangle,
\]

A quick calculation reveals \([T^n, b_{(-1)}]|0\rangle = n! b_{(-1-n)}|0\rangle\). This shows that

\[
e^{zT}b(0)|0\rangle = b(z)|0\rangle,
\]

as announced above when we motivated the axioms for \( T \). According to (1.3.33), we have

\[
c^j(z)|0\rangle = e^{zT}c^j(0)|0\rangle = e^{zT}[a_{(j)} b_{(-1)}]|0\rangle = [a_{(j)} b_{(-1)}](z)|0\rangle.
\]

This motivates the notation for the OPE coefficients

\[
c^j(z) \equiv [a_{(j)} b](z)
\]

which we will adopt from now on. One also uses the notation \( a(z)_{(j)} b(z) \) for the same field. From these considerations it is clear that \( a_{(n)} \) annihilates the state.

---

3As \( T \) does not commute with \( a_{(j)} \), we can in general (for \( z \neq 0 \)) not use this reasoning to compare \([a_{(j)} b](z)\) to \( a_{(j)} \cdot b(z)\).
\( b := b(0)|0\rangle \) for \( n \gg 0 \).

Now, we shall proceed with the proof that the VA axioms as given above imply the quantum field property. We shall show that for sufficiently large \( n \), \( a_{(n)} \) annihilates any vector of the form

\[
b = a^{i_1}_{(n_1)} \cdots a^{i_k}_{(n_k)}|0\rangle.
\]

(1.3.36)

To do this, we construct a field

\[
b(z) := Y(b, z) \in \mathcal{F},
\]

(1.3.37)
such that \( b(0)|0\rangle = b \). From \( Y(b, w) \in \mathcal{F} \) it then follows that \( b(w) \) is local with \( a(z) \), and hence \( a_{(n)} \) annihilates \( b \).

**State-field correspondence** For the construction of \( b(z) \), consider first

\[
a^{1}_{(n_1)}a^{2}_{(n_2)}|0\rangle \in V.
\]

(1.3.38)

If \( n_2 \geq 0 \), this vector vanishes. If \( n_2 < -1 \) we may replace the field \( a^2(z) \) with the derivative \( \partial_z^{-1-n_2}a^2(z) \); then \( a^2_{(n_2)} \) is an integer multiple of \( [\partial_z^{-1-n_2}a^2]_{(-1)} \), and for notational simplicity we shall thus assume that \( n_2 = -1 \). If \( n_1 \geq 0 \), the vector (1.3.38) is created by an OPE coefficient \( c^{n_1}(w) \) of fields in \( \mathcal{F} \), acting on the vacuum. Hence, \( c^{n_1}(w) \in \mathcal{F} \), which is the field in \( \mathcal{F} \) associated to this vector,

\[
Y(a^{1}_{(n_1 \geq 0)}a^{2}_{(-1)}|0\rangle, w) := c^{n_1}(w).
\]

(1.3.39)

If \( n_1 = -1 \), we introduce the **normal ordered product**

\[
: a(z)b(z) : = a(z)_- b(z) + b(z)a(z)_+,
\]

(1.3.40)

where \( a(z)_+ \) consists of the terms with nonnegative powers of \( z \) and thus annihilates the vacuum. The corresponding state coming from the normal ordered product acting on the vacuum is \( a_{(-1)}b \). Thus we have

\[
Y(a^{1}_{(-1)}a^{2}_{(-1)}|0\rangle, w) = : a^{1}(w)a^{2}(w) :.
\]

(1.3.41)

For \( n_{(1)} < -1 \), we note

\[
a_{(-n-1)}b = \frac{1}{n!} : [\partial_z^n a(z)]b(z) : |0\rangle_{z=0}.
\]

(1.3.42)

The normal ordered products are assumed to be fields in \( \mathcal{F} \), and create the vectors \( a^{1}_{(n_1)}a^{2}_{(-1)}|0\rangle \) with \( n_1 < 0 \). This shows that for \( b \) as given in (1.3.36) with \( k = 2 \),
there is indeed a field $Y(b, z)$ in $\mathcal{F}$ which creates $b = Y(b, 0)|0\rangle$. For general $b$, we proceed by induction. Consider

$$a_{(n_3)}^2 a_{(n_2)}^2 a_{(n_1)}^1 |0\rangle = a_{(n_3)}^2 Y\left(a_{(n_2)}^2 a_{(n_1)}^1 |0\rangle, 0\right)_{(-1)} |0\rangle. \quad (1.3.43)$$

The argument that this vector has a corresponding field is identical to the argument presented above for $k = 2$. By induction on the number $k$ of occurring factors, it follows that indeed there exists such a $b(w) \in \mathcal{F}$, with the announced properties. Thus, the definition of a VA as given above is equivalent to the VA axioms given in [23].

**Field-state bijection** We saw that for every vector $v \in V$, there is a field $v(z) \in \mathcal{F}$ such that $v(0)|0\rangle = v$. We shall now show that the field $v(z)$ corresponding to $v \in V$ is unique, thus establishing a field-state bijection. The correspondence between fields and states is linear. Thus it remains to prove $b(z) = 0$, whenever $b = b(z)|0\rangle_{z=0} = 0$. Then, from (1.3.33), it follows that

$$e^{zT} b(0)|0\rangle = b(z)|0\rangle = 0. \quad (1.3.44)$$

It remains to show that $b(z)$ annihilates any vector $a$. By locality,

$$(z - w)^{N_{ab}} b(z)a(w)|0\rangle = 0. \quad (1.3.45)$$

Put $w = 0$, then $z^{N_{ab}} b(z)a = 0$. Therefore, the field $b(z)$ is zero.

This field-state correspondence is often taken as part of an equivalent set of VA axioms. If one chooses to relax the condition that $\mathcal{F}$ is complete, then the field state correspondence states that for every VA, $\mathcal{F}$ can be uniquely extended to $\bar{\mathcal{F}}$ such that there is a bijection between fields in $\bar{\mathcal{F}}$ and vectors in $V$. If equivalence between VA’s is defined in terms of these enlarged sets of fields, then every VA as defined in [23] is equivalent to a VA as defined in [33].

### 1.3.6 Hilbert space structure

A Hilbert space $\mathcal{H}$ is isomorphic to its topological dual [14], i.e. the set of all bounded linear functionals $\mathcal{H} \to \mathbb{C}$. Alternatively, a vector space with a scalar product is a Hilbert space iff all Cauchy sequences (with the scalar product norm) converge to a vector in $\mathcal{H}$. In particular, all finite dimensional vector spaces are Hilbert spaces. The set of square integrable functions (wave functions in quantum mechanics) is a Hilbert space. It is quite commonly taken as an axiom that the state space of a QFT is a Hilbert space (see Wightman axioms). However, this is
an axiom for mathematical convenience, and is not directly motivated with physics: no experiment can determine whether an infinite sequence of states converges to something which is still a state. We shall prove that a VA in which not all fields commute, has a state space which cannot be given a Hilbert space structure. (It can be seen already from results in [63] that a VA with a finite-dimensional state space is necessarily commutative.) Even the simplest model which we shall consider, $u(1)$, is described by a VA in which the fields do not commute. In fact, we are only interested in VA’s where the Virasoro algebra is present; thus fields do not commute. Hence, we cannot expect that the state space of a string model is a Hilbert space. Indeed, we shall consider objects in the (topological) dual of the state space (Ishibashi- and boundary states), which are not dual to vectors of our state space. If they were, they would have finite a norm induced by the scalar product.

The proof presented below is similar to the proof that every continuous function on a compact interval is uniformly continuous. The strategy is as follows. For the Hilbert space completion $\tilde{V}$ of $V$, we would need to include coherent states, that is infinite combinations of vectors (1.3.24) of different grades. Suppose $a(z) \in \mathcal{F}$ is a field that does not commute with all other fields in $\mathcal{F}$. We shall prove that in $\tilde{V}$, the field $a(z)$ does not satisfy locality, by constructing a coherent state $v \in \tilde{V}$ such that $a_{(n)}v \neq 0$ for all $n > 0$. This contradicts the statement that $V$ can be given a Hilbert space structure if not all fields in $\mathcal{F}$ commute. We shall first prove a lemma, similar to Proposition 2.1.6 of [70]. Let us fix a field $a(z)$ which does not commute with all fields $b(w) \in \mathcal{F}$.

**Lemma:** There is no positive integer $n$ such that for all $b(w) \in \mathcal{F}$,

$$ (z - w)^n [a(z), b(w)] = 0. \quad (1.3.46) $$

Assume $N > 0$ is the smallest integer with this property. We shall prove that for any $b(w) \in \mathcal{F}$, also $(z - w)^{N-1} [a(z), b(w)] = 0$, which is a contradiction. This is done as follows. We may consider the $w$-derivative of (1.3.46) with $n = N$,

$$ -N(z - w)^{N-1} [a(z), b(w)] + (z - w)^N [a(z), \partial_w b(w)] = 0. \quad (1.3.47) $$

The second term vanishes, due to the assumption that $N$ satisfies (1.3.46). Therefore the first term vanishes, in contradiction to our assumption on $N$. Thus, an integer satisfying (1.3.46) would have to vanish, and thus $a(z)$ would have to commute with all fields $b(w) \in \mathcal{F}$. This completes the proof of the lemma.

The lemma shows that we may pick a series $\{b^n(z) \in \mathcal{F}\}$ of fields with increasing $n$ such that $n$ is the smallest integer with the locality property (1.3.46). By field-
state correspondence, the vectors \( b^n = b^n(0)|0\rangle \) are linearly independent (which in particular shows that the space of states \( V \) has infinite dimension). Thus if we have a scalar product, we may perform Gram-Schmidt orthonormalization, and we may without loss of generality assume that the \( b^n \) are orthonormal. Out of this sequence, we define the coherent state

\[
b := \sum_{n>0} \frac{1}{\sqrt{2^n}} b^n.
\]

(1.3.48)

The norm of this vector is not greater than 1 (some of the terms may be missing, but there are infinitely many terms in the sum). The vector \( b \) is such that for any \( m > 0, a_{(m)} b \neq 0 \), violating the quantum field property. Further, to the vector \( b \) there corresponds a field \( b(w) \) which is not local wrt \( a(z) \). This proves that if the space of states \( V \) of a VA is a Hilbert space, then all fields in \( \mathcal{F} \) commute.

### 1.3.7 Vertex algebra modules

In physics it is common to assume a Hilbert space structure on the state space, at the same time as fields are supposed not to commute. In all examples we shall consider here (which are related to string theory), the fields do not commute. The arguments in the previous section thus imply that the state space is not a Hilbert space. What is called the space of states in a CFT context is not quite the same as the space of states that underlies the VA. In CFT, the primary fields are related to modules of vertex algebras, cf. [68].

A module \( M \) over a VA \( V \) is a vector space with a translation operator \( T_M \), and one associates to each \( v \in V \) a representation field \( Y_M(v, z) \) which acts on \( M \). On the fields \( Y_M(v, z) \) one imposes the following conditions [85],

\[
Y_M(|0\rangle, z) = \text{id}_M
\]

(1.3.49)

\[
[T_M, Y_M(v, z)] = \partial_z Y_M(v, z)
\]

(1.3.50)

\[
Y_M(v_1, z)Y_M(v_2, w) = Y_M(Y(v_1, z-w)v_2, w).
\]

(1.3.51)

The motivation for the first two of these axioms is quite obviously by analogy with VA's, where we have corresponding properties. The last property can also be motivated by such an analogy; it can be shown [33] that

\[
Y(v_1, z)Y(v_2, w) = Y(Y(v_1, z-w)v_2, w).
\]

(1.3.52)

The axioms of a VA module imply locality of the fields \( Y_M(v, z) \), see [33] Proposition 5.1.2. This implies that the operator product of fields \( Y_M(v, z) \) cannot have branch
cuts (fractional order singularities) [63] Corollary 2.2. However, the fields $Y_M$ and $Y_{M'}$ with $M \neq M'$ are not required to be local with each other.

In a CFT based on the vertex algebra $V$, a primary field is an equivalence class of isomorphic irreducible $V$-representations, cf. [85]. Every VA has at least one module, namely itself. This module always appears as part of the field content in the chiral CFT; it is the identity under the fusion product (which will be discussed section 2.3.1).

### 1.3.8 Borcherds identity

To round up our discussion on VA’s, we shall briefly derive some useful formulae (which are well known in the literature, and which shall be applied later). We begin with proving the Borcherds identity

\[ a(z)b(w)i_{z,w}(z-w)^n - b(w)a(z)i_{w,z}(z-w)^n = \sum_{j \geq 0} (a(n+j)b)(w) \partial^j_w \delta(z, w) \frac{1}{j!}. \]

(1.3.53)

In case $n \geq 0$, we have $i_{z,w}(z-w)^n = i_{w,z}(z-w)^n$ which shows that the left hand side (LHS) of (1.3.53) is proportional to the commutator of the fields. If instead $n < 0$, we can use (1.3.5) to rewrite

\[ LHS = [a(z), b(w)]i_{z,w}(z-w)^n + b(w)a(z)[i_{z,w}(z-w)^n - i_{w,z}(z-w)^n]. \]

(1.3.54)

In both cases, the LHS is local, i.e. $(z-w)^N LHS = 0$ for $N \gg 0$. Thus, there exists a decomposition $LHS = \sum c^j(w) \partial^j_w \delta(z, w)$ with the summation range $0 \leq j \leq N_{ab}$. From [63], Proposition (2.1),

\[ (z-w) \partial^{j+1}_w \delta(z, w) = (j+1)\partial^j_w \delta(z, w), \]

(1.3.55)

follows (together with (1.3.12) and (1.3.13)) that the coefficients are given by

\[ c^j(w) = \frac{1}{j!} \text{Res}_z[(z-w)^j LHS]. \]

(1.3.56)

For $j + n \geq 0$, we use $i_{z,w}(z-w)^{j+n} = i_{w,z}(z-w)^{j+n}$ to see that the second term in (1.3.54) vanishes, and the result (1.3.53) follows from the definition of the OPE coefficients (1.3.29),

\[ \text{Res}_z[(z-w)^j LHS] = \text{Res}_z ((z-w)^{n+j}[a(z), b(w)]) = [a(n+j)b](w). \]

(1.3.57)
For \( j + n < 0 \) we make use of Cauchy’s formulas

\[
\operatorname{Res}_z a(z) \frac{n!}{(z-w)^{n+1}} = \partial_w^n a(w)^
\]

(1.3.58)

\[
\operatorname{Res}_z a(z) \frac{n!}{(z-w)^{n+1}} = -\partial_w^n a(w)_,
\]

(1.3.59)

to write

\[
[a_{n+j}]b(w) = \frac{1}{(-1-n-j)!} [\partial^{-1-n-j} a(w)] b(w) :
\]

(1.3.60)

\[
= \frac{1}{(-1-n-j)!} [\partial^{-1-n-j} a(w)]_+ b(w) + \frac{1}{(-1-n-j)!} b(w)[\partial^{-1-n-j} a(w)]_- 
\]

\[
= \operatorname{Res}_z (a(z) i_{z,w} (z-w)^{n+j}) b(w) - b(w) \operatorname{Res}_z (a(z) i_{w,z} (z-w)^{n+j}),
\]

which completes the proof of the Borcherds identity (1.3.53).

The field-state correspondence allows us to consider the n:th product \( a_n b \), for each \( n \in \mathbb{Z} \). By applying both sides of (1.3.53) to a vector \( c \in V \), one can find the following analogue of the Jacobi identity

\[
\sum_{j \geq 0} (-1)^j \binom{n}{j} [a(m+n-j)(b(k+j)c) - (-1)^n b(n+k-j)(a(m+j)c)]
\]

\[
= \sum_{j \geq 0} \binom{m}{j} (a_{n+j}) b_{(m+k-j)c}.
\]

(1.3.61)

This leads to another definition of a vertex algebra: the data are \( V, |0\rangle \in V \), and for each \( n \in \mathbb{Z} \) we have a product \( V \times V \to V \). The product is required to satisfy the Jacobi identity (1.3.61), and we have the vacuum axiom

\[
a_{(n)}|0\rangle = \begin{cases} a & \text{if } n = -1 \\ 0 & \text{if } n \geq 0 \end{cases},
\]

(1.3.62)

and the ‘translational invariance’

\[
|0\rangle_{(n)} a = \delta_{n,-1}.
\]

(1.3.63)

The translational operator is recovered by reading

\[
Ta = a_{(-2)}|0\rangle
\]

(1.3.64)

as defining its action on \( a \). Translational invariance of the vacuum is guaranteed by (1.3.63) which in particular implies \( |0\rangle_{(-2)} |0\rangle = 0 \).

As an example, let \( (V, \cdot) \) be a commutative associative algebra with unit 1. Set \( 1 = |0\rangle \) and define the n:th product \( a_n b = \delta_{n,-1} a \cdot b \). This defines a VA with \( T = 0 \). In fact, all vertex algebras with \( T = 0 \) are of this form.
1.3.9 Noncommutative Wick formula

Let $\lambda$ be a formal variable. Denote $F_\lambda^z = \text{Res}_z e^{\lambda z}$, the formal Fourier transform. We use the bracketed subscript to distinguish the $n$’th component $a_{(n)} = \text{Res}_z (z^n a(z))$ from the Fourier component

$$a_\lambda = F_\lambda^z a(z) = \text{Res}_z (e^{\lambda z} a(z)) = \sum_{n \geq 0} \frac{\lambda^n a_{(n)}}{n!}. \quad (1.3.65)$$

The $n$’th product from the OPE coefficients is written $[a_{(n)} b]$, to distinguish it from the $\lambda$-bracket, or $\lambda$-product $[a_\lambda b] = \sum_{n \geq 0} \frac{\lambda^n [a_{(n)} b]}{n!}$. \quad (1.3.66)

Note that $[a_{(n)} b]_{(-1)} |0\rangle = a_{(n)} \cdot b_{(-1)} |0\rangle$, see (1.3.34). It follows that $(a_\lambda) b = (a_\lambda b)$. However, at $z \neq 0$, we have in general $[a_{(n)} b](z) \neq a_{(n)} \cdot b(z)$, why $(a_\lambda) \cdot b(z) \neq [a_\lambda b](z)$. In a different (and sometimes less confusing) notation, the last inequality reads $(a_\lambda) \cdot Y(b, z) \neq Y([a_\lambda b], z)$. We will use the product $ab$ without any index, which denotes the normal ordered product

$$ab = a_{(-1)} b = :ab:. \quad (1.3.67)$$

One can show that $F_\lambda^z \partial_z^j \delta(z, w) = e^{\lambda w} \lambda^j$. Applying the Fourier transform $F_\lambda^z$ to the Borcherds identity (1.3.53) with $n = 0$, we find

$$(a_\lambda) b(w) - b(w) (a_\lambda) = e^{\lambda w} [a_\lambda b](w). \quad (1.3.68)$$

Letting both sides of this equation act on a vector $c \in V$, and comparing terms of power $w^0$ gives

$$a_\lambda(bc) = [a_\lambda b] c + b((a_\lambda) c) + \int_0^\lambda d\mu \ [a_\lambda b]_\mu c. \quad (1.3.69)$$

This identity is called the noncommutative Wick formula. The term with the formal integral can be thought of as a quantum correction, and may be rewritten as follows.

$$\int_0^\lambda d\mu [a_\lambda b]_\mu c = \int_0^\lambda d\mu \sum_{n \geq 0} [a_\lambda b]_{(n)} \frac{\lambda^n}{n!} c = \sum_{n \geq 0} [a_\lambda b]_{(n)} \frac{\lambda^{n+1}}{(n+1)!} c. \quad (1.3.70)$$
The $\lambda$-product satisfies the three axioms of a \textbf{Lie super conformal algebra}, which read

\begin{align}
(Ta)_\lambda b &= -\lambda [a, b] \quad \text{(sesquilinearity)} \quad (1.3.71) \\
a_\lambda b &= -b_{-T-\lambda} a \quad \text{(skewcommutativity)} \quad (1.3.72) \\
a_\lambda [b, c] &= b_\mu [a, c] - [a, b]_{\lambda+\mu} c \quad \text{(Jacobi identity).} \quad (1.3.73)
\end{align}

In (1.3.72), we use the notation

\[ [a_{(\lambda+T)}b] := \sum_{n \geq 0} \frac{(\lambda + T)^n [a_n] b}{n!} . \quad (1.3.74) \]

Yet another equivalent definition of a VA reads as follows \cite{11}: The data are \((V, |0\rangle \in V, T)\), with the $\lambda$-product, and the $-1$:th product $ab$. The axioms are as follows: The $\lambda$-product satisfies the three axioms of a Lie conformal (super) algebra as stated above, and the noncommutative Wick formula (1.3.69). We also require for the $-1$'th product $ab = a_{(-1)} b$

\begin{align}
ab - ba &= \int_{-T}^{0} a_\lambda b \, d\lambda \\
(ab)c - a(bc) &= \sum_{j \geq 0} [a_{(-j-2)} (b_{(j)} c) + b_{(-j-2)} (a_{(j)} c)] \quad . \quad (1.3.76)
\end{align}

These properties are called \textit{quasicommutativity} and \textit{quasiassociativity}, respectively. This is to be compared with a \textit{Poisson Vertex algebra} where the data are the same, but the normal ordered product is commutative and associative (instead of being quasi-), and with the commutative Wick formula

\[ a_\lambda : b c : = : a_\lambda b : c + : b a_\lambda c : . \quad (1.3.77) \]

Note that this definition of a VA as a quasi- Poisson algebra does not assume locality, but other concepts, which are perhaps more familiar.

One can derive (1.3.75) for a VA by taking the $z^0$ component of the \textit{skewsymmetry} relation \cite{63}, eq. (4.2.1)

\[ a(z) b = e^{zT} b(-z) a , \quad (1.3.78) \]

see \cite{11} eq. (6.9). One proves the quasiassociativity relation (1.3.76) by noting (see \cite{11} Lemma 4.2)

\[ (ab)c = : ab :_{(-1)} c = \text{Res}_z z^{-1} (a_{+}(z) b(z) c + b(z) a_{-}(z) c) . \quad (1.3.79) \]
1.4 Chiral and full CFT

There are properties of CFT that are not encoded in the VA axioms. For instance, the conformal Ward identities are not implied by the VA axioms. VA’s deal with chiral CFT, which means that one considers the $z$ and $\bar{z}$ dependence of the correlators separately. In the simplest case of CFT on the complex plane, one can multiply these together to obtain full correlation functions. It was argued that string theory requires correlators to exist on world sheets of all genera. On surfaces of general topology, the construction of correlation functions is more involved.

The fields and states in a CFT are commonly taken to satisfy the VA axioms. The solutions to the chiral Ward identities are called conformal blocks. A conformal block depends on the positions $z_i$ of the insertions, as well as on the shape of the surface on which we study the (chiral) CFT. These data on which the conformal block depends, constitute the moduli space $\mathcal{M}$. A conformal block is a section over $\mathcal{M}$.

To remind the reader of the concept of a section, we shall briefly recall the construction of the Möbius bundle. To each point on

$$S^1 = \{ \phi \in [0, 2\pi] \mid 0 \sim 2\pi \} ,$$

(1.4.1)

attach an interval $[-1, 1]$ in such a way that at $\phi = 0 \sim 2\pi$, the point $-1_{\phi=0}$ is identified with the point $1_{\phi=2\pi}$ and vice versa. This is the famous Möbius band, or what one might call the Möbius bundle over $S^1$. A section on this bundle associates to each $\phi$ a number $s(\phi) \in [-1, 1]$, in such a way that $s(0) = -s(2\pi)$. Thus, a section on the Möbius bundle is in general not a function on $S^1$. However, the square of any section is a function on $S^1$, and can be integrated.

1.4.1 The complex double

Previously, we promoted the ($\mathbb{R}$-valued) coordinates $\sigma$ and $\tau$ to be $\mathbb{C}$ valued. For CFT on more general surfaces than the plane, we do not have global coordinates on the world sheet. Thus, the splitting of the correlator into its $z$- and $\bar{z}$ dependent parts is more complicated.

Given a world sheet $\Sigma$ with local coordinate $z$, one constructs the complex double $\hat{\Sigma}$, which is an oriented two-manifold without boundary. Now $z, \bar{z}$ are different coordinates of different points on this double, except if they are points on the boundary of the world sheet. We shall assume that the fields are such that we can consider fields $\Phi(z)$ and $\bar{\Phi}(\bar{z})$ independently. This assumption is used in all cases where
there is a known construction of correlators on all world sheets.

From the complex double, the world sheet can be reconstructed by identifying points which are related by the action of a certain mapping \( \sigma \);

\[
\Sigma = \hat{X}/\sigma .
\]  

(1.4.2)

If the world sheet has boundaries, the points on this boundary are fixed by the action of \( \sigma \). For example, the complex double of a disc is a sphere; two discs glued together along the boundary. Then \( z \) is a coordinate on the southern hemisphere and \( \bar{z} = \sigma(z) \) is a coordinate on the northern part. The complex double of a sphere is the disconnected sum of two spheres with \( z \) the coordinate on one of them and \( \bar{z} = \sigma(z) \) the coordinate on the other. On \( \hat{X} \) we allow for analytic coordinate transformations of the two different (for bulk fields) coordinates \( z, \bar{z} \).

On this complex double one talks about chiral CFT and conformal blocks which are sections in vector bundles over moduli space \( \mathcal{M} \), cf. [35]. The word block is used to distinguish these objects from correlation functions in full CFT. It is on the level of chiral CFT one introduces the chiral (vertex) algebra which describes additional symmetries of the theory.

### 1.4.2 Chiral blocks

On the double we will have fields \( \Psi_{\lambda}(z) \) labeled \( \lambda \) at \( z \), and other fields \( \Psi_{\mu}(\bar{z}) \) labeled \( \mu \) at \( \bar{z} \), which correspond to one single field on the world sheet which generally must have two labels; \( \Psi_{\mu,\bar{\nu}}(z) \). Fields \( \Psi_{\lambda}(z) \) on the boundary \( z = \sigma(z) \) will have only one label \( \lambda \) and insertion point on the double, and hence one label on \( \Sigma \); hence we denote it as \( \Psi_{\lambda}(z) \) on the world sheet as well. In general, we let \( p_i \) denote the insertion point(s) of the field(s) that through the field-state correspondence are marked with the state space vector(s) \( w_i \).

Given \( n \) bulk- and \( m \) boundary fields on the world sheet, we have a double \( \hat{X} \) with \( 2n + m = N \) marked points \( \vec{p} \) and \( N \) labels \( \hat{X} \). Conformal blocks \( \beta \) are linear functionals on the states \( v_1 \otimes \cdots \otimes v_N \),

\[
\langle \Phi(p_1, v_1) \cdots \Phi(p_N, v_N) \rangle := \beta_{\hat{X}, \vec{p}}(v_1 \otimes \cdots \otimes v_N).
\]  

(1.4.3)

The set of conformal blocks (also called chiral blocks) specify the theory under consideration. The conformal blocks are subject to invariance conditions, stated in the language of vertex algebras (see [33] for a coordinate independent description). We shall follow [85] and give a coordinate- dependent description (which is less abstract
than the coordinate-independent one).

Suppose that the vectors $v_i$ are states in representations $\mathcal{H}_{\lambda_i}$ of some algebra $\mathfrak{g}$ with generators $J^a_n$, corresponding to the current $J^a(z) = \sum_n J^a_n z^{-n-1}$. Let $f$ be holomorphic on $\hat{X} \setminus \vec{p}$, and meromorphic on $\hat{X}$ such that its expansion around $p_i$ is $f = \sum_n a^i_n t^n_i$, where $t_i$ is the local coordinate at $p_i$. The current $J^a(z)$ and the function $f$ have an action on the vector $v_1 \otimes \cdots \otimes v_N$ via

$$J^a \otimes f := \sum_{i=1}^N 1 \otimes \cdots \otimes \left( \sum_n a^i_n J^a_n \right) \otimes \cdots \otimes 1,$$

(1.4.4)

where the bracket sits at the $i$th position. The Ward identities state that the $\beta_{\hat{X} \setminus \vec{p}}$ are invariant under the action of $J^a \otimes f$. As an example, consider the sphere $\hat{X} = \mathbb{C} \cup \{\infty\} = S^2$ with two removed points $p_1 = 0$ and $p_2 = \infty$. Local coordinates are $t_1 = z$ and $t_2 = 1/z$. The function $f = z^m$ has coefficients $a^1_n = \delta_{n,m}$ and $a^2_n = \delta_{n,-m}$. In this case, we have an action of $J^a \otimes z^m = J^a_n \otimes 1 + 1 \otimes J^a_n$, and the Ward identity reads

$$\beta_{S^2 \setminus \{0,\infty\}} \circ \left( J^a_n \otimes 1 + 1 \otimes J^a_n \right) = 0.$$  

(1.4.5)

This equation is important for our future considerations of boundary states. We can rewrite it in the Dirac bracket-notation as

$$\langle \langle \beta_{S^2 \setminus \{0,\infty\}} \mid (J^a_n \otimes 1 + 1 \otimes J^a_n) \rangle \rangle = 0.$$  

(1.4.6)

It is customary to write $\langle \langle \beta \rangle \rangle$ instead of simply $\langle \beta \rangle$ to remind of the fact that the boundary block is not dual to a state in the state space $\mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_2}$. Nevertheless, one often finds in the literature the notation $|\beta\rangle$, which may be viewed as a vector with infinite norm (which we shall come back to).

The above discussion of the Ward identities assumes that we consider the current $J^a(z)$ as a vector density on the world sheet, with $\Delta_a = 1$. We shall also consider fields that are tensors $J^a(z)$ of conformal weight $\Delta_a$, which means that we integrate $J^a(z)(dz)^{\Delta_a}$, in which case the Ward identity reads

$$\langle \langle \beta \rangle \mid (J^a_n \otimes 1 - (-1)^{\Delta_a} 1 \otimes J^a_n) \rangle \rangle = 0.$$  

(1.4.7)

### 1.4.3 CFT on the upper half plane

As an example, let us consider CFT on the upper half plane. By the Riemann mapping theorem, the upper half plane can be analytically mapped into the unit
disc, whose complex double is the sphere. In the context of conformal field theory, the set of fields always contains the energy momentum tensor

\[ T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \in \mathcal{F}, \tag{1.4.8} \]

where the coefficients \( L_n \) are required to satisfy the Virasoro algebra (1.2.3), and the translational operator appears as \( T = L_{-1} \). The energy momentum tensor has conformal weight \( \Delta = 2 \), thus (1.4.7) reads

\[ \langle \langle \beta_{\mathbb{S}^2 \backslash \{0, \infty\}} \rangle (L_m \otimes 1 - 1 \otimes L_{-m}) = 0. \tag{1.4.9} \]

One can also arrive to (1.4.9) via the following (semiclassical) considerations. We have a field \( T(z) \), which together with \( \bar{T}(\bar{z}) \) generates the conformal transformations. Here, \( \bar{z} \) is an independent world sheet coordinate as in (1.1.6). We wish to consider CFT on the upper half plane, hence \( z \) and \( \bar{z} \) are coordinates on \( \mathbb{H} \), which is \( \mathbb{C} \) with everything under the real line removed. If we glue these half planes together along \( \mathbb{R} \) (this is dubbed the doubling trick in [72], see also [90]), then we get one copy of \( \mathbb{C} \) with \( T(z) = \bar{T}(\bar{z}) \) when \( z, \bar{z} \in \mathbb{R} \). In the quantum theory, we require \( T(z)|B\rangle = \bar{T}(\bar{z})|B\rangle \) when \( z, \bar{z} \in \mathbb{R} \), which then implies \( L_n|B\rangle = \bar{L}_n|B\rangle \). Here, \( |B\rangle \) is referred to as the boundary state, and plays a similar role as the vacuum when we consider CFT on \( \mathbb{C} \). In the string literature, \( \bar{L}_n \) is referred to as the right moving Virasoro generator, and corresponds to \( 1 \otimes L_{-n} \) in the notation of (1.4.9) (while \( L_n \) corresponds to \( L_n \otimes 1 \)). Thus, one talks about the Ward identities as relating the left- and right moving algebras.

### 1.4.4 Modular invariance

From intuition in QFT one might expect the information about field content on closed strings to be stored in the torus partition function, i. e. the one loop amplitude without insertions. The study of this particular amplitude is a central topic in CFT.
Figure 1.2: The moduli space $M \subset \mathbb{C}$ of the torus.

Inequivalent tori are parameterized by the choice of a parameter $\tau$ on the complex upper half plane. This description is somewhat redundant: tori whose $\tau$ differ by the action of

\begin{align*}
S & : \tau \mapsto -\frac{1}{\tau} \\
T & : \tau \mapsto \tau + 1
\end{align*}  \hspace{1cm} (1.4.10)

(and the group $PSL(2, \mathbb{Z})$ these generate) are analytically equivalent (having the same complex structure). Thus we must require that the torus partition function is invariant under the modular group $PSL(2, \mathbb{Z})$. Modular invariance of the torus partition function also appears as an essential ingredient in the proof of the factorization constraints, which are essential to prove that correlators exist on all world sheets for certain CFT’s [30, 42].

1.4.5 Factorization

For CFT’s that are used in string theory, we impose that correlators exist on all world sheets (the word ‘all’ can depend on the context, sometimes one considers only closed and oriented world sheets, and so on). We want the scattering amplitudes to be invariant under certain deformations of the world sheet. These deformations can be used to deform a world sheet until it consists of two parts which are nearby disconnected. It is often assumed that this deformation can be extrapolated to the point where the world sheet actually disconnects into two (simpler) parts. As is illustrated in Figure 3, one can use this to reduce the world sheet to a set of simpler world sheets. We can isolate the parts of the world sheet $X$ that are related to boundaries (and cross-caps when one discusses un-oriented theories) from the
insertion points of in- and outgoing states, and one can eliminate handles. When doing such a cut, one inserts a complete set of states. The amplitude for the more complicated world sheet one started from, is then written as a sum of products of amplitudes of the parts.

Factorization can be performed in different ways, and this leads to the possibility to reduce a complicated amplitude into different sets of simpler amplitudes. For consistency, we require that the correlation function is independent on the particular factorization procedure. This leads to the so called factorization constraints, cf. [69] and references therein.

\[ \sum_{\mu \nu} \]

Figure 1.3: Factorization of a 0-point torus with one hole into a one-point disc and a three-point sphere. The first arrow is just a topological deformation to indicate where the cuts are to be made in the actual factorization, which is the second arrow. The sums are over charge conjugate pairs of labels at points marked \( X \), where \( \bar{\mu} = (\mu_l^\dagger, \mu_r^\dagger) \) and \( \mu = (\mu_l, \mu_r) \).

On the level of full CFT, all world sheets with insertions can be factorized into a set of elementary world sheets, the set of elementary world sheets depending on the kind of world sheets that are allowed; if boundaries are allowed there will be one-point correlators on the disc et. c. If we allow for un-orientable world sheets as well, there will also appear cross-caps (the Möbius band glued together with a disc along its boundary is a cross-cap, cf. [8,9]). This is summed up in the following table:

<table>
<thead>
<tr>
<th>Allowed world sheets:</th>
<th>Elementary world sheets:</th>
</tr>
</thead>
<tbody>
<tr>
<td>un-orientable, with boundaries</td>
<td>1-point disc, 1-point cross cap, 1,2,3 -point sphere</td>
</tr>
<tr>
<td>orientable, with boundaries</td>
<td>1-point disc, 1,2,3 -point sphere</td>
</tr>
<tr>
<td>un-orientable, without boundaries</td>
<td>1-point cross-cap, 1,2,3 -point sphere</td>
</tr>
<tr>
<td>orientable, without boundaries</td>
<td>1,2,3 -point sphere</td>
</tr>
</tbody>
</table>
In chiral CFT, the doubles of all these world sheets are 3-point spheres (if one insertion is the vacuum, the corresponding point can be omitted).

1.5 Conformal Field Theory

To summarize, a conformal field theory has fields which transform under the Virasoro algebra (1.2.3). Further properties of the currents (which are the fields that generate the symmetries) are encoded in the VA axioms. Other fields that occur in the theory are modules over conformal VA’s [33]. In string theory, we shall require existence of correlators on all world sheets (with arbitrary number of handles, sometimes also holes and crosscaps): If we discard point particle - quantum gravity because the perturbation theory is not renormalizable, we should replace it with a theory where it is at least shown that correlation functions exist on all surfaces. One can prove that correlators exist at all orders [30] under certain assumptions, which we shall impose on the models that are considered below. We shall impose rationality, which is the statement that the VA representation category is semisimple. For (most of) our purposes, it is sufficient to know that rationality implies that there are finitely many primary fields (there are usually infinitely many descendants, which are derivatives of these). Rationality also implies that the state space can be graded in such a way that there are finitely many states at each grade [26,33].
2 Some CFT models

We begun the discussion of CFT by considering an example; the Polyakov action, which describes strings propagating on a flat target space (which admits global coordinates $X^\mu$). We wish to describe strings propagating on curved spaces, for which we need to write down more complicated actions, see (2.2.11). We consider the Wess-Zumino-Witten (WZW) model, which describes strings propagating on a Lie group. Instead of global coordinates, we shall use mainly expressions written down in terms of group elements to define our theories. Then we go on to describe some related models, namely adjoint cosets and orbifolds of Lie groups.

Note that the word ‘model’, or ‘theory’ as used here is quite different from what is often meant with the word ‘model’ in the string literature. In the present thesis, we shall not (directly) discuss super-symmetric models, and distinguish different models by their chiral (bosonic) symmetry algebra. In the string theory literature, one talks for example about for instance the type IIA and type IIB models (or theories) which are different implementations of super-symmetry to a CFT with chiral algebra $u(1)$ [72]. In the last few years, it has become clear that the relevant concept to talk about is landscapes with billions of string ‘vacua’, rather than five ‘models’ [80,91].

2.1 Free Boson

The Polyakov model, which describes a string in a Minkowski space or on a flat torus, was used as a motivation to introduce certain concepts in CFT. The model is also called the free boson, because it is described by non-interacting bosonic fields on the world sheet. Here, we shall revisit the model with the more sophisticated tools that were discussed in previous sections.
2.1.1 Free boson VA

A VA is specified by the $\lambda$-product of its fields, or equivalently, by the commutators of the modes. The fields $\alpha \equiv \partial X$ that appear in the Polyakov action generate the $u(1)_k$-vertex algebra with $\lambda$-product

$$\alpha \lambda \alpha = \lambda.$$  \hfill (2.1.1)

Equivalently, the following $n$-products can be used to specify the VA:

$$\alpha^{(1)} \alpha = 1, \quad \alpha^{(n \neq 1)} \alpha = 0.$$  \hfill (2.1.2)

We regard these relations as part of the physical input to our theory, in some sense, these relations specify the *model* under consideration. These operator products can be derived from the Polyakov action with canonical quantization; by calculating the Poisson brackets for $\partial X$ and its conjugate momentum. Alternatively, one can calculate the $XX$ OPE by functional integration, cf. [72], chapter 2. From the OPE’s follow the following commutation relations for the modes:

$$[\alpha^{(n)}, \alpha^{(m)}] = \text{Res}_{z,w} (z^n \alpha(z) w^m \alpha(w) - w^m \alpha(w) z^n \alpha(z))$$

$$= \text{Res}_{z,w} \left( z^n w^m \sum_l [\alpha^{(l)} \alpha^{(w)}] \partial_2^l \delta(z,w) \frac{1}{l!} \right)$$

$$= \text{Res}_{z,w} \left( z^n w^{m-1} \sum_{k \in \mathbb{Z}} k z^{k-1} \frac{w^k}{w^k} \right)$$

$$= m \delta_{m,-n}.$$  \hfill (2.1.3)

If we use quasiassociativity (1.3.75) to investigate the normal ordered product, we find that only the term with $j = 1$ contributes to the difference

$$(\alpha \alpha) \alpha - \alpha (\alpha \alpha) = 2 \alpha^{(-3)} \alpha = T^2 \alpha.$$  \hfill (2.1.4)

Since the right hand side does not vanish, the VA product is highly non-associative. Introduce the *Virasoro field* $L = \frac{1}{2} \alpha \alpha$. Using the noncommutative Wick formula (1.3.69), we can calculate

$$\alpha \lambda L = \frac{1}{2} \alpha \lambda (\alpha^{(-1)} \alpha)$$

$$= \frac{1}{2} \left( [\alpha \lambda \alpha] \alpha + p(a,b) \alpha [\alpha \lambda \alpha] + \int_0^\lambda d \mu [\alpha \lambda \alpha] \mu \alpha \right)$$

$$= \frac{1}{2} \left( [\lambda]^{(-1)} \alpha + \alpha^{(-1)} \lambda \right) + \int_0^\lambda d \mu [\lambda] \mu \alpha$$

$$= \lambda \alpha.$$
Note that the number \( \lambda \), if interpreted as a field, has only one component, namely the one with index \( n = -1 \). From skewcommutativity (1.3.72) follows immediately that \( L_\lambda \alpha = (T + \lambda)\alpha \). One can now deduce that

\[
L_\lambda L = (T + 2\lambda)L + \frac{\lambda^3}{12}.
\]  

(2.1.6)

The factor \( 1/12 \) comes from the term which makes the noncommutative Wick formula noncommutative;

\[
\frac{1}{2} \int_0^\lambda (\lambda - \mu) d\mu = \frac{\lambda^3}{12}.
\]  

(2.1.7)

From the \( \lambda \)-products displayed here, we can reproduce the commutator

\[
[L(z), \alpha(w)] = \partial_w \alpha(w) \delta(z, w) + \alpha(w) \partial_w \delta(z, w).
\]  

(2.1.8)

It is instructive to carry out this calculation in detail; by comparing (1.3.34) with (1.3.66) we see that the \( j \)-th coefficient in the commutator expansion is given by

\[
c^j(z|j) = [L_{(j)}\alpha](z) = j! \text{Res}_\lambda (\lambda^{-j-1}[L\alpha](z)) = j! \text{Res}_\lambda (\lambda^{-j-1}(T + \lambda)\alpha(z)) = j! \text{Res}_\lambda (\lambda^{-j-1}T\alpha(z)) + j! \text{Res}_\lambda (\lambda^{-j-1}\lambda\alpha(z)) = \delta_{j,0} \partial \alpha(z) + \delta_{j,1} \alpha(z).
\]  

(2.1.9)

Similarly, one can show that

\[
[L(z), L(w)] = \partial_w L(w) \delta(z, w) + 2L(w) \partial_w \delta(z, w) + \frac{1}{12} \partial_w^3 \delta(z, w),
\]  

(2.1.10)

which implies that the modes \( L_m := L_{(m+1)} \) satisfy the Virasoro commutation relations (1.2.3) with \( c = 1 \). The central charge for the tensor product of \( D \) free bosons is \( D \), and models a string propagating in a target space with \( D \) (flat) dimensions. We shall see that in string theories that describe strings in curved spacetimes, the central charge is in general not equal to the dimension of the target space.

### 2.1.2 T-duality

Consider a free boson \( X(z) \) compactified on a circle of radius \( R \); \( X(z) + 2\pi R = X(z) \), where \( \partial X = \alpha \) generates the VA (2.1.3) whose characters shall be given below, see (2.1.14). We can re-introduce the world sheet coordinates \( \sigma, \tau \) by recalling (1.1.6).
A winding excitation is characterized by an integer \( m \) such that \( X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi Rm \), and we can write (cf. [78])

\[
X(\tau, \sigma) = X_L(\tau, \sigma) + X_R(\tau, \sigma) \tag{2.1.11}
\]

\[
X_L(\tau, \sigma) = \frac{1}{2} q_L + \sqrt{\frac{\alpha'}{2}} p_L (\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau + \sigma)}
\]

\[
X_R(\tau, \sigma) = \frac{1}{2} q_R + \sqrt{\frac{\alpha'}{2}} p_R (\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau - \sigma)}
\]

\[
p_L = \sqrt{\frac{\alpha'}{2}} \left( p + \frac{R}{\alpha'} m \right) = \sqrt{\frac{\alpha'}{2}} \left( \frac{n}{R} + R \frac{R}{\alpha'} m \right)
\]

\[
p_R = \sqrt{\frac{\alpha'}{2}} \left( p - \frac{R}{\alpha'} m \right) = \sqrt{\frac{\alpha'}{2}} \left( \frac{n}{R} - \frac{R}{\alpha'} m \right).
\]

Note that the OPE of \( X \) with itself will have logarithmic singularities and is thus not a field of a VA. Likewise, the momentum \( p \) with quantized eigenvalues \( n/R \) is not an operator in a VA.

If \( R = \sqrt{\alpha'} \) the ‘self-dual radius’ (\( \alpha' \) is a constant called Regge slope, cf. [72]), each winding excitation \( m \) will contribute as much to the energy of a string state as a momentum excitation \( n \). For general radii, we have the T-duality

\[
R \leftrightarrow \frac{\alpha'}{R} \quad n \leftrightarrow m. \tag{2.1.12}
\]

In super string theory on flat target spaces, T-duality is an important concept; the type IIA super string theory is T-dual with the type IIB super string [72]. There are similar dualities also in certain curved string theories, meaning that sigma models on different target spaces are described by the same CFT.

### 2.1.3 Rational free boson

At the self-dual radius, and at rational multiples thereof, we have an extended chiral algebra [24]. There are a finite number of inequivalent VA representations, and the corresponding conformal field theories are rational (there are finitely many primary fields).

The zero point function on the torus can be written (by factorization) as a two-point function on the sphere. This correlation function is commonly referred to as the partition function. In the ‘charge conjugate theory’, the partition function is
given by

\[ Z = \sum_i |\chi_i(\tau)|^2, \tag{2.1.13} \]

which is called the charge conjugation invariant (there may be other invariants as well). The characters are

\[ \chi_i(q) = q^{-c/24} \text{Tr}_i q^{L_0} = q^{-c/24 + h_i} \sum_{n=0}^{\infty} d_i(n) q^n, \tag{2.1.14} \]

where \( d_i(n) \) is the number of states at grade \( n \) in the representation \( i \) and \( q = e^{2\pi i \tau} \).

Under the modular transformation \( T \), the characters transform as

\[ \chi_i(\tau + 1) = e^{2\pi i (h_i - c/24)} \chi_i(\tau) \tag{2.1.15} \]

which shows that the diagonal partition function is \( T \)-invariant. Under \( S \)-transformations the characters transform as

\[ \chi_i\left(-\frac{1}{\tau}\right) = \sum_j S_{ij} \chi_j(\tau). \tag{2.1.16} \]

When the number of primary fields is \( 2k \), as it is for the compactified free boson at radius \( R = \sqrt{2k} \), the \( S \)-matrix reads

\[ S_{nm} = \frac{1}{\sqrt{2k}} e^{-i\pi nm}. \tag{2.1.17} \]

The modular matrix is unitary, hence the partition function is modular invariant.

### 2.2 Strings in Lie groups

We shall begin our discussion of strings propagating in curved spacetimes by introducing the Wess-Zumino-Witten (WZW) model cf. [39, 50], which describes a string propagating on a Lie group. We begin by recalling some facts about such groups and the functions that can be defined on them. Then we briefly describe the path integral formulation of the WZW model. After that, we describe the WZW model in more algebraic language. Having done all that, we shall study other CFT models that are constructed from the WZW models by dividing out certain subgroups.

Roughly, a **Lie group** is a set of continuous symmetries. The probably most well known example of a Lie group is the set of rotations of 3-dimensional space, this Lie group is called \( SO(3,\mathbb{R}) \). This set is a group and a \( \mathbb{R} \)-manifold. The set of
infinitesimal rotations is a real Lie algebra and is denoted \( \mathfrak{so}(3, \mathbb{R}) \), which inherits its commutation relations from the Lie group. Regarding the algebra as a vector space over complex numbers, one obtains \( \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{s}(2, \mathbb{C}) \). To every complex Lie algebra \( \mathfrak{g} \) corresponds different real Lie algebras with different signatures of the metric (Killing form). The one whose metric has only negative eigenvalues is called the compact real form and is the Lie algebra of a compact real Lie group [45].

**Functions** Recall that Fourier's theorem states that the functions \( e^{i2\pi nx} \) for \( n \in \mathbb{Z} \) form a dense orthogonal basis of the vector space of \( L^2 \) functions on the circle with the integral norm. This means that every square integrable \( f \in L^2(S^1) \) can be approximated to arbitrary accuracy (in the integral norm) by a finite linear combination of functions \( e^{i2\pi nx} \). For Lie Groups, there is an analogous theorem, the Peter-Weyl theorem (see for instance [17, 45]). The theorem states that for any compact Lie group \( G \), the vector space \( L^2(G) \) has a dense basis formed by the matrix elements of the representation matrices \( R^\lambda(g) \), where \( \lambda \) labels all the unitary highest weight representations of \( G \) (we call such weights \( \lambda \) dominant integral), and \( g \in G \) is the argument of the function. Every function \( f(g) \) can to arbitrary accuracy be approximated by a finite linear combination of matrix elements of \( R^\lambda(g) \);

\[
f(g) = \sum_{\lambda, \mu, \nu} \sqrt{\frac{d_{\lambda}}{|G|}} \hat{f}(v^\lambda_\mu \otimes \tilde{v}^\lambda_{\nu}) \langle v^\lambda_\mu | R^\lambda(g) | \tilde{v}^\lambda_{\nu} \rangle,
\]

(2.2.1)

with \( \{v^\lambda_\mu\} \) a basis of the integrable \( \mathfrak{g} \) module \( \mathcal{H}_\lambda \) of dimension \( d_{\lambda} \), and \( \{\tilde{v}^\lambda_{\nu}\} \) is a basis of the dual representation \( \mathcal{H}^\dagger_\lambda \). The volume of the group manifold \( G \) is denoted \( |G| \). One can think of the \( \hat{f}(v^\lambda_\mu \otimes \tilde{v}^\lambda_{\nu}) \) as Fourier coefficients associated with the basis of functions

\[
R^\lambda_{ij}(g) := \langle v^\lambda_i | R^\lambda(g) | \tilde{v}^\lambda_j \rangle.
\]

(2.2.2)

The equation (2.2.1) is referred to as the Peter-Weyl isomorphism between functions \( f \) on the group manifold and functionals \( \hat{f} \) on the space \( \mathcal{H}_\lambda \otimes \mathcal{H}^\dagger_\lambda \). The basis \( R^\lambda_{ij}(g) \) is complete and orthogonal with respect to the Haar measure;

\[
\langle R^\lambda_{ij}, R^\mu_{kl} \rangle := \int_G \text{d}g R^\lambda_{ij}(g)(R^\mu_{kl}(g))^* = \frac{|G|}{d_{\lambda}} \delta_{ik} \delta_{jl} \delta_{\lambda\mu}.
\]

(2.2.3)

To check the normalization, set \( \lambda = \mu \), sum over \( j = l \) by using unitarity of \( R^\lambda_{ij}(g) \), together with the representation property. The factor \( \sqrt{d_{\lambda}/|G|} \) in the Fourier transformation (2.2.1) implies that the functionals

\[
\hat{f}_{mn} : v^\lambda_\mu \otimes \tilde{v}^\lambda_{\nu} \mapsto \delta_{ij} \delta_{mn} \delta_{\lambda\nu}
\]

(2.2.4)

are orthogonal under the scalar product inherited from (2.2.3).
**Characters** Class functions on the group satisfy $c(g) = c(h^{-1}gh) \forall h \in G$, and the set of characters $\chi_\lambda(g) := \text{tr} R^\lambda(g)$ forms a dense subspace of the space of continuous class functions [17]. It follows from the orthogonality above that the characters are orthogonal with the normalization

$$\langle \chi_\mu, \chi_\nu \rangle = |G| \delta_{\mu\nu}. \quad (2.2.5)$$

Every conjugacy class (with volume $|C_g|$) intersects a maximal torus $T$ at a point $x \in T$, [17]. The class delta distribution $\delta_{C_g}$ is defined by the property that if $c(h)$ is a class function on $G$, then

$$\int_G dh \delta_{C_g}(h)c(h) = c(g) \int_{C_g} dh = c(g)|C_g|. \quad (2.2.6)$$

Such a distribution can be expanded in terms of characters;

$$\delta_{C_g}(h) = \sum_\lambda c_\lambda \chi_\lambda(h), \quad \text{with} \quad \langle \chi_\mu, \delta_{C_g} \rangle = |G| c_\mu, \quad (2.2.7)$$

from the orthogonality relation (2.2.5). One can also evaluate the scalar product by integration as in (2.2.6);

$$\langle \chi_\mu, \delta_{C_g} \rangle = \int_G dh \chi_\mu(h) \delta_{C_g}(h) = \chi_\mu(g)|C_g|. \quad (2.2.8)$$

Therefore the following identity holds:

$$\sum_\lambda \chi_\lambda(g)\chi_\lambda(h) = \frac{|G|}{|C_g|} \delta_{C_g}(h). \quad (2.2.9)$$

### 2.2.1 WZW action

We would like to describe strings propagating in a Lie group $G$. The action

$$\int_{\Sigma} \text{tr} (g^{-1}\partial g)(g^{-1}\bar{\partial} g),$$

with $g : \Sigma \to G$, is proportional to the Polyakov action, upon introduction of local coordinates. However, this action does not give rise to a conformal theory (at the quantum level, see [95, 96]). To this action, one adds a so-called WZ term which ensures holomorphic factorization, which is used to prove conformal invariance [95, 96]. The WZW model can be described with an action (following the conventions of [39], page 176)

$$S^{WZW}(g) = \frac{k}{16\pi} \int_{\Sigma} \text{tr} (g^{-1}\partial g)(g^{-1}\bar{\partial} g) + \frac{k}{24\pi} \int_B \tilde{g}^* \chi. \quad (2.2.11)$$
The symbol $\text{tr}$ denotes the trace over the Lie algebra index, or more generally the invariant bilinear form on $\mathfrak{g}$, which we shall denote $(\cdot, \cdot)$. In the second term of the right hand side, commonly denoted $SWZ$, $B$ is a 3-manifold with $\partial B = \Sigma$, $\hat{g}$ is a map $B \rightarrow G$ and $\chi$ is a certain 3-form. Such an action is in general not well defined, one may for instance have different choices of $B$ such that $\partial B = \Sigma$. It turns out that the action is in many cases defined modulo integers, which does not affect the path integral - that depends on $e^{-2\pi i S}$ - as long as the level $k$ is integer. (In more general cases, the WZ-term is defined in terms of so-called gerbe holonomies [48].)

In canonical quantization, the string states are constructed as vectors in highest weight representations of a certain affine Lie algebra $\mathfrak{g}$. The compact real form of the horizontal subalgebra $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$ is the Lie algebra of $G$, and the eigenvalue of the central element $K \in \mathfrak{g}$ in this representation is the same integer $k$ that appears in the action. Accordingly, we shall take a certain vertex algebra (which we shall introduce later, see (2.2.17)), as our starting point for describing the WZW model. From the Lie algebra point of view, the requirement $k \in \mathbb{Z}$ comes from requiring that the horizontal Lie algebra representation is integrable, which means it is a representation of the Lie group as well.

### 2.2.2 Algebraic formulation

Given a finite dimensional simple algebra $\mathfrak{g}$ with a bilinear form $(\cdot, \cdot)$ one can construct a VA as follows. First define the affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K,$$

which is a central extension of the loop algebra $\mathfrak{g}[t, t^{-1}]$ with basis vectors $at^n|_{n \in \mathbb{Z}}$ and Lie brackets inherited from $\mathfrak{g}$, cf. [62]. The collection of quantum fields includes the central element $K$ and formal polynomials in $z$ with coefficients $at^n$ that are basis elements of the loop algebra;

$$\mathcal{F}^0 = \left\{a(z) = \sum_{n \in \mathbb{Z}} (at^n) z^{-n-1}|_{a \in \mathfrak{g}, K}\right\}.$$

The affine commutation relations imply the following commutation relations for the fields;

$$[a(z), b(w)]_{\mathcal{F}^0} = [a, b] \delta(z, w) + (a, b) \partial_w \delta(z, w) K.$$

In the sequel we will omit the subscripts on the commutators and bilinear forms whenever it is obvious from the input arguments what it should be. As this commutator is annihilated by $(z - w)^2$, the set $\mathcal{F}^0$ is local. The coefficients of $a(z) \in \mathcal{F}^0$
span \mathfrak{g}. The OPE coefficients \([a_{(n)}]b(z)\) of fields in \(F^0\) are also local with all fields in \(F^0\). Thus we shall include all such coefficients in \(F\). We assume that derivatives of fields in \(F^0\) are also included in \(F\), as well as normal ordered products. The translational operator \(T\) is defined by its action on the formal variable \(z\) as

\[ Ta(z) = \partial_z a(z). \tag{2.2.15} \]

We construct the vector space \(V\). Denote

\[ \mathfrak{g}^- = \text{Span} \{ a_{(n)} \mid n \geq 0 \} \tag{2.2.16} \]

the Borel subalgebra in \(\mathfrak{g}\) generated by the nonnegative modes of the field \(a(z) \in F^0\) acting on the vacuum. Let \(T(\mathfrak{g})\) be the tensor algebra of \(\mathfrak{g}\), consisting of all linear combinations of juxtapositions of elements in \(\mathfrak{g}\). Denote \(U(\mathfrak{g})\) the quotient of the tensor algebra, and the ideal generated by \([x, y] - xy + yx\). The space \(U(\mathfrak{g})\mathfrak{g}^- \subset U(\mathfrak{g})\) is an ideal in \(U(\mathfrak{g})\). Now define the vector space \(V\) as the following quotient,

\[ V := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g}^- \equiv \{ [u], u \in U(\mathfrak{g}) \text{ and } [v] = [w] \Leftrightarrow v - w \in U(\mathfrak{g})\mathfrak{g}^- \}. \]

Our vacuum vector \(|0\rangle\) will be the tensor unit \(C \in V\). The data \((V, |0\rangle, T, F)\) as defined above, satisfy the axioms of a vertex algebra. This is called the universal enveloping vertex algebra of \((\mathfrak{g}, F, T)\), and is commonly denoted \(V(\hat{\mathfrak{g}}, F)\).

From the universal enveloping VA, we construct the following VA. In an irreducible representation of \(\hat{\mathfrak{g}}\), the central element \(K\) acts as a constant \(k \in \mathbb{C}\). For any \(k \in \mathbb{C}\) we can define the state space

\[ V^k(\hat{\mathfrak{g}}) = V(\hat{\mathfrak{g}}, F)/(K - k). \tag{2.2.17} \]

The considerations for \(\mathfrak{u}(1)\) in section 2.1.1 can be generalized to the affine Lie VA (2.2.17) at level \(k\) associated with the finite dimensional simple or abelian Lie algebra \(\mathfrak{g}\) by the **Sugawara construction**, as follows. Given a basis \(\{a_i\}\), we choose a dual basis \(\{b_i\}\) by requiring \((a_i, b_j) = \delta_{ij}\) where \((\cdot, \cdot)\) is the non-degenerate symmetric bilinear form. We define the field

\[ L(z) = \frac{1}{k + g^\vee} \sum_j : a_j(z) b_j(z) :, \tag{2.2.18} \]

which is a Virasoro field. Here, \(g^\vee\) denotes the dual Coxeter number of the horizontal Lie algebra \(\hat{\mathfrak{g}}\). When \(\mathfrak{g}\) is abelian, \(\hat{\mathfrak{g}}\) is a direct sum of free bosons as described above. Then, the modes \(L_{(m)}\) satisfy the Virasoro commutation relations (1.2.3) with \(c = \dim \mathfrak{g}\). If \(\mathfrak{g}\) is simple, we have the same commutation relations but with

\[ c = \frac{k \dim \mathfrak{g}}{k + g^\vee}. \tag{2.2.19} \]
From the CFT point of view we need to specify more than the chiral and the Virasoro algebra. When $G$ is simply connected, the action (2.2.11) gives a charge conjugation torus partition function with characters of the algebra, combining left and right movers in a trivial way. The CFT turns out to be rational, implying that there only exists a finite number of primary fields that generate a finite number of $\mathfrak{g}_k$ representations. The state spaces can be decomposed as

$$\bigoplus_{\lambda \in P_k} \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}^\dagger.$$  \hspace{1cm} (2.2.20)

Here $P_k$ denotes the finite set of allowed (unitary irreducible highest weight) representations $\lambda$ at level $k$, and $\mathcal{H}_{\lambda}^\dagger$ is the charge conjugate representation. Each such representation space $\mathcal{H}_{\lambda}$ is also a Virasoro representation by the Sugawara construction (2.2.18). The normal ordered product of fields was defined above (1.3.40), thus the modes of (2.2.18) are

$$L_n := \frac{\kappa_{ab}}{2(k + g^\vee)} \sum_m :J^a_{n-m} J^b_m: ,$$  \hspace{1cm} (2.2.21)

where $J^a_m$ are the generators of $\mathfrak{g}_k$, and $\kappa_{ab}$ is the Killing form of $\mathfrak{g}$. The normal ordered product of modes is such that $:A_n B_m: = :B_m A_n:$, and we have $\langle 0|:A_n B_m: = 0 = :A_n B_m: |0 \rangle$ whenever one of $A_n, B_m$ kills the vacuum. In the limit of infinite level all irreducible highest weight representations are allowed and the horizontal subspace

$$\bigoplus_{\lambda} \bar{\mathcal{H}}_{\lambda} \otimes \bar{\mathcal{H}}_{\lambda}^\dagger \subset \bigoplus_{\lambda \in P_\infty} \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}^\dagger$$ \hspace{1cm} (2.2.22)

can be identified with the set of functions on $G$ via the Peter-Weyl isomorphism (2.2.1). Each closed string state $v \in \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}^\dagger$ is specified by three quantum numbers $\lambda, n, m$ where $n$ labels a state in $\mathcal{H}_{\lambda}$ and $m$ labels a state in $\mathcal{H}_{\lambda}^\dagger$. It is customary to associate to the state $v_{\lambda nm}$ a function $R^\lambda_{nm}(g)$ as given in (2.2.2).

### 2.2.3 SU(2)

The simplest example of a WZW model is the $\mathfrak{sl}(2)_k$ model which describes a string model with target space $SU(2) \cong S^3$ (as a manifold) with volume [10]

$$V = 2\pi^2 (k\alpha')^{3/2}.$$  \hspace{1cm} (2.2.23)

This volume can be reconstructed by writing the embedding fields $g$ in (2.2.11) in local coordinates, and comparing with the Polyakov action.
All group elements of $SU(2)$ can be obtained in the following way. Introduce the Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(2.2.24)

Note that $\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i \epsilon_{ijk} \sigma_k$. For any $\vec{n} \in \mathbb{R}^3$ define $\sigma_n := \vec{n} \cdot \vec{\sigma}$. From now on we assume that $\vec{n}$ is a unit vector. Then for any $\psi \in \mathbb{R}$ one gets

$$
grad(\psi, \vec{n}) := \exp(i\psi \sigma_n) = \cos \psi \mathbf{1} + i \sin \psi \sigma_n.
$$

(2.2.25)

We get all $g \in SU(2)$ as $g = g(\psi, \vec{n})$ by taking $\vec{n}$ any unit vector in $\mathbb{R}^3$, and $\psi$ in the range $[0, \pi]$. The metric on the unit three-sphere is

$$
d\psi^2 = \left( d\psi^2 + \sin^2 \psi \, ds^2_{S^2} \right),
$$

(2.2.26)

where $ds^2_{S^2}$ is the two-sphere metric for the spherical polar coordinates $\theta, \phi$. Geometrically, $\psi$ is a parameter that labels conjugacy classes, while the unit vector $\vec{n}$ lives on $S^2$ and parameterizes points on the conjugacy class. Parameterizing $S^2$ in the standard way by angles $\theta$ and $\phi$ we have

$$
\vec{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta),
$$

(2.2.27)

so that

$$
\sigma_n = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}.
$$

(2.2.28)

CFT The $\mathfrak{sl}(2)_k$ characters $\chi_j(\tau)$ form an $SL(2, \mathbb{Z})$ representation with

$$
S_{ji} = \frac{2}{k+2} \sin \left( \frac{(j+1)(i+1)}{k+2} \pi \right),
$$

(2.2.29)

which can be derived with the character formula, see [39] chapter 2. The primary fields are labeled $j$, with conformal weights

$$
\Delta(j) = \frac{j(j+2)}{4(k+2)}.
$$

(2.2.30)

2.2.4 Remarks on volume quantization

One expects the matter content of a string model to be determined by (among other things) the target space geometry. We also expect the matter content to be the same everywhere in the universe. In the phenomenology literature, one talks about
volume stabilization, cf. [15] and references therein, see also [56] for a review from a string theory perspective. In the framework of rational CFT, the volume of the extra dimensions is quantized and need not be stabilized.

There are approaches to quantum gravity other than string theory. One such approach is called loop quantum gravity (LQG), (see [87, 92] for an introduction), and there it appears that volumes (and areas) are quantized. With some optimism, one might hope that both string theory and LQG can be used to describe quantum gravity, which gives us yet another reason to expect that volumes are at least stabilized (but perhaps better quantized) in string theory.

2.3 Simple Currents and Orbifolds

Another class of target spaces in which we study propagating strings are the orbifolds. They are constructed out of a parent target space $M$ by dividing out a discrete group $\Gamma$ which acts on $M$. That is, we construct the orbifold $O$ as the set of equivalence classes

$$O = \{ [m] \mid m \in M, [m] = [\gamma m], \forall \gamma \in \Gamma \}. \quad (2.3.1)$$

A popular example of an orbifold is the Lie group $SO(3)$, which can be constructed from $SU(2)$ by dividing out the action of $\mathbb{Z}_2$ whose nontrivial element acts as $g \mapsto zg = e^{i\pi \sigma_3}g$, with $\sigma_3$ the Cartan generator of $\mathfrak{su}(2)$. One can construct the corresponding CFT as a so-called simple current extension of $\mathfrak{su}(2)$, as shall be described below. In many cases, one can construct CFT’s describing strings in certain orbifolds with simple current techniques. However, there are also cases where this is not (directly) possible. For example, one can construct other orbifolds from $SU(2)$ by dividing out larger groups than $\mathbb{Z}_2$, such orbifolds are called lens spaces. As we shall see, the $\mathfrak{su}(2)$ CFT has only a $\mathbb{Z}_2$ simple current group, hence general lens spaces cannot be described as simple current extensions of $\mathfrak{su}(2)$.

2.3.1 Fusion Rules

We would like to specify the operator products of the currents in the simple current theory we wish to construct. The most important information from the OPE’s is contained in the fusion rules, which are important also for other reasons. The fusion rules specify the multiplication in what is called the fusion ring, cf. [39] Ch. 5. The objects which are multiplied in this ring are called conformal families. A conformal family $\mu$ is a primary field $\phi_\mu$ (which appears in the partition function) together
with its descendants. The fusion rule

$$\lambda \ast \mu = \sum_{\nu} N_{\lambda \mu}^{\nu} \nu$$  \hspace{1cm} (2.3.2)

says that OPEs between fields in the conformal families $\lambda, \mu$ contains fields from those conformal families $\nu$ for which $N_{\lambda \mu}^{\nu} \neq 0$. In the language of VA’s, the currents are those fields which describe the symmetries of the theory, and these are the objects that are multiplied. The concept of a conformal family translates into a module of that VA, and the fusion rules are dimensions of intertwiner spaces between modules.

The fusion coefficients can be calculated from the modular $S$-matrices with the Verlinde formula [58, 93]

$$N_{ij}^k = \sum_n \frac{S_{in}^{s_{j \mu} = s_{kn}} S_{kn}^{s_{0 0}}}{S_{0n}^{s_{0 0}}}.$$  \hspace{1cm} (2.3.3)

### 2.3.2 Simple currents

In the present section, we shall define simple currents and review some important properties of these, see [39], chapter 5, for further information. A simple current $J$ is a conformal family, whose fusion product with any other field in the model gives precisely one term on the right hand side; $J \ast \mu = \lambda$, where we also denote $\lambda = J\mu$. Equivalently, one can also define the set of simple currents as those fields $J$ which have an inverse $J^\vee$ such that $J \ast J^\vee = 0$, where 0 is the vacuum field such that $0 \ast i = i$, and $Q_0(i) = 0$. The product of two simple currents is again a simple current, and simple currents are invertible. If the CFT is rational, the representation of the chiral symmetry algebra is semi-simple, which will always be the case in the sequel. In rational theories we have only a finite set of conformal families [26]. Hence, applying $J$ to $\phi$ sufficiently many times must at some point give back $\phi$. Therefore, the set of simple currents forms a finite group. Moreover, it is an abelian group. Every finite abelian group can be written as

$$G \cong \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_i},$$  \hspace{1cm} (2.3.4)

with $N_i$ a divisor of $N_{i+1}$, cf. [12].

The set of simple currents in a CFT can be determined (if the modular $S$-matrix is known) by the fact that they have unit quantum dimension

$$\text{qdim}(J) := \frac{S_{J,0}}{S_{0,0}}.$$  \hspace{1cm} (2.3.5)
If $J$ is a simple current, one can show that there exists an inverse $J^\vee$ under the fusion product with $q\text{dim}(J^\vee) = q\text{dim}(J)$. The quantum dimension of the vacuum is $q\text{dim}(0) = 1$ and the quantum dimensions are multiplicative, thus

$$1 = q\text{dim}(J \ast J^\vee) = q\text{dim}(J) q\text{dim}(J^\vee).$$

(2.3.6)

This shows that simple currents have unit quantum dimension. One can also show that any field with unit quantum dimension is a simple current.

### 2.3.3 Simple current extensions

We would like a theory which is (in some sense) invariant under $G$, thus we define a vacuum character $\chi_{\Omega}$ as

$$\chi_{\Omega'} = \sum_{g \in G} \chi_{g\Omega}. \quad (2.3.7)$$

To construct a modular invariant partition function for a theory with such a vacuum character is nontrivial, and is in almost all known cases done with simple current methods (other invariants are called exceptional). Given a charge conjugation (or diagonal) modular invariant torus partition function in, say a WZW model, we can construct non-diagonal invariants. In some cases, the characters of the old theory are combined in such a way that the new theory has as many characters as the old one, in which case one talks about permutation invariants. It is called an extension when the new theory has fewer characters than the old theory, which indicates that the chiral algebra is (in some sense) larger. It turns out that we have an extension if the simple currents have integer conformal weights, and if the partition function is symmetric.

The **monodromy charge** of a conformal family with respect to a certain simple current is defined (modulo integers) in terms of the conformal weights of the primaries as

$$Q_J(i) := \Delta_J + \Delta_i - \Delta_{J_i}. \quad (2.3.8)$$

Since the Virasoro operators change the conformal dimensions with integer steps, the fractional part of a conformal weight is well defined for a conformal family. Thus, the above definition makes sense.

The construction of the partition function goes as follows. Let $G \cong \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_l}$ be a simple current group where all elements can be written as $J = J_1^{j_1} \cdots J_l^{j_l}$. The
matrix with nonzero entries

\[ M_{\lambda^+,J\lambda} = \prod_{i=1}^{l} \delta^l \left( Q_{J}(\lambda) + \sum_{j=1}^{l} X_{ij} \beta_j \right) , \]  

(2.3.9)

where

\[ \delta^l(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{else} \end{cases} , \]  

(2.3.10)

gives a modular invariant partition function iff the matrix \( X_{ij} \) is chosen with the following restrictions [67] (note that only the fractional part of \( X \) is relevant for (2.3.9)). Define the matrix of relative monodromies as \( R_{ik} = Q_{J}(J_k) \mod \mathbb{Z} \). The diagonal elements should be defined modulo \( 2\mathbb{Z} \), which can be achieved by requiring [81,83]

\[ 2\Delta(J_k) = (N_k - 1)R_{kk} \mod 2\mathbb{Z} . \]  

(2.3.11)

Now we require

\[ X + X^t = R , \quad \gcd(N_i,N_j)X_{ij} \in \mathbb{Z} . \]  

(2.3.12)

The constraint \( X + X^t = R \) determines the symmetric part of \( X \), which is sometimes referred to as the discrete torsion matrix. The anti-symmetric part of \( X \) is called discrete torsion, and is constrained (but not always determined) by the second requirement of (2.3.12). (The same ambiguity arises in the sigma model description where the WZ-term is formulated in terms of gerbe holonomies [48].) Then we can write down a partition function

\[ Z = \sum_{\lambda \in \mathcal{G}} M_{\lambda^+,J\lambda} \chi_{\lambda^+} \bar{\chi}_{J\lambda} , \]  

(2.3.13)

which is modular invariant. Let \( \mathcal{G} \) be an extension simple current group, and let \( [\lambda] \) be an orbit \( \mathcal{G}\lambda \). \( \mathcal{S}_\lambda \subset \mathcal{G} \) is the subgroup that fixes \( \lambda \), and \( \mathcal{U}_\lambda \subset \mathcal{S}_\lambda \) is the set of simple currents for which the twist \( F_\lambda \) (see [43,59]) with \( \mathcal{S}_\lambda \) is trivial;

\[ F_\lambda(\mathcal{U}_\lambda,\mathcal{S}_\lambda) = 1 . \]  

(2.3.14)

One important property of the untwisted stabilizer \( \mathcal{U}_\lambda \) is that it has square integer index in \( \mathcal{S}_\lambda \), which implies \( \mathcal{U}_\lambda = \mathcal{S}_\lambda \) if \( |\mathcal{S}_\lambda| \) is not divisible by a squared integer (which is the case in the applications we consider). Since \( \mathcal{U}_\lambda \) is abelian, its group characters \( \chi_{\mathcal{U}_\lambda} \) are \( |\mathcal{U}_\lambda| \)th roots of unity. Let \( \psi_\lambda = 1, ..., |\mathcal{U}_\lambda| \) and define an action of \( J \in \mathcal{G} \) on
these by $J\psi_\lambda = \psi_{J\lambda}$. The characters of the extended theory will now be labeled by orbits $[\lambda, \psi_\lambda]$ with

$$J \cdot (\lambda, \psi_\lambda) = (J \star \lambda, \psi_{J\star\lambda}) \sim (\lambda, \psi_\lambda).$$

(2.3.15)

Expressed in terms of the new characters

$$\chi_{[\lambda, \psi_\lambda]} = \sqrt{|S_\lambda|} \sum_{J \in G/S_\lambda} \chi_{J\lambda},$$

(2.3.16)

the partition function (2.3.13) is diagonal.

### 2.3.4 Orbifolds of SU(2)

Many, but not all, geometric orbifolds can be described as simple current CFT’s. We shall consider orbifolds of the $SU(2)_k$ WZW model, whose conformal families are labeled by integers $0 \leq j \leq k$. There is a field $k$ which generates a simple current group isomorphic to $\mathbb{Z}_2$, with fusion rule $k \star j = k - j$. Then, the simple current extension (often called the D-type partition function) gives the $SO(3)_k$ WZW model (which exists when the level is even [94, 95]).

The CFT $SO(3)_k$ can be described as a simple current extension of $SU(2)_k$ when $k = 4l$, and as a permutation invariant when $k = 4l + 2$. The primary fields in the covering $SU(2)_k$ are labeled $j = 0, 1, ..., k$. The simple current group is $G = \mathbb{Z}_2 = \{k, 0\}$. The nontrivial element acts via fusion as $k \star j = k - j$ (and has a fixed point $j_F = k/2$). The relative monodromy of a field $j$ with respect to the current $k$ is given by $Q_k(j) = j/2$. (Here we see that the level needs to be even; suppose $2 \nmid k$, then $2 \nmid R_{kk} \equiv R$ and $4 \nmid X_{kk} \equiv X$. This is in contradiction to the second requirement (2.3.12).) The relative monodromy is $Q_k(k) = k/2$ and the requirement (2.3.11) says that $R_{kk} = k/2 \text{ mod } 2$. Thus, when $4|k$, (which means 4 divides $k$) (2.3.12) gives $X = 0$ and when $4 \nmid k$ (which means 4 is not a divisor of $k$), $R = 1$ and $X = 1/2$. In other words, $X = k/4$ modulo integers. Now (2.3.9) is

$$M_{j,J\star j} = \delta^1 \left( j/2 + k/4 \cdot \beta \right).$$

(2.3.17)

Thus, when $4|k$, only fields with $2|j$ contribute to the partition function, which means that there are less fields than we started with, and one calls the modular invariant (2.3.13) an extension. Note that the field $j$ is paired with $k \star j = k - j$ (in the sense $M_{j,k\star j} \neq 0$), hence the field identification $j \sim k - j$. 

When $4 \nmid k$, fields with $2 \mid j$ are paired only with themselves, and fields with $2 \nmid j$ are paired only with their ‘simple current partners’ $k - j$. Thus, the modular invariant has as many characters as the diagonal $SU(2)$ invariant and is called a permutation invariant.

Of course, one can construct target space orbifolds by dividing out any $\mathbb{Z}_n$, and it is possible to describe strings propagating on these orbifolds [55]. Even though these CFT’s are not simple current extensions of $SU(2)$, one can construct these CFT’s as simple current extensions of a $PF \times U(1)$ CFT, as described in section 2.5 below (and in [1]).

2.3.5 Functions on orbifolds

Suppose we know the set of functions on $M$, the covering of the orbifold, and want to investigate the functions on the orbifold $O = M/\Gamma$. Functions on the orbifold are functions on $M$ that are invariant under $\Gamma$. Thus, we require for $\gamma \in \Gamma$, that $f(\gamma x) = f(x)$. In WZW models where the Peter-Weyl theorem provides a basis of the space of functions on $M$, this information may be useful to find a basis of functions in the orbifold.

For example, we can find the functions on $SO(3)$ by requiring $R^i_{mn}(e^{i\pi H} g) = R^i_{mn}(g)$, which holds when $2 \mid m$ (which implies $2 \mid j$ and $2 \mid n$). No linear combination of odd basis functions $R^i_{mn}(g)$ can be invariant under the action of $\Gamma \cong \mathbb{Z}_2$. By the Peter-Weyl theorem on $SU(2)$, it follows that any $\mathbb{Z}_2$-invariant function on $SU(2)$ can be written as a linear combination of the $R^i_{mn}(g)$ with even integer indices. Hence a basis of functions on $SO(3)$ consists of the projections of the $SU(2)$-basis functions $R^i_{mn}(g)$ with even labels. With the understanding that the closed string basis states $v^i_m \otimes \bar{v}^i_n$ correspond to the basis functions $R^i_{mn}(g)$ we arrive again at the selection rule $2 \mid j$. Note that when $4 \nmid k$ there are characters in the partition function (those with $2 \nmid j$) which do not correspond to functions on the orbifold. In this example, precisely those states which are paired with themselves in the partition function are those which correspond to functions on the orbifold.

In general orbifolds of Lie groups, we expect the selection rule in the simple current construction to allow those state $v^\lambda_m \otimes \bar{v}^\mu_n$ for which the corresponding function $R^\lambda_{mn}(g)$ on $M$ projects to the orbifold $M/\Gamma$. States of the form $v^\lambda_m \otimes \bar{v}^\mu_n$ with $\lambda \neq \mu$ are not expected to correspond to any function at all.
2.3.6 Simple Currents in WZW models

The center $Z$ of a group $G$ is the set of elements which commute with all of $G$;

$$z \in Z \iff (zg = gz \quad \forall \; g \in G),$$  \hspace{1cm} (2.3.18)

or simply $zG = Gz$. The center is a finite abelian group. One can show that the set of simple currents $\mathcal{G}$ of a WZW model is isomorphic to the center of the corresponding Lie group, $Z \cong \mathcal{G}$, see [24] section 14.2.3, in all cases except $E_8$ at level $k = 2$. Let $H_J$ be the element in the dual weight lattice corresponding to the simple current $J$, namely

$$H_J = (J\hat{\omega}_0, H) \hspace{1cm} (2.3.19)$$

with $\hat{\omega}_0$ as in [24] eq. (14.60), and $H$ is a vector of Cartan generators such that $(J\hat{\omega}_0, H)\lambda = (J\hat{\omega}_0, \lambda)\lambda$ for any weight $\lambda$. The action of $J$ on $\hat{\omega}_0$ can be read off table 14.1 in [24] (see also our discussion in section 2.4.3). The isomorphism between the center of $G$ and the set of simple currents is given by

$$J \in \mathcal{G} \leftrightarrow e^{2\pi iH_J} \in Z. \hspace{1cm} (2.3.20)$$

In the expansion $\theta = \sum_{i=1}^r a_i \alpha^{(i)}$ of the highest root (see [45]) in terms of the simple positive roots $\alpha^{(i)}$, the coefficients $a_i$ are called Coxeter labels. The fundamental weights $\Lambda^{(i)}$ are the duals of the simple positive roots; $(\Lambda^{(i)}, \alpha^{(j)\vee}) = \delta_{ij}$ where $\alpha^{\vee} = 2\alpha/(\alpha, \alpha)$. Those fundamental weights $\Lambda^{(i)}$ for which $a_i = 1$ are called cominimal, see [39] (3.5.53) for a list. One can show, that in all WZW models except $E_8$ level 2, the simple currents are given by the cominimal weights [39].

Another characterization of the set of simple currents is as a subset of the outer automorphisms of the extended Dynkin diagrams [24], section 14.2.1.

It follows from the simple current-center isomorphism (2.3.20) that $\Lambda^{(i)} - \Lambda^{(j)} = \omega(\Lambda^{(k)})$ with $\omega \in W$, where $W$ is the (affine) Weyl group. Recall that the affine Weyl group is strictly larger than the horizontal Weyl group $\bar{W}$, apart from reflections it also contains translations. In the sequel, we shall need the following lemma: For those $\Lambda^{(i)} \neq \Lambda^{(j)}$ that are cominimal, $\Lambda^{(i)} - \Lambda^{(j)} = \bar{\omega}(\Lambda^{(k)})$ with $\bar{\omega} \in \bar{W}$, where $\bar{W}$ is the horizontal Weyl group. All corresponding $\alpha_k$ have equal length. We shall consider also the vacuum $\Lambda^{(0)}$ as a cominimal weight.
For general Lie algebras, we have $w_{max}(\Lambda) = -\Lambda^+$. The longest element of the Weyl group is its own inverse. Further, if $-1 \in \hat{W}$, then $w_{max} = -1$. It follows that $w_{max}(\Lambda) = -\Lambda$ iff $\Lambda = \Lambda^+$.

We begin with $A_r$, where we use the ON basis of the weight space. The Weyl group acts on the fundamental weights by permutation on their coefficients [45]. We have

$$\Lambda(j) = \sum_{i=1}^{j} e_i - \frac{j}{r+1} \sum_{i=1}^{r+1} e_i,$$  \hspace{1cm} (2.3.21)

and we shall see that the difference of these is a permutation of the labels. Consider first $-\Lambda(j)$, we have $-\Lambda(j) = w_{max}(\Lambda^+_{(j)})$, and $\Lambda^+_{(j)}$ is among the cominimal weights $\Lambda(j')$. Thus, it follows that $-\Lambda(j) = w(\Lambda(j'))$. We begin with the case $j > l$,

$$\Lambda(j) - \Lambda(l) = \sum_{i=l+1}^{j} e_i - \frac{j-l}{r+1} \sum_{i=1}^{r+1} e_i,$$  \hspace{1cm} (2.3.22)

which is a permutation of the coefficients of the fundamental weight $\Lambda(j-l)$. For $j < l$,

$$\Lambda(j) - \Lambda(l) = - \sum_{i=j+1}^{l} e_i + \frac{l-j}{r+1} \sum_{i=1}^{r+1} e_i,$$  \hspace{1cm} (2.3.23)

which is a permutation of the coefficients of $-\Lambda(l-j) = w_{max}(\Lambda_{(r+1+j-l)})$. All $\alpha_k$ have equal length (all dots are black in the Dynkin diagram). Thus the lemma holds for $A_r$.

$E_7$ has a $\mathbb{Z}_2$ simple current group, in which case the proof is particularly simple. The lemma holds if we can show $-\Lambda(6) = \omega(\Lambda(k))$, and indeed, $-\Lambda(6) = \omega_{max}(\Lambda(6))$, see [65], page 57. Thus, the lemma holds for $E_7$.

The same argument can also be applied to $E_8$ (at level $k = 2$), where $-1 \in \hat{W}_{SO(16)} \subset \hat{W}E_8$ [65]. Thus the lemma holds for $E_8$.

For $B_r$, the simple current group $Z \cong \mathbb{Z}_2$ is generated by $\Lambda_{(1)}$, and $-1 \in W$ [65]. The same argument can be applied to $C_r$. All occurring $\alpha_k$ have equal length. Thus, the lemma holds for $C_r$ and $B_r$. 


For $D_r$, a basis permutation is an element of the Weyl group \([45]\), and in fact also an even number of sign changes are allowed. If \(2 | r\), then \(-1 \in W\), which is the case we shall begin with. The cominimal weights are
\[
\Lambda_{(1)} = (1, 0, 0, \ldots, 0) , \quad \Lambda_{(r-1)} = \frac{1}{2} (1, 1, \ldots, 1) , \quad \Lambda_{(r)} = \frac{1}{2} (1, 1, \ldots, 1, -1) .
\]
Subtracting the last two from each other gives \(\pm e_r\) which is a permutation of labels of \(\pm \Lambda_{(1)}\). Subtracting \(\Lambda_{(1)}\) from \(\Lambda_{(r)}\) or \(\Lambda_{(r-1)}\) gives
\[
\frac{1}{2} (-1, 1, \ldots, \pm 1)
\] (2.3.24)
which can be brought back to either \(\Lambda_{(r)}\) or \(\Lambda_{(r-1)}\) by a Weyl element. All occurring \(\alpha_k\) have equal length. Thus the lemma holds for \(D_{2|r}\).

Now to \(D_r\) with odd \(r\). \(-\Lambda_{(1)} = \omega(\Lambda_{(1)})\), because there are zeroes whose sign we may change at no cost. Let \(\omega \in \bar{W}\) change the first \(r - 1\) (even number) signs, then \(\omega(-\Lambda_{(r-1)}) = \Lambda_{(r)}\). The same Weyl element \(\omega\) gives \(\omega(-\Lambda_{(r)}) = \Lambda_{(r-1)}\).

\[
\Lambda_{(r)} - \Lambda_{(r-1)} = (0, \ldots, 0, -1) = w(\Lambda_{(1)})
\]
\[
\Lambda_{(r)} - \Lambda_{(1)} = \frac{1}{2} (-1, 1, \ldots, 1, -1) = w(\Lambda_{(r-1)})
\]
\[
\Lambda_{(r-1)} - \Lambda_{(r)} = (0, \ldots, 0, 1) = w(\Lambda_{(1)})
\]
\[
\Lambda_{(r-1)} - \Lambda_{(1)} = \frac{1}{2} (-1, 1, \ldots, 1) = w(\Lambda_{(r)})
\]
\[
\Lambda_{(1)} - \Lambda_{(r)} = \frac{1}{2} (1, -1, \ldots, -1, 1) = w(\Lambda_{(r)})
\]
\[
\Lambda_{(1)} - \Lambda_{(r-1)} = \frac{1}{2} (1, -1, \ldots, -1) = w(\Lambda_{(r-1)}) ,
\]
for some Weyl elements denoted \(\omega\). Thus the lemma holds for all \(D_r\).

The algebras \(E_8\) at level \(k \neq 2\), \(F_4\) and \(G_2\) (at all levels) do not have any non-trivial simple currents.

\(E_6\) has a \(\mathbb{Z}_3\) SC group with cominimal weights \(\Lambda_{(1)}\) and \(\Lambda_{(5)}\). In terms of fundamental Weyl reflections \(w_i(\lambda) = \lambda - \lambda' \alpha_i\) on can show (following an algorithm suggested in [18]) that
\[
\Lambda_{(1)} - \Lambda_{(5)} = w_5 w_4 w_3 w_2 w_6 w_3 w_4 w_5(\Lambda_{(5)}) .
\]
One can also show that \(-\Lambda_{(1)} = w(\Lambda_{(5)})\), from which follows (with the inverse Weyl transformation) that \(-\Lambda_{(5)} = w(\Lambda_{(1)})\). From these facts, it follows (by linearity) that
\[
\Lambda_{(5)} - \Lambda_{(1)} = w(-\Lambda_{(5)}) = w'(\Lambda_{(1)}) .
\]
This ends the proof of the lemma.
2.4 Coset theories

A large class of CFT’s are the coset models. For example, the Ising model and all the other minimal Virasoro models (with \(c < 1\)) can be realized as coset models, see section 2.4.5. To construct these models we shall use the tools of simple current constructions discussed in the previous section.

The coset target space is described as follows. Let \(H \subset G\) be an embedded subgroup in \(G\). We shall consider target spaces which are adjoint cosets, that is, the elements are equivalence classes

\[
[g] := \{g' \in G \mid g' = hgh^{-1}, \ h \in H\} \equiv \text{Ad}_h(g).
\]  

The collection of these equivalence classes is denoted \(G/\text{Ad}(H)\). To describe the set of functions on the coset it is convenient to give an alternative description of the target space as a certain coset of \(G \times H\) \([32]\)

\[
G/\text{Ad}(H) = \frac{G \times H}{H \times H}.
\]  

The equivalence relation divided out on the right hand side is

\[
(g, h) \sim (ugv^{-1}, uhv^{-1}), \quad \forall \ u, v \in H.
\]  

We shall denote this description of the coset the \(lr\)-description (as opposed to the \(Ad\)-description where elements of \(Q\) are given in (2.4.1)). That these descriptions are equivalent can be seen as follows. To each element \([g, h] \in G \times H/\text{Ad}(H)\) we associate an element \([gh^{-1}] \in G/\text{Ad}(H)\), and to \([g] \in G/\text{Ad}(H)\) we associate the element \([g, e] \in G \times H/\text{Ad}(H)\). This really is a well defined prescription, i.e. \([g] = [wgw^{-1}]\ in G/\text{Ad}(H)\) maps to one and the same equivalence class \([g, e] = [wgw^{-1}, e] \in G \times H/\text{Ad}(H)\). Likewise the mapping \([g, h]) \mapsto [gh^{-1}]\ is really a mapping for the equivalence classes. It is also the inverse mapping to the one above,

\[
[(g, h)] \mapsto [gh^{-1}] \mapsto [(gh^{-1}, e)] = [(g, h)],
\]  

2.4.1 Functions on the coset

The functions on the coset \(Q\) can be described as the functions on \(G \times H\) that are invariant under the action of \(H \times H\) as given in (2.4.3). Accordingly, the projection \(\pi_{lr}^*f \in \mathcal{F}(Q)\) of a function \(f \in \mathcal{F}(G\times H)\) is given by

\[
\pi_{lr}^*f([g, h]) := \frac{1}{|H|^2} \int_{H \times H} du \, dv \, f(ugv, uhv),
\]
where $|H| = \int_{H}du$ is the volume of $H$. A function $f \in \mathcal{F}(G \times H)$ is $H_t \times H_r$-invariant iff $\pi_{lr}^* f = f$. The space $\mathcal{F}(G \times H)$ is spanned over $\mathbb{C}$ by the functions

$$D_{mn,ab}^{\Lambda,\lambda} := D_{mn}^{\Lambda} D_{ab}^{\lambda^*},$$

with $m, n \in \{1, 2, \ldots, d_{\Lambda}\}$ and $a, b \in \{1, 2, \ldots, d_{\lambda}\}$, the entries of the corresponding representation matrices (the complex conjugation on $D^{\lambda}$ is chosen for later convenience). We shall investigate the behavior of the basis functions (2.4.6) under the projection (2.4.5). We shall see that among the functions (2.4.6), only those $D_{mn,ab}^{\Lambda,\mu}$ give rise to non-zero functions on $\mathcal{Q}$ for which the irreducible $H$-representation $R_{\mu}$ with highest weight $\mu$ occurs in the decomposition of the irreducible $G$-representation $R_{\Lambda}$ with highest weight $\Lambda$ as a $H$-representation, a property that we will indicate by writing $\mu \prec \Lambda$. All other functions $D_{mn,ab}^{\Lambda,\lambda}$ have vanishing projection to $\mathcal{F}(\mathcal{Q})$. In short, the space $\mathcal{F}(\mathcal{Q})$ is spanned by the functions $\pi_{lr}^* D_{mn,ab}^{\Lambda,\lambda}$ with $\lambda \prec \Lambda$. The irreducible $G$-representation $R_{\Lambda}$ decomposes as a direct sum of irreducible $H$-representations $R_{\mu}$ in terms of the representation matrices. Namely, upon suitable basis choices, for $h \in H$ the representation matrix $D_{\Lambda}$ decomposes into blocks along the diagonal,

$$D_{\Lambda}(h) = \bigoplus_{\lambda \prec \Lambda} \bigoplus_{\ell=1}^{b_{\Lambda,\lambda}} D_{\Lambda}(\ell)(h),$$

(2.4.7)

where the summation is over all irreducible $H$-representations that appear in the branching of $R_{\Lambda}$, counting multiplicities, and where the symbol $D_{\Lambda}(\ell)$ denotes the matrix block within the big matrix $D_{\Lambda}$ that corresponds to the $\ell$th occurrence of $R_{\lambda}^{\lambda}$ in $R_{\Lambda}$. Thus the labels $\lambda^{(\ell)}$ enumerate the matrix blocks, while $\lambda$ labels (equivalence classes of) irreducible representations. For a general choice of orthonormal bases in the representation spaces of $G$ and $H$, the equality (2.4.7) holds up to a similarity transformation. The matrix elements are then related by

$$D_{mn}^{\Lambda}(h) = \sum_{\lambda} \sum_{\ell=1}^{b_{\Lambda,\lambda}} \sum_{a, b = 1}^{d_{\lambda}} e_{m,a}^{\lambda < \Lambda; \ell} c_{n,b}^{\lambda < \Lambda; \ell} D_{ab}^{\lambda}(h).$$

(2.4.8)

The numbers $e_{m,a}^{\mu < \Lambda; \ell}$ appearing here are the coefficients in the expansion

$$e_{m}^{\Lambda} = \sum_{\mu, \ell, a} e_{m,a}^{\mu < \Lambda; \ell} \beta_{\ell}^{(\ell)} \otimes e_{a}^{\mu}$$

(2.4.9)

of the vectors $e_{m}^{\Lambda}$ in the chosen basis of $\mathcal{H}_{\Lambda}$ as a linear combination of vectors $e_{a}^{\mu}$ in the chosen bases of the irreducible $H$-modules $\mathcal{H}_{\mu}$ and $\beta_{\ell}^{(\ell)}$ in bases of the multiplicity
spaces $B_{\Lambda;\lambda}$ (in particular, $c_{m,a}^{\mu;\Lambda;\ell} \mu \prec \Lambda$ vanishes unless $\mu \prec \Lambda$). Being coefficients of a basis transformation between orthonormal bases, the $c_{m,a}^{\mu;\Lambda;\ell}$ form unitary matrices
\[
\sum_{\mu,\ell,a} c_{m,a}^{\mu;\Lambda;\ell} (c_{n,a}^{\mu;\Lambda;\ell})^* = \delta_{mn}.
\] (2.4.10)

Using (2.4.8) together with the representation property and the orthogonality relations for the representation matrices, we get
\[
\pi^*_{\ell} D_{mn,ab}^{\Lambda;\lambda}(g, h) = \frac{1}{|H|^2} \int_{H \times H} \sum_{p,q,d,e} D_{mp}^\lambda(u) D_{pq}^\lambda(g) D_{qg}^\lambda(v) D_{ad}^\lambda(u)^* D_{de}^\lambda(h)^* D_{eb}^\lambda(v)^* 
= \frac{1}{d^\lambda} \sum_{\ell',=1}^{b_{\Lambda;\lambda}} \sum_{m,a} c_{m,a}^{\lambda;\Lambda;\ell'} (\sum_{\ell',=1}^{d^\lambda} \sum_{c,d=1}^{\lambda;\Lambda;\ell'} c_{m,a}^{\lambda;\Lambda;\ell'} D_{\Lambda;\lambda}(g, h)).
\] (2.4.11)

The right hand side of (2.4.11) is non-zero iff the coefficients $c_{m,a}^{\lambda;\Lambda;\ell'}$ are non-zero, thus the horizontal representations must be embedded; $\lambda \prec \Lambda$, for the function to project.

To determine how many linearly independent functions we have, let us consider a fixed pair $\lambda \prec \Lambda$ and a fixed pair of degeneracy labels $\ell$ and $\ell'$. Note that the large bracket on the right hand side of (2.4.11) does not depend on $m, n, a, b$. Since $c_{m,a}^{\mu;\Lambda;\ell}$ does not depend on $(g, h)$, the functions on the right hand side with different labels $m, n$ and $a, b$ are proportional (for a fixed pair of degeneracy labels $\ell$ and $\ell'$). Hence each allowed pair $(\Lambda, \lambda)$ of dominant integral G- and H-weights gives rise to (at most) $b_{\Lambda;\lambda}^2$ functions on $Q$. That these functions are linearly independent follows from the observation that they are linearly independent functions on $G \times H$ (which project to the coset), see the discussion after (2.4.14).

The expression (2.4.11) simplifies when one makes the following adapted Gelfand-Zetlin type basis choice: First select arbitrary orthonormal bases in all $\mathcal{H}_\mu$ and then, for each $\Lambda$, a basis of $\mathcal{H}_\Lambda$ consisting of basis vectors of the $\mathcal{H}_\mu$, counting multiplicities. This corresponds to formula (2.4.7) holding exactly, not only up to a similarity transformation (i.e. as an equality of matrices rather than just as an equality between linear transformations). With this adapted basis choice the branching coefficients can be written $c_{m,a}^{\mu;\Lambda;\ell} = \delta_{\mu\lambda} \delta_{\mu\lambda}^\Lambda$, and thus we have
\[
D_{mn,\ell'}^\Lambda(h) = \delta_{\ell,\ell'} D_{m,\ell'}^{\Lambda}(h),
\] (2.4.12)

where the labels $\tilde{m}_\ell$ and $\tilde{n}_\ell$ are the row and column labels of the matrix block $D^{\Lambda(\ell)}$ in $D^\Lambda$ that according to (2.4.7) correspond to the row and column labels $m$ and $n$. 
of the big matrix. When \( m \) and \( n \) are not part of one and the same block labeled \( \ell \), then \( D^{\Lambda}_{mn}(h) = 0 \), as a consequence of \( h \in H \). For general arguments \( g \in G \) the matrix \( D^{\Lambda}(g) \) has non-vanishing entries off the blocks \( \lambda \) along the diagonal. Thus, it makes sense to write expressions like \( D^{\Lambda}_{m_n} h \). To illustrate the notation; suppose \( \Lambda \) can be decomposed into one isomorphism class of \( H \)-reps \( \lambda \) which occurs twice, then

\[
D^{\Lambda}_{m_n}(g) = \left( \begin{array}{cc}
D^{\Lambda}_{\hat{m}_1 \hat{n}_1}(g) & D^{\Lambda}_{\hat{m}_1 \hat{n}_2}(g) \\
D^{\Lambda}_{\hat{m}_2 \hat{n}_1}(g) & D^{\Lambda}_{\hat{m}_2 \hat{n}_2}(g)
\end{array} \right)
\]

(2.4.13)

(which is block-diagonal for \( g \in H \)). With the Gelfand-Zeitlin basis choice, the result (2.4.11) reduces to

\[
\pi^{\star}_{ll'} D^{\Lambda,\lambda}_{mn,ab}(g, h) = \begin{cases}
d^{-2}_\lambda \sum_{\ell,\ell'} \delta_{a \hat{m}_\ell} \delta_{b \hat{n}_{\ell'}} f^{\lambda,\Lambda}_{\ell,\ell'}(g, h) & \text{for } \lambda \prec \Lambda, \\
0 & \text{else}
\end{cases}
\]

(2.4.14)

with

\[
f^{\lambda,\Lambda}_{\ell,\ell'}(g, h) = \sum_{p,q=1}^{d_\lambda} \sum_{c,d=1}^{d_\lambda} \delta_{\hat{a}_\ell, q} \delta_{\hat{b}_{\ell'}, p} D^{\Lambda,\lambda}_{pq,cd}(g, h)
\]

\[
= \sum_{c,d=1}^{d_\lambda} D^{\Lambda,\lambda}_{\hat{c}_{\ell}, \hat{d}_{\ell'}, cd}(g, h)
\]

(2.4.15)

In particular, when the branching rule for \( \Lambda \) is multiplicity-free, then formula (2.4.11) reduces to

\[
\pi^{\star}_{ll'} D^{\Lambda,\lambda}_{mn,ab}(g, h) = \delta_{a \hat{m}} \delta_{b \hat{n}} d^{-2}_\lambda \sum_{p,q} D^{\Lambda,\lambda}_{pq, pq}(g, h)
\]

for \( \lambda \prec \Lambda \). (This applies e.g. to all branching rules in the description of the unitary Virasoro minimal models as \( \mathfrak{so}(2) \oplus \mathfrak{so}(2) / \mathfrak{so}(2) \) coset models, for which the coefficients \( c^{\mu - \lambda} \) are just ordinary Clebsch–Gordan coefficients.)

In order to convince ourselves that the functions \( f^{\lambda,\Lambda}_{\ell,\ell'}(g, h) \) are linearly independent, let us revisit the example (2.4.13). Then the function (2.4.15) picks up contributions from only one of the four blocks on the right hand side of (2.4.13). Further, we know from the Peter-Weyl theorem that all these blocks are independent, considered as function on \( G \). Thus, the \( f^{\lambda,\Lambda}_{\ell,\ell'}(g, h) \) are independent viewed as functions on \( G \times H \), and since they are also invariant under the \( l-r \) action, they are linearly independent on the coset. This shows that each horizontal embedding \( \lambda \prec \Lambda \) gives rise to precisely \( b^2_{\Lambda,\lambda} \) functions on the coset.

\[1\text{Note that in [3], there is a mistake in the expression (3.15) corresponding to equation (2.4.15) above, which is due to a mix-up in the notation (3.14) of [3], and which makes it difficult to see that the functions } f^{\lambda,\Lambda}_{\ell,\ell'}(g, h) \text{ as given in [3] would be independent. Concretely, the summation in [3] equation (3.15) is up to } d_\lambda \text{ which would imply that all four blocks contribute in the case of (2.4.13) above.} \]
2.4.2 Coset CFT

Let us assume that both $\hat{g}$ and $\hat{h}$ are simple, and that the levels for $\hat{g}$ and $\hat{h}$ are $k$ and $l$, respectively. The numbers $k$ and $l$ must be chosen such that the embedding exists, $l = i_{h \mapsto g}k$ where $i_{h \mapsto g}$ is the embedding index (the Dynkin index of the embedding $h \mapsto g$). The chiral algebra $C(V_k(\hat{g}), V_l(\hat{h}))$ of the coset model is the commutant of the vertex algebra associated to $\hat{h}$ in the vertex algebra associated to $\hat{g}$,

$$\mathcal{A} = C(V_k(\hat{g}), V_l(\hat{h}))$$

$$:= \left\{ v \in V_k(\hat{g}) \left| Y(h, z)v \in V_k(\hat{g})[[z]] \quad \forall \, h \in V_l(\hat{h}) \right. \right\}.$$

The symbol $V[[z]]$ denotes the set of power series in $z$ (only positive powers!) with coefficients in $V$. The condition implies that all fields in $\mathcal{A}$ commute with those in $V_l(\hat{h})$: if the vector $Y(v, z - w)v_2$ only contains positive powers of $z - w$, then the right hand side of (1.3.52), $Y(Y(v, z - w)v_2, w)$, converges in the limit $z = w$. But if the left hand side of (1.3.52), $Y(v_1, z)Y(v_2, w)$, is non-singular, and the two fields $Y(v_1, z)$ and $Y(v_2, w)$ commute.

It turns out that the commutant $\mathcal{A}$ is again a VA, cf. [33]. One can show that it is a module of the Virasoro algebra, with central charge given by

$$c(\mathcal{A}, k, l) = c(\hat{g}, k) - c(\hat{h}, l).$$

(2.4.17)

Our main interest here is in the characters of this algebra. The representations of $\mathcal{A}$ can be decomposed into $\hat{h}$-representation spaces as

$$\mathcal{H}_{\Lambda}^g = \bigoplus_{\mu} B_{\Lambda, \mu} \otimes \mathcal{H}_{\mu}^h,$$

(2.4.18)

where $B_{\Lambda, \mu}$ are representation spaces for the coset chiral algebra. Note that this is an embedding of affine algebras, different from the embedding of horizontal algebras considered in (2.4.9). The characters come naturally from the character decomposition

$$\chi^g_{\Lambda}(\tau) = \sum_{\mu} b_{\Lambda, \mu}(\tau) \chi^h_{\mu}(\tau),$$

(2.4.19)

where $b_{\Lambda, \mu}(\tau) = \chi^{g/h}_{\Lambda, \mu}(\tau)$ are called branching functions. We want to interpret them as characters of the coset algebra. Some of these branching functions vanish, and others coincide.

Further, the left and right moving characters must somehow be combined into a modular invariant partition function. The solution in most cases, except for so-called
Maverick cosets [27, 37], is to do a simple current extension of the $G \times H^*$ theory, where $H^*$ is a certain CFT related to $H$ (the modular tensor category describing the primary fields of $\hat{H}$ is the dual of the MTC associated to $H$ [36], see also [60] and [64]).

The theory of simple currents then gives the right selection rules, field identifications, and a modular invariant partition function [82]. The identification simple current group $G_{id}$ to take is the set of all simple currents $J = (J^g, J^h)$ in the product theory such that the difference between their conformal weights $\Delta(J^g) - \Delta(J^h)$ is integer for generic levels (we shall discuss the identification group in section 2.4.3). This gives field identifications (2.3.15) and selection rules (2.3.9) from the theory of simple current extensions. As we want this model to be right-left symmetric, we do not expect discrete torsion. Therefore, there is no choice in the matrix $X$ in (2.3.12), and from (2.3.9) one can read off the selection rules.

We saw in (2.4.11) that each allowed pair $(\Lambda, \lambda)$ provides us with $b_{\Lambda,\lambda}^2$ functions on $Q$. Since these are $H\times H$-invariant linearly independent functions on $G\times H$, they are linearly independent functions on $Q$. In analogy with the Peter-Weyl correspondence between open string states and functions on the Lie group, it is then natural to identify the $b_{\Lambda,\lambda}^2$-dimensional space spanned by these functions with the tensor product $B_{\Lambda,\lambda} \otimes B_{\Lambda,\lambda}^*$. The factors are the branching space $B_{\Lambda,\lambda} = \mathbb{C}^{b_{\Lambda,\lambda}}$ and its dual space. The latter, in turn, is naturally isomorphic to the branching space $B_{\Lambda^+,\lambda^+}$ for the dual modules. Thus we arrive at a description of $\mathcal{F}(Q)$ as

$$\mathcal{F}(Q) \cong \bigoplus_{\Lambda < \Lambda} B_{\Lambda,\lambda} \otimes B_{\Lambda^+,\lambda^+}.$$  

This is nothing but the analogue of the the Peter-Weyl isomorphism

$$\mathcal{F}(G) \cong \bigoplus_{\Lambda} \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda^+$$  

for the functions on the target space $G$ of the $\mathfrak{g}$-WZW model.

Note that the horizontal branching spaces $B_{\Lambda,\lambda}$ are not subspaces of the branching spaces $B_{\hat{\Lambda},\hat{\lambda}}$ that appear in the branching rules (2.4.18) of $\hat{\mathfrak{g}}$-representations. As a consequence of field identification, a few specific states present in the $B_{\Lambda,\lambda}$ can be absent in the $B_{\hat{\Lambda},\hat{\lambda}}$. For instance, for the parafermions (see section 6 below), there is precisely one such state: at level k there exist k allowed fields labeled as $(k, n)$, while the number of horizontally allowed pairs of the form $(k, n)$ is $k+1$. 
As a sigma model, the coset theory can be obtained from the $G$ WZW model by gauging the adjoint action of a subgroup $H \subset G$. This is done by adding a gauge fixing term in the action [51].

2.4.3 The identification group

The simple currents in $G_{id}$ correspond to elements in $Z_G \cap H$. Here $Z_G$ denotes the center of $G$, the set of elements that commute with all elements of $G$. For later considerations, the following consequence of that fact will be important. An action of $J$ on the conjugacy classes $C^G_\Lambda$ of $G$ labeled $\Lambda$ can be defined as $J : C^G_\Lambda \mapsto C^G_{J \Lambda}$. To $(J, j) \in G_{id}$ corresponds an element $(g_J, h_J) \in Z_G \cap H \times Z_H$ [83]. Let us consider the action of simple currents $J$ of the $g$-WZW model on the conjugacy classes $C^G_\Lambda \subset G$ that is obtained by mapping $C^G_\Lambda$ to

$$C^G_\Lambda : g(J) := \{ gg_J \mid g \in C^G_\Lambda \} \subset G, \quad (2.4.22)$$

with $g(J)$ as in (2.3.20).

We shall determine $g_J \Lambda$. The action of the generators of the simple current group (denoted $a$ and $\tilde{a}$) are given in [24], Table 14.1. We see by inspection that $a \cdot \hat{\omega}_0 = a[1,0,...,0]$ \footnote{For $D_{r=2l+1}$ with simple current group $Z_4$, the generator is denoted $\tilde{a}$ and $\tilde{a}^2 = a$ in [24] with $a$ as given for $D_{r=2l}$. We shall denote the generator of the $Z_4$ by $a$ instead, and hope that this causes no confusion.} are given by

$$
egin{align*}
A_r, B_r, D_{r=2l} & : \quad [0, 1, 0,..., 0] \\
C_r, D_{r=2l+1} & : \quad [0,...,0,1] \\
E_6, E_7 & : \quad [0,...,0,1,0].
\end{align*}
(2.4.23)

The only simple current group with two generators is the one for $D_{r=2l}$ (which is $Z_2 \times Z_2$), and the generator $\tilde{a}$ acts as

$$\tilde{a}[1,0,...,0] = [0,...,0,1]. \quad (2.4.24)$$

In our notation, the generators $\Lambda(J)$ of the (cyclic) simple current groups of $A_r, B_r$ (see [39] (3.5.53)) are $\Lambda(1)$ and it follows from [24] (14.98) that in these cases, the horizontal part of $J \Lambda$ can be written as

$$J^\pi \Lambda = w(J) w_o(\Lambda) + k \Lambda(J). \quad (2.4.25)$$

Here $w_o$ is the longest element of the Weyl group $W$ of $g$ and $w(J)$ the longest element of the subgroup of $W$ that is generated by all simple Weyl reflections except...
the one corresponding to the simple root that is dual to \( \Lambda_{(J)} \).

We can also see that the relation (2.4.25) holds also for the generator \( \Lambda_{(1)} \) of \( D_{r=2l} \). The other generator is \( \Lambda_{(r)} \) (which is denoted \( \tilde{a} \) in [24]) and again, (2.4.25) holds. For \( C_{r}, D_{r=2l+1} \) the simple current generator is \( \Lambda_{(J)} = \Lambda_{(r)} \) and (2.4.25) holds as well. For \( E_{6}, E_{7} \) we can take \( a = \Lambda_{(r-1)} \) and again, (2.4.25) holds. Since (2.4.28) is our only application of (2.4.25), it is sufficient to have proved (2.4.25) for the simple current group generators \( J \).

The Weyl vector is fixed under \( w_{\circ} \) and \( w_{\circ} J(\rho) = \rho - g^{\vee} \Lambda_{(J)} \). It follows that

\[
J \pi \Lambda + \rho = w(J) w_{\circ} (\Lambda + \rho) + (k + g^{\vee}) \Lambda_{(J)},
\]

(2.4.26)

and thus the group elements (2.3.20) satisfy

\[
g_{J \pi \Lambda} = (w(J) w_{\circ}(g_{\Lambda})) g_{(J)}.
\]

(2.4.27)

Since Weyl transformations do not change the conjugacy class of a group element, this implies that

\[
C_{A}^{G} \cdot g_{(J)} = C_{J \pi \Lambda}^{G}.
\]

(2.4.28)

Thus the simple current group \( J_{g} \) acts in a natural way via the fusion product on conjugacy classes of \( G \). Analogously, the simple current group \( J_{h} \) of the \( \bar{h} \)-WZW model acts via the fusion product on conjugacy classes of \( H \), as \( C_{A}^{H} \mapsto C_{A}^{H} \cdot h_{(J)} = C_{J \pi \Lambda}^{H} \), and hence \( J_{g} \times J_{h} \) acts on the conjugacy classes (see (3.5.1)) of \( G \times H \) as

\[
C_{A,\Lambda} \mapsto C_{J \pi \Lambda,\pi \Lambda} = C_{A,\Lambda} \cdot (g_{(J)}, h_{(J)}).
\]

(2.4.29)

In particular, owing to \([g, h] = [gh^{-1}, e] \) in \((G \times H)/(H_{x} \times H_{r})\), it follows that for \( g_{(J)} = h_{(J)} \) we have

\[
\pi_{l_{r}}(C_{J \pi \Lambda,\pi \Lambda}) = \pi_{l_{r}}(C_{A,\Lambda})
\]

(2.4.30)

for the coset branes which will be given in (3.5.7). That is, if the simple current \((J, J) \in J_{g} \times J_{h} \) is an identification current, then it acts trivially on the coset branes (3.5.7). The set (3.5.1) is invariant under field identification iff \( g_{J} h_{J}^{-1} \) acts as the identity when acting on \( C_{A}^{G} \) from the right. This is almost, but not quite, the same as saying \( g_{J} = h_{J} \); if for example \( C_{A}^{G} \) is the equatorial conjugacy class in \( SU(2) \) with \( \Lambda = k/2 \), we can have \( g_{J} h_{J}^{-1} = -e \). However, it seems that requiring that \( g_{J} h_{J}^{-1} \) acts as the identity when acting on \( C_{A}^{G} \) for all \( C_{A}^{G} \) implies \( g_{J} = h_{J} \). Further, we expect that only under exceptional circumstances we can have that \( g_{J} \neq h_{J} \) but still \( g_{J} h_{J}^{-1} \) acts as the identity when acting on \( C_{A}^{G} \). As we shall see in the sequel, the conjugacy class \( C_{A}^{G} \) in \( SU(2) \) with \( \Lambda = k/2 \) is quite exceptional.
2.4.4 The parafermion coset

A nontrivial example is the parafermion model [54] based on the $u(1)_k \subset sl(2)_k$ coset construction, where both theories are at the same level $k$. The primary fields in the theory are labeled by pairs of integers $(j, n)$ where $j$ is the $sl(2)_k$ label $0 \leq j \leq k$ and $n$ is the $u(1)_k$ label $0 \leq n < 2k$. The nontrivial simple current of $sl(2)$ is labeled $k$ with conformal weight $\Delta(k) = k/4$, which acts under fusion as $k \ast j = k - j$. All currents labeled $m$ in $u(1)$ are simple, with conformal weights $\Delta(m) = m^2/4k$ and fusion $m \ast n = m + n \mod \mathbb{Z}_{2k}$. The nontrivial simple current of $su(2)$ labeled $k$ corresponds to the element $-1 \in G \cap H$. The requirement that a generic difference between conformal weights is integer means $m = k$, so the identification group $G_{id} = \mathbb{Z}_2$ is generated by $(k, k)$, which gives the field identification

\[(j, n) \sim (k - j, n + k)\]  \hfill (2.4.31)

These pairs of labels have the same branching function and describe the same field [24]. The conformal dimensions of the Parafermion fields are given by

\[
\Delta(j, n) = \begin{cases} 
\frac{j(j+2)}{4(k+2)} - \frac{n^2}{4k} & \text{for } n \leq k \\
\frac{j(j+2)}{4(k+2)} - \frac{(n-2k)^2}{4k} & \text{for } n > k.
\end{cases}
\]  \hfill (2.4.32)

When we consider monodromy charges, and in most other cases where the conformal dimensions are used, we are only interested in the conformal weights up to integers. Then these expressions are equal and we can simply think of the conformal dimensions $\Delta(j, n)$ as the difference of the building block conformal dimensions; $\Delta(j, n) = \Delta(j) - \Delta(n) \mod \mathbb{Z}$. The selection rule $2 | j + n$ says that if the labels have different parities, the branching function must vanish and that that pair of labels does not correspond to any field in the coset theory. The selection rule can be derived by viewing the coset as a simple current extension of $G \times H^*$. To construct the partition function (2.3.13), we need to determine the matrix (2.3.9). Note that all monodromies involving the vacuum vanish; $Q_{(0,0)}(j, n) = 0$, and

\[
Q_{(k,k)}(j, n) = \frac{\Delta(k, k) + \Delta(j, n) - \Delta(k - j, k + n)}{2}.
\]  \hfill (2.4.33)

The product in (2.3.9) contains only one factor; $l = 1$. The discrete torsion matrix is in this case just a number $X_{(k,k)(k,k)}$, which by (2.3.12) vanishes. Thus, the field $(j, n)$ appears in the partition function only if the selection rule

\[2 | j + n\]  \hfill (2.4.34)

is satisfied.
Geometry

The target space of the parafermion theory can be described as follows.

We parameterize, as in section 2.2.3, the points on the manifold $SU(2) \cong S^3$ as

$$g = g(\psi, \theta, \phi) = \exp(i\psi\sigma_\vec{n}) = \cos \psi \mathbb{1} + i \sin \psi \sigma_\vec{n}. \quad (2.4.35)$$

Here $\vec{n}$ is a point on the unit two-sphere with the spherical polar coordinates $\theta$ and $\phi$, and $\sigma_\vec{n} := \vec{n} \cdot \vec{\sigma}$ with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices (2.2.24) with relations $\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k$. Then the coordinate ranges are $\psi \in [0, \pi]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$, and taking the radius of the three-sphere to be $\sqrt{k}$, the metric is

$$ds^2 = k (d\psi^2 + \sin^2 \psi d\theta^2 + \sin^2 \psi \sin^2 \theta d\phi^2),$$

so that $|G| = 2\pi^2 k^{3/2}$ and $|H| = 2\pi^2 k^{1/2}$.

The non-trivial $su(2)_k$ simple current $j = k$ corresponds to the element

$$g_{(k)} = -\mathbb{1} \in G \cap H = U(1) \quad (2.4.36)$$

with $U(1) = \{e^{i\sigma_3} \mid t \in [0, 2\pi]\} \subset SU(2)$, while for the primary fields of $u(1)_k$ (all of which are simple currents) we have $h_{(n)} = e^{in\sigma_3/k}$. The non-trivial simple current of the $su(2)$ theory acts on $SU(2)$ as

$$g = g(\psi, \theta, \phi) \mapsto gg_{(j)} = -g = g(\pi-\psi, \pi-\theta, \pi+\phi), \quad (2.4.37)$$

and the simple currents $\lambda$ of the $u(1)$ theory act on the maximal torus as multiplication by $e^{i\lambda \sigma_3/k}$, i.e. as a rigid rotation by the angle $\lambda \pi/k$.

The representation matrices for $H = u(1)_k$ are the numbers

$$D^n(e^{it\sigma_3}) = e^{int}. \quad (2.4.38)$$

The horizontal branching spaces $B_{j,n}$, where now $j \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$, are one-dimensional if $2|j+n$ and $|n| \leq j$, and are zero else. Every orbit of the identification group (2.4.31) contains precisely one horizontally allowed pair, except for the orbit of the identity field, for which both representatives $(0, 0)$ and $(k, k)$ are horizontally allowed pairs.

The subgroup $H = U(1)$ whose adjoint action is gauged is the maximal torus, which in the parametrization (2.4.35) is given by

$$H = \{e^{it\sigma_3} \mid t \in [0, 2\pi]\} = \{g(\psi, \theta, \phi) \mid \phi = 0, \theta = 0, \pi\}. \quad (2.4.39)$$

Direct calculation (using the notation of section 2.2.3) shows that for all $\vec{m}, \vec{n} \in S^2$ and all $s, \psi \in \mathbb{R}$ we have

$$\exp(is\sigma_m) \exp(i\psi\sigma_n) \exp(-is\sigma_m) = \cos \psi \mathbb{1} + i \sin \psi \tau(s, \vec{m}, \vec{n}) \quad (2.4.40)$$
with
\[
\tau(s, \vec{n}, \vec{n}^\prime) := \cos^2 s \sigma_{n'} + i \sin s \cos s [\sigma_n, \sigma_{n'}] + \sin^2 s \sigma_n \sigma_{n'} \sigma_n .
\] (2.4.41)

One checks that \((\tau(s, \vec{n}, \vec{n}^\prime))^2 = \mathbb{1}\), and hence \(\tau(s, \vec{n}, \vec{n}^\prime) = \sigma_{n''}\) for some \(\vec{n}'' = \vec{n}''(s, \vec{n}, \vec{n}^\prime)\), so that
\[
\exp(is\sigma_m) \exp(i\psi\sigma_n) \exp(-is\sigma_m) = \exp(it\sigma_{n''}) ,
\] (2.4.42)

Thus the value of \(\psi\) remains unchanged when conjugating with any group element (i.e., indeed parameterizes conjugacy classes). In other words, it provides a good coordinate on the quotient of \(G\) by (the adjoint action of) any one-parameter subgroup \(H \cong U(1)\). This is of course not surprising, since the adjoint action of any subgroup preserves conjugacy classes.

Now to find another coordinate on the quotient \(Q = SU(2)/U(1)\), take, without loss of generality, \(\vec{n} = \hat{e}_3\) in (2.4.42). One then obtains
\[
\tau(s, \hat{e}_3, \vec{n}) = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} e^{2is} \\ \sin \theta e^{i\phi} e^{-2is} & -\cos \theta \end{pmatrix} .
\] (2.4.43)

We conclude that the coordinate \(\theta\) is unaffected by the gauging \(g \sim hgh^{-1}\) and is thus (also) a good coordinate on the quotient. In contrast, the other coordinate \(\phi\) on \(S^2\) gets shifted by \(-2s\) and hence is gauged away. To summarize, the quotient by the adjoint action of \(U(1)\) is parametrized by the coordinates \(\psi\) and \(\theta\) of \(SU(2)\). From the form of the metric (2.2.26) we see that at \(\psi = 0, \pi\), that is at the endpoints of the interval over which \(\psi\) ranges, the space degenerates to a point. Hence the coset \(PF\) is topologically a disc, parameterized by \(\psi \in [0, \pi]\) and \(\theta \in [0, \pi]\).

In terms of the coordinates
\[
x := \frac{\pi}{2} \cos \psi \quad \text{and} \quad y := -\frac{\pi}{2} \cos \theta \sin \psi ,
\] (2.4.44)

\(Q_{PF}\) is the set of points \((x, y) \in \mathbb{R}^2\) in the range
\[
x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] , \quad y \in \left[-\sqrt{\left(\frac{\pi}{2}\right)^2 - x^2} , \sqrt{\left(\frac{\pi}{2}\right)^2 - x^2}\right] ,
\] (2.4.45)
i.e. a (round) disk of diameter \(\pi/2\). Embedding this disk \(D_{\pi/2}\) in its covering space \(SU(2)\) at \(\phi = 0\) describes it as the set of group elements
\[
g(x, y) = g(\arccos \frac{2x}{\pi}, \arccos \frac{-2y}{\sqrt{\pi^2 - 4x^2}} , 0)
\] (2.4.46)
\[= \frac{2x}{\pi} \mathbb{1} - \frac{2y}{\pi} i\sigma_3 + \frac{1}{\pi} \sqrt{\pi^2 - 4x^2 - 4y^2} i\sigma_1 .\]
Parafermion functions  The functions on the parafermion coset can be characterized as those functions on \( G = SU(2) \) that are invariant under the gauged \( H = U(1) \)-action. For the basis functions \( D^\Lambda_{mn} \) of \( SU(2) \) this means
\[
D^\Lambda_{mn}(g) = D^\Lambda_{mn}(hgh^{-1}),
\]
which holds true iff \( n = m \). In the notation of (2.4.14), we denote the \((1 \times 1)\) \( H \)-matrices \( D^\Lambda_{\lambda \lambda} \); each \( \lambda \prec \Lambda \) occurs with multiplicity one. Thus, the prediction of (2.4.14) agrees with the direct analysis (2.4.47).

2.4.5 The diagonal coset

Another coset which will be discussed is
\[
Q = \frac{SU(2) \times SU(2)}{Ad(SU(2))},
\]
which is the set of equivalence classes \([g_1, g_2] \) with the equivalence relation
\[
(g_1, g_2) \sim (gg_1g^{-1}, gg_2g^{-1}),
\]
where \( g, g_1, g_2 \in SU(2) \). Recall (2.4.35) that we parameterize group elements \( g \in SU(2) \) as
\[
g(\psi, \theta, \phi) = \cos \psi \mathbf{1} + i \sin \psi \bar{n} \cdot \bar{\sigma}.
\]
Here \( \bar{n} = \bar{n}(\theta, \phi) \in S^2 \) is given by the standard spherical coordinates \( \theta \) and \( \phi \). Note that we have
\[
g^{-1} = g(-\psi, \theta, \phi) = g(\psi, \pi - \theta, \pi + \phi) = g^\dagger.
\]
The group manifold \( SU(2) \) can be written as a semidirect product of the conjugacy classes \( S^2 \) times the interval. For any \([g_1, g_2] \) we can choose a representative of \( g_1 \sim t_1 \) in \( T_W = T/W = [0, \pi] \), where \( T \) is a maximal torus and \( W \cong \mathbb{Z}_2 \) is the (horizontal) Weyl group. It remains to determine where we can choose a representative of \( g_2 \) with the remaining ‘gauge freedom’
\[
S_{t_1} := \{ g \in SU(2) \mid gt_1g^{-1} = t_1 \}.
\]
The stabilizers at the center \( Z = \{ \pm e \} \) are given by \( S_{\pm e} = G \), and for other \( t_1 \in T_W/Z \) we have \( S_{t_1} \cong T \) (which we shall show below, see (2.4.55)). This gives the fibration
\[
Q = \left\{ (t, q) \mid t \in T_W, \quad q \in T_W \quad \text{if } t = \pm e, \quad q \in G/Ad(T) \quad \text{else} \right\}.
\]
In section 2.4.4, we saw that the geometry of the parafermion coset $PF = G/\text{Ad}(T)$ can be described as a round disc (see also [71]). It can also be described as disc with two ‘corners’. Thus, we may think of $Q$ as a pillow (with four corners), or as a pinched cylinder [32] where the corners are pairwise connected by two singular lines. The description above has an arbitrariness in that we could have equally well fixed $\psi_2 \in T_W$. Then, we could have a pinched cylinder with the singular lines going between other pairs of corners, instead. Either way, (2.4.48) has four corners. In the bulk, it is three-dimensional. We choose as coordinates the angles $\psi_1$, $\psi_2$ and $\theta$, where $\theta$ may be either $\theta_1$ or $\theta_2$. At the edges of the pillow, $\psi_1, \psi_2 = 0, \pi$, the coordinate $\theta$ has no meaning. The ambiguity in $\theta$ is the only ambiguity, we cannot get rid of $\psi_1$ and $\psi_2$ as coordinates (assuming a fixed choice of coordinates on the covering). In any case, the coordinates $\phi_1$ and $\phi_2$ are gauged away.

**Stabilizers** We can in a natural way think of $Q$ as embedded in its covering $G \times G$, via the above described coordinates. The stabilizer of the $\text{Ad}(G)$-action on an element $(t, q) \in Q \subset G \times G$ is given by those $g \in G$ for which

$$(t, q) = (gtg^{-1}, gqq^{-1}), \quad \Leftrightarrow \quad g \in S_{(t,q)} = S_t \cap S_q. \tag{2.4.54}$$

The stabilizers at the exceptional points $\pm e$ are $S_{\pm e} = G$. Since every element $q \in G$ can be brought to $T$ via conjugation; $cqc^{-1} = t_q \in T$ with an element $c \in G$, we can show that the generic (non-exceptional) stabilizers are given by

$$S_q = c^{-1}Tc, \tag{2.4.55}$$

with $c \in G$ a fixed element (depending on $q$). Assume $q \notin T$ and let $s \in S_{t_q}$, which equals $T$, then

$$(c^{-1}sc)q(c^{-1}sc)^{-1} = c^{-1}s t_q s^{-1}c = c^{-1}t_q c = q. \tag{2.4.56}$$

This shows $c^{-1}Tc \subset S_q$. Likewise, one can show $S_q \subset c^{-1}Tc$: Suppose $a \in S_q$, then

$$aqa^{-1} = q \quad \Rightarrow \quad caqa^{-1}c^{-1} = t_q$$

$$\Rightarrow \quad ca(c^{-1}t_q c)a^{-1}c^{-1} = t_q$$

$$\Rightarrow \quad cac^{-1} \in S_{t_q} = T,$$

and it follows that $c^{-1}Tc = S_q$. Indeed, one may choose $T$ to be any maximal torus, and one could equally well have defined $T = S_q$. Note that for $c \neq \pm e$, we have $c^{-1}Tc \cap T = Z$, the center of the group. The analysis is summed up in the following table.
Note that the choice of the edges was arbitrary, so they must indeed have the same stabilizer as the rest of the boundary of $Q$ (except for the corners). The corners are the most singular points, and have the largest stabilizer.

### CFT

The identification group is the one that geometrically corresponds to $\mathbb{Z}_G \cap H$. Thus, the identification group is generated by $(-e, -e) \in G \times G$, which corresponds to $\mathbb{Z}_2 = (0, 0, 0), (k, l, k + l)$. The conformal weight of the simple current generator $(k, l, k + l)$ is $(k + l)/2$, and $X = (k + l)/2$. The monodromy of a field $(a, b, c)$ with respect to this current is

$$Q_{(k,l,k+l)}(a, b, c) = \Delta(a, b, c) + \Delta(k, l, k+l) - \Delta(k-a, l-b, k+l-c).$$

For a field $(a, b, c)$ to be paired with itself in the partition function, the monodromy is required to be integer. The field $(a, b, c)$ is paired with its ‘$(k, l, k+l)$-conjugate’ $(k-a, l-b, k+l-c)$ iff $a + b - c = k + l \mod 2$. The identification group has a fix-point if both $k$ and $l$ are even, which then is $(k/2, l/2, k/2 + l/2)$. The resulting CFT naturally carries the structure of a Virasoro representation with central charge $c_Q = c_G - c_H$. In our case, we can see that if $l = 1$, this yields the Virasoro minimal models. (Note that we have a theory with $c < 1$ that describes a string propagating in a three-dimensional space.)

### 2.5 Lens spaces

Geometrically, a Lens space is an orbifold $SU(2)/\mathbb{Z}_l$. String theories, described as sigma models on such target spaces, were first studied in [55]. For the theory to make sense, we need the integer $l$ to divide the level of the $SU(2)_k$ WZW model we start with. Lens spaces will in the sequel be denoted $L_{k,k_1}$, where $l = k_1 | k$. The partition function of this theory was first found in [55], where it arose in the context of describing stringy black holes from a sigma model perspective. The D-branes in these Lens spaces were discussed, and to some extent classified in [71]. Later, the
partition function was reconstructed with simple current methods [1]. This allowed for a more detailed study of its boundary conditions, as we shall explain.

### 2.5.1 Partition function

One can obtain the Lens space partition functions starting from the tensor product theory $PF_k \otimes u(1)_k$, where we label the fields $(j, m, n)$. The first two integers $(j, m)$ are parafermion labels as described above, and $n$ is the $u(1)_k$ label. The $PF_k \otimes u(1)_k$ CFT has a $G \cong \mathbb{Z}_k \times \mathbb{Z}_{k_1}$ simple current group, with $k = k_1 k_2$ for some integer $k_2$. This realization of the group $G$ is generated by currents labeled $J_1 = (0, 2, 2)$ and $J_2 = (0, 0, 2k_2)$, which act under fusion as $(0, 2, 2) \ast (j, m, n) = (j, m+2, n+2)$, and $(0, 0, 2k_2) \ast (j, m, n) = (j, m, n+2k_2)$. Choosing the discrete torsion

$$X = \begin{bmatrix} 0 & -2/k_1 \\ 0 & -k/k_1^2 \end{bmatrix}, \quad (2.5.1)$$

gives us, via the simple current construction, the Lens space modular invariant

$$Z(\mathcal{L}_{k,k_1}) = \sum_{j=0}^{k-1} \left( \sum_{n-n'=0 \mod 2k_2 \atop n+n'=0 \mod 2k_1} \chi_{j,n}^{PF_k} \chi_{n'}^{U(1)_k} \right) \bar{\chi}_j^{SU(2)_k}. \quad (2.5.2)$$

In particular, the Lens space $\mathcal{L}_{k,1}$ is $SU(2)$, because in this case (2.5.2) simply gives back the defining equation for the $PF_k$ characters (compare (2.4.19)); the expression in the large bracket is

$$\sum_{n-n'=0 \mod 2k \atop n+n'=0 \mod 2} \chi_{j,n}^{PF} \chi_{n'}^{U(1)} = 2^{-1} \sum_{n=0}^{2k-1} \chi_{j,n}^{PF} \chi_{n}^{SU(1)} = \chi_j^{SU(2)}. \quad (2.5.3)$$

(The selection rule fixes the parity of $n$, and since $n = n' \mod 2k$ the constraint $n + n' = 0 \mod 2$ is redundant.)

The (for general $k_1$) nontrivial discrete torsion (2.5.1) causes left- and right-moving currents to be combined in an asymmetric way in the partition function. When the theory is asymmetric, it cannot be used to describe strings with un-orientable world sheets, since there is no well defined notion of left and right moving when there is no notion of in- and out side. In the literature, as for example in [41], results (such as boundary coefficients) are stated for general (oriented and un-oriented) world sheets, including such results which in fact hold true for more general discrete torsion [59], which can only appear on oriented world sheets.
2.5.2 T-duality

From the relation \( \chi_{-n}^{U(1)} = \chi_{n}^{U(1)} \), one sees that the Lens space partition function (2.5.2) is invariant under exchanging \( k_1 \) with \( k_2 \). This is interpreted as a T-duality (along the Cartan subalgebra \( U(1) \) in \( SU(2) \)) between the \( L_{k,k_2} \) and the \( L_{k,k_1} \) theory [71].

The fundamental group of \( SU(2) \cong S^3 \) is trivial, and therefore, there cannot be winding excitations. Thus, if T-duality is interpreted as an exchange of momentum- and winding excitations (as it is for the free boson), the T-dual lens space \( L_{k,k} = SU(2)/\mathbb{Z}_k \) is forbidden to have momentum excitations. The partition function is

\[
Z(L_{k,k}) = \sum_{j=0}^{k} \left( \sum_{n} \chi_{jn}^{PF_k} \chi_{2k-n}^{U(1)_k} \right) \bar{\chi}_{j}^{SU(2)_k}. \tag{2.5.4}
\]

We would like to interpret the momentum excitations as those functions on \( SU(2) \) that are invariant under the orbifold group \( \mathbb{Z}_k \), and thus project to functions on the orbifold. The orbifold group is generated by \( g_k = e^{2\pi i/kH_3} \) with \( R_{mn}^{g_k}(gg_k) = e^{2\pi in/k}R_{mn}(g) \). Thus, only the functions \( R_{mk}^k(g) \) correspond to momentum excitations on \( L_{k,k} = SU(2)/\mathbb{Z}_k \). We can think of the state \( \chi_{jm}^{PF_k} \chi_{m}^{U(1)_k} \chi_{jn}^{PF_k} \chi_{n}^{U(1)_k} \) in the partition function as corresponding to the function \( R_{mn}^k(g) \) on the group \( SU(2) \). The states \( \chi_{jm}^{PF_k} \chi_{m}^{U(1)_k} \chi_{jn}^{PF_k} \chi_{n}^{U(1)_k} \) with \( m \neq m' \) cannot in an obvious way be associated to functions on the group. Thus, one might try to postulate that these are winding excitations. The only state in the partition function that corresponds to a \( \mathbb{Z}_k \) invariant function is the one that corresponds to the function \( R_{kk}^k(g) \). Thus, there is only one momentum excitation in the partition function (2.5.4) of \( SU(2)/\mathbb{Z}_k \), and the other states are winding excitations. This is difficult to match with the T-dual \( SU(2) \), which has no winding excitations, only momentum excitations.

A similar analysis can be done with the \( SO(3) \) partition function, which is

\[
Z(L_{k,2}) = \sum_{j=0,2,...}^{k} \left( \sum_{n=0 \mod 2} \chi_{jn}^{PF_k} \chi_{n}^{U(1)_k} \right) \bar{\chi}_{j}^{SU(2)_k} \tag{2.5.5}
\] 

\[
+ \sum_{j=0}^{k} \left( \sum_{n=k/2 \mod 2} \chi_{jn}^{PF_k} \chi_{n+k}^{U(1)_k} \right) \bar{\chi}_{j}^{SU(2)_k}.
\]

The first sum contains the momentum excitations and the second sum contains what we might think of as being both momentum- and winding excitations (\( SO(3) \) has a
The $\mathbb{Z}_2$ fundamental group). The T-dual partition function is

$$Z\left(L_{k,k/2}\right) = \sum_{j=0}^{k} \left( \sum_{n=0 \text{ mod } 2} \chi^{PF_k, U(1)_k}_{jn} \chi^{U(1)_k}_{-n} \right) \bar{\chi}^{SU(2)_k}_j$$

(2.5.6)

$$+ \sum_{j=0}^{k} \left( \sum_{n=k/2 \text{ mod } 2} \chi^{PF_k, U(1)_k}_{jn} \chi^{U(1)_k}_{-n+k} \right) \bar{\chi}^{SU(2)_k}_j.$$

In the first sum, we have something that corresponds to a function when $n = k$, which forces $j = k$, which gives us the functions $D^k_{km}$. In the second sum, we have the possibility $n = k/2$ or $n = -k/2$, both force $4|k$ and those states with $j \geq |n|$ can correspond to a function. Thus we would also have the functions $D^j_{\pm k/2,m}$. We conclude that since the counting of functions and momenta here depends on whether $4|k$ or not, it is again difficult in this situation to interpret T-duality as an exchange of winding and momentum excitations.

### 2.6 CFT and geometry

Different target space can give the same CFT, as is the case when we have T-dualities. Thus, there is no one-one relation between target space and geometry. It seems that if we think of string theory as a CFT on the world sheet, the notion of target space is an ‘emergent’ concept.

If we would like to regard string theory as ‘fundamental’, it makes sense to say that aspects of our world that cannot be ‘seen’ by strings, are not present. This point of view has important consequences to the concept of spacetime and target space. Suppose we live in a world which is described by a string theory, and suppose further that the semiclassical intuition that ‘the string seeks to minimize the area that it sweeps out as it propagates through target space’ makes sense. Then it follows that strings cannot come arbitrarily close to each other, which introduces a fundamental length scale into physics, beyond which distances cannot be measured. This is a property which manifolds do not have.

Thus we should not take for granted that the target space - our spacetime - is a manifold. (Of course, we expect the target space should to be such that it looks like a manifold ‘at large distances’.) It is believed that target spaces can be described by non-commutative geometry, see for example [20, 38] and references therein. Non-commutative geometries are described by algebras, which we think of as the algebra of functions on the target space. On a manifold, the product in the corresponding
algebra is usually given by point-wise multiplication of (complex- or real valued) functions, and is commutative. If we only have a finite set of functions, we must define some other product so that the algebra closes.

Whether we choose to interpret target spaces as non commutative geometries, or if we choose to think of the concept of target spaces as emerging in some limit ($k \to \infty$ for the WZW model), we must not forget that geometric interpretations of CFT are problematic and indirect.
3 D-branes and Boundary states

In the remainder of this thesis we shall discuss CFT on open world sheets, and applications to open string theories. Open strings have end points, and when these end points are constrained to move on a subset of target space, this subset is called a D-brane. The introduction of D-branes allows for more possibilities to build realistic string models. For example, in (non-compact) Minkowski space, photons are open string states. Closed strings cannot give rise to spin-one particles due to level-matching [72]. (However, as was known already to Kaluza and Klein (see [66]), spin-1 fields can appear in circle compactifications of a spin-2 field.) Another reason to introduce D-branes is that D-branes provide a mechanism for the emerging effective four-dimensional theory to have gauge groups. Thus, we would like to consider strings with charges, that interact with such gauge groups. If we consider a propagating string which splits into two, we may ask what happens to the charge. It appears that the only places where the charges can reasonably be, are at the string end points. Thus, we postulate that the string end point carries a label $1 \leq i \leq N$, describing some kind of charge. One can argue that amplitudes are invariant under $U(N)$ [73], which then is the gauge group. It is also possible to find other gauge groups, cf. [97].

We have T-duality for closed strings, and would also like to have it for open strings. Classically, we think of T-duality as a relation between momentum and winding excitations. If the end points of the open strings move freely, it is difficult to see how we at all can have winding excitations for open strings: the string tension would cause the string to ‘unwrap’ the circle it winds. With D-branes, we can fix the open string end points and still have T-duality.

One way to describe D-branes is in terms of Dirichlet boundary conditions that are imposed on the embedding fields in the sigma model. In this setting, it is very clear where the string end points are located. It is less clear, however, whether the resulting boundary CFT is really conformal. Alternatively, one can describe
D-branes in terms of conformal blocks. In that description, the boundary CFT is manifestly conformal. However, in that description it is less clear where the string end points really are located. This question whether these descriptions agree is the main topic of the present thesis. We shall see that for $u(1)$ models, the description with the sigma model is in perfect agreement with the description in terms of boundary blocks. In WZW and related models, there is a certain disagreement which we shall discuss. Thus, it seems that CFT formulated in terms of boundary blocks is not equivalent to CFT as described in terms of path integrals.

### 3.1 Boundary Functionals

We shall describe D-branes in terms of one point correlators on the disc. These correlators can be described as functionals $\langle\langle B_\alpha |$ acting on the states $|v\rangle$. These $\langle\langle B_\alpha |$ are commonly denoted boundary states, and are subject to various conditions, which shall be discussed below. We shall see that in many cases, the solutions to all these constraints can in a natural way be associated to certain functions $f_\alpha$ on target space $M$. These functions are then interpreted as the shapes of the boundary states. In case the shape $f_\alpha$ is a delta distribution located at some set $D \subset M$, we can in a natural way interpret the boundary state $\langle\langle B_\alpha |$ as describing a D-brane located at $D$.

#### 3.1.1 Ward identities

The one-point correlator on the disc $D$ is a special case of (1.4.3),

$$
\langle \Phi(p, v_\lambda \otimes v_\mu) \rangle_{D|p} := B_{S^2\setminus\{p', p''\}}(v_\lambda \otimes v_\mu).
$$

(3.1.1)

The right hand side is a two point conformal block on the sphere $S^2$ which is the double of the disc, with two marked points $p', p''$ related by $p' = \sigma(p'')$ removed; hence the notation $S^2 \setminus \{p', p''\}$. In Dirac notation we write this correlator as $\langle\langle B |\Phi_{\lambda, \mu}\rangle$. To think of $\langle\langle B |$ as a state makes as much sense as thinking of the Dirac delta function as a function, neither of them is normalizable, but when acting on states that have finite norm they give meaningful results.

A conformal 2-point block on the sphere (recall the discussion in section 1.4) $\hat{X} = S^2$ is a mapping

$$
B : \mathcal{H} \otimes \mathcal{H}' \rightarrow \mathcal{M}(S^2 \setminus 2) \ltimes \mathbb{C}.
$$

(3.1.2)

It maps the state space to a $\mathbb{C}$-bundle $\mathcal{M}(S^2 \setminus 2) \ltimes \mathbb{C}$ over the moduli space $\mathcal{M}(\hat{X} \setminus 2)$ of $S^2$ with two marked points. We shall think of $S^2$ as $\mathbb{C} \cup \{\infty\}$ and take the two
points to be 0 and ∞. Other choices of insertion points are related to this one by conformal transformations, thus the moduli space is a point and the boundary state is a functional

$$\langle\langle B \rangle \rangle : \mathcal{H} \otimes \mathcal{H}^\vee \to \mathbb{C}. \quad (3.1.3)$$

The boundary state $B$ is required to be conformally invariant, which gives the Ward identity (1.4.9)

$$\langle\langle B \rangle \rangle \circ (L_n \otimes 1 - 1 \otimes L_{-n}) = 0. \quad (3.1.4)$$

If we have an additional current $J^a$ of conformal weight $\Delta_a$, we also require that the Ward identity (1.4.7)

$$\langle\langle B \rangle \rangle \circ (J^a_n \otimes 1 - (-1)^{\Delta_j} 1 \otimes J^a_{-n}) = 0 \quad (3.1.5)$$

is satisfied. We shall regard this as defining symmetry preserving boundary states. One can relax the latter condition and still have conformal boundary conditions, as long as (3.1.4) is satisfied with $L_n$ the generators of the Virasoro algebra. In the literature, the requirement for symmetry preserving boundary states (3.1.5) is by some authors replaced by introducing an automorphism $\omega$,

$$\langle\langle B \rangle \rangle \circ (J^a_n \otimes 1 - (-1)^{\Delta_j} 1 \otimes \omega(J^a_{-n})) = 0. \quad (3.1.6)$$

These more general boundary states with a twist $\omega$ are sometimes still called symmetry preserving, cf. [76].

### 3.1.2 Cardy constraints

The Ward identities are linear equations, and the space of solutions has as many dimensions as there are primary fields in the theory. A natural basis of the space of solutions to the Ward identities are the Ishibashi [61] blocks $\langle\langle I_i \rangle \rangle$, that act as $\langle\langle I_i \rangle \rangle \cdot v_j \otimes v_k \sim \delta_{ij} \delta_{ik}$ on vectors $v_j \otimes v_k \in \mathcal{H}_i \otimes \mathcal{H}^\vee_k \subset \mathcal{H} \otimes \mathcal{H}^\vee$. There are also non-linear relations which the boundary states must satisfy and which are not satisfied by the Ishibashi blocks, accordingly we shall take as boundary states $\langle\langle B_\alpha \rangle \rangle$ certain linear combinations of the Ishibashi blocks $\langle\langle I_i \rangle \rangle$. We require that the set of correlators satisfy the factorization constraints discussed in section 1.4.5. For the boundary states that we discuss here, factorization constraints imply the Cardy conditions [69]. We require that we obtain a sensible open string partition function from considering the time-space rotated process where one closed string is created at a brane and annihilated at another brane, as illustrated in figure 4. This leads
to nonlinear equations singling out certain linear combinations $\langle\langle B\rangle\rangle$ in the space spanned by the conformal blocks,

$$\langle\langle B_\alpha \rangle\rangle = \sum_\lambda R_{\alpha\lambda} \langle\langle I_\lambda \rangle\rangle.$$  \hspace{1cm} (3.1.7)

It is these linear combinations with the reflection coefficients $R_{\alpha\lambda}$ [90] that are the boundary states.

Figure 3.1: The Annulus. The picture to the left describes a closed string created at a brane, propagating in the direction of the arrow to another brane. (For illustration, the first brane has been made invisible, and the second is drawn as a plane.) This is (on the world sheet) related, via a modular S-transformation, to an open string ending at the same two branes, circling as indicated by the arrow on the picture to the right.

Consider a closed string propagating from one D-brane described by the boundary state $\langle\langle B_\alpha \rangle\rangle$ with boundary condition $\alpha$ to a boundary state $\langle\langle B_\beta \rangle\rangle$. The amplitude for this propagation is given by $\langle\langle \alpha | e^{Ht/2} \beta \rangle\rangle$, where $| e^{Ht/2} \beta \rangle$ is a time-propagated boundary state. We require that this overlap also can be interpreted as a partition function for an open string with endpoints on each of these branes. With $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i / \tau}$,

$$Z_{\alpha\beta}(q) = \langle\langle \alpha | \tilde{q}^{H/2} \beta \rangle\rangle = \sum_\nu N_{\alpha\beta}^\nu X_\nu(q),$$ \hspace{1cm} (3.1.8)

with $H$ the Hamiltonian. The right hand side is an open string (annulus) partition function, hence we demand that:

I) the identity appears only once, and only when $\alpha = \beta$ (concretely, $N_{\alpha\beta}^0 = \delta_{\alpha\beta}$), and

II) all coefficients are nonnegative integers, $N_{\alpha\beta}^0 \in \mathbb{Z}$. 
These conditions I and II are often called the Cardy conditions, and are satisfied if the boundary coefficients are given by certain combinations S-matrices [19]. Such boundary states are called Cardy states, and can be shown to be part of a set of factorizable correlators [30, 42]. There are often other solutions to the constraints I and II, but they are not guaranteed to satisfy the factorization constraints. Indeed, there are counterexamples of nonnegative integer matrices $Z_{\alpha \beta}$ that satisfy Cardy’s conditions but do not satisfy factorization constraints [47].

The Verlinde formula guarantees that the Cardy constraints are satisfied for the symmetry preserving boundary states if the boundary coefficients are given by (appropriately normalized) modular $S$- matrix elements

$$R_{nm} = \frac{S_{nm}}{\sqrt{S_{0m}}}.$$  

(3.1.9)

3.1.3 The shape of a boundary state

When we talk about CFT’s that have a target space interpretation, it is interesting to see if the boundary states can be thought of as localizing the string endpoints to certain subsets of target space. As the symmetry preserving boundary states are built out of dual string states, we want to relate $\langle\langle B|\rangle\rangle$ to a (Fourier transformed) function on target space $T$, sending $g \in T$ to what we call $\langle\langle B|g\rangle\rangle$. It is this function that we will see converges to a delta function in certain cases.

To think of the object $\langle\langle B|g\rangle\rangle$ as an amplitude leads to the idea that $|\langle\langle B|g\rangle\rangle|^2$ should be a probability distribution. However, when $\langle\langle B|g\rangle\rangle$ converges to a delta distribution, its square does not converge to anything that can be interpreted as a probability distribution. Neither does it make sense to think of $\langle\langle B|g\rangle\rangle$ as it stands as a probability distribution; as we shall see it takes negative - and sometimes even complex - values. It appears that the only sensible interpretation of the function $\langle\langle B|g\rangle\rangle$ is as a (generalized) Fourier transformation of the boundary functional - and nothing more.

If it makes sense to think of the boundary state as a function on some target space, one calls this function the shape, or profile, of the brane. From the sigma model perspective, D-branes are introduced in terms of Neumann- and Dirichlet boundary conditions, which localize the string endpoints to certain subsets of the target space. Recall that a Dirichlet boundary condition fixes the value of the field at the boundary, that is $X^\mu(\sigma, \tau)$ on the boundary of the world sheet is not varied when one derives the Euler-Lagrange equations. Neumann boundary conditions
means that \( \partial_\sigma X^n(\sigma, \tau) = 0 \) on the boundary of the world sheet. Both of these conditions relate the right moving excitations to left moving excitations (intuitively one can understand this by picturing the boundary of the world sheet as reflecting incoming waves).

There is no obvious reason why the Cardy boundary states would give something that corresponds to Neumann- or Dirichlet boundary conditions on string endpoints. We shall see that in flat target spaces, the boundary states are shaped like delta distributions, and indeed correspond to Neumann- or Dirichlet boundary conditions. In WZW models at finite level, these shapes are smooth functions, which in the large level limit converge to delta distributions.

### 3.2 D-branes in flat spaces

When the target space is a circle, the symmetry algebra is \( \mathfrak{u}(1)_k \), with commutators given by equation (2.1.3). There is a procedure to associate functions to boundary functionals, which as we shall see gives either a delta distribution or a constant function, for the Dirichlet- or Neumann boundary state, respectively (see [25]). It is customary to introduce the Ishibashi ‘state’ \(|I\rangle\rangle\) which is the Hermitian conjugate of the solution of the Ward identity (but not a state in any Hilbert space, as it has infinite norm, see (3.2.7)). Let us assume that \(|I\rangle\rangle\) is in (some completion of) the closed string state space, and that it can be written as some operator \(K\) acting on some state \(|\cdot\rangle\) at grade zero;

\[
|I\rangle\rangle = K |\cdot\rangle. \tag{3.2.1}
\]

The set of solutions to the Ward identities is a vector space. We shall consider linear combinations of Ishibashis \(|I\rangle\rangle\), so let us at this point consider those Ishibashis which descend from the vacuum state \(|p_L, p_R\rangle\). We begin with the (Hermitian conjugate of) the Neumann Ward identity

\[
\langle\langle B| \circ (\alpha_n \otimes 1 + 1 \otimes \alpha_{-n}) = 0. \tag{3.2.2}
\]

This differs from (3.1.5) (which we shall refer to as the Dirichlet Ward identity) by a sign. In the literature, one often finds the notation \(\tilde{\alpha}\) for the generators of the right moving algebra, and the Neumann Ward identity can be written as \(\langle\langle B| \circ (\alpha_n + \tilde{\alpha}_{-n}) = 0\). This is one equation for each integer \(n\). With \(n = 0\), the Neumann Ward identity now implies that \(p_L = -p_R\), thus the total momentum of this boundary block vanishes, and it is a constant function.
For the Dirichlet Ward identity $\langle\langle B|\circ(\alpha_n - \tilde{\alpha}_{-n}) = 0$ with $n = 0$, we have

$$p_L = p_R = \frac{1}{2}p = \frac{m}{2R},$$

(3.2.3)

where $R$ is the radius of the circle. Accordingly, it is customary to label the Ishibashis with quantum numbers $m$ such that

$$\langle\langle I_m \rangle\rangle = \langle m, m | K^\dagger.$$

(3.2.4)

We may without loss of generality assume that $K$ does not contain annihilation operators. Indeed, if $K$ contains a term with an annihilation operator, we may use the commutation relations (2.1.3) to move this operator to the right. For the $m \neq 0$ Dirichlet Ward identities we need $[\alpha_m, K]|n, n\rangle = -\tilde{\alpha}_{-m}K|n, n\rangle$. This holds if $[\alpha_m, K] = -\tilde{\alpha}_{-m}K$, which is satisfied by

$$K = \prod_{n>0} e^{-\frac{1}{n}\alpha_n\tilde{\alpha}_{-n}}.$$  

(3.2.5)

(It is not clear from the argument presented above that we find a unique solution.)

To the Ishibashi block with momentum $m$, it is natural to associate a function

$$\langle\langle I_m | x \rangle\rangle = Ne^{\frac{2\pi}{x}},$$

(3.2.6)

where $N$ is some normalization constant. If we choose $N = 1/\sqrt{2\pi R}$, the Ishibashis are orthonormal with the usual integration measure on the circle. The normalization of the Ishibashi block (3.2.4) is

$$\langle\langle I_m | e^{-\frac{2\pi}{x}(L_0 + L_0^{-1/12}) I_n}\rangle\rangle = \delta_{mm} \chi_n(\frac{2i}{t}).$$

(3.2.7)

Note that this is an affine Lie algebra character; when $t \to \infty$ its value is the dimension of the representation, which is infinite. In a rational free boson at level $k$, the momenta are labeled $m = 0, \ldots, 2k - 1$ and the algebra is extended so that $m = 2k$ is a descendant of $m = 0$. The characters are

$$\chi^r_m = \sum_{n \in \mathbb{Z}} \chi_{m+2kn},$$

(3.2.8)

and accordingly, the Ishibashis are

$$\langle\langle I_m^r \rangle\rangle = \sum_{n \in \mathbb{Z}} \langle\langle I_{m+2kn} \rangle\rangle.$$  

(3.2.9)

Thus, the Ishibashi block does not have a definite non-vanishing momentum, but we can still use our intuition from quantum mechanics to postulate that it has the shape

$$I_m^r(x) = \sum_{n \in \mathbb{Z}} I_{m+2kn}(x) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} e^{\frac{2\pi i}{x}(m+2kn)x}.$$  

(3.2.10)
The Cardy constraints (section 3.1.2) say that the boundary states are certain linear combinations of Ishibashis, with boundary coefficients $B_{nm}$ such that

$$\langle\langle B_n|q^{L_0+L_0-1/12} B_m\rangle\rangle$$

is a non-negative integer linear combination of characters. With the $S$-matrix given by (2.1.17) and the boundary coefficient given by (3.1.9), the boundary states are written in terms of Ishibashis as

$$\langle B_n\rangle = k^{-\frac{1}{2}} \sum_{m=1}^{2k} e^{i\pi n^{m}} \langle I_m\rangle .$$

We can now calculate the overlap (3.2.11)

$$\langle\langle B_n|q^{L_0+L_0-1/12} B_{n'}\rangle\rangle = \sum_{m} \frac{1}{\sqrt{k}} e^{i\pi m(n-n')} \chi_{m} \left( \frac{2i}{t} \right)$$

and see that it indeed satisfies the Cardy conditions. With $R = \sqrt{2k}$, the shape of the brane is

$$\langle x|B_n\rangle = \frac{1}{\sqrt{2\pi R k}} \sum_{m \in \mathbb{Z}} \exp \left( -2\pi i m \left( \frac{n}{2k} - \frac{x}{R} \right) \right) = \frac{1}{\sqrt{\pi R}} \delta \left( \frac{n}{R} - x \right).$$

Note that the form of the operator $K$ is not essential to the analysis of the shape. What is essential is that we associate the function $e^{ipx}$ to the state with momentum $p$.

### 3.3 D-branes in WZW models

We shall go on to discuss D-branes in WZW models. At finite level, there is a finite number of Ishibashis, to each of which we assign a smooth function (3.3.14).

A finite linear combination of smooth functions is again a smooth function, thus the shape of the boundary state is not a delta distribution. We shall discuss some alternative ways to assign functions to Ishibashi blocks. To have delta function -shaped boundary states, we would need something like (3.2.10) in the WZW model. It is perhaps tempting to try something like (3.3.27) below, which is problematic for reasons that shall be discussed.

**Lagrangian Description** D-branes are described as objects which have tension, see [71]. In order to have such objects with non vanishing volume which are stable, one needs to have some stabilizing flux (unless the brane wraps a topological
defect). Such a flux is provided by the WZ-term, which is needed for holomorphic factorization, see section 2.2.1. Such a term can only be defined (uniquely up to an integer ambiguity) in case the target space satisfies certain conditions. Recent work [48] shows that this can be done for oriented world-sheets, by introducing gerbe module structures (in more general settings one needs to define it in term of bundle-gerbes). For un-oriented strings, the question is settled only under certain conditions [84]. In any case, there are restrictions which constrain the set of allowed sub-manifolds where branes may be localized.

3.3.1 Boundary states

The Ishibashi blocks obeying the Ward identities (3.1.5) and therefore preserving the \( g \)-algebra, are labeled by the irreducible integrable highest weights \( \lambda \in P_k \) at level \( k \). They are towers of affine lowering modes acting on primary fields also labeled \( \lambda \in P_k \), with ‘normalization’

\[
\langle\langle I_\lambda | q L_0 \otimes 1 \otimes L_0 - c/12 | I_\mu \rangle \rangle = \delta_{\lambda,\mu} \chi_\mu(q^2),
\]

where \( \chi_\mu \) is a \( g \)-character (which is infinite at \( q = 1 \)). Apart from the normalization above, we need to know how these Ishibashi dual states act on elements in the state space. Recall that the space of closed oriented string states consists of two subspaces, called left-moving and right-moving. In the WZW model, the left moving state space \( \mathcal{H}^\mu \) is built out of affine generators \( J^a_n \) acting on a unique vacuum, which is annihilated by all positive mode generators [45]. This gives a graded vector space, where the grade zero subspace is called the horizontal subspace \( \mathcal{H}^0_\mu = \mathcal{H}^\mu \). Since the horizontal subalgebra \( \tilde{\mathfrak{g}} \subset \mathfrak{g} \) preserves the grading, the horizontal subspace is mapped onto itself by the action of \( \tilde{\mathfrak{g}} \). Moreover, each subspace with fixed grade \( \mathcal{H}^\mu_n \) is finite-dimensional. The right moving state space \( \mathcal{H}^{\mu\dagger} \) is similar, with the conjugate representation of \( \tilde{\mathfrak{g}} \) acting on the dual vectors in \( \mathcal{H}^{\mu\dagger}_n \) at each grade. On basis vectors

\[
v^\mu_m \otimes \tilde{v}^\mu_n \in \mathcal{H}^\mu \otimes \mathcal{H}^{\mu\dagger} \subset \bigoplus_{\nu \in P_k} (\mathcal{H}^\nu \otimes \mathcal{H}^{\nu\dagger}),
\]

where \( \{v^\mu_m\} \) is a basis of \( \mathcal{H}^\mu \), and \( \{\tilde{v}^\mu_n\} \) with \( \tilde{v}^\mu_n(v^\mu_m) = \delta_{mn} \) is a basis of the dual space \( \mathcal{H}^{\mu\dagger} \), the Ishibashi block acts as

\[
\langle\langle I_\lambda | v^\mu_m \otimes \tilde{v}^\mu_n = \delta_{\lambda,\mu} \delta_{m,n}. \ 
\]

The boundary states of the WZW model that also satisfy the factorization constraints are labeled, just like the Ishibashi blocks, by the allowed (irreducible highest
weight) affine representations at level $k$. A Cardy-type boundary state $\langle \langle B_\alpha \rangle \rangle$ with $\alpha \in P_k$ is a linear combination of the Ishibashi blocks $\langle \langle I_\lambda \rangle \rangle$ with coefficients given by combinations of modular $S$-matrix elements:

$$\langle \langle B_\alpha \rangle \rangle = \sum_\lambda \frac{S_{\lambda,\alpha}}{\sqrt{S_{0,\lambda}}} \langle \langle I_\lambda \rangle \rangle .$$

(3.3.4)

### 3.3.2 Regular D-branes

We analyze the shape of boundary states in general WZW models, and show that the shapes converge to certain delta distributions in the large level limit. A boundary state is given as a combination of Ishibashi functionals, given by (3.3.4) with the normalization of the Ishibashis as in (3.3.1). Using instead the more general normalization

$$\langle \langle I_\lambda \rangle \rangle |q^L_0 \otimes 1 \otimes L_0 - c/12 || I_\mu \rangle \rangle = \delta_{\lambda,\mu} \chi_\mu(q^2) N^2_\lambda ,$$

(3.3.5)

the boundary state is

$$\langle \langle B_\alpha \rangle \rangle = \sum_{\lambda \in P_k} \frac{S_{\lambda,\alpha}}{N_\lambda \sqrt{S_{0,\lambda}}} \langle \langle I_\lambda \rangle \rangle .$$

(3.3.6)

We can express the quotient of $S$-matrices in terms of group characters. A useful formula is [39] eq. (2.7.29);

$$S_{\lambda', \lambda} = S_{\lambda, \lambda'} = a_\lambda \chi_{\lambda'}(\tilde{\lambda}) \quad \tilde{\lambda} := \frac{2 \pi i}{k + g^\vee} (\lambda + \rho) ,$$

(3.3.7)

where $a_\lambda$ is a normalization constant [39] eq (2.7.30), and

$$S_{0,\lambda} = S_{\lambda 0} = a_\lambda .$$

(3.3.8)

The character in (3.3.7) is the horizontal Lie algebra character, from which the group character (which is what we are interested in) can be obtained via exponentiating the argument. (The exponential map is surjective on a compact connected Lie group, see [17], page 24.) Setting $N_\lambda = \sqrt{S_{0,\lambda}}$, we get the boundary coefficient

$$\frac{S_{\lambda,\alpha}}{S_{0,\lambda}} = \chi_\alpha(e^{\tilde{\lambda}}) ,$$

(3.3.9)

with $\tilde{\lambda}$ as in (3.3.7). Note that this equation relates the boundary coefficient to geometric data (the group character); this is the crucial step to the geometric analysis of the WZW boundary states in [29].
Using that the modular $S$-matrix is symmetric, one can write this as

$$\frac{S_{\lambda\alpha}}{S_{0\lambda}} = \frac{S_{\alpha\lambda}}{S_{0\alpha}} \frac{S_{0\alpha}}{S_{0\lambda}} = \chi_\lambda(e^\alpha) \frac{S_{0\alpha}}{S_{0\lambda}} \quad (3.3.10)$$

We have [39] (2.7.29) and [45] (13.32)

$$\frac{S_{0\lambda}}{S_{00}} = \frac{a_\lambda}{a_0} = \prod_\beta \sin \left( \frac{(\lambda + \rho, \beta)}{k + g^\vee} \pi \right) \quad \rightarrow \quad \prod_\beta \frac{(\lambda + \rho, \beta)}{(\rho, \beta)} = d_\lambda. \quad (3.3.11)$$

The product is over $r$ factors, $\beta$ ranges over the simple positive roots. With arbitrary normalization, the boundary coefficient in the large level limit is

$$\frac{S_{\lambda\alpha}}{N_\lambda \sqrt{S_{0\lambda}}} = \frac{d_\alpha}{d_\lambda} \chi_\lambda(h_\alpha) \quad h_\alpha := \exp \left( \frac{2\pi i}{k + g^\vee} (\alpha + \rho) \right). \quad (3.3.12)$$

The Ishibashi block $\langle\langle I_\lambda |$ with the normalization (3.3.5) does not act as (3.3.3), but as

$$\langle\langle I_\lambda | v^\mu_m \otimes \tilde{v}^\mu_n = N_\lambda \delta_{\lambda,\mu} \delta_{m,n}, \quad (3.3.13)$$

and can therefore be interpreted (2.2.1) as a function

$$I_\lambda(g) = N_\lambda \sqrt{\frac{d_\lambda}{|G|}} \chi_\lambda(g) \quad (3.3.14)$$

on the group manifold $G$. These Ishibashi functions are satisfy the normalization

$$\int_G dg I_\lambda(g) I_\mu(g) = N_\lambda^2 \delta_{\lambda,\mu} |d_\lambda|. \quad (3.3.15)$$

Hence the shape of the brane in the large level limit is

$$B_\alpha(g) = \frac{d_\alpha}{|G|} \sum_\lambda \chi_\lambda(h_\alpha) \chi_\lambda(g) = \frac{d_\alpha}{|C_\alpha|} \delta_{C_\alpha}(g) \quad (3.3.16)$$

using (2.2.9). This equation completes the argument [29] that the boundary states are delta functions in the large level limit.

### 3.3.3 Exceptional Branes

We shall argue that there are no boundary states at exceptional conjugacy classes. Note that it is not the integrable highest weights $\Lambda$ in the Weyl chamber $W$ that (directly) give the locations of the conjugacy classes on the maximal torus where the branes are localized; there is a shift and a compression

$$\Lambda \mapsto \hat{\Lambda} := \frac{k}{k + g^\vee} (\Lambda + \rho) \quad (3.3.17)$$
The mapping (3.3.17) of the fundamental Weyl alcove $W$ for $su(3)$ at level 3. The left picture shows the alcove $W$, with the dots indicating the location of the integral weights, while the right picture shows its image $\tilde{W}$ under (3.3.17) inside the original region.

The left picture shows the alcove $W$ for $su(3)$ at level 3. The right picture shows its image $\tilde{W}$ under (3.3.17) inside the original region. This maps $W$ to the interior of itself as illustrated in figure 3.2. The points on the boundary $\partial W$ are those weights $\tilde{\Lambda}$ whose associated Cartan subalgebra elements are mapped by the exponential mapping to points on exceptional conjugacy classes of $SU(3)$. The region $\tilde{W}$ (the grey area on the right hand side) is mapped to generic conjugacy classes on the group. Hence the boundary coefficients (3.3.9) are related to characters at regular points of a maximal torus.

$SU(2)$ In the $sl(2)_k$ WZW model, the primary fields $\lambda \in P_k$ are labeled by nonnegative integers $l \leq k$, and the symmetry preserving boundary states are given by

$$|B_a\rangle = \sum_{0 \leq l \leq k} \frac{\sqrt{2}}{k+2} \frac{\sin \left( \frac{(l+1)(a+1)}{k+2} \pi \right)}{\sqrt{\sin \left( \frac{l+1}{k+2} \pi \right)}} |I_l\rangle.$$  

This boundary state is often denoted as $C(A,a|$ where the $C$ stands for Cardy, and the $A$ for $A$-type, as opposed to the B-type branes discussed later, see section 3.4.2.

The Ishibashi block $\langle\langle I_l |$ can be related to a function on the group $SU(2)$ via the Peter-Weyl isomorphism; the functional $\langle\langle I_l | : v^l_m \otimes \bar{v}^l_n \mapsto \delta_{mn}$ is identified via the Peter-Weyl isomorphism (2.2.1) with the function

$$I_l(g) = \sum_{m<l} \sqrt{\frac{l+1}{V}} \langle v^l_m | R^l(g) | \bar{v}^l_m \rangle = \sqrt{\frac{l+1}{V}} \chi_l(g).$$

The sum is over $m = -l, -l+2, ..., l$, that is over $m$ that label all the basis vectors in the (horizontal) module $\mathcal{H}^l$. The character $\chi_l(g)$ is a class function, hence the shape of the brane only depends on an angle $\psi$ along a maximal torus $U(1) \subset SU(2)$. This
gives the following shape of the brane:

\[
C\langle A, l | g \rangle = \sqrt{\frac{2}{k+2}} \sum_{l'} \frac{\sin((l'+1)\hat{\psi}) \sin((l'+1)\psi)}{\sin \psi \sqrt{\sin \left(\frac{(l'+1)\pi}{k+2}\right)}}
\]  \hspace{1cm} (3.3.20)

\[
\hat{\psi} = \frac{(l+1)\pi}{k+2} \hspace{1cm} \text{(3.3.21)}
\]

with \(g \in SU(2)\) parameterized as in (2.4.50). This is nothing but the precise version of the formula (D.4) of [71] which (for finite levels) describes the branes as centered around 2-dimensional conjugacy classes. There is some disagreement in the literature as to whether there are 0-dimensional branes centered at the exceptional conjugacy classes \(\pm 1 \in SU(2)\) [71,88], (and some references therein) at infinite level, or if there do not exist such exceptional branes [29].

In section 3.3.2, we showed that the function (3.3.20) behaves for large \(k\) as

\[
C\langle A, l | g \rangle \longrightarrow (l+1) \frac{\delta_{C_{g_l}}}{|C_{g_l}|} = (l+1) \delta_T^{g_l},
\]  \hspace{1cm} (3.3.22)

with \(g_l\) as in (3.3.12). The volume of the conjugacy class is \(|C_{g_l}| = 4\pi k \sin^2 \psi_l\) with \(\psi_l\) as in (3.3.21), and decreases as \(k^{-1}\) at large level. With the expression (3.3.20) for the shape of the boundary state we can give a more explicit argument for the large level behavior than the argument that led to (3.3.16).

We see from (3.3.12) that when \(k \rightarrow \infty, h_{\alpha} \rightarrow 1\). Thus for \textit{any} fixed, level-independent \(l\) the brane \(C\langle A, l | g \rangle\) approaches the D0-brane located at \(g = 1\). Since \(C\langle A, l | g \rangle\) only depends on the coordinate \(\psi\), it is natural to regard it as a function on the maximal torus, parameterized (modulo the Weyl group) by the angle \(\psi \in [0, \pi]\), rather than on SU(2). When expressing \(B^{SU(2)}_A\) as a function of \(\psi\), we must take into account the volume of \(C_{g_l}\); thus when we want to visualize the way the brane tends to a delta function, we should study

\[
\tilde{B}^{SU(2)}_l(\psi) := |C_{g_l}| B^{SU(2)}_l(\psi) = 4\pi k \sin^2 \psi C\langle A, l | g \rangle
\]  \hspace{1cm} (3.3.23)

rather than \(C\langle A, l | g \rangle\) itself. In other words: we want to show that the boundary state is a class delta function in the sense of (2.2.6), thus when we restrict to the variable \(\psi\) on the torus, we must compensate for the volume of the conjugacy class.

Also, when we want to account for the growing radius \(r = \sqrt{k}\) of the group manifold \(S^3\) (and hence of the torus), we must measure distances as seen by a
"comoving" observer on the group, i.e. use the scaled variable \( \sqrt{k} \psi \). However, as we are particularly interested in the vicinity of \( \psi = 0 \), it is indeed convenient to introduce in addition the 'blow-up' coordinate \( a := (k+2)\psi / \pi = \psi / \psi_0 \). In terms of this parameter we have

\[
\tilde{B}_l^{SU(2)} (\psi) \bigg|_{\psi = a \psi_0} = 2^{7/4} k^{1/4} \pi^{1/4} \sin(a \psi_0) \sum_{l'=0}^{k} \sqrt{\psi_{l'}} \sin((l+1)\psi_{l'}) \sin(a \psi_{l'}) .
\]

In the limit \( k \to \infty \) the \( l' \)-summation turns into \((k / \pi \text{ times})\) an integration over \( t = \psi_{l'} \):

\[
\lim_{k \to \infty} \tilde{B}_l^{SU(2)} (\psi) = 2^{7/4} \sqrt{\frac{\pi}{k}} a \int_0^{\pi} f_l(a; t) \, dt
\]

with

\[
f_l(a; t) := \sqrt{\frac{t}{\sin t}} \sin((l+1)t) \sin(at) .
\]

For any \( \Lambda \) the integral \( F_\Lambda(a) := \int_0^{\pi} f_\Lambda(a; t) \, dt \) is a continuous function of \( a \) independent of \( k \); it follows from the general arguments in section 3.3.2 that the center of mass is at \( a = l+1 \), i.e. at \( \psi = \psi_l \). According to (3.3.25), in terms of the coordinate \( a \), asymptotically at large level the shape grows with the level uniformly as \( \sqrt{k} \). Indeed, the shape stabilizes already at small level; this is illustrated in figure 3.3 for \( l = 0 \), i.e. for the brane closest to the exceptional conjugacy class \( \{1\} \), and for \( l = 5 \).

In terms of the parameter \( \sqrt{k} \psi = a / \sqrt{k} \) which accounts for the growing radius, the width of the peak of the function \( F_l \) shrinks with the level as \( \sqrt{k} \), and so does the distance from its peak to the origin, as well as the distance between the peaks for any two different branes; the area under the peak stabilizes for large \( k \). Since the distance between the peaks for different \( l \), and between peak and origin, shrinks exactly at the same rate as the width of the peaks, even at arbitrarily large level we can distinguish the individual branes and distinguish their location from the origin. It is in this sense that in the large limit the branes keep being well separated from each other, and also well separated from the exceptional conjugacy class. To put it more sloppily, even at arbitrarily large level the branes insist on being well located at \( \psi_\Lambda \simeq l/k \) rather than at zero.

**General WZW models**  For general WZW models, we expect a similar behavior, namely that as \( k \) grows, the shape of any brane labeled \( \Lambda \) approaches the singular
Figure 3.3: The brane shape (3.3.25) as a function of the blow-up variable $a$ for $l = 0$ and $l = 5$ at levels $k = 10, 100$ and $10000$. For $k = 10$ and $k = 100$ the whole range $[0, k+2]$ of $a$ is shown; for $k = 10000$ the range $[0, 12]$ is displayed instead, in order to facilitate comparison with the diagrams in the first row.

points at the same pace as the width of the peak shrinks. Therefore, the peak never hits these singular points and if some branes with fixed highest weight are considered as singular in the large level limit, so are all branes with fixed highest weight in that limit.

3.3.4 Alternative function assignments

One might wonder if it is possible to associate functions to Ishibashi blocks in some way different from (3.3.14), such that branes are sharply peaked at finite level.
The boundary state in a rational theory is built from a finite number of Ishibashis, thus if it is to be sharply peaked the Ishibashi functions cannot be smooth. Recall that in \( U(1) \), each Ishibashi corresponds to an infinite number of identified momenta (3.2.10). In WZW models, if there is a corresponding identification of highest weights, we would expect something like a subgroup of the (affine) Weyl group \( W' \).

Recall that the horizontal Weyl group \( \bar{W} \subset W' \) consists of reflections and rotations, this is not quite what we want for our field identification; we want the part of \( W' \) that takes a Weyl chamber and duplicates it to fill out the Weyl alcove. This would correspond to the situation where we sum over all \( \lambda \in P_\infty \) in (3.3.6). Thus, we try

\[
I_{\Lambda} (g) = \sum_{\omega \in W \setminus \bar{W}} \sqrt{N_{\omega \Lambda}} \chi_{\omega \Lambda} (g).
\] (3.3.27)

Unfortunately, this function ruins the orthogonality of the Ishibashis for \( \Lambda \) on the boundary of the Weyl chamber: In \( SU(2) \), we would have

\[
I_j (g) = \sum_{n \geq 0} \sqrt{N_{j+nk}} \chi_{j+nk} (g),
\] (3.3.28)

and it is easy to see that \( I_0 (g) \) and \( I_k (g) \) are not orthogonal (unless we let some of the \( N_n \) vanish, which seems rather unnatural). Another (not very natural) ansatz would be

\[
I_j (g) = \sum_{n \geq 0} \sqrt{N_n} \chi_{j+n(k+a)} (g).
\] (3.3.29)

We see that for any integer \( a \geq 1 \), the Ishibashis with \( 0 \leq j \leq k \) are orthogonal. It remains to find a sequence \( \{N_n\}_n \) so that the normalization (3.3.15) is restored. Such a sequence would obviously have to decrease, which violates our original thought that the different terms in (3.3.27) should be on equal footing, as are those in (3.2.10).

It is tempting to try to reproduce instead the full Ishibashi block normalization (3.3.5) with \( N_\lambda = 1 \) and try something like

\[
\langle g|q^{L_{\mu} \otimes 1 + 1 \otimes L_0 - c/12} I_\lambda \rangle \rangle = \sum_{n=0}^{\infty} q^{2n} \sum_{\mu \prec_n \lambda} \mathcal{I}_n^\mu (g).
\] (3.3.30)

for some to be determined functions \( \mathcal{I}_n^\mu \). Here \( \mu \prec_n \hat{\Lambda} \) labels all horizontal submodules at level \( n \). These functions should be such that

\[
\langle \langle I_\mu | q^{L_{\mu} \cdots} I_\lambda \rangle \rangle = \int_G \text{d}g \langle \langle I_\mu | g \rangle \langle g | q^{L_{\mu} \otimes 1 + 1 \otimes L_0 - c/12} I_\lambda \rangle \rangle
\]

\[
= \delta_{\mu \lambda} \sum_{n=0}^{\infty} D_n^\lambda q^{2n} = \delta_{\mu \lambda} \chi_\lambda(q^2),
\] (3.3.31)
with $D^\lambda_n$ the number of weights in the representation $\hat{\lambda}$ at grade $n$. It is difficult to see how Ishibashi functions $I_\lambda$ and $I_\mu$ in (3.3.30) with $\lambda \neq \mu$ can be made orthogonal; it is common that the same horizontal lie algebra module $m$ occurs as a submodule of different affine modules $\lambda$ at grade $n \neq 0$. Thus, the different Ishibashi functions are not automatically orthogonal but must be made orthogonal by (probably quite unnatural) choices of the $I^\mu_n(g)$.

It seems that there are no good alternatives to the procedure (3.3.14), which takes only the horizontal part of the Ishiashi blocks and assigns to it the function prescribed by the Peter-Weyl isomorphism (2.2.1). This is expressed by saying that the boundary state is probed by the tachyons (which are at grade $n = 0$). One might try to take instead of the horizontal part, the part of the Ishibashi block at grade $n = 1$ [29], which gives results qualitatively similar to those obtained here. In particular, at finite level each boundary state is associated to a finite sum of smooth functions.

### 3.4 Branes in geometric orbifolds

We consider the shape of boundary states in a target space which is a $\Gamma$-orbifold of $G$, where $\Gamma$ is a finite group. The Ishibashi blocks in the corresponding CFT are constructed with simple current methods [41, 30], and we would like to associate functions to these. When the orbifold can be constructed with simple current methods and the simple current group $G \cong \Gamma$ has no fixed points, we can average out the action of $\Gamma$ on the Ishibashi functions on $G$. (Or, equivalently, one averages the functions associated to the boundary states over the action of the orbifold group.) This gives functions that are manifestly invariant under the orbifold group action, and therefore descend to functions on the orbifold space $G/\Gamma$.

#### 3.4.1 Simple current construction of Boundary states

The boundary states for CFT’s of simple current type have been built in [41]. More precisely, in [41] only the case of trivial discrete torsion was studied, because it allows for discussing also amplitudes on non-orientable world sheets. However, the results from [41] relevant to the present discussion are also applicable for non-trivial discrete torsion [59].
We first describe the Ishibashi blocks, in the language of the parent (covering) theory. The boundary blocks correspond to primaries that are combined with their charge conjugate in the partition function of the extended theory. We denote $G$ the simple current group and $S_\lambda \subseteq G$ is the stabilizer of $\lambda$. Applying the results of [41, 59], it follows that in terms of the parent theory the Ishibashi blocks are labeled by pairs $(\mu, F)$ with $F \in S_\mu$ that satisfy the requirement

$$Q_J(\mu) + X(J, F) \in \mathbb{Z}, \quad \forall \ J \in \mathcal{G},$$

with $X(J_{1}^{a_{1}} J_{2}^{b_{1}}, J_{1}^{b_{2}}, J_{2}^{a_{2}}) := \sum_{ij} a_i X_{ij} b_j$, where $J_i$ are generators of the simple current group.

The boundary states are labeled by $\mathcal{G}$-orbits on the set of all chiral labels of the unextended theory, possibly with multiplicities. More concretely, they are labeled by pairs $[\rho, \psi_\rho]$ with $\rho$ a representative of a $\mathcal{G}$-orbit and $\psi_\rho$ a $C_\rho$-character. Here $C_\rho \subseteq S_\rho$, the central stabilizer, is a subgroup in the stabilizer of square index [46].

The boundary coefficients $B_{(\mu, F), [\rho, \psi]}$ that appear in the expansion

$$|A, [\rho, \psi]\rangle = \sum_{j, r, F} B_{(\mu, F), [\rho, \psi]} |A; j, r, F\rangle \rangle$$

of the boundary states in terms of Ishibashi blocks are given by [41, 59]

$$B_{(\mu, F), [\rho, \psi]} = \sqrt{\frac{|G|}{|S_\rho| |C_\rho|}} \frac{S_{\mu, \rho}^F}{S_{\Omega, \mu}} \alpha_F S_{\mu, \rho}^* \psi(F)^*.$$  

(3.4.3)

These boundary states are symmetry breaking, as opposed to the symmetry preserving boundary states that are labeled by the primary fields of the (orbifold) theory.

In the closed string channel the annulus amplitude with boundary conditions $a, b$ is

$$A_{a}^{b}(t) = \langle a | e^{-\frac{2\pi}{T} (L_0 \otimes 1 + 1 \otimes L_0 - c/12)} | b \rangle.$$  

(3.4.4)

In the open string channel, we can expand the amplitude in terms of characters as

$$A_{a}^{b}(t) = \sum_{\nu} A_{a}^{\nu} b \chi_{\nu}(e^{2\pi i T}).$$

For a simple current construction, the annulus coefficients $A_{a}^{\nu} b$ appearing in this expansion are given by [59]

$$A_{a}^{\nu} b = \sum_{(\mu, J)} B_{(\mu, J), [a, \psi_a]} B_{(\mu, J^c), [b, \psi_b]} S_{\mu}.$$

(3.4.5)
3.4.2 Boundary states in Lens spaces

We shall investigate the shape of the boundary states in the lens space models. The parent theory is $PF \times u(1)$. We shall use the results of section 3.4.1 and the notation of section 2.5 to describe the boundary states, as was done in [1]. In section 3.4.3, we shall discuss some problems that arise when the simple current group has fixed points. This is for example the case in the lens space $SO(3)$. These problems were neglected in [1], thus we shall only be interested in those parts of the geometric analysis in [1] which do not deal with fixed points.

In our case, the stabilizer subgroups of the simple current group are trivial, that is $S_\mu = \{ \Omega \} \equiv \{(0,0,0)\}$, except when $k_1$ is even in which case we have $S_{(k/2,n,m)} = \{ \Omega, K \} \cong \mathbb{Z}_2$, generated by $K = (0,k,0)$. We first concentrate on the trivial element $\Omega$ of the stabilizers. We have $X(J,\Omega) = 0$ and the requirement (3.4.1) is $Q_J(j,n,m) = 0$, $\forall J \in G$. This means $m = 0 \mod k_1$ and $m = n \mod k$. Thus we have Ishibashi labels of the type

$$(\mu, \Omega) \quad \text{with} \quad \mu = (j,rk_1,rk_1) \quad \text{and} \quad \Omega = (0,0,0), \quad (3.4.6)$$

with $0 \leq r < 2k_2$ and $0 \leq j \leq k$. For a pair $(\mu, K)$ with $\mu = (k/2,n,m)$ the requirement (3.4.1) is $n = m \mod k$ and $m = -k/2 \mod k_1$. The corresponding Ishibashi labels are

$$(\mu, K) \quad \text{with} \quad \mu = (k/2,k/2+rk_1,k/2+rk_1) \quad \text{and} \quad K = (0,k,0) \sim (k,0,0), \quad (3.4.7)$$

with $0 \leq r < 2k_2$. When $k_1$ is even we have $2k_2$ Ishibashi labels of this kind. The total number of Ishibashi labels is then

$$\# I = \begin{cases} (k+4)k_2 & \text{if} \ 2|k_1 \\ (k+1)k_2 & \text{else} \end{cases}, \quad (3.4.8)$$

We label the boundary blocks by $|A; j, r, F\rangle = |A; j, rk_1\rangle^{PFk}|A; rk_1\rangle^{U(1)k}$ with $F = \Omega, K$ for the boundary blocks in (3.4.6) and (3.4.7) respectively. Following [71], we use the label $A$ to indicate that they preserve the full $PF \otimes U(1)$ symmetry, as opposed to the ones discussed below, labeled $B$. We normalize the $A$-type boundary blocks as

$$\langle A; j', r', F'|q^{L_0+1}L_0-c/12|A; j, r, F\rangle = \delta_{jj'}\delta_{rr'}\delta_{FF'}\chi_{j,rk_1}(q^2)^{PFk} \chi_{rk_1}(q^2)^{U(1)k}. \quad (3.4.9)$$

Next, we determine the labels of the boundary states. In the case at hand, the stabilizer has at most two elements, from the discussion in section 3.4.1 it follows that $C_\rho = S_\rho$ for all $\rho$. For cyclic groups, the values of the character are roots of
unity of order $|C_ρ|$, hence they are signs for our considerations. Orbits whose fields do not appear in the torus partition function correspond to boundary conditions that break at least part of the (maximally extended) bulk symmetry.

In all cases except $2|k_2, 2|k_1$, the $ρ$ labels for the orbits can be represented by

$$ρ = (j, n, n + s) \quad \text{with} \quad 0 \leq j \leq \lfloor k/2 \rfloor,$$

where $\lfloor k/2 \rfloor$ is the integer part of $k/2$. The index $n \in \{0, 1\}$ appearing satisfies the parafermion selection rule $2|j + n$, and $0 \leq s \leq 2k_2 - 1$. We will sometimes condense the notation and suppress the dummy index $n$, and instead label boundary states by $[j, s, ψ]$, with the group character displayed only when it is nontrivial. All $PF \otimes U(1)$ fields lie in such an orbit; the only subtlety is that some of the orbits (3.4.10) can actually be identical. This happens iff the simple current $(0, 2, 2)^{k/2}(0, 0, 2k_2) \frac{k_1 - 1}{2} = (0, k, k_2)$ is in $G$, which due to field identification acts as $(0, 0, k_2)$ on the label $(k/2, n, s)$. In that case, the given range of $s$ must be modified.

For this current to appear we need $2|k_1$ and $2|k$. Then, the orbits are labeled by $[j, n, n + s]$ with $0 \leq j < k/2$, and $0 \leq s \leq 2k_2 - 1$, and there is a second kind of orbits labeled by $[k/2, n, n + s]$ now with $0 \leq s \leq k_2 - 1$. One can check that the number of boundary states equals the number of Ishibashi blocks, as predicted from the general theory (see e.g. [13, 46]).

The boundary coefficients are given by (3.4.3) where the indices $μ = (jmn), ρ = (j'm'n')$ are $PF \otimes U(1)$ indices and $α_F$ is a phase, with $α_0 = 1$ [59] (6.17). To determine $α_K$, we need [59] (6.77) $α_Jα_F = β(J)$ and at present, $K^c = K$ (the charge conjugate of a simple current is again a simple current, and the identity is its own conjugate). We also need [59], after (6.53), $β(J)^2 = e^{-2πihJ}$. Since $h_K = k/4$, when $4|k$ we take $α_K = 1$ and $α_K = e^{iπ/4}$ for $4\not{|} k$. $SF$ is the modular transformation matrix for 1-point blocks with insertion $F$ on the torus, in particular $S^Ω = S$. A current $F$ gives rise to nontrivial matrix elements $S^F_{μρ}$ only if both $μ$ and $ρ$ are fixed by $F$ [44]. The $PF \otimes U(1)^* - S$ matrix reads

$$S_{(jmn), (j'm'n')}^{PF \otimes U(1)} = 2S_{j,j'}^{SU(2)}S_{n,n'}^{U(1)}(S_{m,m'}^{U(1)})^*, \quad (3.4.11)$$

where the modular $S$-matrices on the right hand side are given by (2.2.29) and (2.1.17).

In case $k_1$ is odd, the stabilizers are all trivial and $ψ(F) = ψ(Ω) = 1$. All boundary states $|A, j', s⟩$ that are related to trivial $C_ρ$-characters can be expressed
as

\[ |A, j', s' \rangle = \sqrt{k_1} \sum_{j=0,1,\ldots,k \atop r=0,1,\ldots,k_2-1} \frac{S^{SU(2)}_{jj'}}{\sqrt{S^{SU(2)}_{0,j}}} e^{-i\pi rk_1j'} |A; j, r, \Omega \rangle. \quad (3.4.12) \]

In case \( k_1 \) is even, there are nontrivial stabilizers and we must take into account the character \( \psi_\rho \) of \( C_\rho \). The boundary coefficients (3.4.3) depend on whether the stabilizing current \( J \) is trivial or not. For \( J = \Omega \), the boundary coefficients are just as above, so for \( 0 \leq j' < [k/2] \), we can express the boundary state in terms of boundary blocks again as in equation (3.4.12). When \( j' = k/2 \) we have also a summation over the Ishibashi blocks appearing in (3.4.7),

\[ |A, k/2, s, \psi \rangle = \frac{\sqrt{kk_1}}{2} \left( \sum_{r=0,1,\ldots,k} \frac{S_{j+rk_1,j+rk_1},(k/2,n,n+s)}{\sqrt{S_G,j+rk_1,j+rk_1}} |A; j, r, \Omega \rangle \right) \quad (3.4.13) \]

\[ + \sum_{r=0,1,\ldots,k_2-1} \alpha K \psi(K) S^K_{k/2, k/2+rk_1, k/2+rk_1},(k/2,n,n+s)} \frac{S^{SU(2)}_{k/2, k/2+rk_1, k/2+rk_1}}{S_G(k/2, k/2+rk_1, k/2+rk_1)} |A, k/2, r, K \rangle \right), \]

where we omitted the superscript \( PF \otimes U(1) \) on the S-matrices, because this can be recognized from the form of the (multi-)labels that appear as indices. The \( S^K \) matrix appearing in (3.4.13) can be factorized in its \( SU(2) \) and \( U(1) \) parts just like the ordinary \( S \) matrix,

\[ S^K_{k/2,n',m'},(k/2,n,m) = \frac{1}{k} S^{SU(2)}_{k/2,k/2} e^{i\pi (n'-m')m} = \frac{1}{k} D e^{i\pi (n'-m')m} \quad (3.4.14) \]

with \([43]\ D = e^{-3\pi ik/8}\). These branes are called ‘fractional’ branes, reflecting the additional factor of \( 1/2 \) in (3.4.13), as opposed to the ‘regular’ ones (3.4.12) which involve only a summation over the boundary blocks in (3.4.6).

**B-type branes** As noted in section 2.5 by inspection of the lens space partition function (2.5.2) and the relation \( \chi_n^{U(1)} = \chi_{-n}^{U(1)} \), the T-dual (along the Cartan subalgebra \( U(1) \) in \( SU(2) \)) of the \( L_{k,k_2} \) theory is the \( L_{k,k_1} \) theory. The T-duals of the A-type branes constructed above in \( L_{k,k_2} \) give a new type of branes in \( L_{k,k_1} \), called B-type branes [71]. These can be studied by regarding the boundary blocks of \( L_{k,k_1} \) as tensor products of the boundary blocks of the parafermion and free boson theories. Indeed, since T-duality amounts to changing the sign of the \( U(1) \) label \( n \) in the left-moving field labeled \( (j, m, n) \), leaving the \( PF \) labels unchanged, the B-type boundary blocks can be written as

\[ |B; j, rk_2, rk_2, \Omega \rangle^{L_{k,k_1}} = |A; j, rk_2 \rangle^{PF_k} |B; rk_2 \rangle^{U(1)_k}. \quad (3.4.15) \]
where the label $B$ on the right is to remind us that we have to switch sign on the left-moving momentum, $|B; rk_2⟩⟩^{U(1)k} := |−rk_2, rk_2⟩⟩^{U(1)k}$. The B-type branes in the $L_{k,k}$ theory are T-dual to A-type branes in $L_{k,k}$2. This gives for the non-fractional branes

$$|B; j′, s′⟩⟩_{L_{k,k}} = \sqrt{k_2} \sum_{j=0,1,\ldots,k_1-1} \frac{S^{SU(2)}_{j,j′}}{S^{SU(2)}_{0,j}} e^{-\frac{i\pi j′ k_2}{k}} |A; j, rk_2⟩⟩^{PF_k} |B; rk_2⟩⟩^{U(1)k},$$

with the range of $s′$ as before but $k_1$ interchanged with $k_2$. Likewise, for the fractional branes which arise iff $2|k_2$, we get from performing T-duality on (3.4.13) with $k_1$ interchanged with $k_2$,

$$|B; k/2, s, ψ⟩⟩_{L_{k,k}} = \frac{\sqrt{k_2}}{2} \left( \sum_{j=0,1,\ldots,k_2-1} S_{(j/2,k/2+rk_2), (j/2,n+n+k)} |A; j, rk_2⟩⟩^{PF_k} |B; rk_2⟩⟩^{U(1)k} + \sum_{r=0,1,\ldots,2k_1-1} \frac{\alpha K ψ(K) S^{K}_{(k/2,k/2+rk_2), (k/2,n+n+k)}}{S_{(k/2,k/2+rk_2), (k/2,n+n+k)}} |A; \frac{k}{2}+rk_2⟩⟩^{PF_k} |B; \frac{k}{2}+rk_2⟩⟩^{U(1)k} \right).$$

**SU(2)** Now let $k_1 = 1$ and $k_2 = k$, in which case we recover the $SU(2)_k$ theory. We get more boundary blocks and boundary states than discussed in [71], who found the boundary states with $s = 0$. The boundary states (3.4.12) with $s = 0$ are the Cardy states for $SU(2)$ symmetry preserving boundary states (3.3.18). When allowing for all boundary blocks obtained above, we get all $PF \otimes U(1)$ preserving boundary states.

**Remark** We emphasize the fact that the simple current construction yields directly the correct boundary states, in particular the appropriate set of boundary blocks and reflection coefficients. In other approaches, like the one of [71], the reflection coefficients are determined by imposing the NIMrep properties of the annulus coefficients. Here, NIMrep stands for non-negative integer matrix representation, which is the same as the Cardy conditions discussed in section 3.1.2. Cardy studied these conditions only for symmetry preserving boundary states that are labeled by the primary fields of the theory [19]; one can study NIMrep conditions in a more general setting. The NIMrep conditions are necessary, but not sufficient for the CFT to make sense; indeed, many NIMreps are known which can not appear in any consistent CFT (see e.g. [47]). The simple current construction can be shown to yields NIMreps that are physical, i.e. do belong to a consistent CFT [42]; thus the boundary conditions studied here (and hence in particular those also discussed
in [71]) are indeed physical.

We proceed to calculate the annulus coefficients (3.4.5)

\[ A^\nu_{a,\psi_a} = \sum_{(\mu, J)} B_{(\mu, J)[a, \psi_a]} B^*_{(\mu, J)[b, \psi_b]} S^\nu_{\mu}. \]  

(3.4.16)

This formula depends on the choice of discrete torsion through the restrictions on the summation over Ishibashi labels. In case \( k_1 \) is odd, the result is

\[ A^{(j', n', m')}_{[a, s_a]} = \delta^{2k_2} (s_a - s_b + n' - m') N^j_{j_a j_b}, \]

(3.4.17)

where \( N^j_{j_a j_b} \) are the SU(2)_k fusion rules, and \( \delta^n(m) = 1 \) if \( n | m \) and \( \delta^n(m) = 0 \) if \( n \nmid m \).

Now we consider the case \( 2|k_1 \). If none of the orbits \( a, b \) is a fixed point, we get

\[ A^{(j', n', m')}_{[a, s_a]} = \delta^{2k_2} (s_a - s_b + n' - m') \left( N^j_{j_a j_b} + N^{k/2 - j'}_{j_a j_b} \right). \]

(3.4.18)

In case precisely one of the boundary labels is a fixed point, the computation is similar, since the appearing \( S^K \) matrix elements vanish, and

\[ A^{(j', n', m')}_{[k/2, s_a, \psi_a]} = \delta^{2k_2} (s_a - s_b + n' - m') N^j_{k/2 j_b} \]

(3.4.19)

(which does not depend on \( \psi_a \)). In case both orbits \( a, b \) are fixed points we have to include both types of Ishibashi labels in the summation. We get

\[ A^{(j', n', m')}_{[k/2, s_b, \psi_b]} = \frac{1}{2} \delta^{2k_2} (s_a - s_b + n' - m') \left( N^j_{k/2 j_b} + 1 \right)^{s_a - s_b + n' - m'} \psi_a \psi_b \sin((j' + 1)\pi). \]

Note that this is always a non-negative integer, since the fusion rule \( N^j_{k/2 j_b} \) is non-vanishing exactly when \( \sin(\pi(j' + 1)/2) \) is, and \( 2|(s_a - s_b + n' - m') \) due to the Kronecker delta in the prefactor.

To sum up, the annulus coefficients are

\[ A^\nu_{a b} = \begin{cases} 
\delta \times N^j_{j_a j_b} & \text{if } 2|k_1 \\
\delta \times (N^j_{j_a j_b} + N^{k/2 - j'}_{j_a j_b}) & \text{if } 2|k_1 \text{ and } j_a, j_b \neq k/2 \\
\delta \times N^j_{k/2 j_b} & \text{if } 2|k_1 \text{ and } j_b \neq k/2, \quad j_a = k/2 \\
\delta \times \frac{1}{2} \delta^1(j') \left( 1 + \psi_a \psi_b (-1)^{j' + \frac{1}{2}(s_a - s_b + n' - m')} \right) & \text{if } 2|k_1 \text{ and } j_a, j_b = k/2 
\end{cases} \]

with \( \delta \equiv \delta^{2k_2} (s_a - s_b + n' - m') \), \( \nu \equiv (j', n', m') \), \( a \equiv [j_a, s_a, \psi_a] \) and \( b \equiv [j_b, s_b, \psi_b] \). This generalizes the results of [71] who considered boundary states with \( s = 0 \), and where the details on the amplitude with \( 2|k_1 \) and \( j_a, j_b = k/2 \) where not carried out explicitly. All amplitudes depend only on the difference \( s_a - s_b \), which is consistent with the results below that indicate that the branes with \( s \neq 0 \) correspond to averaging over the \( \mathbb{Z}_{k_1} \) orbits of twisted [29] conjugacy classes in the covering space.
Recall that the boundary blocks are linear functionals $B_j: H_j \otimes H_{j^\dagger} \to \mathbb{C}$, and we restrict this action to the horizontal submodules $\bar{H}_j \otimes \bar{H}_{j^\dagger} \subset H_j \otimes H_{j^\dagger}$. In the large $k$ limit, all $j$ are allowed and we can identify the boundary blocks with functions $\tilde{B}_j$ on the group manifold through the Peter-Weyl isomorphism

$$\tilde{B}_j'(g) = \sum_{j,m_L,m_R} \sqrt{\frac{j+1}{V}} B_j' (v^j_{m_L} \otimes \tilde{v}^j_{m_R}) \langle v^j_{m_L} | R^j(g) | \tilde{v}^j_{m_R} \rangle,$$

(3.4.20)

with $\{v^j_m\}$ a basis of the $\mathfrak{sl}(2)$ module $\bar{H}_j$ with highest weight $j$, and $V$ the volume of the group manifold $SU(2)$. Recall that the lens space $L_{k_1}$ is the set of equivalence classes of $SU(2)$ group elements with the equivalence relation $g \sim e^{2\pi i k_1 H} g$. A function on $SU(2)$ is independent of the choice of representative of such a class, and therefore a function on the lens space $f(v^j_{m_L} \otimes \tilde{v}^j_{m_R}) \neq 0$, iff $m_L = 0 \bmod k_1$. For notational simplicity, we then still write a group element, instead of an equivalence class, as argument for such a function. As in our previous analysis of branes in WZW models, we take the part of the boundary state that acts on the tachyon sector and convert that to a function via the Peter-Weyl isomorphism. We expect similar expressions when the graviton, dilaton and Kalb-Ramond fields are used as probes as in [29].

The Ishibashi blocks for the lens space CFT are expressed as $B^\mathcal{L}_{mn} = B^{PF}_{jm} B^{U(1)}_n$ in terms of $PF$ and $U(1)$ Ishibashi blocks. In order to interpret this as a function on the lens space, we decompose vectors in $\bar{H}_j \otimes \bar{H}_{j^\dagger}$ as

$$v^j_n \otimes \tilde{v}^j_m = w^j_{(m+n)/2} \epsilon_{n,-m},$$

(3.4.21)

so that $B^{U(1)}_q (\epsilon_{n,-m}) = \delta_{q,n} \delta_{q,m}$, and we set

$$B^{PF}_{j',rk_1}(w^j_{(m_L+m_R)/2}) = \delta_{j,j'} \delta_{2rk_1,m_L+m_R} + \delta_{j,k-j'} \delta_{2rk_1-2k,m_L+m_R},$$

(3.4.22)

where field identification of the boundary blocks is taken into account. We motivate the decomposition (3.4.21) and the action (3.4.22) by its naturality, and by noting that it gives results which in many cases make sense. We think of $\epsilon_{n,-m}$ as the part of $v^j_n \otimes \tilde{v}^j_m$ that carries the $u(1)$-charge (the minus sign is due to the fact that $m$ is the charge in the conjugate representation). We have the factor 1/2 in the index of $w^j_{(m+n)/2}$ because we want to think of the pair $j, (m + n)/2$ as a parafermion index, and the second entry must not exceed $2k$.

To analyze the shape of the regular A-branes, we want to interpret (3.4.12) as a function on the lens space. We can make use of the Peter-Weyl isomorphism
(2.2.1) to associate to the Ishibashi block \( \langle A; j', r, \Omega \rangle = B_{j', r, k_1}^{PF} B_{r, k_1}^{U(1)} \) the function
\[
\sqrt{\frac{j'+1}{V}} D_{r, k_1, r, k_1}^j (g) \text{ on } SU(2) \text{ which is indeed invariant under the orbifold group and thus projects to a function on the orbifold. The shape of the regular A-branes is}
\]
\[
\tilde{B}_{j, s}^j (g) = \sqrt{k_1} \sum_{j'=0, 1, \ldots, k}^{r=-k_2-k_2+1} \tilde{S}_{j, j'} \tilde{D}_{j'}^{j'} (g_s) = \tilde{B}_{j, 0} (g_s) ,
\]
where we introduce the shorthands
\[
\tilde{S}_{j, j'} := S_{j, j'}^{SU(2)} \sqrt{\frac{j'+1}{S_{0, j'}^{SU(2)} V}} \quad \text{and} \quad g_s := e^{-\frac{is}{k_1} H} g .
\]
The matrix element \( D_{m n}^j \) is taken to vanish unless \( |m|, |n| \leq j \). We interpret \( s \) as parameterizing a rotation of the brane. Since \( 0 \leq s < 2k_2 \) except for the exceptional case, we see that the exponent in \( e^{i\frac{is}{k_1} H} \) has the range \( 0 \leq i\frac{is}{k_1} H < 2i\pi \frac{H}{k_1} \), where \( \exp \left( \frac{i\pi s}{k_1} H \right) \) is the generator of the orbifold group.

For the A-branes on \( SU(2) \), i.e. the lens space with \( k_1 = 1 \), the shape reduces to \( \tilde{B}_{j, s}^j (g) = \sum_{j'} \tilde{S}_{j, j'} \chi_{j'} (g_s) \). For \( s = 0 \), these branes are (almost) the standard Cardy branes described in [71]. There is one term missing in the \( j' = k \) character, which comes from the fact that for \( j = k \), the term with \( n = 2k \) (or equivalently, with \( n = -k \)) is missing in (2.5.3).

At finite level, the support of the profile, which one would like to interpret as the world volume of the brane, is in fact the whole target space. But the profile is peaked at a conjugacy class (if \( s = 0 \)) respectively a tilted conjugacy classes (if \( s \neq 0 \)) of \( SU(2) \), so that at finite level one can think of it as a smeared brane at the (tilted) conjugacy class [7, 29, 49, 89].

The action of the fractional Ishibashi block (based on the decomposition (3.4.21) and the analogue of (3.4.22)) would be
\[
\langle \langle \frac{k}{2}, \frac{k}{2} + r k_1, \frac{k}{2} + r k_1 \rangle | \cdot \tilde{v}^j_n \otimes \tilde{v}^j_m \rangle = \delta_{2, j} \left( \delta_{2 r k_1 + k, m + n} + \delta_{2 r k_1 - k, m + n} \right) \delta_{r k_1 - k/2, n} \delta_{r k_1 + k, m} .
\]
The Peter-Weyl isomorphism would associate to this Ishibashi block a function which is not invariant under the orbifold group when \( 2 \nmid k_2 \). In [1] we imposed invariance under the orbifold group (by restricting the summation in (2.2.1) to only include the orbifold-invariant basis functions), which gave us vanishing Ishibashis when \( 2 \nmid k_2 \). This is quite unsatisfactory. Thus, one should probably try approaches different
from the decomposition (3.4.21) and the implicit interpretation of that in (3.4.25). An alternative procedure to associate functions to fractional Ishibashis is given in the following section.

The geometrical analysis in [1] of the B-type branes was based on the assumption that one could simply neglect those Ishibashi blocks that do not correspond to functions on the lens space, and we refrain from presenting that analysis here.

### 3.4.3 Fractional Ishibashi functions

When we consider D-branes or boundary states on geometric orbifolds, the standard procedure (as applied in the previous section, see also [1, 71]) is to start with D-branes or boundary states in the covering theory and average out the orbifold action, which gives something that projects to the orbifold. When there are fixed points, the procedure to average out the orbifold action on the Ishibashi functions gives Ishibashi functions of which some are vanishing, or associates the same function to different Ishibashis. This mismatch is unsatisfactory, because the Ishibashis should be orthogonal [46], and it is natural to expect the same for the associated functions.

The origin of this mismatch is the fact that in the CFT description, the Ishibashis are not labeled with the same labels as the primary fields (to which functions are naturally associated). Instead we have pairs \((\Lambda_f, J)\) as discussed in section 3.4.1. Recall that \(\Lambda_f\) is a label of a primary field in the covering theory, \(J\) is a simple current used to define the orbifold, and \(J \ast \Lambda_f = \Lambda_f\) for \(\Lambda_f\) a fixed point.

For concreteness, we consider the orbifold \(SU(2)/\mathbb{Z}_2 = SO(3)\). The two elements of the simple current group \(\mathbb{Z}_2\) are denoted 0 and \(k\), and their action under fusion is \(0 \ast j = j\) and \(k \ast j = k - j\). The simple current group has a fixed point under the simple current action; \(k \ast k/2 = k/2\). We face two problems:

I) At level \(k = 4l + 2\), the label \(m = k/2\) occurs as (part of) an allowed Ishibashi label, namely \((k/2, k)\), but the function associated to this Ishibashi block by the afore mentioned averaging procedure vanishes.

II) If instead \(k = 4l\), we have two allowed Ishibashi labels \((k/2, k)\) and \((k/2, 0)\), which are both associated to the same function with the averaging procedure.

In both cases we run into trouble with the orthogonality of the Ishibashis [46]

\[
\langle\langle I_{m,j} | q^{L_0 + L_0 - c/12} I_{n,K}\rangle\rangle = \delta_{nm} \delta_{JK} \chi_n(q^2). \tag{3.4.25}
\]
In the Lagrangian description, one associates the structure of a gerbe module to the branes [48]. Branes that are supported at the same sub-manifolds may differ by nothing more than their gerbe module structure, and this happens precisely when there are fixed points in the CFT description. The main purpose of the discussion in this section is to find a description that resolves the ambiguity II at finite values of the level. A procedure to associate functions to Ishibashi blocks is suggested for rank $r = 1$, (3.4.28). The ansatz (3.4.30) is natural because it makes use of the isomorphism (as discussed in section 2.3.6) between the center $Z \subset G$ and the relevant simple current group $G$. Further, it provides a universal procedure to associate functions to Ishibashi blocks such that the functions are (in many cases, including infinite series) orthogonal, as required. We also propose a way to resolve the ambiguity I. In section 3.4.4, the implications of (3.4.28) are investigated. In section 3.5.5, we discuss applications to the coset model $SU(2)_k \times SU(2)_l/SU(2)_{k+l}$.

Choose a maximal torus $H \subset G$, and let $h \in \mathfrak{h}^* = \text{Lie}(H)^*$. For $x \in \mathbb{C}$, a representation function at $g e^{ixh}$ is related to its value at $g$ as

$$D^\Lambda_{ab}(ge^{ixh}) = \langle \Lambda; a|R^\Lambda(ge^{ixh})| \Lambda; b \rangle = e^{ix(b,h)}D^\Lambda_{ab}(g),$$

(3.4.26)

where $a, b$ label states in the representation $\Lambda$ and $(\cdot, \cdot)$ is the Killing form (where we take the weight part of the label $b$, in case there are multiplicities). Recall that the functions $D^\Lambda_{ab}$ are orthogonal in all three indices with the usual scalar product (integration with the Haar measure). A character at this argument can be written as

$$\chi_\Lambda(ge^{ixh}) = \sum_{a < \Lambda} e^{ix(a,b)}D_{aa}^\Lambda(g).$$

(3.4.27)

The sum is over all states $a$ (with multiplicities) in the representation $\Lambda$.

When $k = 4l$, the fixed point corresponds to a primary field in $SO(3)$ and we have Ishibashis which are 2-fold degenerate. These are labeled by pairs $(\Lambda, J)$ with the degeneracy $J \in \{0, k\} = \{k^n\}$ with $n \in \{0, 1\}$ and $\Lambda = k/2$ is the fixed point. Recall that $|\Lambda| = \Lambda + 1$ is the dimension of the (horizontal) $\mathfrak{su}(2)$-representation with highest weight $\Lambda$. One can associate to the Ishibashi block labeled $(\Lambda, J)$ the function

$$\langle g|I_{\Lambda,J}\rangle := \sqrt{\frac{1}{|G|}} \chi_\Lambda(ge^{in\sigma_3^{1/[2n]}}),$$

(3.4.28)

with unit normalization. Here, $|G| = 2\pi^2(k\alpha')^{3/2}$ is the volume of the group $G = SU(2)$. For both values of $n$, (3.4.28) defines a function on $SO(3)$. We shall see that
they are not only linearly independent, but in fact orthogonal. Consider the scalar product (with respect to the Haar measure) of two Ishibashis with different labels \( n = 0 \) and \( n' = 1 \),

\[
\int_G \frac{\langle \langle I_{\Lambda,0} | g \rangle \langle \langle I_{\Lambda,1} \rangle \rangle}{|G|} = \frac{1}{|G|} \int_G \sum_{a,b<\Lambda} e^{i\alpha_a/|\Lambda|} D_{aa}^\Lambda (g) D_{bb}^\Lambda (g)^* = \frac{1}{|\Lambda|} \sum_{a<\Lambda} e^{i\alpha_a/|\Lambda|} = 0 . \tag{3.4.29}
\]

The sum is over the weights in the representation, \( a = -\Lambda, -\Lambda+2, \ldots, \Lambda-2, \Lambda \), so we have a sum over roots of unity \( e^{i\alpha_a/|\Lambda|} \). Ishibashis with different degeneracy labels are now orthogonal, as they should be, see [46] eq. (4.40). This is the prescription to resolve degenerate Ishibashi functions that will be applied in section 3.4.4.

The simple currents of the WZW models (with one exception, which occurs for \( E_8 \)) are labeled by \( k \) times the fundamental weights \( \Lambda (i) \) with \( a_i = 1 \) [40]. These are the cominimal weights, cf. the table in [39], p 203. The \( \Lambda (i) \) are the highest weights of the fundamental representations, and \( \Lambda (0) \) is the unit element in the fusion ring, hence also of the simple current group. \( E_8 \) at level \( k = 2 \) has an additional simple current \( \Lambda (7) \). In all cases except \( E_8 \) at level \( k = 2 \), there exists an isomorphism \( G \cong Z(G) \) given by \( g^\xi (j) = \exp (\xi i H_{\Lambda(j)}) \) with \( \xi = 2\pi \), see section 2.3.6. When \( \Lambda = \Lambda_f \) is a fixed point \( \Lambda (i) \ast \Lambda_f = \Lambda_f \), it makes sense to require \( I_{(\Lambda_f,\Lambda (0))} (g) = \chi_{\Lambda_f} (g) \). We also require that we have an action of the simple currents on the Ishibashis, \( I_{(\Lambda_f,\Lambda (0))} (g) = I_{(\Lambda_f,\Lambda (0))} (gg^\xi (j)) \), as in the case without fixed points [3]. In the light of the simple current - center isomorphism, it would seem natural to set \( I_{(\Lambda_f,\Lambda (0))} (g) = \chi_{\Lambda_f} (gg^\xi (j)) \). However, as \( g^\xi (j) \in Z(G) \), \( \chi_{\Lambda_f} (gg^\xi (j)) \) is a class function, which we know is a linear combination of characters. We require that the Ishibashi functions are linearly independent. Thus, we associate functions

\[
I_{(\Lambda_f,\Lambda (i))} (g) := \sqrt{\frac{1}{|G|}} \chi_{\Lambda_f} \left( g e^{iH_{\Lambda (i)} \xi} \right) , \quad \xi \neq 2\pi \tag{3.4.30}
\]

to the Ishibashi blocks labeled \( (\Lambda_f,\Lambda (i)) \). We shall now investigate whether the prescription (3.4.30) leads to Ishibashi functions which are also orthogonal (with the right choice of \( \xi \)), in which case the basis for Ishibashis suggested in [46] is indeed orthogonal. The scalar product between two Ishibashis with \( \Lambda_f \) a fixed point, is

\[
\langle I_{(\Lambda_f,\Lambda (i))} (g), I_{(\Lambda_f,\Lambda (j))} (g) \rangle = \sum_{a<\Lambda, b<\Lambda'} e^{i(a,\Lambda (i)) \xi} e^{-i(b,\Lambda (j)) \xi} \int_G D_{aa}^\Lambda (g) D_{bb}^\Lambda (g)^* = \frac{1}{|\Lambda_f|} \tilde{\chi}_{\Lambda_f} (\Lambda (i) \xi - \Lambda (j) \xi) . \tag{3.4.31}
\]
Thus, the questions whether these Ishibashis are orthogonal reduces to a calculation of the corresponding horizontal Lie algebra character $\tilde{\chi}_{\Lambda_f}(\Lambda(i)\xi - \Lambda(j)\xi)$.

When we consider the fixed point $\Lambda = m\rho$ (which appears in all Lie groups at level $k = mg^\vee$), it is useful to rewrite the Lie algebra character with the Weyl character formula

$$\langle I_{(\Lambda,\Lambda(i))}(g), I_{(\Lambda,\Lambda(j))}(g) \rangle = \frac{1}{|\Lambda|} \frac{\sum_{\sigma \in W} |\sigma| \exp \left( i\xi \rho, \sigma(\Lambda(k)) \right)}{\sum_{\sigma \in W} |\sigma| \exp \left( i\xi \sigma(\Lambda(i) - \Lambda(j)) \right)},$$

where $|\sigma|$ is the sign of the Weyl group element $\sigma$. If $\Lambda(i) = \Lambda(j)$, we have a character evaluated at the origin, which takes the value $|\Lambda|$. Otherwise, $\Lambda(i) - \Lambda(j) = \omega(\Lambda(k))$ for some fundamental weight $\Lambda(k) \neq 0$, and some Weyl group element $\omega \in W$, as shown in section 2.3.6.

We use this fact to proceed with our calculation of the Ishibashi function scalar product,

$$\langle I_{(\Lambda,\Lambda(i))}(g), I_{(\Lambda,\Lambda(j))}(g) \rangle = \frac{1}{|\Lambda|} \frac{\sum_{\sigma \in W} |\sigma| \exp \left( i\xi (m+1) \rho, \sigma(\Lambda(k)) \right)}{\sum_{\sigma \in W} |\sigma| \exp \left( i\xi \rho, \sigma(\Lambda(k)) \right)}$$

We can now use the denominator identity to rewrite both the denominator and the numerator,

$$\langle I_{(\Lambda,\Lambda(i))}(g), I_{(\Lambda,\Lambda(j))}(g) \rangle = \frac{1}{|\Lambda|} \prod_{\alpha > 0} \sin \left( \frac{1}{2} \xi (\alpha, \Lambda(k)) \right) \prod_{\alpha > 0} \sin \left( \frac{1}{2} \xi (\alpha, \Lambda(k)) \right)$$

The product is over the positive roots. Some factors in both products vanish because $(\alpha^j, \Lambda(k)) = 0$ for simple $\alpha^j$ with $j \neq k$. This happens at both sides of the fraction, thus we shall ignore those factors. The other factors where $\alpha$ contains $\alpha^k$ are the ones which do not vanish for all $\xi$. We shall choose $\xi$ such that

$$\sin \left( \frac{1}{2} (m+1)\xi (\alpha^k, \Lambda(k)) \right) = 0 \quad \text{and} \quad \sin \left( \frac{1}{2} \xi (\alpha^k, \Lambda(k)) \right) \neq 0.$$

Then the Ishibashis have the desired orthogonality. We set

$$\xi = \frac{2\pi}{(m+1)(\alpha^k, \Lambda(k))}.$$
functions, which describe the shape of the D-branes. This will be displayed in detail
for $SU(2)/\mathbb{Z}_2 = SO(3)$ in section 3.4.4.

As an example of a fixed point that is not of the form $\Lambda = m\rho$, we consider $SU(4)$
at level $k = 4$ where the simple current $\Lambda_{(2)}$ has the fixed point $\Lambda_f = 2\Lambda_{(0)} + 2\Lambda_{(2)}$, the
horizontal part of which is $\bar{\Lambda} = (0, 2, 0)$. The corresponding horizontal Lie
algebra character is

$$\bar{\chi}_{(0,2,0)}(i\xi\Lambda_{(2)}) = 10 + 2\cos(2\xi) + 8\cos(\xi),$$

(3.4.37)

which does not vanish for any (real) value of $\xi$. (To calculate such characters, [98]
is useful.) As the simple current group has only two elements, and precisely one of
the functions (3.4.30) associated to the fixed point Ishibashi labels is a class func-
tion, it is clear that the two functions are linearly independent. But to obtain an
orthogonal basis of Ishibas, it appears that we must take a linear combination
of $\{\Lambda_{(0)}, \Lambda_{(2)}\}$ to label the degeneracy. The simple current $\Lambda_{(2)}$ also has the fixed
point $\Lambda_f = 2\Lambda_{(1)} + 2\Lambda_{(3)}$; for this fixed point, the corresponding horizontal character
does vanish with a suitable choice of $\xi$. Thus, $\{\Lambda_{(0)}, \Lambda_{(2)}\}$ provides in this case an
orthogonal basis of degeneracy labels.

That the Ishibas can be made orthogonal for $\Lambda_f = (2, 0, 2)$, but not for $\Lambda_f =$
$(0, 2, 0)$, is related to the fact that the former representation has more states (84
instead of 20). Since the states in the representations are distributed on regular
polygons, a large representation will typically have a smaller percentage of states
orthogonal to the axis $\Lambda_{(2)}$. Thus, we expect the failure of the Ishibashi functions to
be orthogonal in the $(0, 2, 0)$ case to be a small level phenomenon, which is supported
by an analysis of $SU(6)$ at level $k = 2$. There, the current $\Lambda_{(3)}$ has 3 fixed points
of dimensions 20, 84 and 84. The relevant horizontal Lie algebra character for the
fixed point $\Lambda(20) = (0, 0, 1, 0, 0)$ is

$$\bar{\chi}_{\Lambda(20)}(\Lambda_{(2)}) = 12 + 8\cos(6\xi),$$

(3.4.38)

which has no zeroes. Another fixed point is $\Lambda(84) = (0, 1, 0, 0, 1)$, with character

$$\bar{\chi}_{\Lambda(84)}(\Lambda_{(2)}) = 4e^{i10\xi} + 32e^{i4\xi} + 36e^{-i2\xi} + 12e^{-i8\xi}.$$  

(3.4.39)

By numerical analysis, we see that this character can be made to vanish by an appro-
 priate choice of $\xi$. The other representation of dimension 84, with $\Lambda = (1, 0, 0, 1, 0)$,
has the same character (3.4.39) (this follows from the Weyl character formula,
see [45] eq. (13.30), and the form of the Weyl group of $A_5$).
The missing function  As already mentioned, in $SO(3) = SU(2)/\mathbb{Z}_2$ at level $k = 4l + 2$, we get a label $\kappa$ for the Ishibashi block which does not correspond to a primary field in the quotient theory, and thus cannot in an obvious way be related to a function on the quotient space. However, we can take a fundamental domain (of the $\Gamma$-action) $D \subset M$ in the covering space $M$ (which in our applications is a Lie group, but the considerations are general). On $M$, we do have a natural Ishibashi function $I^{M}_{\kappa}$ for that label. Now we can take the restriction $I^{M}_{\kappa} |_D$, and extend it to a function $\tilde{I}_\kappa$ on $M$ that is invariant under the orbifold group $\Gamma$ (and which is different from the function $I^{M}_{\kappa}$). This function projects to a candidate for the Ishibashi function $I^Q_{\kappa}$ on the quotient $Q = M/\Gamma$. Since the integration $\langle I_{\kappa}, I_{\Lambda} \rangle_Q$ can be lifted to $D$, and if $\langle I_{\kappa}, I_{\Lambda} \rangle_D = 0$ for $\kappa \neq \Lambda$, the Ishibashi functions are orthogonal. We shall spell out explicitly how this works for $M = SU(2)$ and $\Gamma = \mathbb{Z}_2$ in the next section, and we shall see that the suggested function is indeed orthogonal to the other Ishibashi functions on $SO(3)$.

### 3.4.4 Fractional boundary states in SO(3)

We apply the results of the previous section to the lens space $SO(3) = SU(2)/\mathbb{Z}_2$. The Ishibashis are labeled $(m, J)$ with

$$m = J \ast m, \quad Q_\mathcal{G}(m) + X(\mathcal{G}, J) \in \mathbb{Z}.$$  \hspace{1cm} (3.4.40)

Since $\mathcal{G}$ is cyclic, the discrete torsion matrix $X$ vanishes except for $X(k, k) = \frac{1}{2} Q(k) = \frac{k}{2} \mod \mathbb{Z}$. When $k = 4l + 2$, we have discrete torsion in the sense $X \neq 0$ (but not in the sense of a freedom of choice in $X$). There are two types of Ishibashi labels in the simple current construction; the regular type

$$(m, 0) \quad \text{with} \quad m \in 2\mathbb{Z},$$  \hspace{1cm} (3.4.41)

and one Ishibashi block of exceptional type

$$\left(\frac{k}{2}, k\right) \quad \text{with} \quad k \in 2\mathbb{Z}.$$  \hspace{1cm} (3.4.42)

Both species are always present, and the first type Ishibashis are labeled by the allowed primary fields. The Ishibashis are expected to be orthogonal to each other, \cite{46}, and therefore the function cannot only depend on more than the first entry in the pair $(m, J)$. Thus, the associated function cannot simply be the character $\chi_m(g)$ averaged over the orbifold group; instead (3.4.28) is suggested. Alternatively, we take (3.4.30) with $\xi$ given by (3.4.36), which reduces to (3.4.28) in this case. Recall that the group manifold $SO(3) = SU(2)/\mathbb{Z}_2$ is the set of equivalence classes

$$[g] = [zg] \quad g \in SU(2) \quad z = -1 = e^{i\pi \sigma_3}.$$  \hspace{1cm} (3.4.43)
The stabilizer of this identification is trivial for all \( g \). A function on \( SO(3) \) is a function on the covering for which \( f(zg) = f(g) \). In the case \( k = 4l + 2 \) we have

\[
D_{m/n}^{k/2}(zg) = -D_{m/n}^{k/2}(g).
\] (3.4.44)

Simply averaging a sum of these over the orbifold group gives an everywhere vanishing function. Therefore, the group character \( \chi_{k/2}(g) \) cannot be projected to \( SO(3) \) by this averaging procedure. Instead, we may proceed as follows: for each \( g \in SO(3) \) pick a representative \( g \) that lies in the upper hemisphere of \( SU(2) \), which is the set of points with \( \psi < \pi/2 \) in the parametrization (2.4.50). The character \( \chi_{k/2} \) (which is an odd function of \( \psi \) on \( SU(2) \)), can be modified to an even function \( \tilde{\chi}_{k/2} \) on \( SU(2) \) by defining \( \tilde{\chi}_{k/2} := \chi_{k/2} \) on the upper hemisphere, and \( \tilde{\chi}_{k/2} := -\chi_{k/2} \) on the lower hemisphere. Note that \( \chi_{k/2} = 0 \) on the equatorial plane. To the Ishibashi block labeled \( (k/2, k) = (2l + 1, k) \), we associate

\[
I_{(2l+1,k)}([g]) = \sqrt{\frac{1}{|G|}} \tilde{\chi}_{k/2}(g).
\] (3.4.45)

This function is orthogonal to all \( SO(3) \)-characters: the allowed \( SO(3) \)-characters are of the form \( \chi_{2n} \) and are even functions as well, thus we may evaluate the scalar product by just integrating over the upper hemisphere. Then we can use invariance of the measure to obtain

\[
\langle \tilde{\chi}_{k/2}, \chi_{2l} \rangle = \int_0^{\pi/2} d\mu_\psi \chi_{k/2}(\psi)\chi_{2l}(\psi) - \int_{\pi/2}^\pi d\mu_\psi \chi_{k/2}(\psi)\chi_{2l}(\psi) = 0.
\]

A similar calculation reveals that \( \langle \tilde{\chi}_{k/2}, \tilde{\chi}_{k/2} \rangle = \langle \chi_{k/2}, \chi_{k/2} \rangle \).

**Boundary states** Recall that the boundary labels are orbits \([j, \psi_j]\), where \( j \) is an \( SU(2) \) label that represents a \( G \cong \mathbb{Z}_2 \) orbit, and \( \psi_j \) is a character of \( C_j \subset S_j \). The subgroup \( C_j \) of the stabilizer \( S_j \) is in our case given by \( C_j = S_j \). Further, \( C_{k/2} = \mathbb{Z}_2 \), and all other \( C_j \) are trivial. Hence \( \psi_j \) is a degeneracy label that takes values \( \pm 1 \). A list of boundary labels is (suppressing trivial labels)

\[
[j, \psi_j] = [0], [1], ..., [k/2 - 1], [k/2, 1], [k/2, -1].
\] (3.4.46)

The boundary states \([j]\) with odd \( j \) do not correspond to primary fields of \( SO(3)_{2k} \) and are interpreted as being symmetry-breaking. There are \( k/2 + 2 \) states in this list, just as many as there are Ishibashis (3.4.41) and (3.4.42). The boundary states are linear combinations of the characters with coefficients

\[
B_{(m,J),(j,\psi)} = \sqrt{\frac{|G|}{|S_j||C_j|}} \sqrt{\frac{1}{S_{m/J}} \psi_j(J)^*} = \sqrt{2} \sqrt{\frac{\alpha J S_{m/J}}{|S_j|}} \sqrt{\frac{1}{S_{0,m}}} \psi_j(J)^*,
\] (3.4.47)
which are known as the boundary coefficients. The matrix $S^J$ is different from the modular matrix $S$ only if $m = j = k/2$. In that case, $S^k_{k/2,k/2} = \frac{1}{k} e^{-3\pi ik/8}$. The phase $\alpha_J$ can be taken to be $\alpha_k = e^{i\pi/4}$ when $k = 4l + 2$ and unity in all other cases.

**Shape of the boundary states** Now we are ready to compute some of these boundary shapes. At level $k = 4l$, we have a fractional boundary state with $j = k/2 = 2l$. With (3.4.28), the shape of the fractional boundary state is

$$B_{[2l,\psi]}(g) = \frac{1}{\sqrt{2}} \sum_{(m,J)} S^J_{m,2l} \psi(J) I_{(m,J)}(g)$$

$$= \frac{1}{\sqrt{2}} \sum_{m=0,2,...,k} S^J_{m,2l} \chi_m(g) + \psi(4l) \frac{e^{-3\pi i l/2}}{4l \sqrt{S_{0,2l}}} \chi_2(e^{i\pi/(2l+1)} g),$$

where $\psi(k) = \pm 1$. The shift $g \mapsto e^{i\pi/(2l+1)} g$ is interpreted as a tilting of the conjugacy class by an angle $\pi/(2l + 1)$. In the description of [71] (and [1]) the $\psi$-dependent last term does not contribute with a full group character.

In the large level limit, the fractional boundary states (3.4.48) correspond to two fractional branes wrapping the non-orientable $\mathbb{RP}^2$ (cross-cap) conjugacy class in $SO(3)$ [21] that comes from the antipodal identification of the equatorial conjugacy class of $SU(2)$. Note that in the sum $B_{[2l,+]} + B_{[2l,-]}$ of the boundary states (3.4.48), the contributions from the tilted conjugacy class cancel. This matches with the observation [21] that at large level, the composite of two different D-branes on the equator gives a $S^2$-brane wrapping the $\mathbb{RP}^2$.

### 3.5 Branes in Coset theories

The following discussion about D-branes in coset models coincides to large extent with the papers [3] and [4]. We discuss D-branes in coset models at large level and the shape of the boundary states at finite level. We also discuss two examples of cosets; the parafermion coset $S(2)/U(1)$ in section 3.5.3, and the diagonal $SU(2)$ coset in sections 3.5.4 and 3.5.5.

#### 3.5.1 Branes at large level

The description of coset models given in section 2.4, together with the knowledge about D-branes in WZW models can be combined to draw conclusions about the geometry of D-branes in coset models at large level. As usual, we consider only the boundary conditions that preserve the full chiral symmetry of the coset model. For
these boundary conditions, at large level the branes on $G \times H$ are concentrated on the sub-manifolds

$$C_{\Lambda,\lambda} := \{(g, h) \in G \times H, \mid g \in C_{\Lambda}^G, h \in C_{\lambda}^H\} = C_{\Lambda}^G \times C_{\lambda}^H \subset G \times H.$$  

Here

$$C_{\Lambda}^G := \{g'g_{\Lambda}g^{-1} \mid g' \in G\}$$  

is the conjugacy class in $G$ of the group element

$$g_{\Lambda} := \exp \left( \frac{2\pi i}{k + g^\vee H} H_{\Lambda + \rho} \right).$$  

Recall that $\rho$ and $g^\vee$ are the Weyl vector and dual Coxeter number of $g$, respectively, and $H_{\Lambda + \rho} \in \mathfrak{g}_0$ is the element of the Cartan subalgebra dual to $\Lambda + \rho \in \mathfrak{g}_0^\ast$.

Analogously, $C_{\lambda}^H = \{h'h_{\lambda}h'^{-1} \mid h' \in H\}$ is the conjugacy class in $H$ of

$$h_{\lambda} := \exp \left( \frac{2\pi i}{k' + h^\vee H'} H'_{\lambda + \rho'} \right),$$  

with $\rho'$ and $h^\vee$ the Weyl vector and dual Coxeter number of $\tilde{h}$. Further, the labels $\Lambda$ and $\lambda$ are (the horizontal parts of) those of the primary fields of the $g$- and $\tilde{h}$-WZW models, i.e. dominant integral weights satisfying $(\Lambda, \theta) \leq k$ and $(\lambda, \vartheta) \leq k'$, with $\theta$ and $\vartheta$ the highest roots of $g$ and $\tilde{h}$, respectively.

Let us remark that instead of (3.5.1) we may equally well use the sets

$$C_{\Lambda,\lambda}^{-} := \{(g, h^{-1}) \in G \times H, \mid g \in C_{\Lambda}^G, h \in C_{\lambda}^H\} \subset G \times H.$$  

(3.5.5)

to describe the subsets on which the $G \times H$-branes are concentrated. Indeed, this just amounts to choosing a different labeling, owing to

$$C_{\Lambda,\lambda}^{-} = C_{\Lambda,\lambda}^{+}$$  

(3.5.6)

with $\lambda^+$ the $\tilde{h}$-weight charge conjugate to $\lambda$. The equality (3.5.6) holds because $h \in C_{\Lambda}^H$ iff $h^{-1} \in C_{\lambda}^{H^\ast}$, which in turn follows from the fact that the map $\lambda \mapsto -\lambda^+$ is a Weyl transformation (namely the one corresponding to the longest element of the Weyl group of $\tilde{h}$) and that the Weyl vector is self-conjugate.

In the description (2.4.3) of $Q$, we can obtain the branes on the target space $Q$ of the coset model as the projections

$$\pi_{ilr}(C_{\Lambda,\lambda}) = \{[g, h] \mid (g, h) \in C_{\Lambda,\lambda}\}$$  

(3.5.7)
of the sets (3.5.1). In the alternative description (2.4.1), one deals instead with projections of the sets \([28, 49, 32, 31]\)
\[
\tilde{C}_{\Lambda,\lambda} := \{ gh^{-1} \in G \mid g \in C^G_{\Lambda}, h \in C^H_{\lambda} \subset G \} \subset G.
\] (3.5.8)
These subsets are \(\text{Ad}(H)\)-invariant, i.e. satisfy \(u \tilde{C}_{\Lambda,\lambda} u^{-1} = \tilde{C}_{\Lambda,\lambda}\) for all \(u \in H\), and hence they trivially project on \(Q\), i.e.
\[
\pi_{\text{Ad}}(\tilde{C}_{\Lambda,\lambda}) = \{ [gh^{-1}] \in Q \mid g \in C^G_{\Lambda}, h \in C^H_{\lambda} \} \subset Q.
\] (3.5.9)
It follows directly from the existence of the bijection \(\varpi\) defined after (2.4.3) that these descriptions of the coset branes at large level are equivalent:
\[
\pi_{\text{Ad}}(\tilde{C}_{\Lambda,\lambda}) = \pi_{|r}(C_{\Lambda,\lambda}^{-1}).
\] (3.5.10)
Expressed in terms of functions on \(Q\), the discussion above amounts to the statement that the shape of a brane is a delta function on the subset (3.5.10). This description is adequate in the limit of large level, whereas at any finite value of the level the shape of the brane is smeared about this subset. The extent of localization increases with the level; this will be analyzed quantitatively in the next section. Recall that in the WZW case the limit yields \(D_G = \delta_{C^g_{g \Lambda}}^{C^g_{g \Lambda}}\) on a conjugacy class \(C^g_{g \Lambda}\) of the group, see (3.3.16). The corresponding function on the coset is
\[
D_{\Lambda,\lambda}([g,h]) = \sum_{[\Lambda',\lambda']} \frac{S^{|g \Lambda'|}[\Lambda',\lambda']}{{S^{|g \Lambda'|}[0,0]}} I_Q^{C_{g \Lambda}^{-1}}(u) D_H^{(|g \Lambda'|,\lambda').
\] (3.5.11)
One of the two integrations over \(H\) is trivial, and the other can be performed with the help of the identity \(\delta^G_{C^g_{g \Lambda}}(uvw^{-1}h^{-1}g) = \delta^G_{C^g_{g \Lambda}}(uh^{-1}g)\), which for \(u, v, h \in H\) and \(g \in G\) is valid as an equality of functions on \(Q\). The result is
\[
D_{\Lambda,\lambda}([g,h]) = |C^g_{g \Lambda}|^{-1} \delta^G_{C^g_{g \Lambda}}(h^{-1}g) = |C^g_{g \Lambda}|^{-1} \delta^G_{C_{g \Lambda}^{-1}}(h^{-1}g).
\] (3.5.12)

### 3.5.2 Branes at finite level

The symmetry preserving boundary states \(B^Q\) of the coset theory are naturally labeled by the labels of primary fields, and when expressing them in terms of the Ishibashi functionals (or boundary blocks) \(I^Q\), which form bases of the spaces of conformal blocks for the one-point correlators on the disk, the coefficients are given by the modular \(S\)-matrix. In the absence of field identification fixed points, the \(B^Q\) are labeled by the \(J_{g/h}\)-orbits \([\hat{\Lambda}, \hat{\lambda}]\), and are related to the Ishibashi blocks as
\[
B^Q_{[\hat{\Lambda}, \hat{\lambda}]} = \sum_{[\Lambda', \lambda']} \frac{S_{[\Lambda,\lambda],[\Lambda',\lambda]}}{\sqrt{S_{[\Lambda',\lambda'],[0,0]}}} I^Q_{[\Lambda', \lambda']},
\] (3.5.13)
In the presence of fixed points we would have more complicated boundary conditions (see equation (3.4.3)), as well as more complicated Ishibashi functions (see section 3.4.3). We slightly abuse notation by writing $\Lambda$ in place of $\hat{\Lambda}$ in the subscripts of $S$-matrices. Also (in the absence of fixed points), the coset $S$-matrix is expressible through the modular $S$-matrices of the $g$- and $\bar{h}$-WZW models as

$$S_{[\Lambda',\lambda'],[\Lambda,\lambda]} = |J_{g/\bar{h}}| \cdot S_{\Lambda,\lambda}^g \cdot S_{\Lambda,\lambda}^{\text{b}*},$$  \hspace{1cm} (3.5.14)$$

where on the right hand side arbitrary representatives ($\hat{\Lambda}, \hat{\lambda}$) and ($\hat{\Lambda}', \hat{\lambda}'$) of the orbits $[\hat{\Lambda}, \hat{\lambda}]$ and $[\hat{\Lambda}', \hat{\lambda}']$ are chosen [52, 82].

We would like to associate functions on the target space to the Ishibashi functionals $I^Q$, and thereby to the boundary functionals (3.5.13). In the $G \times H$ WZW models, we take

$$I_{\Lambda,\lambda}^{G \times H}(g, h) = \sqrt{\frac{d\lambda}{|G|} \frac{d\Lambda}{|H|}} \cdot \chi_{\Lambda}(g) \cdot \chi_{\lambda}(h) = \sqrt{\frac{d\lambda}{|G|} \frac{d\Lambda}{|H|}} \sum_{m,a} D_{\Lambda,\lambda}^{\Lambda,\lambda}(g, h).$$  \hspace{1cm} (3.5.15)$$

Here we have chosen a different convention on phases for the Peter-Weyl mapping $f_{\bar{h}}^{\Lambda,\lambda}$ of the $\bar{h}$-WZW model, in order to conform with the fact that in the algebraic description instead of the $\bar{h}$-WZW model it is the ‘complex conjugate theory’ $\bar{h}$ that matters.

By comparison with the discussion of $\mathcal{F}(Q)$ in section 2.4.1, we expect that in order to obtain the shape of Ishibashi blocks of the coset model, we should consider the projection

$$\sqrt{\frac{d\Lambda}{|G|} \frac{d\lambda}{|H|}} \sum_{m,a} \pi_{\bar{L}}^a \cdot D_{mm,aa}^{\Lambda,\lambda}$$  \hspace{1cm} (3.5.16)$$

$$= \sqrt{\frac{d\Lambda}{|G|} \frac{d\lambda}{|H|}} \sum_{\ell,\ell'=1}^{b_{\Lambda,\lambda}} \sum_{m,p,q=1}^{d_{\lambda}} \sum_{a,b,c=1}^{d_{\Lambda}} c_{m,a}^{\lambda<\Lambda;\ell} \cdot c_{m,a}^{\lambda<\Lambda;\ell'} \cdot c_{q,c}^{\lambda<\Lambda;\ell} \cdot c_{p,b}^{\lambda<\Lambda;\ell'} \cdot D_{pq,bc}^{\Lambda,\lambda}$$

of the functions (3.5.15) to the coset manifold $Q$ (in the second equality, the unitarity (2.4.10) of the matrices $c$ is used).

The ($G \times H$-preserving, $Q$-breaking) coset Ishibashi blocks (in the absence of fixed points) are labeled by pairs ($\hat{\Lambda}, \hat{\lambda}$) satisfying (3.4.1). There is, in general, no bijection between allowed highest $g \oplus h$-weights and allowed highest $\hat{g} \oplus \hat{h}$-weights.
To the former, we can naturally associate functions, by the results in section 2.4.1, and the latter label the \((Q\)-symmetry breaking Ishibashis). Inspection of various examples indicates that every (by (3.4.1)) allowed \(J_{g/h}\)-orbit contains at least one representative whose horizontal projection is a horizontally allowed pair (in the sense \(\Lambda \prec \Lambda\)), and that orbits with more than one such representative are rare: For example, in the parafermion case we have the label \((\hat{\Lambda}, \hat{\lambda}) = (0,0) \sim (k,k)\) where both \((0,0)\) and \((k,k)\) are horizontally allowed. Thus it is not clear to which function we must associate the Ishibashi label \((\hat{\Lambda}, \hat{\lambda}) = (0,0) \sim (k,k)\), it could be \(D_{00}^k\) or it could be \(D_{kk}^0\), see (2.4.14). A natural choice (in the absence of fixed points) is to take the average over the simple current group;

\[
P_{\hat{\Lambda},\hat{\lambda}}^Q([g,h]) := \sqrt{\frac{|H|}{|G|}} \sum_{(J, \ell) \in J_{g/h}} \sqrt{d_{J\Lambda^*}d_{\ell^* \lambda}} \sum_{m,a} \pi_{jl}^* D_{m_m, a_a}^{J_{\ell^* \lambda^* \Lambda^*}}(g, h). \tag{3.5.17}
\]

Thus, there is a natural choice to associate functions to \(Q\)-symmetry preserving Ishibashi blocks (labeled by orbits \([\Lambda, \lambda]\)), but not to the general \(G \times H\)-preserving ones (labeled by \((\Lambda, \lambda)\) satisfying (3.4.1)).

The branes, i.e. the shapes of the boundary states, of the coset model, are then given by

\[
B_{\hat{\Lambda}, \hat{\lambda}}^Q([g,h]) = \sqrt{\frac{|H|}{|G|}} \sum_{[\Lambda, \lambda]} \sqrt{S_{[\Lambda, \lambda]}^{[\Lambda, \lambda]}} \sum_{(J, \ell) \in J_{g/h}} \sqrt{d_{J\Lambda^*}d_{\ell^* \lambda}} \sum_{p, q} c_{p, q}^{\ell^* \Lambda^* < J, \ell^* \lambda^*} c_{p, b}^{\ell^* \Lambda^* < J, \ell^* \lambda^*} D_{p, q, b, c}^{J_{\ell^* \lambda^* \Lambda^*}}(g, h)
\]

\[
= \frac{|J_{g/h}|}{|G|} \sqrt{\sum_{\Lambda} \frac{d_{\lambda}^{\Lambda}}{S_{\lambda,0}^{\Lambda}}} \sum_{p, q} D_{p, q}^{\Lambda}(g) \sum_{\hat{\lambda}} \frac{S_{\hat{\lambda},0}^{\Lambda}}{\sqrt{d_{\lambda}^{\Lambda}}} \delta_{\lambda \prec \Lambda} \sum_{\ell} D_{\ell, q, \ell, c}^{\Lambda}(h).
\]

Here in the second equality we switched to use adapted bases, and combined the summation over allowed orbits and over the identification group to a summation over pairs \((\hat{\Lambda}, \hat{\lambda})\). A horizontal embedding gives a unique affine embedding \([39]\) section 2.8, thus \(\lambda \prec \Lambda\) implies \(\hat{\lambda} \prec \hat{\Lambda}\) and the latter pair labels a primary field of \(Q\), and therefore a symmetry-preserving Ishibashi block. Thus we can sum over all pairs, including those which are forbidden by the selection rules, because forbidden pairs do not contribute owing to the presence of \(\delta_{\lambda \prec \Lambda}\).

The formula (3.5.18) is our result for the shape of symmetry preserving branes in a coset model without field identification fixed points. It expresses the coset branes entirely in terms of quantities for \(\hat{\Lambda}, \hat{\lambda}\) and \(G, H\). Notice that field identification is
built in through the summation over the identification group in (3.5.17). Still, in the formula (3.5.18) the only explicit remnant of field identification is the overall factor of $|J_{g/h}|$; such a simplification will certainly no longer arise in models with field identification fixed points. It is not easy to check directly that in the limit of large level, the shape of the boundary states converge to delta functions centered on the conjugacy classes (3.5.10). However, we strongly expect these results to agree, so that the process of taking the large level limit commutes with the process of forming the coset.

There is an interesting class of coset models in which the formula (3.5.18) simplifies considerably – the case that $H = T$ is a maximal torus of $G$, and hence $\mathfrak{h} = \mathfrak{g}_0$ the Cartan subalgebra of $\mathfrak{g}$. These coset models are known as generalized parafermions \cite{53}. It turns out that in this case the representation matrices appearing in the formula can be combined to characters of $G$.

All irreducible $T$-representations are one-dimensional, with representation matrices the numbers $D^{\lambda}(e^{H_p}) = e^{(\lambda, \mu)}$. The primary fields of the $\mathfrak{h}$-theory are labeled by the weight lattice of $\mathfrak{g}$ modulo $k$ times the root lattice, so that their number $N_T$ is $k^{\text{rank} \mathfrak{g}}$ times the number of weight-conjugacy classs of $\mathfrak{g}$, i.e. $N_T = k^{\text{rank} \mathfrak{g}} |Z(G)|$ (see \cite{24}, eq. (14.322)).

The modular S-matrix of the $\mathfrak{g}_0$-theory is

$$S^\mathfrak{g}_0_{\lambda,\lambda'} = N_T^{-1/4} e^{-2\pi i \langle \lambda, \lambda' \rangle / k},$$

and the identification group $J_{\mathfrak{g}_0/h}$ is isomorphic as an abelian group to the group $J_{\mathfrak{g}}$ of simple currents of $\mathfrak{g}$. It follows that

$$B^Q_{\mathfrak{h}_0}((g, h)) = |J_{\mathfrak{g}}| N_T^{-1/4} \sqrt{\left| \frac{\mathcal{H}}{|G|} \right|} \sum_{\lambda} \frac{\sqrt{d_{\lambda}}}{{S^\mathfrak{g}_{\lambda,\lambda}}} \sum_{\lambda' < \lambda} \sum_{\ell} e^{2\pi i \langle \lambda, \lambda' \rangle / k} D^\lambda_{pp}(g) D^{\lambda*}_{pp}(h)$$

$$= |J_{\mathfrak{g}}|^{3/4} k^{-\text{rank} \mathfrak{g}/4} \sqrt{\left| \frac{\mathcal{H}}{|G|} \right|} \sum_{\lambda} \frac{\sqrt{d_{\lambda}}}{{S^\mathfrak{g}_{\lambda,\lambda}}} \chi_{\lambda}(e^{2\pi i \tau \lambda / k} h^{-1} g)$$

(3.5.20)

Here in the first line $\ell$ labels the occurrences of the weight $\lambda$ in $\Lambda$, and we use the short-hand $p \equiv (\lambda; \ell)$. Thus $\chi_{\lambda} = \sum_{\rho} D^\lambda_{p\rho}$. The second equality holds because the diagonal entries of the representation matrices $D^\lambda$ satisfy $D^\lambda_{pp}(gh) = D^\lambda_{pp}(g) D^\lambda(h)$ for any $g \in G$ and $h \in T$.

From (3.5.20) it is easy to see that the considerations in section 3.3.2 generalize to show that the boundary state function in the large level limit indeed converges to a delta function. Thus, we see explicitly in this particular case that the processes of forming the coset and that of taking the large level limit, commute.
3.5.3 Parafermion branes

The parafermions constitute the simplest case $G = SU(2)$ of the $H = T$ situation, in which the formula (3.5.18) reduces to (3.5.20). Plugging in the data for the parafermions into (3.5.20), we obtain

$$B_{[\Lambda, \lambda]}^{PF}(g, e^{i\tau \sigma_3}) = \frac{|J_{g/b}| \sqrt{|H|}}{(k(k+2))^{1/4} \sqrt{|G|}} \sum_{\Lambda=0}^{k} \sqrt{\Lambda+1} \sin \left( \frac{(\Lambda+1)(\Lambda'+1)\pi}{k+2} \right)$$

$$\times \sum_{\lambda'=-\Lambda}^{\Lambda} e^{i\pi \lambda'/k} D_{\lambda \lambda}(g) e^{-i\lambda t}$$

$$= \sqrt{\frac{2}{\pi k (k(k+2))^{1/4}}} \sum_{\Lambda=0}^{k} \sqrt{\Lambda+1} \sin \left( \frac{(\Lambda+1)(\Lambda'+1)\pi}{k+2} \right) X_{\Lambda} (e^{i\sigma_3(-t+\pi \lambda'/k)} g).$$

While the right hand side of (3.5.21) is written as a function of $g' = e^{i\sigma_3(-t+\pi \lambda'/k)} g$, i.e. like a function on the group $SU(2)$, it is indeed a function on $Q_{PF}$, since $g'(x, y)$ as given by (2.4.46) just projects to $(x, y) \in Q$. Also, as the group characters $X_{\Lambda}$ depend only on a single variable, each brane actually depends on a definite combination of $x$ and $y$. Concretely, with our choice of coordinates they are constant along some direction on the disk. This is most directly seen for branes with $\lambda = 0$; they depend only on $x$, but not on $y$. Further, the factor $e^{i\pi \lambda \sigma_3/k} \in U(1)$ amounts to a rigid rotation of $\partial D_{\pi/2}$ by an angle $\pi \lambda/k$ about its center, so that the straight line for the brane labeled by $(\Lambda, \lambda+\lambda')$ is obtained from the one for the brane $(\Lambda, \lambda')$.
by such a rotation (see also [31]). We illustrate this behavior in figure 3.4 for a brane at level 10. The same observation also shows that the branes are mapped to themselves by the action of the non-trivial identification current.

Figure 3.5: The shape $\tilde{B}^{PF}$ of the brane $(\frac{k-4}{6}, 0)$ as a function of the coordinate $x$, at four different values of the level. At large level, the brane is concentrated at $x = \frac{5\pi}{12}$.

In the large level limit, according to (3.5.12) the shape of each brane converges to a multiple of the delta function on the projection of the product of the relevant conjugacy classes, i.e. of the class of $e^{i\sigma_3} g$ in SU(2) and of the point $e^{i\tau_3\lambda_3/k} \in U(1)$. By the previous remarks, this yields just a delta function on a straight line in $D_{\pi/2}$; these lines are shown, for level $k = 6$, in figure 3.6.

At finite level the branes are peaked along these straight lines, but they are smeared significantly about these subsets. As an illustration, in figure 3.5 we display the shape of the brane $(\Lambda, 0)$ as it evolves with the level, plotted as a as a function of the coordinate $x$, for $\Lambda = \Lambda(k)$ chosen such that $\Lambda(k)+1 = (k+2)/6$. Since we draw the shapes as a function of the variable along the straight line, rather than on the disk, we must account for the different extension in perpendicular direction by an appropriate measure factor; in the case at hand, this amounts to replacing $B^{PF}(x)$ by $\tilde{B}^{PF}(x) = 2(\frac{\Lambda^2}{x^2} - x^2)^{1/2} B^{PF}(x)$.
Figure 3.6: The straight lines on $Q_{PF} = D_{\pi/2}$ at which the parafermion branes at level $k = 6$ are centered: (a) The branes $(\Lambda, 0)$. (b) The branes $(\Lambda, 1)$. (c) All 21 branes. Note that some of the branes meet on the boundary $\partial D_{\pi/2}$, but that this is not the generic situation.

### 3.5.4 D-branes in the diagonal coset

In the large level limit, the D-branes in the coset

$$Q = \frac{SU(2) \times SU(2)}{Ad(SU(2))} \quad (3.5.22)$$

are localized at sets $\pi_{Ad}(C_{abc})$. In this section, we consider $SU(2)$ as a three-sphere with unit radius. We denote

$$C_x := \{g(\psi, \theta, \phi) \in G \mid \psi = x\} \quad (3.5.23)$$

and use the following operation on pairs of conjugacy classes

$$C_x * C_y := \{g = g_x g_y \in G \mid g_x \in C_x, \ g_y \in C_y\} \quad (3.5.24)$$

The set $C_x * C_y$ is invariant under $Ad(G)$. Furthermore, introduce the sets

$$C_{abc} := (C_a \times C_b) \ast C_c \quad (3.5.25)$$

$$:= \{(g_a g_c, g_b g_c) \in G \times G \mid g_a \in C_a, \ g_b \in C_b, \ g_c \in C_c\} \subset G \times G.$$  

These sets are the ones that project to the coset branes. Note that they are fixed under the adjoint action;

$$(g, g) C_{abc} (g^{-1}, g^{-1}) = C_{abc}. \quad (3.5.26)$$

By studying the sets $C_{abc}$, we shall discuss the dimensionalities and other properties of the branes in $Q$. First an observation. The simplest example of a set which is
invariant under the diagonal Ad-action is the orbit of a point \((p_1, p_2) \in G \times G\). If this is a generic point (projecting to the bulk of \(Q\)), the stabilizer under the Ad\((G)\)-action is \(Z_2\), thus the orbit \(\text{Ad}_G(p_1, p_2)\) is three-dimensional in \(G \times G\). The geometric picture of this orbit is that \((\psi_1, \psi_2)\) are fixed, \((\phi_1, \phi_2)\) take all values and we have a line in the \((\theta_1, \theta_2)\)-plane. If instead \((p_1, p_2) \in Z \times Z\) is in the center, the orbit \(\text{Ad}_G(p_1, p_2)\) is zero-dimensional, and projects to a corner in \(Q\). Thus, we shall need not only the dimensionality of \(C_{abc}\) to find the dimensionality of \(\pi_{\text{Ad}}(C_{abc})\), we also need the information about the stabilizers.

**Corners** From (2.4.51) we learn that conjugacy classes \(C_x \subset SU(2)\) equal their own inverses; if \(g \in C_x\), then \(g^{-1} \in C_x\). For the \(\psi_1 = 0\) edge to be contained in \(C_{abc}\) it is necessary that \(a = c\) because otherwise, \(C_c\) does not contain the inverse of \(C_a\). The equation (2.4.51) tells us that this condition also is sufficient. Thus, the edge defined by \(\psi_1 = 0\) is reached if and only if \(a = c\). For reaching the edge \(\psi_1 = \pi\), we are faced with the following task: Given \(g \in C_a\), we would like to have a \(g' \in C_c\) such that \(g(g')^{-1} = -1\). This is equivalent to that \(C_c\) contains the inverses of \((-1)C_a = C_{\pi-a}\). But then, \(C_c = C_{\pi-a}\). As argued above, the edges per se are not interesting, but it is useful to note that a corner is where two edges meet (now viewing \(Q\) as a pillow, rather than a pinched cylinder). The analysis described above leads to the following conclusions about the four corners (whose \(\theta\)-coordinates are irrelevant).

- \((0, 0) \in \pi_{\text{Ad}}(C_{abc})\) if and only if \(a = b = c\).
- \((0, \pi) \in \pi_{\text{Ad}}(C_{abc})\) if and only if \((a = c\) and \(b = \pi - c\)).
- \((\pi, 0) \in \pi_{\text{Ad}}(C_{abc})\) if and only if \((a = \pi - c)\) and \(b = c\).
- \((\pi, \pi) \in \pi_{\text{Ad}}(C_{abc})\) if and only if \((a = \pi - c)\) and \(b = \pi - c\).

In particular, if a brane contains two corners, it automatically contains all four. The brane which corresponds to the fixed point of the identification group is \(\pi_{\text{Ad}}(C_{fff})\), with \(f = \pi/2\). This is the unique brane that contains all four corners, and is therefore in some sense the most degenerate brane. We shall show that the fixed point brane fills up all of \(Q\). However, it is not the only brane that is three-dimensional.

**Dimensionality of the fractional branes** It is instructive to first show that

\[
C_f \ast C_f = G.
\]

(3.5.27)

Note that (3.5.27) is equivalent to

\[
\forall \ g \in G, \ \exists \ u, w \in C_f : \ uw = g.
\]

(3.5.28)
The product of two elements $u$ and $w$ on the equatorial conjugacy class

$$C_f := \{ g \in G \mid \psi_g = \frac{\pi}{2} \} \subset G$$

(3.5.29)

is computed using (2.4.50).

$$uw = \bar{n}_u \cdot \bar{n}_w 1 - i \bar{\sigma} \cdot (\bar{n}_u \times \bar{n}_w)$$

(3.5.30)

Take an arbitrary element $g \in G$. The following shows that there exist $u, w \in C_f$ such that $uw = g$. Choose $\bar{n}_u \in S^2$ and $\bar{n}_w \in S^2$ in the plane perpendicular to $\bar{n}_g$. Choose the angle between them such that $\bar{n}_u \cdot \bar{n}_w = \cos \psi_g$, and so that $\bar{n}_u \times \bar{n}_w$ points in the same direction as $\bar{n}_g$. Since the product $uw$ is a group element, we do not even need to check that $|\bar{n}_u \times \bar{n}_w| = \sin \psi_g$, and hence $uw = g$. This proves the claim (3.5.27).

The following argument shows that

$$C_{fff} = G \times G,$$

(3.5.31)

which implies that the corresponding brane in the coset is spacefilling,

$$\pi_{Ad}(C_{fff}) = Q.$$  

(3.5.32)

First note that this is not an immediate consequence of (3.5.27), because the same factor $w \in C_f$ occurs at two places; we need to prove that for all $g_1, g_2 \in G$, we can choose $u, v, w \in C_f$ so that $uw = g_1$ and $vw = g_2$. A quick glance at (3.5.30) reveals that the choice of $g_1$ forces $\bar{n}_w \perp \bar{n}_1$. Given $w$, one may choose $u \in C_f$ such that $uw = g_1$. Now it remains to prove that with $\bar{n}_w$ fixed in the plane $\perp \bar{n}_1$ through the origin, and for any $g_2 \in G$, we can find $v \in C_f$ so that $vw = g_2$. We need $\bar{n}_w$ also to lie in the plane through the origin $\perp \bar{n}_2$. The two planes mentioned intersect at no less than two points on the sphere $C_f$. Choose $\bar{n}_w$ one of these points. Then choose $\bar{n}_v \perp \bar{n}_2$ such that $\bar{n}_v \cdot \bar{n}_w = \cos \psi_2$. It now follows $vw = g_2$. This proves the claim (3.5.31) and (3.5.32).

**Dimensionality of generic branes** We proceed to determine the dimensions of more general branes $\pi_{ad}(C_{abf})$. The following argument shows that in the generic (nondegenerate) situation, that is when none of the labels $a, b$ takes value $0, \pi$, the set $C_{abf}$ is six- dimensional in $G \times G$, while the corresponding brane is three-dimensional in $Q$. One can parameterize conjugacy classes as

$$u = c_a 1 + i s_a \bar{n}_u \times \bar{\sigma} \in C_a,$$

(3.5.33)
where \( c_\alpha \equiv \cos \alpha \) and \( s_\alpha \equiv \sin \alpha \) are fixed and \( \vec{n}_u \) ranges over \( S^2 \). With an analogous parametrization for \( v \in C_b \), we need for \( g_1 = uw \) and \( g_2 = vw \) the following conditions,

\[
\begin{align*}
\cos \psi_1 &= s_\alpha \vec{n}_u \cdot \vec{n}_w & \sin \psi_1 &= c_\alpha \vec{n}_w + s_\alpha \vec{n}_u \times \vec{n}_w \quad \text{(3.5.34)} \\
\cos \psi_2 &= s_\beta \vec{n}_v \cdot \vec{n}_w & \sin \psi_2 &= c_\beta \vec{n}_w + s_\beta \vec{n}_v \times \vec{n}_w
\end{align*}
\]

We immediately observe that in order for \((g_1, g_2) \in C_{abf}\), it is necessary that \( \cos \psi_1 \leq |\sin a| \), and that \( \cos \psi_2 \leq |\sin b| \). Assume that \( \cos \psi_1 < |\sin a| \), and that \( \cos \psi_2 < |\sin b| \), which is possible because we required \( a, b \) to be different from 0, \( \pi \). (This implies \( \vec{n}_u \cdot \vec{n}_w \neq 1 \) and \( \vec{n}_v \cdot \vec{n}_w \neq 1 \.) The rest of the argument aims to show that we now are in the interior of a six-dimensional region.

Given allowed choice of angles \( \psi_1, \psi_2 \), we must keep the scalar products \( \vec{n}_u \cdot \vec{n}_w \) and \( \vec{n}_v \cdot \vec{n}_w \) fixed accordingly. We are still free to choose the vector \( \vec{n}_w \) anywhere on \( S^2 \), and we are free to rotate the vectors \( \vec{n}_u \times \vec{n}_w \) and \( \vec{n}_v \times \vec{n}_w \) around the axis \( \vec{n}_w \).

Let us decide for a certain \( \vec{n}_1 \). With the freedom we have, we can certainly find vectors \( \vec{n}_u \) and \( \vec{n}_w \) such that \( \cos a \vec{n}_w + \sin a \vec{n}_u \times \vec{n}_w \) points in the direction of \( \vec{n}_1 \). Since \( uw \) is a group element, it is automatic that \( |\sin \psi_1 \vec{n}_1| = |\cos a \vec{n}_w + \sin a \vec{n}_u \times \vec{n}_w| \). Thus, we require that the three vectors \( \vec{n}_1, \vec{n}_w \) and \( \vec{n}_u \times \vec{n}_w \) lie in the same plane, and we fix the angle between them in this plane. We may rotate this plane, thus, we may rotate \( \vec{n}_w \) around the axis \( \vec{n}_1 \). Furthermore, the angle between these vectors is different from zero, because we assumed \( \vec{n}_u \cdot \vec{n}_w \neq 1 \), which implies \( \vec{n}_u \times \vec{n}_w \neq 0 \).

As we rotate the above mentioned plane, \( \vec{n}_w \) sweeps out a (nondegenerate) circle, around each choice of \( \vec{n}_w \), the vector \( \vec{n}_u \times \vec{n}_w \) sweeps out another nondegenerate circle. Thus, the vector \( \vec{n}_2 \) is allowed to reach a 2-dimensional subset of \( S^2 \). This completes the proof; subject to the constraints \( \cos \psi_1 < |\sin a| \) and \( \cos \psi_2 < |\sin b| \), \( g_1 = uw \) can take any value on the group, with enough freedom left to let \( g_2 \) sweep out a 3-dimensional subset of the group (varying \( \psi_2 \) and \( \vec{n}_2 \)). Thus, the generic set \( C_{abf} \subset G \times G \) is six-dimensional. It follows that \( \pi(C_{abf}) \subset Q \) is three-dimensional.

Let \( a, b, c \) be an arbitrary non-degenerate triple of labels. These branes are 3-dimensional as well. We have

\[
\begin{align*}
g_1 &= (c_a c_c - s_a s_c \vec{n}_u \cdot \vec{n}_w) \mathbf{1} - i (s_a s_c \vec{n}_u \times \vec{n}_w + c_a c_c \vec{n}_w + c_c s_a \vec{n}_u) \cdot \tilde{\sigma} \\
g_2 &= (c_b c_c - s_b s_c \vec{n}_v \cdot \vec{n}_w) \mathbf{1} - i (s_b s_c \vec{n}_v \times \vec{n}_w + c_b c_c \vec{n}_w + c_c s_b \vec{n}_v) \cdot \tilde{\sigma}
\end{align*}
\]

So assume \( \psi_1, \psi_2 \) are within the appropriate range, and fix the scalar products \( \vec{n}_u \cdot \vec{n}_w \) and \( \vec{n}_v \cdot \vec{n}_w \) accordingly. The three vectors

\[
c_c s_q \vec{n}_u \; , \; c_a s_c \vec{n}_w \; , \; s_q s_c \vec{n}_u \times \vec{n}_w
\]  

(3.5.35)
span a polyhedron, whose one corner is at the origin, and the opposite corner is at $\bar{n}_1$. We may rotate this polyhedron around these two points, and the rest of the proof carries over analogously, showing that also these branes project to something three-dimensional in the coset $Q$.

**Dimensionalities of degenerate branes** In section 3.3.3, we argued that symmetry-preserving branes in WZW models are non-degenerate. Nevertheless, we shall for the remainder of this section study the corresponding degenerate conjugacy classes in $Q$. The branes $\pi_{Ad}(C_{abc})$ where the second index $b = 0, \pi$ is ‘degenerate’ were studied in [32],1 where 2-dimensional branes were found. Let us take $b = 0$ (the case $b = \pi$ is analogous).

$$
\begin{align*}
 u &= c_a 1 + i s_a \bar{n}_u \cdot \bar{\sigma} & v &= 1 & w &= c_c 1 + i s_c \bar{n}_w \cdot \bar{\sigma} \\
 g_1 &= (c_a c_c - s_a s_c \bar{n}_u \cdot \bar{n}_w)1 - i(s_a s_c \bar{n}_u \times \bar{n}_w + c_a s_c \bar{n}_w + c_c s_a \bar{n}_u) \cdot \bar{\sigma} \\
 g_2 &= c_c 1 - i s_c \bar{n}_w \cdot \bar{\sigma}
\end{align*}
$$

We immediately see that the branes are localized along the slice $\psi_2 = c$. The last of the above equations fixes $\bar{n}_w = \bar{n}_2$. As before, $\psi_1$ has a certain allowed range.

For fixed $\psi_1$, note that the scalar product $\bar{n}_u \cdot \bar{n}_w$ is fixed. Thus, $\bar{n}_1$ takes a one-dimensional subset of values on $S^2$. To sum up, the set $C_{00c}$ is four-dimensional. If we project out the coordinates $\theta_1$ and $\phi_1$ on the set $C_{00c}$, then $\bar{n}_2$ takes values in a circle. Since we need to project out $\phi_2$, the coordinate $\theta_2$ is fixed. Thus, these branes $\pi_{Ad}(C_{00c})$ are at most one-dimensional, extending only in the coordinate $\psi_1$.

If we also take $a = 0$, $g_1 = c_c 1 - i s_c \bar{n}_w \cdot \bar{\sigma}$, also $\psi_1$ is fixed; $\psi_1 = c = \psi_2$. Also note that $\bar{n}_1 = \bar{n}_w = \bar{n}_2$. Thus, $C_{00c} \subset G \times G$ is two-dimensional and must project to a region where the stabilizer is at least one-dimensional, that is at the boundary of $Q$, which is at $\theta = 0, \pi$. Thus, the branes $\pi_{Ad}(C_{00c})$ are pointlike.

If we consider $C_{a00}$, we have $g_1 = 1c_a - i s_a \bar{n}_u \cdot \bar{\sigma}$, $g_2 = 1$. This projects to a point $\psi_1 = a$, $\psi_2 = 0$ on the edge of $Q$.

The case left to consider is $C_{ab0}$. We have $g_1 = c_a 1 - i s_a \bar{n}_u \cdot \bar{\sigma}$, and $g_2 = c_b 1 - i s_b \bar{n}_v \cdot \bar{\sigma}$. Thus, $C_{ab0}$ is four-dimensional in the covering, and projects to a line that meets the boundary of the coset at both ends $\theta = 0, \pi$.

---

1To interpret these branes as branes in the Virasoro minimal model is not without complications. In order to do that, we need to fix one of the levels to the value $l = 1$, which forbids us to use results for the large level limit.
Some remarks  In the non-simply connected Lie group $SO(3) = SU(2)/\mathbb{Z}_2$, there are two fractional branes supported at the un-orientable conjugacy class $\mathbb{R}P^2 \subset SO(3)$. The fundamental group of this conjugacy class is $\mathbb{Z}_2$, and the two branes correspond to different choices of gerbe modules [48]. In the CFT description, these fractional branes correspond to boundary states that are labeled by the fixed point of the simple current group $\mathbb{Z}_2$ of the $SU(2)$ WZW model. In the model (2.4.48) that was investigated here, we have a comparable situation: The CFT description at levels $k, l, k + l$ involves a $\mathbb{Z}_2$ identification group, with the generator acting as

$$(a, b, c) \mapsto (k-a, l-b, k+l-c) .$$

(3.5.36)

The fixed point boundary states have been shown to be supported at the whole target. Other branes reach at most one of the singular points (corners). If there is a corresponding gerbe (module) structure for cosets, it seems to be unknown in the literature. Hopefully, the present investigations can be used as a guideline for future work in this direction.

3.5.5 Boundary blocks in the diagonal coset

Most of the analysis of the boundary state geometry is in the case without fixed points a straightforward application of earlier results. In the presence of fixed points, we need something like (3.4.28) to associate functions to the degenerate Ishibashis. One can find the Ishibashi and boundary labels by following the standard procedure. The self-monodromy of the non-trivial identification current is two times its conformal weight, which is always integer. Hence the discrete torsion matrix $X$ vanishes. Therefore, the Ishibashis are labeled by tuples $(m, J)$ where the monodromy of $m$ with respect to $J$ vanishes, and $J \ast m = m$. Thus, the first species of Ishibashis are simply the primary fields allowed by (2.3.9). For even levels $k, l \in 2\mathbb{Z}$ there will be an additional fractional Ishibashi block labeled

$$(m; K) = (k/2, l/2, k/2+l/2; k, l, k+l) .$$

(3.5.37)

We consider the (adjoint) coset (2.4.48) as a set of equivalence classes, see [32],

$$[g, h]_{tr} = [ugv, uhv]_{tr}, \quad u, v \in H .$$

(3.5.38)

To the Ishibashis labeled $(a, b, c; 0)$ we associate a function which is invariant under the coset action;

$$I_{(a, b, c; 0)}[ug_1v, ug_2v, uhv] = I_{(a, b, c; 0)}[g_1, g_2, h] ,$$

(3.5.39)
which is achieved by the function (3.5.17). Applied to the present context, the formula (3.5.17) involves the Clebsch-Gordan coefficients of $SU(2)$. (For notational simplicity, we shall consider the $G \times H$-preserving Ishibashi blocks, which means we take (3.5.17) without averaging over the simple current group.) In our notation, $c$ is contained in $a \times b$ if $c = |a-b|, \ldots, a+b$, and the magnetic quantum number $\gamma$ is in the representation $c$ if $|\gamma| \leq c$, with multiplicity given by the Clebsch-Gordan coefficient $c_{\gamma,\alpha,\beta}^c$. Recall that the Clebsch-Gordan coefficients are defined by

$$c_{j,m,j_1,m_1,j_2,m_2} \equiv \langle j,m | (|j_1,m_1\rangle \otimes |j_2,m_2\rangle).$$

(3.5.40)

Note that $c_{\gamma,\alpha,\beta}^c = 0$ if $\gamma \neq \alpha + \beta$. Also note that $\sum_{\alpha,\beta,\gamma} |c_{\gamma,\alpha,\beta}^c|^2 = d_a d_b = d_c$. The regular Ishibashi function after integrating out the coset action is

$$I_{Q}^{a,b,c}[g_1, g_2, h] = \sqrt{\frac{|G_{k+l}|^5}{|G_k||G_l|}} \frac{1}{d_c} \times \sum_{\delta,\epsilon \prec \alpha,\mu \prec \nu \prec b} D_{\delta\epsilon}^{a}(g_1) D_{\mu\nu}^{b}(g_2) D_{\delta + \mu, \epsilon + \nu}^{c}(h) \ast (c_{\delta + \mu, \epsilon, \nu}^{c \times a \times b}) \ast (c_{\epsilon + \nu, \epsilon, \nu}^{c \times a \times b}).$$

(3.5.41)

The shape of the fractional Ishibashi block is the analogous projection of the product of twisted characters,

$$I_{K}^{a,b,c}[g_1, g_2, h] = \sqrt{\frac{|G_{k+l}|^5}{|G_k||G_l|}} \frac{1}{d_c} \times \sum_{\delta,\epsilon \prec \alpha,\mu \prec \nu \prec b} D_{\delta\epsilon}^{k/2}(g_1 e^{i \sigma_3 2\pi/k}) D_{\mu\nu}^{l/2}(g_2 e^{i \sigma_3 2\pi/l}) D_{\delta + \mu, \epsilon + \nu}^{(k+l)/2}(he^{i \sigma_3 2\pi/(k+l)}) \ast (c_{\delta + \mu, \epsilon, \nu}^{c \times a \times b}) \ast (c_{\epsilon + \nu, \epsilon, \nu}^{c \times a \times b}).$$

(3.5.42)

The Ishibashi function (3.5.42) is orthogonal to the other Ishibashi functions (3.5.41), because the scalar product can be lifted to the covering $G \times G \times G$ where orthogonality follows from previous considerations. The shapes of the boundary states can now be calculated by using the boundary coefficients given in [41]. The results of this calculation are more complicated than enlightening; we refrain from presenting them here.
Conclusions and outlook

The present thesis is based on four published articles, [1], [3], [4] and [5]. The first chapter of this thesis is to some extent based on the lecture notes [6] from the summer school in Bad Honef, July 2005. Most of the contents of these lecture notes appeared in the introduction to [23], except the discussion of Hilbert space structures (see section 1.3.6), which is due to the present author. The proof that the given definition of a VA agrees with the one in [6] and [23], is new in this thesis.

The first article is [1] on lens spaces and their branes; sections 2.5 and 3.4.2 are based on this article. The discussion about T-duality in section 2.5 is more extensive here than it was in [1]. Both authors took part in doing the calculations and writing the paper. Jürgen Fuchs and Christoph Schweigert contributed with many helpful remarks.

The second article [3] is on boundary states and branes in coset models, on which sections 2.4.1 - 2.4.4, 3.3.3, and 3.5.1 - 3.5.3 are based. The results in section 3.3.3 and to some extent 3.5.1, 2.4.3 and 2.4.1 had appeared already in the licentiate thesis [2]. The remaining calculations in [3] are due to both authors.

The third article [4] is on D-branes in the diagonal coset, on which sections 2.4.5 and 3.5.4 are based. Jürgen Fuchs and to some extent also Christoph Schweigert checked the calculations and the proofs, as well as the written text.

Some of the claims in [1] are not included in this thesis for reasons that are indicated in the text, namely that the way to associate functions to Ishibashi blocks used in [1] gives unsatisfying properties of the obtained Ishibashi functions. By better understanding exactly why the decomposition (3.4.21) and its interpretation does not work, we could perhaps learn something interesting about coset models in general.

The theory of coset models is quite extensively discussed in the literature, but
some questions remain open. One thing which is unclear (at least to the present author), is if we can see that the commutant $VA$ (2.4.16) is equivalent to some orbifold $VA$ of $G \times \bar{H}$. In such case, it would be interesting to learn more about the $VA\bar{H}$ (see [60] for some directions), and perhaps one could be able to classify the Maverick cosets, and learn more about coset CFT’s with fixed points.

The aim of fourth article [5] was to learn more about the Ishibashi blocks in the presence of fixed points of simple current groups. This helps in particular to understand fixed points of identification groups, and is thus a step towards a better understanding of boundary states in cosets with fixed points. The calculations were checked by Jürgen Fuchs who also contributed with many helpful remarks. Section 3.4.3, 3.4.4, and 3.5.5 are based on this article, and section 2.3.6 contains a proof that was omitted in [5].
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Conformal Field Theory and D-branes

The main topic of this doctoral thesis is D-branes in string theory, expressed in the language of conformal field theory. The purpose of string theory is to describe the elementary particles and the fundamental interactions of nature, including gravitation as a quantum theory. String theory has not yet reached the status to make falsifiable predictions, thus it is not certain that string theory has any direct relevance to physics. On the other hand, string theory related research has led to progress in mathematics.

We begin with a short introduction to conformal field theory and some of its applications to string theory. We also introduce vertex algebras and discuss their relevance to conformal field theory. Some classes of conformal field theories are introduced, and we discuss the relevant vertex algebras, as well as their interpretation in terms of string theory.

In string theory, a D-brane specifies where the endpoint of the string lives. Many aspects of string theory can be described in terms of a conformal field theory, which is a field theory that lives on a two-dimensional space. The conformal field theory counterpart of a D-brane is a boundary state, which in some cases has a natural interpretation as constraining the string end point. The main focus of this thesis is on the interpretation of boundary states in terms of D-branes in curved target spaces.