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Scalar fields on star graphs

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C-level thesis

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Abstract

A star graph consists of a vertex to which a set of edges are connected. Such an object can be used to, among other things, model the electromagnetic properties of quantum wires. A scalar field theory is constructed on the star graph and its properties are investigated. It turns out that there exist Kirchoff's rules for the conserved charges in the system leading to restrictions of the possible type of boundary conditions at the vertex. Scale invariant boundary conditions are investigated in detail.

Sammanfattning

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1 Introduction

The notion of a graph is a general mathematical concept. A graph consists of a set of elements which is connected by some relations. This concept has found applications in many different areas of science, engineering and also social science [9]. Some examples from very different fields of science and engineering of things that can be modeled by graphs are street networks, network of neurons and the structure of databases.

In this thesis a particular type of graph will be studied, namely the quantum graph. This is a graph considered not only as a purely combinatorical object, but as a one-dimensional singular variety equipped with a self-adjoint differential operator [13, 14].

Some reviews of graphs in general and quantum graphs in particular can be found in [9, 13, 14, 15].

1.1 Quantum graphs

One reason to study quantum graphs is that they occur naturally as simplified models in mathematics, chemistry and engineering. The first application in physics was probably in the context of free electron models for organic molecules by Pauling [20]. Other historical notes can be found in [13]. In more recent years, the progress of nanotechnology has for example made it possible to construct quantum wires. The thickness of these wires are of the order of a few nanometers which makes them essentially one-dimensional. In order to construct electronics, several wires need to be interconnected at a junction. Such junctions has been synthesized by growing Y-junction single-walled carbon nanotubes [7]. Modelling of the electronic behaviour at such small scales makes the understanding of electric conduction at the quantum limit is necessary. Quantum graphs provide a useful approximation of such systems and are therefore an important theoretical tool. Some of the most interesting physical problems concern the charge and spin transport. These problems can be investigated by using quantum field theory and is therefore a motivation to construct quantum fields on graphs [1]. Some examples of theoretical studies of the electrical properties of quantum wires can be found in [16] and [6].

A graph $\Gamma$ consists of a set of vertices $V = \{v_i\}$ and a set of edges $E = \{e_i\}$ which connect these vertices. A graph is said to be a metric graph, if each edge $e_i$ is assigned a positive length $l_{e_i} \in (0, \infty]$. This additional structure makes $\Gamma$ a topological and metric object (see section 2 of [15]). A quantum graph is a metric graph with the additional structure of a differential operator on $\Gamma$. This operator is usual (but not always) required to be self-adjoint.

An example of a graph with 4 vertices and 12 edges is shown in figure 1. There are 6 internal and 6 external edges. The external edges can be thought of as connected to vertices at infinity. The figure does not show any loops or multiple edges between vertices but such objects are also allowed. Note that the crossing of two of the edges at the point where there is no vertex does not not have any mathematical meaning. It is only a byproduct of putting the mathematical concept of a graph into what is colloquially referred to as a graph.
1.2 Star graphs

One special type of graph is the *star graph*, which in a way can be thought of as a building block of a more general graph. A star graph $\Gamma$ consists of only one vertex together with $n$ edges as shown in figure 2. A natural way to assign coordinates to the edges of a star graph is to use the pair $(x, i)$, where $i$ refer to the edge $e_i$ and $x$ is the distance from the vertex. The star graph will be the type of graph that is studied in this thesis.

![Figure 2: A star graph $\Gamma$ with $n$ edges.](image)

1.3 Quantum field theory on graphs

The topic on quantum field theory on graphs has recently attracted much attention. Besides from a purely theoretical interest, this is largely motivated by the physics of quantum wires as was hinted in section 1.1. In quantum field theory, the vertices can be interpreted as pointlike defects, which are characterized by a scattering matrix $S$ [4]. The scattering matrices under consideration in this thesis concern only defects which only admits scattering states. For a discussion of bound states see [4] and [5].

At the vertices of the graph, the field will have to satisfy boundary conditions. Finding admissable boundary conditions for the vertices is an important part in developing QFT on
graphs. A systematic approach for finding boundary conditions can be found in [11]. Regarding the physical relevance of different boundary conditions a discussion can be found in [8].

A common way to approach the electromagnetic properties of a quantum graph is to develop a theory of bosonic fields and then express the interacting fermionic fields in terms of the bosonic ones via the process of bosonization. This is done, in for example [1, 2, 3, 4]. This motivates the study of scalar fields on quantum graphs.

1.4 Overview

This thesis discusses mainly some topics presented in the article [1] by Bellazzini, Burrello, Mintchev and Sorba. The thesis is a recollection of existing knowledge and does not introduce any new results, but does give some details which do not seem to be given anywhere in the litterature, e.g. section 3.5.

The organization of the thesis is as follows. Section 2 describes some specific properties of quantum fields on star graphs. The section specifically discusses how the choice of boundary conditions is affected by symmetries of the theory. In section 3 basic concepts of the theory of scalar fields on star graphs are developed. Special interest are devoted to scale invariant boundary conditions. Kirchoff's rules are constructed for different conserved currents and it is seen that they cannot, in general, be satisfied simultaneously. Section 4 is devoted to the Fock and Gibbs representations of an algebra which appears in the study of the scalar field. The representations are defined and their two-point expectation values are calculated. There are also some mentioning of possible applications of the representations.
2 Characteristic features of QFT on the star graph $\Gamma$

In quantum field theory, symmetries play a fundamental role. When doing QFT on graphs, some new features come into play. This section explores the additional conditions that must hold for a conserved current if the corresponding charge is to be conserved.

On a star graph $\Gamma$, let $\{j_{\nu}(t, x, i) \mid \nu = t, x\}$ be a conserved current, that is

$$\partial_t j_t(t, x, i) - \partial_x j_x(t, x, i) = 0.$$ (2.1)

Taking the time derivative of the corresponding charge gives

$$\partial_t \sum_{i=1}^n \int_0^\infty dx j_i(t, x, i) = \sum_{i=1}^n \int_0^\infty dx \partial_x j_x(t, x, i) = -\sum_{i=1}^n j_x(t, 0, i).$$ (2.2)

This implies that charge conservation holds only if Kirchhoff’s rule

$$\sum_{i=1}^n j_x(t, 0, i) = 0$$ (2.3)

holds. This condition imposes restrictions on the interaction at the junction and hence restricts the choice of possible boundary conditions.

Different conserved currents generate non-equivalent Kirchhoff’s rules which may be in contradiction for general boundary conditions. This means that for a system which exhibits this property, there are no boundary conditions which conserve all the corresponding charges. One therefore has to choose which charges to conserve. An explicit example of this is given in section 3.4. This means that not all symmetries on the line can be preserved on the star graph. It should be noted however, that for star graphs with an even number of edges, there exist boundary conditions for which the edges can be rearranged so that the graph is equivalent to a bunch of independent lines. These are called exceptional boundary conditions and for these, the theory on the star graph coincides with the theory on the line.

It follows from the above discussion that a crucial point for QFT on star graphs is to select boundary conditions. The rest of this section shows a general criterion for how to find the right boundary conditions. It follows closely the reasoning in section 2 of [1].

Consider systems that are invariant under time translations. The conserved current is the energy momentum tensor $\theta$. For the components $\{\theta_{tt}(t, x, i), \theta_{tx}(t, x, i)\}$ the corresponding Kirchhoff’s rule is

$$\sum_{i=1}^n \theta_{tx}(t, 0, i) = 0.$$ (2.4)

Using (2.2) it follows that this rule selects all boundary conditions for which the Hamiltonian

$$H = \sum_{i=1}^n \int_0^\infty dx \theta_{tt}(t, x, i)$$ (2.5)
is time independent.

Under certain assumptions [1] it can be deduced that the time-independent Hamiltonian is symmetric. To ensure unitary time evolution, $H$ must admit self-adjoint extensions. As outlined in [1] this is satisfied for systems which are invariant under time-reversal, i.e. there exists an anti-unitary operator $T$, which implements time-reversal according to

$$THT^{-1} = H.$$  \hspace{1cm} (2.6)

The requirement that the Hamiltonian admits self-adjoint extensions leads to further restrictions on the boundary conditions. This will be seen explicitly in the next section.
3 Scalar fields on the star graph

In this section, a concrete example of a scalar field is used to investigate some basic concepts regarding quantum field theory on a star graph $\Gamma$ with $n$ edges.

3.1 The field $\varphi$

Introduce the scalar field on $\Gamma$ with a relativistic dispersion relation [2] which is defined by the equation of motion

$$\left(\partial_t^2 - \partial_x^2 + m^2\right) \varphi(t, x, i) = 0, \quad x > 0, \ i = 1, \ldots, n.$$  \hspace{1cm} (3.1)

In the massless case this becomes

$$\left(\partial_t^2 - \partial_x^2\right) \varphi(t, x, i) = 0, \quad x > 0, \ i = 1, \ldots, n.$$  \hspace{1cm} (3.2)

The field is subject to the initial conditions

$$[\varphi(0, x_1, i_1), \varphi(0, x_2, i_2)] = 0,$$  \hspace{1cm} (3.3)

$$[(\partial_t \varphi)(0, x_1, i_1), \varphi(0, x_2, i_2)] = -i \delta_{i_1,i_2} \delta(x_1 - x_2),$$  \hspace{1cm} (3.4)

and the vertex boundary condition

$$\sum_{j=1}^{n} [A_{ij} \varphi(t, 0, j) + B_{ij} (\partial_x \varphi)(0, j)] = 0, \quad \forall t \in \mathbb{R}, \ i = 1, \ldots, n,$$  \hspace{1cm} (3.5)

where $A$ and $B$ are two $n \times n$ complex matrices. Obviously, multiplying the matrices $A$ and $B$ with any invertible matrix $C'$ define equivalent boundary conditions. As mentioned in the introduction, a differential operator is needed to make the graph $\Gamma$ a quantum one. The operator $-\partial_x^2$ fits this purpose and is self-adjoint on $\Gamma$ provided that [11]

$$AB^\dagger - BA^\dagger = 0,$$  \hspace{1cm} (3.6)

where $^\dagger$ denotes the Hermitian conjugate, and that the composite matrix $(A, B)$ has rank $n^1$.  

A problem with the parametrization of the boundary condition via the matrices $A$ and $B$ is due to the fact that the matrices $(A, B)$ and $(C'A, C'B)$ define equivalent boundary conditions. This means that the pair $(A, B)$ is not uniquely defined by the boundary condition. By rewriting the matrices $A$ and $B$ in terms of a unitary matrix $U$ and a matrix $C$ where $C$ is the inverse of $C'$, the description becomes unique in the sense that there is a bijection between boundary conditions and unitary matrices $U$. This argument is explained in detail in the paper by Harmer [10], see also remark 6 in [15].

\footnote{There are also other equivalent ways of expressing the self-adjointness, see theorem 5 in [15].}
Using the results by Harmer, the application of the constraint \((3.6)\) gives the following general form of on \(\{A, B\}\) \([10]\):

\[
A = C(1 - U), \quad B = -\frac{i}{k_0} C(1 + U)
\]  

(3.7)

where \(C\) and \(U\) are the matrices described above and \(k_0 \in \mathbb{R}\) is a dimensional constant. Although it should be apparent from \((3.6)\), the expression \((3.7)\) makes it very clear that the matrices \(A\) and \(B\) are not independent of each other if the operator \(-\partial_x^2\) is to be self-adjoint.

It is clear from equations \((3.5)\) and \((3.7)\) that a diagonal \(U\) corresponds to decoupled boundary conditions in the sense that for each edge, there is no contribution to the boundary condition from the fields at the other edges. Such a graph corresponds to the picture given in figure 3.

![Figure 3: A decoupled star graph with \(n\) edges.](image)

The equation of motion \((3.1)\) with the given initial and boundary conditions can be quantized. The problem has a unique solution which is \([2]\)

\[
\varphi(t, x, i) = \int_{-\infty}^{\infty} \frac{dk}{2\pi \sqrt{2\omega(k)}} \left[ a_{i}^\dagger(k) e^{i(\omega(k)t - kx)} + a_i(k) e^{-i(\omega(k)t - kx)} \right],
\]

(3.8)

where \(\omega(k)\) is the relativistic dispersion relation

\[
\omega(k) = \sqrt{k^2 + m^2}.
\]

(3.9)

In the massless case, the dispersion relation becomes \(\omega(k) = |k|\) and the scalar field is then expressed as

\[
\varphi(t, x, i) = \int_{-\infty}^{\infty} \frac{dk}{2\pi \sqrt{2|k|}} \left[ a_{i}^\dagger(k) e^{i(|k|t - kx)} + a_i(k) e^{-i(|k|t - kx)} \right].
\]

(3.10)

The set \(\{a_i(k), a_i^\dagger(k), k \in \mathbb{R}\}\) generates an algebra which corresponds to the boundary conditions \((3.5)\). This is an associative algebra with an identity element \(\mathbb{1}\), which satisfies the relations

\[
a_{i_1}(k_1) a_{i_2}(k_2) - a_{i_2}(k_2) a_{i_1}(k_1) = 0
\]  

(3.11)

\[
a_{i_1}^\dagger(k_1) a_{i_2}^\dagger(k_2) - a_{i_2}^\dagger(k_2) a_{i_1}^\dagger(k_1) = 0
\]  

(3.12)

\[
a_{i_1}(k_1) a_{i_2}^\dagger(k_2) - a_{i_2}^\dagger(k_2) a_{i_1}(k_1) = 2\pi \left[ \delta_{i_1 i_2} \delta(k_1 - k_2) + S_{i_1 i_2}(k_1) \delta(k_1 + k_2) \right] \mathbb{1}
\]  

(3.13)
and

\[ a_i(k) = \sum_{j=1}^{n} S_{ij}(k)a_j(-k), \quad a_i^\dagger(k) = \sum_{j=1}^{n} a_j^\dagger(-k)S_{ji}(-k) \quad (3.14) \]

This algebra is a special case of the boundary and the reflection-transmission algebras which has been studied in [17, 18]. It can be shown [4], that the field (3.10) satisfies the boundary condition (3.5) and the initial condition (3.3). To ensure that also (3.4) is satisfied, the following condition is needed:

\[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} S_{ij}(k) = 0. \quad (3.15) \]

The interpretation of this condition is that the system described by the field \( \varphi \) doesn’t allow bound states [4].

The \( S \)-matrix in the preceding equations is given in terms of the boundary matrices. The explicit form is [11]

\[ S(k) = -[A + ikB]^{-1} [A - ikB] \quad (3.16a) \]

or equivalently by using (3.7)

\[ S(k) = - \left( \mathbb{I} - U \right) + \frac{k}{k_0} \left( \mathbb{I} + U \right) \right]^{-1} \left( \mathbb{I} - U \right) - \frac{k}{k_0} \left( \mathbb{I} + U \right). \quad (3.16b) \]

The invertibility of the individual factors of equations (3.16) is established in lemma 2.3 of [11].

It follows from (3.16b) that the \( S \)-matrix fully characterizes the boundary conditions and therefore the vertex. The vertex can therefore be seen as a sort of point-like defect. This also means that the boundary conditions can be reconstructed from the \( S \)-matrix via the process of inverse scattering [3].

The physical interpretation of the \( S \)-matrix is that the diagonal elements, \( S_{ii} \) is the reflection amplitude on the edge \( E_i \), and the off-diagonal elements \( S_{ij}, \ i \neq j \) is the transmission amplitude from the edge \( E_i \) to the edge \( E_j \). As an example it is readily seen that for a situation like the one in figure 3 in which the boundary conditions (3.7) is described by a diagonal unitary matrix, the resulting \( S \)-matrix will also be diagonal.

The following theorem establishes some basic properties of the \( S \)-matrix.

**Theorem 3.1.** The \( S \)-matrix (3.16) satisfies unitarity

\[ S(k)^\dagger = S(k)^{-1}, \quad (3.17) \]

and hermitian analyticity

\[ S(k)^\dagger = S(-k), \quad (3.18) \]

which combined gives

\[ S(k)S(-k) = \mathbb{I}. \quad (3.19) \]

The last expression is consistent with the constraints (3.14).
Proof. From (3.16a) it follows that
\[ S(k)\dagger = -[A\dagger + ikB\dagger] [A\dagger - ikB\dagger]^{-1} \] (3.20)
and
\[ S(k)^{-1} = -[A - ikB]^{-1} [A + ikB]. \] (3.21)
Equating (3.20) and (3.21) and multiplying from the left with \((A - ikB)\) and multiplying from the right with \((A\dagger - ikB\dagger)\) gives
\[ (A - ikB) (A\dagger + ikB\dagger) = (A + ikB) (A\dagger - ikB\dagger) \] (3.22)
which reduces to
\[ AB\dagger - BA\dagger = 0. \] (3.23)
This is exactly the condition (3.6) that the matrix \(AB\dagger\) is self-adjoint and is therefore satisfied. Thus, \(S(k)\dagger = S(k)^{-1}\).

To prove hermitian analyticity (3.18) it suffices to see that
\[ S(-k) = -[A - ikB]^{-1} [A + ikB] = S(k)^{-1} = S(k)\dagger. \] (3.24)

On \(\varphi\), time-reversal is implemented via
\[ T\varphi(t, x)T^{-1} = \varphi(-t, x), \] (3.25)
where \(T\) is the time-reversal operator.

By requiring that \(\varphi\) is Hermitian and invariant under time-reversal, there exists an invertible matrix \(C\) such that \(\{CA, CB\}\) are real matrices [3]. Since multiplication by \(C\) defines equivalent boundary conditions and leaves the \(S\)-matrix invariant, it follows that in this case the matrices \(\{A, B\}\) can be assumed to be real without loss of generality.

For \(A, B\) real, the condition (3.6) takes the form
\[ AB^t - BA^t = 0, \] (3.26)
where \(t\) denotes transposition. The \(S\)-matrix will now satisfy the additional condition
\[ S(k)^t = S(k), \] (3.27)
which is proved in a similar way as unitarity was proved in theorem 3.1.
3.2 Scale invariance

An interesting subset of systems which can be described by the above fields and boundary conditions are the scale invariant systems. There are mainly two reasons for paying some closer attention to such systems. Firstly, scale invariance simplify a lot of calculations which makes it possible to explore them in greater detail. Secondly, scale invariance is closely related to the notion of critical points and universality. A critical point is a notion which is used in different areas of physics to describe the conditions at which a phase boundary ceases to exist. An example from thermodynamics is that for a substance there exists a combination of pressure and temperature at which the distinction between liquid and gas phases ceases to exist. Universality means that different systems shows the same behavior at a critical points even though they may behave very differently at off-critical points.

A scale invariant system is invariant under the following transformation

\[ t \mapsto \rho t, \quad x \mapsto \rho x, \quad \rho > 0. \]  

(3.28)

In momentum space (3.28) induces

\[ k \mapsto \rho^{-1} k, \]  

(3.29)

which means that a system is scale invariant if and only if

\[ S(k) = S(\rho^{-1} k), \quad \forall k \in \mathbb{R}, \]  

(3.30)

which implies that \( S \) is independent of \( k \). From (3.17), (3.18) and (3.27) it follows that any scale invariant matrix obeys

\[ S^\dagger = S^{-1}, \quad S^\dagger = S, \quad S^t = S. \]  

(3.31)

For a scale invariant system the expression for the \( S \)-matrix (3.16a) can be rewritten in the form

\[ (A + i k B) S = - (A - i k B). \]  

(3.32)

Taking the hermitian conjugate gives

\[ S^\dagger (A^\dagger - i k B^\dagger) = - A^\dagger - i k B^\dagger. \]  

(3.33)

Thus it follows that each column of \( A^\dagger \) is either zero or an eigenvector of \( S^\dagger \) with eigenvalue \(-1\) and each column of \( B^\dagger \) is zero or an eigenvector of \( S^\dagger \) with eigenvalue \(1\). Moreover, since the composite matrix \((A, B)\) has rank \( n \), there must be exactly \( p \) non-zero columns of \( A^\dagger \) and \( n - p \) non-zero columns of \( B^\dagger \) and these columns do not coincide. Returning to \( A \) and \( B \) this means that \( A_i \neq 0 \iff B_i = 0 \), where \( A_i \) denotes the \( i \)'th row. From the above it follows that a \( n \)-dimensional scale invariant \( S \)-matrix has \( p \) eigenvectors with eigenvalue \(-1\) and \( n-p \) eigenvectors with eigenvalue \(1\).

The scale invariance therefore imposes restrictions on the form of the boundary condition matrices \( A \) and \( B \) [3]. Each equation in the boundary condition can only involve the functions or...
the derivatives but not both. This is not surprising since the boundary condition (3.5) involves a dimensional parameter \( k \), and for the system to be scale invariant the terms must decouple. For the matrices \( A \) and \( B \) it follows that for every non-zero row in \( A \), the corresponding row in \( B \) must be zero. The condition that the composite matrix \((A, B)\) has rank \( n \) implies that for a scale invariant system, \( \text{rank}_A = p \) and \( \text{rank}_B = n-p \). The limiting cases \( p=n \) and \( p=0 \) leads to Dirichlet and Neumann boundary conditions respectively.

The self-adjointness of \( AB^\dagger \) is also satisfied. It follows from (3.31) and (3.33) that

\[
AB^\dagger = -ASS^\dagger B^\dagger = -AB^\dagger. \tag{3.34}
\]

Hence

\[
AB^\dagger = 0. \tag{3.35}
\]

### 3.3 Examples of scale invariant systems

The limiting cases \( p=n \) and \( p=0 \) described above lead to the trivial \( S \)-matrices \( S = -\mathbb{1} \) and \( S = \mathbb{1} \) respectively. These systems do not admit any transmission at the boundary. In fact, they correspond to the decoupled boundary conditions described in figure 3.

The non-limiting cases \( 0<p<n \) are more interesting. These systems depend on \( p(n-p) \) parameters [1]. For \( n=2 \) and \( p=1 \), the matrices \( A \) and \( B \) have the form

\[
A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \tag{3.36}
\]

for which the corresponding \( S \)-matrix is

\[
S = \frac{1}{1+\alpha^2} \begin{pmatrix} \alpha^2 - 1 & -2\alpha \\ -2\alpha & 1 - \alpha^2 \end{pmatrix}, \quad \alpha \in \mathbb{R}. \tag{3.37}
\]

In other words, there exists a one-parameter family of scale invariant \( S \)-matrices. A detailed derivation can be found in appendix A.

For a system with \( n=3 \) edges there exist [1] for \( p=1 \) and \( p=2 \) the two-parameter families

\[
S_1(\alpha_1, \alpha_2) = -S_2(\alpha_1, \alpha_2), \tag{3.38}
\]

where

\[
S_2(\alpha_1, \alpha_2) = \frac{1}{1+\alpha_1^2+\alpha_2^2} \begin{pmatrix} \alpha_1^2 - \alpha_2^2 - 1 & 2\alpha_1\alpha_2 & 2\alpha_1 \\ 2\alpha_1\alpha_2 & \alpha_1^2 + \alpha_2^2 - 1 & 2\alpha_2 \\ 2\alpha_1 & 2\alpha_2 & 1 - \alpha_1^2 - \alpha_2^2 \end{pmatrix} \tag{3.39}
\]

and \( \alpha_1, \alpha_2 \in \mathbb{R} \). These \( S \)-matrices can be derived in the similar, albeit more tedious manner than what is done for the \( n=2 \) case in appendix A.
3.4 The dual field $\tilde{\phi}$ and symmetries

In order to describe all relevant quantities of the theory it is helpful to introduce the dual field $\tilde{\phi}$. This field is defined up to a constant in terms of $\phi$ by the relations

$$
\partial_t \tilde{\phi}(t, x, i) = -\partial_x \phi(t, x, i) \quad (3.40a)
$$
$$
\partial_x \tilde{\phi}(t, x, i) = -\partial_t \phi(t, x, i), \quad (3.40b)
$$

where $x > 0$, $i = 1, \ldots, n$. It follows directly that

$$
(\partial_t^2 - \partial_x^2) \tilde{\phi}(t, x, i) = 0. \quad (3.41)
$$

The solution is

$$
\tilde{\phi}(t, x, i) = \int_{-\infty}^{\infty} dk \frac{\epsilon(k)}{2\pi \sqrt{2|k|}} \left[ a_i^\dagger(k) e^{i(|k|t-kx)} + a_i(k) e^{-i(|k|t-kx)} \right], \quad (3.42)
$$

where $\epsilon(k)$ is the sign function. The sign function appears because of factors of the type $k/|k|$ that appear in the expression when the derivation is carried out. A detailed derivation can be found in [5].

As will be seen in section 3.5, there exist applications which make it interesting to investigate what happens under the following transformations:

$$
\phi(t, x, i) \rightarrow \phi(t, x, i) + c, \quad c \in \mathbb{R}, \quad (3.43)
$$
$$
\tilde{\phi}(t, x, i) \rightarrow \tilde{\phi}(t, x, i) + \tilde{c}, \quad \tilde{c} \in \mathbb{R}. \quad (3.44)
$$

The equations (3.2) and (3.40) are obviously invariant under these transformations, which imply that the following currents are conserved:

$$
j_\nu(t, x, i) = \partial_\nu \phi(t, x, i), \quad \nu = t, x \quad (3.45)
$$
$$
\tilde{j}_\nu(t, x, i) = \partial_\nu \tilde{\phi}(t, x, i), \quad \nu = t, x. \quad (3.46)
$$

3.5 Kirchoff’s rules

In most applications of quantum field theory on graphs, the behavior of electrons is studied. An example is the junctions of carbon nanotubes. It is therefore desirable to study the fermionic fields. Fortunately, in 1+1 dimensions interacting fermions can be described in terms of bosons via the process of *bosonization*. The bosonic fields $\phi$ and $\tilde{\phi}$ described above can be used to describe fermions in this context. In this framework, the currents (3.45) and (3.46) control the charge and spin transport, respectively [1]. The relative Kirchoff’s rules as described in section 2 are therefore important to be checked.
By rewriting the expression (3.10) of the field $\varphi(t, x, i)$ as a sum of two integrals, one from $-\infty$ to 0 and one from 0 to $\infty$ and using the general identity
\[
\int_{-\infty}^{0} f(s)\,ds = \int_{0}^{\infty} f(-s)\,ds,
\] (3.47)
the Kirchoff’s rule (2.3) for the current (3.45) becomes
\[
\sum_{i=1}^{n} j_x(t, 0, i) = \sum_{i=1}^{n} \int_{0}^{\infty} \frac{dk}{2\pi \sqrt{2|k|}} \left[ \left( a_i^\dagger(-k) - a_i^\dagger(k) \right) e^{ikt} + \left( a_i(k) - a_i(-k) \right) e^{-ikt} \right].
\] (3.48)

The Kirchoff’s rule is satisfied if and only if the above expression is zero. This is satisfied if
\[
\sum_{i=1}^{n} \left( a_i^\dagger(-k) - a_i^\dagger(k) \right) = 0
\] (3.49a)
and
\[
\sum_{i=1}^{n} \left( a_i(k) - a_i(-k) \right) = 0.
\] (3.49b)

By using the constraints (3.14), the equation (3.49a) can be written as
\[
0 = \sum_{i=1}^{n} \left( a_i^\dagger(-k) - a_i^\dagger(k) \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_j^\dagger(k) S_{ji}(k) - \sum_{i=1}^{n} a_i^\dagger(k)
\] (3.50a)
and (3.49b) becomes
\[
0 = \sum_{i=1}^{n} \left( a_i(k) - a_i(-k) \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}(k) a_j(-k) - \sum_{i=1}^{n} a_i(-k)
\] (3.50b)
where in the last equality it has been used that the $S$-matrix is symmetric. Thus it follows that the current $j_\nu$ satisfies Kirchoff’s rule if
\[
\sum_{j=1}^{n} S_{ij}(k) = 1, \quad \forall i = 1, \ldots, n, \ k \in \mathbb{R}.
\] (3.51)
For the current \( \tilde{j}_\nu \) (3.46) similar calculations shows that Kirchoff’s rule is satisfied if
\[
\sum_{j=1}^{n} S_{ij}(k) = -1, \quad \forall i = 1, \ldots, n, \quad k \in \mathbb{R}. \tag{3.52}
\]
This shows in particular that the Kirchoff’s rules for \( j_\nu \) and \( \tilde{j}_\nu \) cannot be satisfied simultaneously, which means that at most one of the charges
\[
Q = \sum_{i_1}^{n} \int_{0}^{\infty} dx j_i(t, x, i) \quad \tilde{Q} = \sum_{i_1}^{n} \int_{0}^{\infty} dx \tilde{j}_i(t, x, i) \tag{3.53}
\]
can be independent of time.

For the construction of \( S \)-matrices it can be helpful to investigate how the conditions (3.51) and (3.52) translates into restrictions of the boundary conditions, i.e. the form of the matrices \( A \) and \( B \). Following the reasoning in section 2 of [3], this is done by introducing the vector \( \mathbf{v} = (1, 1, \ldots, 1) \). Assuming that (3.51) holds it is easy to show that
\[
\mathbf{v} \in \text{Ker}[S(k) - \mathbb{I}]. \tag{3.54}
\]
In lemma 3.17 of [12] it established that
\[
\text{Ker}[S(k) - \mathbb{I}] = \text{Ker} A. \tag{3.55}
\]
Therefore
\[
0 = (Av)_i = \sum_{j=1}^{n} A_{ij}, \tag{3.56}
\]
i.e. the rows of \( A \) must add up to zero if (3.51) is about to be upheld. Assuming instead that (3.52) is satisfied, an analogous reasoning shows that
\[
\mathbf{v} \in \text{Ker}[S(k) + \mathbb{I}] = \text{Ker} B, \tag{3.57}
\]
where again the equality of the kernels is shown in [12]. In this case the rows of \( B \) must add up to zero analogously to (3.56).

### 3.6 Examples

This section revisits the scale invariant examples of section 3.3 but with the additional assumption that the Kirchoff’s rule for the current \( j_\nu \) is satisfied, i.e. equation (3.51) holds.

For the two-dimensional system described in (3.37), the additional restriction (3.56) results in the \( S \)-matrix
\[
S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.58}
\]
This result is easily obtained by imposing the additional condition on the calculations in appendix A. The S-matrix (3.58) describes a system with complete transmission which actually is the free field on \( \mathbb{R} \).

For the case \( n=3 \) things becomes more interesting. For the case \( p=2 \) (3.39) taking into account the condition (3.51), the S-matrix becomes \[ S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \] (3.59)

This correspond to all boundary conditions of the form
\[
A = \begin{pmatrix} a_{11} & a_{12} & -a_{11} - a_{12} \\ a_{21} & a_{22} & -a_{21} - a_{22} \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{pmatrix}.
\] (3.60)

This system is clearly invariant under edge permutations.

The case \( p=1 \) is somewhat different. Imposing the constraint (3.51) no longer reduces the two-parameter family (3.38) to an isolated case but instead to a one-parameter family. It turns out [1] that the condition (3.51) relates \( \alpha_1 \) and \( \alpha_2 \) in (3.38) as \( \alpha_2 = -(1 + \alpha_1) \). By setting \( \alpha \equiv \alpha_1 \), the resulting S-matrices are on the form
\[
S = \frac{1}{1 + \alpha + \alpha^2} \begin{pmatrix} \alpha + 1 & -\alpha & \alpha(\alpha + 1) \\ -\alpha & \alpha(\alpha + 1) & \alpha + 1 \\ \alpha(\alpha + 1) & \alpha + 1 & -\alpha \end{pmatrix},
\] (3.61)

which corresponds to the boundary conditions
\[
A = \begin{pmatrix} a_{11} & a_{12} & -a_{11} - a_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.
\] (3.62)

The S-matrix (3.61) are also different from the S-matrix (3.59) in the sense that for a generic \( \alpha \), there are no edge permutations under which (3.61) is invariant.
4 Representations of the RT algebra

The algebra defined by generators \( \{a_\alpha(x), a^{\dagger \alpha}(x)\} \) and relations (3.11)-(3.13) is a special case of the exchange algebra \( \mathcal{B}_R \) which is defined in [17, 18]. An exchange algebra \( \mathcal{B}_R \) is an algebra that contains both boundary and bulk generators. For an algebra of this type, the exchange of two bulk generators involves boundary elements in general. This means that the boundary’s impact on the bulk theory manifests itself already on the algebraic level. The detailed boundary conditions are specified on the level of representations [17].

For physical applications, a representation of the algebra must therefore be chosen. Following in the steps of [1, 2] some general results regarding the Fock representation and the Gibbs representation will be stated. For technical details, the reader is referred to [17] and [19] respectively.

4.1 The Fock representation

A Fock representation is characterized by the existence of a cyclic state \( \Omega \) from which all other states can be constructed by applying polynomials in certain creation operators. Moreover the state \( \Omega \) is annihilated by annihilation operators, which typically are hermitian conjugates of the creation operators. For an algebra with generators \( \{a_\alpha(x), a^{\dagger \alpha}(x)\} \), there is no a priori prescription whether \( a_\alpha(x) \) or \( a^{\dagger \alpha}(x) \) take the role of creation operators. For suitable algebras the possible choices can be related by an involution.

The exchange algebra \( \mathcal{B}_R \) is defined in [17] by generators and relations is an associative algebra over \( \mathbb{C} \) with an identity element \( \mathbb{I} \) and two types of generators:

\[
\{a_\alpha(x), a^{\dagger \alpha}(x) : \alpha = 1, \ldots, N, x \in \mathbb{R}^s\} \tag{4.1}
\]

and

\[
\{b_\beta^\alpha(x) : \alpha, \beta = 1, \ldots, N, x \in \mathbb{R}^s\} \tag{4.2}
\]

The generators \( a_\alpha \) and \( b_\beta^\alpha \) are called the bulk and boundary generators respectively. For the special boson field which is discussed in section 3, this reduces to the set \( \{a_\alpha(k), a^{\dagger}_\alpha(k), k \in \mathbb{R}\} \).

Regarding the boundary generator, it is shown section 3 and 4 in [17], that the boundary operators \( \{b_\beta^\alpha(k)\} \) acts as a multiplication by \( S_{\beta\alpha}(k) \). The full set of relations which appear in [17] is therefore reduced to the ones presented in section 3 namely

\[
a_{i_1}(k_1)a_{i_2}(k_2) - a_{i_2}(k_2)a_{i_1}(k_1) = 0 \tag{4.3}
\]

\[
a^\dagger_{i_1}(k_1)a^\dagger_{i_2}(k_2) - a^\dagger_{i_2}(k_2)a^\dagger_{i_1}(k_1) = 0 \tag{4.4}
\]

\[
a_{i_1}(k_1)a^\dagger_{i_2}(k_2) - a^\dagger_{i_2}(k_2)a_{i_1}(k_1) = 2\pi [\delta_{i_1i_2}\delta(k_1 - k_2) + S_{i_1i_2}(k_1)\delta(k_1 + k_2)]\mathbb{I}. \tag{4.5}
\]

A Fock representation of the exchange algebra \( \mathcal{B}_R \) can be built from a very general structure. In the beginning of section 3 of [17] the bare minimum requirement on the algebra and the

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The values. In the Gibbs representation these are given by (4.7). Again this means that it is sufficient to calculate the two-point expectation values. In the Fock representation these are given by (3.10). The Γs have been suppressed. It is therefore sufficient for this purpose to only calculate the two-point expectation values. In the Fock representation these are given by (4.10) and (4.11).

\[ \langle a_i(p) a_j^\dagger(q) \rangle = \frac{1}{1 - e^{-\beta |p|}} 2\pi [\delta_{ij} \delta(p - q) + S_{ij}(p)\delta(p + q)] , \]

\[ \langle a_i^\dagger(p) a_j(q) \rangle = \frac{1}{1 - e^{-\beta |p|}} 2\pi [\delta_{ij} \delta(p - q) + S_{ij}(-p)\delta(p + q)] . \]

4.2 The Gibbs representation

To construct the Gibbs representation, a cyclic state from which all other states can be obtained is chosen. This is similar to what was done in the Fock representation. The difference here is that the ground state of the Gibbs representation involves an infinite number of particles. This models a heat bath which maintains the system at constant inverse temperature \( \beta \). For technical details on how to construct the representation see [19].

As for the Fock representation the computation of a generic correlation function can be done via the iteration (4.7). Again this means that it is sufficient to calculate the two-point expectation values. In the Gibbs representation these are given by

\[ \langle a_i(p) a_j^\dagger(q) \rangle_\beta = \frac{1}{1 - e^{-\beta |p|}} 2\pi [\delta_{ij} \delta(p - q) + S_{ij}(p)\delta(p + q)] , \]

\[ \langle a_i^\dagger(p) a_j(q) \rangle_\beta = \frac{1}{1 - e^{-\beta |p|}} 2\pi [\delta_{ij} \delta(p - q) + S_{ij}(-p)\delta(p + q)] . \]
The two-point correlator of the field $\varphi$ becomes in the Gibbs representation [2]

$$\langle \varphi(t_1, x_1, i_1) \varphi(t_2, x_2, i_2) \rangle_\beta$$

$$= \int_{-\infty}^{\infty} \frac{dk}{4\pi |k|} \frac{e^{-\beta|k|+i|k|(t_1-t_2)} + e^{-i|k|(t_1-t_2)}}{1 - e^{-\beta|k|}} \left[ e^{ik(x_1-x_2)} \delta_{ii_1i_2} + e^{ik(x_1+x_2)} S_{ii_1i_2}(k) \right].$$

(4.12)

The Gibbs representation has in [1, 2] been used to derive the expectation value of some local observables for the field $\varphi$ e.g. the energy density $\theta_{\ell\ell}$. 
5 Summary

In this thesis, notion of a quantum graph has been introduced. The quantum graph has shown to be a valuable tool in the investigation of various phenomena, most notably the electromagnetic properties of quantum wires. The focus has been on star graphs and the new features of quantum field theory which appear on the star graph.

In section 2 it was shown that for a charge to be conserved on a star graph, the corresponding current must obey Kirchoff’s rule (2.3). This imposes restriction on the possible choice of boundary conditions. Moreover, the Hamiltonian has to admit self-adjoint extensions to ensure unitary time evolution of the system.

A scalar field on the star graph was introduced in section 3. The boundary conditions at the vertex was found to have to fulfill the condition (3.6) for the Hamiltonian to be self-adjoint. The boundary conditions can be parametrized in a unique way in terms of a unitary matrix. The scalar field was quantized which introduced an algebra corresponding to the boundary conditions. The scattering matrix corresponding to the vertex was introduced and it was found that there is a direct relationship between the \( S \)-matrix and the boundary conditions. The physical interpretation of the \( S \)-matrix is that the diagonal elements \( S_{ii} \) are the reflection amplitudes at the edge \( E_i \), while the off-diagonal elements \( S_{ij} i \neq j \) are the transmission amplitudes form edge \( E_i \) to edge \( E_j \). The special case of scale invariant systems were treated in detail. For these systems, the theory simplifies and some explicit examples of \( S \)-matrices were given. The introduction of the dual field lead to the situation where there were several conserved currents but the conservation of the corresponding charges lead to incompatible Kirchoff’s rules. This means that a choice has to be made in which charge to conserve. Two of these Kirchoff’s rules were used to show how they restricted the form of the \( S \)-matrix and therefore also on the boundary conditions.

The algebra which resulted from the field (3.10) belongs to a general class of exchange algebras. For these algebras, the exchange of the bulk generators involves boundary elements in general. This means that the boundary’s impact on the bulk theory manifests itself already at the algebraic level. The detailed boundary conditions are then specified at the level of representations. Two such representations were presented in section 4, namely the Fock and the Gibbs representation together with their two-point expectation values.

The representations can be used to compute observables and to construct vertex operators which are used in the process of bosonization. This process is carried out in for example [1, 3].
A Derivation of the \( S \)-matrix for the two-dimensional scale invariant system

The boundary matrices for the two-dimensional scale invariant \( S \)-matrix are

\[
A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad (A.1)
\]

where \( \text{rank}(A, B) = 2 \). From these, it follows from (3.16a) that the \( S \)-matrix is

\[
S = \frac{1}{a_1 b_2 - a_2 b_1} \begin{pmatrix} -(a_1 b_2 + a_2 b_1) & -2a_2 b_2 \\ 2a_1 b_1 & (a_1 b_2 + a_2 b_1) \end{pmatrix}. \quad (A.2)
\]

Imposing the constraint (3.26) it follows that

\[
a_1 b_1 = -a_2 b_2. \quad (A.3)
\]

Assuming \( b_2 \neq 0 \), define

\[
\alpha := -\frac{b_1}{b_2}. \quad (A.4)
\]

By using (A.3) and (A.4), the \( S \)-matrix can be written as

\[
S = \frac{1}{1 + \alpha^2} \begin{pmatrix} \alpha^2 - 1 & -2\alpha \\ -2\alpha & 1 - \alpha^2 \end{pmatrix}. \quad (A.5)
\]

In the case \( b_2 = 0 \), it follows from equation (A.2) that the \( S \)-matrix becomes

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.6)
\]

The same result can be obtained by letting \( \alpha \to \pm \infty \) in equation (A.5). Thus, equation (A.5) can be used for all boundaries of the type (A.1).
References


