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THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Embedding Theorems for Mixed Norm Spaces and Applications

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Abstract This thesis is devoted to the study of mixed norm spaces that arise in connection with embeddings of Sobolev and Besov type spaces. We study different structural, integrability, and smoothness properties of functions satisfying certain mixed norm conditions. Conditions of this type are determined by the behaviour of linear sections of functions. The work in this direction originates in a paper due to Gagliardo (1958), and was further developed by Fournier (1988), by Blei and Fournier (1989), and by Kolyada (2005).

Here we continue these studies. We obtain some refinements of known embeddings for certain mixed norm spaces introduced by Gagliardo, and we study general properties of these spaces. In connection with these results, we consider a scale of intermediate mixed norm spaces, and prove intrinsic embeddings in this scale.

We also consider more general, fully anisotropic, mixed norm spaces. Our main theorem states an embedding of these spaces to Lorentz spaces. Applying this result, we obtain sharp embedding theorems for anisotropic Sobolev-Besov spaces, and anisotropic fractional Sobolev spaces. The methods used are based on non-increasing rearrangements, and on estimates of sections of functions and sections of sets. We also study limiting relations between embeddings of spaces of different type. More exactly, mixed norm estimates enable us to get embedding constants with sharp asymptotic behaviour. This gives an extension of the results obtained for isotropic Besov spaces by Bourgain, Brezis, and Mironescu, and for anisotropic Besov spaces by Kolyada.

We study also some basic properties (in particular the approximation properties) of special weak type spaces that play an important role in the construction of mixed norm spaces, and in the description of Sobolev type embeddings.

In the last chapter, we study mixed norm spaces consisting of functions that have smooth sections. We prove embeddings of these spaces to Lorentz spaces. From this result, known properties of Sobolev-Liouville spaces follow.

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Contents

1. Introduction

This thesis relates to one of the fundamental directions in the theory of function spaces - embeddings and inequalities. We study different structural, integrability, and smoothness properties of functions satisfying certain mixed norm conditions. Conditions of this type are determined by the behaviour of linear sections of functions. Initially they arose in the works of Gagliardo (1958) and Fournier (1988) in connection with embeddings of Sobolev spaces.

1.1. **Antecedent results.** A function $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, is said to belong to the Sobolev space $W_p^1(\mathbb{R}^n)$ if f has usual (weak) derivatives $D_k f \in L^p(\mathbb{R}^n)$ for all $1 \le k \le n$. In 1938 Sobolev proved the following, now classical, theorem.

Theorem 1.1. *Let $n \ge 2$, $1 < p < n$, and $q = np/(n-p)$. If $f \in W_p^1(\mathbb{R}^n)$ then $f \in L^q(\mathbb{R}^n)$ and*

$$\|f\|_q \le c \sum_{k=1}^n \|D_k f\|_p. \tag{1.1}$$

It was first in 1958 that this theorem was extended to the case $p = 1$. This was done independently by Gagliardo and Nirenberg. The next lemma was the central part of Gagliardo's approach (see [17]). We use the notation $\hat{x}_k$ for the vector in $\mathbb{R}^{n-1}$ obtained from a given vector $x \in \mathbb{R}^n$ by removing its kth coordinate.

Lemma 1.2. *Let $n \ge 2$. Assume that the functions $g_k \in L^1(\mathbb{R}^{n-1})$, $k = 1, ..., n$, are non-negative. Then*

$$\int_{\mathbb{R}^n} \prod_{k=1}^n g_k(\hat{x}_k)^{1/(n-1)} dx \le \Big(\prod_{k=1}^n \int_{\mathbb{R}^{n-1}} g_k(\hat{x}_k) d\hat{x}_k \Big)^{1/(n-1)}.$$

Let $f \in W_1^1(\mathbb{R}^n)$. For almost every $x \in \mathbb{R}^n$,

$$|f(x)| \le \frac{1}{2} \int_{\mathbb{R}} |D_k f(x)| dx_k \equiv g_k(\hat{x}_k), \quad k = 1, ..., n. \tag{1.2}$$

As usual, we let p' denote the Hölder conjugate of p, i.e. we set $p' = p/(p-1)$ $(1 \le p \le \infty)$. Applying Lemma 1.2, we obtain

$$\|f\|_{n'} \le \frac{1}{2} \Big(\prod_{k=1}^n \|D_k f\|_1 \Big)^{1/n}.$$

This implies Theorem 1.1 for $p = 1$. However, one can obtain a stronger statement from Lemma 1.2. Let

$$V_k \equiv L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})], \quad 1 \le k \le n$$

be the space with the mixed norm

$$\|f\|_{V_k} \equiv \|\Psi_k\|_{L^1(\mathbb{R}^{n-1})},$$

where

$$\Psi_k(\hat{x}_k) = \operatorname{ess\,sup}_{x_k\in\mathbb{R}} |f(x)|.$$

We say that the L^1-norm is the "exterior" norm of V_k and the L^∞-norm is the "interior" norm. We shall also denote

$$V = \bigcap_{k=1}^{n} V_k. \tag{1.3}$$

Applying Lemma 1.2 to the functions Ψ_k gives the following theorem.

Theorem 1.3. *Let $n \geq 2$. If $f \in \cap_{k=1}^{n} V_k$, then $f \in L^{n'}(\mathbb{R}^n)$ and*

$$\|f\|_{n'} \leq \Big(\prod_{k=1}^{n} \|f\|_{V_k}\Big)^{1/n}.$$

For $f \in W_1^1(\mathbb{R}^n)$, inequality (1.2) gives

$$\|f\|_{V_k} \leq \frac{1}{2}\|D_k f\|_1. \tag{1.4}$$

This estimate and Theorem 1.3 imply inequality (1.1) for $p = 1$.

For a measurable set $E \subset \mathbb{R}^m$, we denote by $\mathrm{mes}_m E$ the Lebesgue measure of E in $\mathbb{R}^m$.

Let $S_0(\mathbb{R}^n)$ be the class of all measurable almost everywhere finite functions f on $\mathbb{R}^n$ such that the distribution function

$$\lambda_f(y) = \mathrm{mes}_n\{x \in \mathbb{R}^n : |f(x)| > y\}$$

is finite for all $y > 0$. Let f^* denote the non-increasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ (the definition is given in Section 2.2). If $0 < q, p < \infty$, then the Lorentz space $L^{q,p}(\mathbb{R}^n)$ is defined as the class of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{q,p} = \Big(\int_0^\infty \big[t^{1/q} f^*(t)\big]^p \frac{dt}{t}\Big)^{1/p} < \infty.$$

For any fixed q, the Lorentz spaces increase as the secondary index p increases (see Section 2.3 below).

It is well known that the left-hand side in (1.1) can be replaced by the stronger Lorentz norm (see [14], [39], [40], and [42]). That is, the following theorem holds.

Theorem 1.4. *Let $n \geq 2$ and $1 \leq p < n$. Set $q = np/(n-p)$. If $f \in W_p^1(\mathbb{R}^n)$, then $f \in L^{q,p}(\mathbb{R}^n)$ and*

$$\|f\|_{q,p} \leq c \sum_{k=1}^{n} \|D_k f\|_p. \tag{1.5}$$

In [16], Fournier proved this theorem for $p = 1$, using the following refinement of Theorem 1.3.

Theorem 1.5. *Let $n \geq 2$. If $f \in \cap_{k=1}^n V_k$, then $f \in L^{n',1}(\mathbb{R}^n)$ and*

$$\|f\|_{n',1} \leq n' \Big(\prod_{k=1}^{n} \|f\|_{V_k} \Big)^{1/n}. \tag{1.6}$$

Observe that for the characteristic function of the unit cube in $\mathbb{R}^n$ we have equality in (1.6). Thus, the constant n' is optimal.

Some extensions of Theorem 1.5 were obtained in the paper [8] due to Blei and Fournier. In particular, it was proved that for any $1 < r \leq \infty$

$$\|f\|_{q,1} \leq c \sum_{k=1}^{n} \|f\|_{V_k^{(r)}}, \tag{1.7}$$

where $q = nr/(nr - r + 1)$ and

$$V_k^{(r)} = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^r_{x_k}(\mathbb{R})] \quad (k = 1, \dots, n).$$

It was shown in [16], [36] that the preceding results give a sharpening of some inequalities for bilinear forms proved by Hardy and Littlewood.

In view of (1.4), Theorem 1.5 immediately implies Theorem 1.5 for $p = 1$. Fournier [16, p. 66] observed that it was not clear how the methods based on mixed norm estimates could be applied to obtain (1.5) also for $1 < p < n$. This problem was studied by Kolyada in [29]. He introduced a scale of more general mixed norm spaces in which the interior norms are defined by conditions on the rearrangements with respect to specific variables. These conditions are expressed in terms of the "weak" spaces Λ^σ.

Let $\sigma \in \mathbb{R}$. Denote by $\Lambda^\sigma(\mathbb{R})$ the class of all functions $f \in S_0(\mathbb{R})$ such that

$$\|f\|_{\Lambda^\sigma} = \sup_{t>0} t^\sigma (f^*(t) - f^*(2t)) < \infty. \tag{1.8}$$

If $0 < \sigma < \infty$ and $r = 1/\sigma$, then $\Lambda^\sigma = L^{r,\infty}$ (where $L^{r,\infty}$ is the Marcinkiewicz space weak-L^r). If $\sigma = 0$, then Λ^σ coincides with the space weak-L^∞ introduced in [4]. If $\sigma < 0$, then (1.8) is a weak version of Lipschitz condition for the rearrangement (see Section 4).

The main result in [29] is the following theorem.

Theorem 1.6. *Let $n \geq 2$. Assume that $1 \leq p < \infty$ and that α_k, $k = 1, ..., n$, are positive numbers such that*

$$\alpha \equiv n\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k}\Big)^{-1} < \frac{n}{p}. \tag{1.9}$$

Set

$$\sigma_k = \frac{1}{p} - \alpha_k, \quad V_k \equiv L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})],$$

and $q = np/(n - \alpha p)$. Suppose that $f \in S_0(\mathbb{R}^n)$ and $f \in \cap_{k=1}^{n} V_k$. Then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|_{q,p} \leq c \prod_{k=1}^{n} \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{1.10}$$

where

$$c = c_n\Big(\prod_{k=1}^{n} (n\alpha_k - \alpha)^{\alpha/(n\alpha_k)}\Big)^{-1/p} \tag{1.11}$$

and c_n depends only on n.

Observe that Theorem 1.6 remains true for $\alpha = n/p$, $q = \infty$ (the space $L^{\infty,p}$ is defined in Section 2.3).

It follows from Theorem 1.6, that the interior L^r-norm on the right-hand side of (1.7) can be replaced by the weaker $L^{r,\infty}$-norm for $1 < r < \infty$, and by the norm in weak-L^∞ (see (4.4) below) for $r = \infty$.

It was proved in [29] (see Lemma 6.4 below) that if a function $f \in L^p(\mathbb{R}^n)$ has a usual (weak) derivative $D_k f \in L^p(\mathbb{R}^n)$, then

$$\|f\|_{L^p_{\hat{x}_k}[\Lambda^{1/p-1}_{x_k}]} \leq c\|D_k f\|_p, \quad 1 \leq p < \infty.$$

Hence, there holds the embedding

$$W^1_p(\mathbb{R}^n) \subset \bigcap_{k=1}^{n} L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p-1}_{x_k}(\mathbb{R})]. \tag{1.12}$$

We now obtain Theorem 1.4 in two steps. The first (and simplest) step is (1.12) and the second step is Theorem 1.6 with $\alpha_1 = \cdots = \alpha_n = 1$. Observe that no smoothness condition is imposed on the functions in the second step.

In [29], Theorem 1.6 was also applied to obtain optimal constants in embeddings of anisotropic Besov spaces.

1.2. **Main objectives.** As we can see, the use of mixed norm estimates clarifies the role of smoothness conditions in the embedding theorems for Sobolev and Besov type spaces. Moreover it was shown in [29], and it will be seen below in Section 6.2 and 6.1, that such estimates provide a flexible method which can be applied to the study of different types of function spaces.

The general objective of this thesis is the further study of mixed norm spaces. More concretely, we shall study the following:

- some general properties of the Fournier-Gagliardo space V (see (1.3)) and the spaces Λ^σ;
- refinements of embeddings of the Fournier-Gagliardo space V;
- a scale of intermediate mixed norm spaces, and intrinsic embeddings in this scale;
- an extension of Theorem 1.6 to more general mixed norm spaces;
- embeddings of mixed norm spaces of functions with smoothness conditions on linear sections.

1.3. **Summary.** In what follows we give a brief description of the main content of this thesis.

Section 2 contains general definitions and known results. In particular, we define and consider some basic properties of the non-increasing rearrangement, the Lorentz spaces, and the iterative rearrangement.

In Section 3 we obtain some properties of the Fournier-Gagliardo space V, defined in (1.3). The most interesting result is the observation that $\|f\|_{V_k}$ has a clear geometric interpretation: it is the n-dimensional measure of the essential projection of the set

$$\{(x,y) \in \mathbb{R}^n \times [0,\infty) : 0 \le y \le |f(x)|\},$$

onto the hyperplane $x_k = 0$ (see Theorem 3.3 below). We also show that V is *not* invariant under rotation.

In Section 4 we study the space Λ^σ, defined in (1.8). As follows from the above, and as we will see in Section 6 below, the spaces Λ^σ have a relevant role in the description of Sobolev-type embeddings. This motivates us to study the basic properties of these spaces. As it was mentioned above, we show that Λ^σ relates to known spaces, in particular the Marcinkiewicz spaces. We also show that $\Lambda^\sigma \subset L^\infty(\mathbb{R})$, for $\sigma < 0$.

Results for approximation of functions in Λ^σ are given in Section 4.2. We will see that approximation in the "norm" on Λ^σ behaves badly. However, we have obtained some positive results on approximation of functions f in this space. Our main result for the space Λ^σ is the following theorem.

Let $C_0(\mathbb{R})$ denote the class of all continuous functions with bounded support in $\mathbb{R}$.

Theorem 1.7. *Let $f \in \Lambda^\sigma$ $(\sigma \in \mathbb{R})$. Then there exists a sequence $\{f_k\}$, $f_k \in C_0(\mathbb{R})$, such that $\{f_k\}$ converges to f in measure and $\|f_k\|_{\Lambda^\sigma} \to \|f\|_{\Lambda^\sigma}$.*

This is in fact a special case of the more general result obtained in Theorem 4.10. Observe that this theorem is similar to known results for approximation in variation (see [49], [24, Section 9.1]).

In Section 5 we prove our main result, Theorem 1.8 below. It is an extension of Theorem 1.6. This section also includes some relevant lemmas.

In Theorem 1.4 all derivatives $D_k f$ belong to the same space $L^p(\mathbb{R}^n)$. Nevertheless, it is quite reasonable to suppose that the functions $D_k f$, $k = 1, \dots, n$, belong to different spaces $L^{p_k}(\mathbb{R}^n)$. Such conditions naturally appear in embedding theory as well as in applications. Furthermore, many authors have studied Sobolev and Besov spaces whose construction involves, instead of L^p-norms, norms in more general spaces - first of all, in the Lorentz spaces.

Therefore it is natural to study mixed norm spaces which are anisotropic not only with respect to interior norms, but also with respect to exterior norms. The main problem considered in this work is to extend Theorem 1.6 to these, more general, mixed norm spaces. This extension is given by Theorem 5.4, and it states in particular the following.

Theorem 1.8. *Let $n \geq 2$, $1 \leq p_1, \dots, p_n, s_1, \dots, s_n < \infty$, and $\alpha_1, \dots, \alpha_n > 0$. Put*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1}, \ p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}, \ \text{and } s = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k s_k}\Big)^{-1}.$$

Assume that $p < n/\alpha$ and put $q = np/(n - \alpha p)$. Set

$$\sigma_k = \frac{1}{p_k} - \alpha_k, \ \text{and } V_k = L^{p_k, s_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})],$$

and assume that

$$\frac{1}{p} - \frac{\alpha}{n} - \sigma_k > 0,$$

for $k = 1, \dots, n$. Suppose that

$$f \in S_0(\mathbb{R}^n) \text{ and } f \in \bigcap_{k=1}^n V_k.$$

Then $f \in L^{q,s}(\mathbb{R}^n)$ and

$$\|f\|_{q,s} \leq c \prod_{k=1}^n \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{1.13}$$

where c depends only on $p_1, \dots, p_n, s_1, \dots, s_n$, $\alpha_1, \dots, \alpha_n$, and n.

We have obtained the constant in (1.13) explicitly. This explicit value is used in Section 6, where we consider applications of Theorem 1.8.

As we will show, Theorem 1.8 holds in the case $p = n/\alpha$ as well.

The proof of Theorem 1.8 is based on the approach given in the works of Kolyada [29] and Kolyada and Pérez [32].

In Section 6 we apply Theorem 1.8 to obtain sharp embedding theorems for anisotropic Sobolev-Liouville spaces and anisotropic Sobolev-Besov spaces. We also study limiting relations between embeddings of spaces of different type. More exactly, mixed norm estimates enable us to get embedding constants with sharp asymptotic behaviour. This gives an extension of the results obtained for isotropic Besov spaces B_p^α by Bourgain, Brezis, and Mironescu [11], and for Besov spaces $B_p^{\alpha_1,\dots,\alpha_n}$ by Kolyada [29].

As follows from the exposition given above, the Fournier-Gagliardo space

$$V = \bigcap_{k=1}^{n} L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})]$$

(from (1.3)) appears naturally in connection with embeddings of the space $W_1^1(\mathbb{R}^n)$. In this work we continue the study of embeddings of the space V.

In Section 7.1 we prove that V is embedded to the modified Lorentz space defined in terms of iterative rearrangements (see Section 2.4). In particular, V is embedded into the space $\mathcal{L}^{n',1}(\mathbb{R}^n)$, which is strictly smaller that $L^{n',1}(\mathbb{R}^n)$. Thus our result is a refinement of Fournier's inequality (1.6). We observe that this result was inspired by embeddings of Sobolev spaces into the Lorentz spaces $\mathcal{L}^{q,p}(\mathbb{R}^n)$ proved in [28].

It is natural to study the intrinsic relations between different mixed norm spaces. In Section 7.2 we introduce a one parameter family of spaces, containing the space V. This scale is formed by the spaces

$$V^p = \bigcap_{k=1}^{n} L^{p,1}_{\hat{x}_k}(\mathbb{R}^{n-1})[L^{r_p,1}_{x_k}(\mathbb{R})],$$

$1 \le p \le (n-1)'$, $r_p = p'/(n-1)$. For $p = 1$ we have $V^1 = V$ and the norms coincide, so the space V is included in the scale V^p. Further, for $1 < p \le (n-1)'$, we prove that $V \subset V^p$, and

$$\|f\|_{V_j^p} \le c\|f\|_{V_j}^{1/r_p'} \prod_{k\neq j} \|f\|_{V_k}^{1/p'}, \quad j = 1, \dots, n, \tag{1.14}$$

for $1 < p \le (n-1)'$, $r_p = p'/(n-1)$, $f \in V$. We obtain also some results concerning the optimality of the estimates for the V^p-norms (Remark 7.16 and Theorem 7.28 below).

Embeddings of the space V^p are closely connected with Theorem 1.8. Namely, using this theorem, we prove

$$\|f\|_{n',1} \le c \prod_{k=1}^{n} \|f\|_{V_k^p}^{1/n}$$

$(1 \le p < n')$ for $f \in V^p$ (when $p = 1$, this is inequality (1.6)).

In Section 7.3 we apply our results for the space V^p to obtain embedding theorems for Sobolev spaces. We prove the inequality

$$\prod_{k=1}^{n} \|f\|_{V_k^p} \le c \prod_{k=1}^{n} \|D_k f\|_1$$

$(1 < p \le (n-1)')$ for all $f \in W_1^1(\mathbb{R}^n)$. In particular this result states that

$$W_1^1(\mathbb{R}^2) \subset V_k^2 = L_{\hat{x}_k}^{2,1}(\mathbb{R})[L_{x_k}^{2,1}(\mathbb{R})], \quad k = 1, 2.$$

This inclusion does not follow from the strong type Sobolev inequality (1.5), but it can be derived from known results for iterative rearrangements (Remark 7.23 below).

In Section 7.4, we consider some limiting relations for the V^p-norm. Recall that inequality (1.14) holds for $1 < p \le (n-1)'$, and that $V^1 = V$. In Theorem 7.26 below, we clarify the limiting behaviour of $\|f\|_{V_k^p}$, as $p \to 1+$.

In Section 7.5 we obtain a result concerning the relations between spaces V^p with different values of p. In particular we see that these spaces do not form a monotone scale.

As is mentioned above, Theorem 1.6 (and its extension - Theorem 1.8) is closely related to embeddings of Sobolev spaces with smoothness not greater than 1. The objective of Section 8 is to obtain an extension of Theorem 1.6 related to Sobolev spaces of higher smoothness. In this case, the mixed norms from Theorem 1.6 are not suitable, since the definition of $\|\cdot\|_{\Lambda^\sigma}$ involves the rearrangement.

Instead we will consider mixed norm spaces with interior norms defined in terms of the modulus of continuity $\omega^r(\cdot, t)$ (see Section 8) of the sections of the function. For simplicity, we study only isotropic mixed norm spaces. Suppose $\lambda > 0$, and let $r = r(\lambda)$ be the smallest integer such that $\lambda < r$. Let $\mathcal{C}^\lambda(\mathbb{R})$ be the space of all measurable functions φ on $\mathbb{R}$ such that

$$\|\varphi\|_{\mathcal{C}^\lambda}^* = \sup_{t>0} \frac{\omega^r(\varphi; t)}{t^\lambda} < \infty.$$

These seminorms will be used as interior norms instead of $\|\cdot\|_{\Lambda^\sigma}$. The mixed norm spaces thus obtained, are denoted

$$U_k^{p,\lambda} = L_{\hat{x}_k}^p(\mathbb{R}^{n-1})[\mathcal{C}_{x_k}^\lambda(\mathbb{R})], \quad (k = 1, \dots, n),$$

for $1 \leq p < \infty$ and $\lambda > 0$. We also put

$$U^{p,\lambda} = \bigcap_{k=1}^{n} U_k^{p,\lambda}.$$

Observe that if $0 < \alpha < 1$, then $\|\varphi\|_{\Lambda^{-\alpha}} \leq \|\varphi\|_{\mathcal{C}^\alpha}^*$ (see Proposition 4.3 below).

Note that using the modulus of continuity of orders higher than $r(\lambda)$ in the definition of $\|\cdot\|_{\mathcal{C}^\lambda}^*$, would yield equivalent seminorms (Remark 8.16 below). We emphasize also that if $\lambda \in \mathbb{N}$, then $r(\lambda) = \lambda+1$. If we considered instead the modulus of continuity of order λ, we would get a stronger (semi) norm. However, in our estimates, these norms will appear on the right hand side, and thus it is better to use weaker norms.

In Section 8.1, we consider the case when $0 < \lambda < (n-1)/p$, and we prove:

Theorem 1.9. *Let $1 \leq p < \infty$, $0 < \lambda < (n-1)/p$, and $q = np/(n-1-\lambda p)$. Suppose that $f \in S_0(\mathbb{R}^n)$, $f \in U^{p,\lambda}$, and that*

$$f^*(t) = O(t^{-1/q}), \quad t \to \infty. \tag{1.15}$$

Then $f \in L^{q,p}(\mathbb{R}^n)$, and

$$\|f\|_{q,p} \leq c \sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}}$$

where c depends only on p, λ, and n.

In the special case $1 \leq p < \infty$ and $0 < \lambda < \min(1, (n-1)/p)$, this theorem can be derived from Proposition 4.3 and Theorem 1.6. Moreover, the assumption (1.15) in Theorem 1.9 can be omitted for these values of p and λ (see Remark 8.6 below).

In Section 8.2, we first give some simple lemmas on the properties of the Steklov averages defined in (8.30). We then prove the following teorem.

Theorem 1.10. *Let $1 \leq p \leq \infty$, and $(n-1)/p < \lambda < \infty$. Suppose that $f \in S_0(\mathbb{R}^n)$ and $f \in U^{p,\lambda}$. Then there exists a bounded and uniformly continuous function $g \in U^{p,\lambda}$, such that $f = g$ a.e. and $\|g\|_{U_k^{p,\lambda}} = \|f\|_{U_k^{p,\lambda}}$, $k = 1, \ldots, n$. Moreover, if $\beta \equiv \lambda - (n-1)/p$ and $s > \beta$, $s \in \mathbb{N}$, then*

$$\omega^s(g;\delta) \leq c\delta^\beta \sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}}$$

for $\delta \geq 0$, where c depends only on p, λ, and n.

We emphasize that this theorem does not hold for $s = \beta$, $\beta \in \mathbb{N}$ (see Remark 8.14).

In Section 8.3 we first prove some basic properties of functions with fractional derivatives. Such generalized derivatives, and the Sobolev-Liouville spaces, are defined in Section 6. The main result in Section 8.3 is the following theorem.

Theorem 1.11. *Let $1 \le p < \infty$, $\alpha > 1/p$, and $\lambda = \alpha - 1/p$. Suppose that $f \in L_p^\alpha(\mathbb{R}^n)$. Then there exists a function $f_0 \in U^{p,\lambda}$ which is equivalent to f and satisfies*

$$\|f_0\|_{U_j^{p,\lambda}} \le c\|D_j^\alpha f_0\|_p, \quad j = 1, \ldots, n, \tag{1.16}$$

where c depends only on n, p, and α.

For the Sobolev-Liouville spaces, inequality (1.16) plays the same role as (1.4) for $W_1^1(\mathbb{R}^n)$.

We apply this theorem to illustrate that the properties of the mixed norm spaces $U^{p,\lambda}$ are consistent with known results for the Sobolev-Liouville spaces $L_p^\alpha(\mathbb{R}^n)$ (see Theorem 8.22 and Theorem 8.23 below).

2. Definitions and auxiliary propositions

This section contains definitions and known results. Section 2.1 contains Hardy type inequalities that we need. In Section 2.2 we define the non-increasing rearrangement of a function and give some of its basic properties. This definition was first given by Hardy and Littlewood [19]. Estimates in terms of rearrangements will be important in the following sections. In Section 2.3 we introduce the Lorentz spaces. In the last section, Section 2.4, we consider iterative rearrangements.

2.1. **Hardy's inequalities.** The next theorem was proved by Hardy (see e.g. [5, p. 124]).

Theorem 2.1. *Let $\alpha > 0$ and $1 \leq p < \infty$. If f is a non-negative measurable function on $\mathbb{R}_+ \equiv (0, \infty)$ then*

$$\Big(\int_0^\infty t^{\alpha-1} \Big(\int_t^\infty f(u)du \Big)^p dt \Big)^{1/p} \leq \frac{p}{\alpha} \Big(\int_0^\infty t^{p+\alpha-1} f(t)^p dt \Big)^{1/p} \tag{2.1}$$

and

$$\Big(\int_0^\infty t^{-\alpha-1} \Big(\int_0^t f(u)du \Big)^p dt \Big)^{1/p} \leq \frac{p}{\alpha} \Big(\int_0^\infty t^{p-\alpha-1} f(t)^p dt \Big)^{1/p}. \tag{2.2}$$

If, as in the above theorem, f is a non-negative measurable function on $\mathbb{R}_+$ and $\alpha > 0$, there hold the obvious inequalities

$$\sup_{t>0} t^\alpha \int_t^\infty f(u)du \leq \frac{1}{\alpha} \sup_{t>0} t^{1+\alpha} f(t) \tag{2.3}$$

and

$$\sup_{t>0} t^{-\alpha} \int_0^t f(u)du \leq \frac{1}{\alpha} \sup_{t>0} t^{1-\alpha} f(t). \tag{2.4}$$

For $u, v > 0$, we let $\Gamma(u)$ and $\mathrm{B}(u,v)$ denote the usual Gamma- and Beta-functions, respectively. Recall that

$$\Gamma(u) = \int_0^\infty e^{-t} t^{u-1} dt \tag{2.5}$$

and

$$\mathrm{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \int_0^1 t^{u-1}(1-t)^{v-1} dt.$$

The next inequality is similar to Hardy's inequality (2.1), but for the case $0 < p < 1$ and for non-increasing functions. It was obtained by Bergh, Burenkov, and Persson (see [6, Corollary 3.7]). We stress that the constant in (2.6) is the best possible.

Theorem 2.2. *Let f be a non-negative non-increasing function on $\mathbb{R}_+$. Suppose that $\alpha > 0$ and $0 < p < 1$. Then*

$$\int_0^\infty t^{\alpha-1}\Big(\int_t^\infty f(u)du\Big)^p dt \le p\mathrm{B}(p,\alpha)\int_0^\infty t^{\alpha+p-1} f(t)^p dt. \tag{2.6}$$

Remark 2.3. A different proof of inequality (2.6) was given in [29, Lemma 2.5], but with the worse constant:

$$c = e\Big(1 + \frac{p}{\alpha}\Big).$$

2.2. **The non-increasing rearrangement.** Let $X \subset \mathbb{R}^n$ be a measurable set and let f be a measurable function on X. For $y \ge 0$ we define the distribution function of f by

$$\lambda_f(y) = \mathrm{mes}_n\{x \in X : |f(x)| > y\}.$$

Observe that λ_f may take the value ∞. If $\mathrm{mes}_n X = \infty$, we let $S_0(X)$ denote the class of all measurable almost everywhere finite functions f on X, for which $\lambda_f(y) < \infty$ for all $y > 0$. Two functions $f \in S_0(X)$, $X \subset \mathbb{R}^n$, and $g \in S_0(Y)$, $Y \subset \mathbb{R}^m$, are said to be equimeasurable if

$$\lambda_f(y) = \lambda_g(y), \quad y \ge 0.$$

A non-negative and non-increasing function f^* on $\mathbb{R}_+$ which is equimeasurable with f is said to be a non-increasing rearrangement of the function $f \in S_0(X)$. We will also assume that f^* is left-continuous on $\mathbb{R}_+$. Under this condition f^* is defined uniquely by (see [33, p. 59]),

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) < t\}. \tag{2.7}$$

We will refer to f^* as the rearrangement of f. We say that a function is rearrangable on X if it belongs to $S_0(X)$. Note that if two rearrangable functions are equimeasurable, then their rearrangements coincide. Further, for $f \in S_0(X)$, there holds the identity (see [12, p. 32])

$$f^*(t) = \sup_E \Big(\inf_{x\in E} |f(x)|\Big), \tag{2.8}$$

where the supremum is taken over all measurable sets $E \subset X$ having measure t. Notice that it is sufficient to take supremum only over all F_σ-sets of measure t in (2.8). Indeed, let $f \in S_0(X)$ and denote

$$g(t) = \sup_A \Big(\inf_{x\in A} |f(x)|\Big),$$

where the supremum is taken over all F_σ-sets $A \subset X$ with $\mathrm{mes}_n A = t$. Obviously $g(t) \le f^*(t)$. Further, for every measurable set $E \subset X$ with $\mathrm{mes}_n E = t$, there exists an F_σ-subset A with $\mathrm{mes}_n A = t$ such that

$$\inf_{x\in E} |f(x)| = \inf_{x\in A} |f(x)|. \tag{2.9}$$

To verify the last property, observe that E contains a sequence $\{x_k\}$ such that

$$0 \leq |f(x_k)| - \inf_{x \in E} |f(x)| \leq \frac{1}{k}.$$

We can assume that also A contains this sequence, and then (2.9) follows. From (2.9), we see that

$$\inf_{x \in E} |f(x)| \leq g(t),$$

for every measurable set $E \subset X$ with $\mathrm{mes}_n E = t$, and thus $f^*(t) \leq g(t)$. This proves that $f^*(t) = g(t)$. In what follows, we will have $X = \mathbb{R}^n$, $X = \mathbb{R}^n_+$, or $X = \mathbb{R}^n \times \mathbb{R}^m_+$ $(n, m \geq 1)$.

Let $f \in S_0(\mathbb{R}^n)$. We now give some basic properties of the rearrangement that will come to use in what follows. Put

$$A_t = \{x : |f(x)| > f^*(t)\},$$

$t > 0$. By the definition of f^* it holds that

$$\mathrm{mes}_n A_t \leq t. \tag{2.10}$$

It is also a consequence of the definition of f^* that the measure of the set $B_t = \{x : |f(x)| \geq f^*(t)\}$ satisfies

$$\mathrm{mes}_n B_t \geq t. \tag{2.11}$$

For each $f \in S_0(\mathbb{R}^n)$ and every scalar $a \in \mathbb{R}$ it is immediate that $af \in S_0(\mathbb{R}^n)$ (the distribution function of af is $y \mapsto \lambda_f(y/a)$, so it is finite). It follows directly from (2.8) that

$$(af)^*(t) = |a| f^*(t), \tag{2.12}$$

for all $t > 0$.

For $f, g \in S_0(\mathbb{R}^n)$ and $t, s > 0$ it holds that (see [33, p. 67])

$$(f+g)^*(t+s) \leq f^*(t) + g^*(s). \tag{2.13}$$

Let $f \in S_0(\mathbb{R}^n)$ and fix $\varepsilon > 0$. Since f^* and f are equimeasurable, we have

$$\mathrm{mes}_1\{t > 0 : f^*(t) > \varepsilon\} = \lambda_f(\varepsilon) < \infty.$$

Since f^* is non-increasing it follows that $f^*(t) \leq \varepsilon$ for all $t > \lambda_f(\varepsilon)$. Thus,

$$\lim_{t \to \infty} f^*(t) = 0. \tag{2.14}$$

We also have

$$\lim_{t \to 0+} f^*(t) = \|f\|_\infty. \tag{2.15}$$

Indeed, let y_0 denote this limit. By (2.10) it holds that

$$\mathrm{mes}_n\{x : |f(x)| > y_0\} \leq \mathrm{mes}_n\{x : |f(x)| > f^*(t)\} \leq t,$$

for all $t > 0$. Thus $\text{mes}_n\{x : |f(x)| > y_0\} = 0$, so that $\|f\|_\infty \le y_0$. Furthermore, (2.11) implies that $\|f\|_\infty \ge f^*(t)$ for all $t > 0$, and therefore $\|f\|_\infty \ge y_0$.

We also mention the following result [33, p. 67]

Proposition 2.4. *If the sequence $\{f_k\} \subset S_0(\mathbb{R}^n)$ converges in measure to the function $f \in S_0(\mathbb{R}^n)$, then $f_k^* \to f^*$ at every point of continuity of f^*.*

Let $C(\mathbb{R}^n)$ denote the class of all bounded continuous functions on $\mathbb{R}^n$. Lemma 2.5 and Lemma 2.6 below, state known elementary properties of the rearrangement of a continuous function.

Lemma 2.5. *Let $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then, for every $t_0 > 0$ there exists a point $x_0 \in \mathbb{R}^n$ such that $f^*(t_0) = |f(x_0)|$.*

Proof. Fix $t_0 > 0$. It is immediate from the definition of f^* that $0 \le f^*(t_0) \le \|f\|_\infty$. First we assume that $f^*(t_0) = 0$. Suppose $|f(x)| > 0$ for all $x \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a compact set having measure t_0. Since $f \in C(\mathbb{R}^n)$ there exists $x_1 \in E$ where

$$f^*(t_0) \ge \inf_{x \in E} |f(x)| = |f(x_1)| > 0,$$

which is a contradiction.

Next we suppose that $f^*(t_0) = \|f\|_\infty$. According to (2.11), it holds that

$$\text{mes}_n\{x : |f(x)| = \|f\|_\infty\} = \text{mes}_n\{x : |f(x)| \ge f^*(t_0)\} \ge t_0 > 0,$$

so there exists $x_0 \in \mathbb{R}^n$ where $|f(x_0)| = \|f\|_\infty = f^*(t_0)$.

The remaining case is when $0 < f^*(t_0) < \|f\|_\infty$. Since $f \in S_0(\mathbb{R}^n)$, we can not have $|f(x)| > f^*(t_0) > 0$ for all $x \in \mathbb{R}^n$. So there exists $x' \in \mathbb{R}^n$ such that

$$0 \le |f(x')| \le f^*(t_0). \tag{2.16}$$

Clearly there also exists a point $x'' \in \mathbb{R}^n$ where

$$f^*(t_0) \le |f(x'')| \le \|f\|_\infty. \tag{2.17}$$

Since f has the intermediate value property, it follows from (2.16) and (2.17) that there exists some x_0 along the line segment from x' to x'' for which $|f(x_0)| = f^*(t_0)$. $\square$

Lemma 2.6. *Let $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then f^* is continuous on $\mathbb{R}_+$.*

Proof. Fix $t_0 > 0$. Assume that f^* is discontinuous at t_0. Since f^* is left-continuous and non-increasing, it follows that

$$y_0 \equiv \lim_{t \to t_0+} f^*(t) < f^*(t_0).$$

So, f^* takes no values in $(y_0, f^*(t_0))$. Let $\tau \in (y_0, f^*(t_0))$ and suppose $|f(x_0)| = \tau$ for some $x_0 \in \mathbb{R}^n$. Since f is continuous, there exists some $\delta > 0$ such that if $x_1 \in \mathbb{R}^n$ and $|x_0 - x_1| < \delta$ then

$$\big|\tau - |f(x_1)|\big| = \big||f(x_0)| - |f(x_1)|\big| < f^*(t_0) - y_0.$$

Therefore

$$\mathrm{mes}_n\{x : |f(x)| \in (y_0, f^*(t_0))\} > 0.$$

But, f and f^* are equimeasurable so

$$\mathrm{mes}_n\{x : |f(x)| \in (y_0, f^*(t_0))\} = \mathrm{mes}_1\{s > 0 : f^*(s) \in (y_0, f^*(t_0))\} = 0,$$

which is a contradiction. Thus, if f^* is discontinuous at t_0, then $|f|$ takes no values in the interval $(y_0, f^*(t_0))$. By (2.11)

$$\mathrm{mes}_n\{x : |f(x)| \geq f^*(t_0)\} \geq t_0 > 0.$$

Again by (2.11) and the equimeasurability of f and f^*,

$$\mathrm{mes}_n\{x : f^*(t_0 + 1) \leq |f(x)| \leq y_0\} =$$

$$= \mathrm{mes}_n\{x : |f(x)| \geq f^*(t_0 + 1)\} - \mathrm{mes}_1\{s > 0 : f^*(s) > y_0\} \geq 1,$$

so $|f|$ takes values greater than $f^*(t_0)$ and values less than y_0. Since f has the intermediate value property, it follows that the whole interval $(y_0, f^*(t_0))$ is in the range of $|f|$. Thus, the assumption that f^* is discontinuous at some point t_0 leads to a contradiction. □

Let f be continuous on a set $E \subset \mathbb{R}^n$. The *total* modulus of continuity of f is the function $\delta \mapsto \omega(f; \delta)$, which is defined for all $\delta > 0$ by

$$\omega(f; \delta) = \sup\{|f(x) - f(y)| : x, y \in E, |x - y| \leq \delta\}. \tag{2.18}$$

The supremum is over all x and y in the domain E of f such that $|x-y| < \delta$. For all $\alpha > 0$ it holds that (see [38, p. 148])

$$\omega(f; \alpha\delta) \leq (\alpha + 1)\omega(f; \delta). \tag{2.19}$$

The inequality stated by the next proposition is known, but we give a simpler proof of it. Similar estimates can be found e.g. in [20], [37] and [25].

Proposition 2.7. *Let $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then*

$$\omega(f^*; \delta) \leq c\,\omega(f; \delta^{1/n}), \tag{2.20}$$

for all $\delta > 0$, where $c = 2v_n^{-1/n} + 1$ and v_n is the measure of the unit ball in $\mathbb{R}^n$.

Proof. By the triangle inequality we have $\omega(|f|;\delta) \leq \omega(f;\delta)$, so we may assume that $f \geq 0$. Fix $0 < t' < t''$ and estimate $f^*(t') - f^*(t'')$. We can assume that $f^*(t'') < f^*(t')$. Let

$$A' = \{x : f(x) = f^*(t')\} \quad \text{and} \quad A'' = \{x : f(x) = f^*(t'')\}.$$

Since $f \in S_0(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, the sets A' and A'' are nonempty by Lemma 2.5. Fix $N \geq 2$. We will show that there exist points $x' \in A'$ and $x'' \in A''$ such that

$$|x' - x''| < 2\frac{N+1}{N-1} v_n^{-1/n} (t'' - t')^{1/n}. \tag{2.21}$$

Let d be the distance from A' to A'', i.e.

$$d = \inf\{|x' - x''| : x' \in A', x'' \in A''\}.$$

If $d = 0$ then $|x' - x''|$ can be chosen arbitrarily small, in particular so small that (2.21) is satisfied. Assume that $d > 0$. Then there exists $x' \in A'$ and $x'' \in A''$ such that $|x' - x''| < (1 + 1/N)d$. Let these points be chosen so that the function $\tau \mapsto f(x'\tau + (1-\tau)x'')$ only takes values in $(f^*(t''), f^*(t'))$ for $\tau \in (0,1)$. Set $\lambda_N = N/(N+1) - 1/2 > 0$. Let B be the ball in $\mathbb{R}^n$ centered at $p = (x' + x'')/2$ of radius $\lambda_N |x' - x''|$. Then $B \cap A' = \emptyset$. Indeed, suppose there exist a point $y' \in B \cap A'$. Then

$$|y' - x''| \leq |y' - \frac{x' + x''}{2}| + |\frac{x' + x''}{2} - x''| <$$

$$< (\lambda_N + \frac{1}{2})|x' - x''| < (\lambda_N + \frac{1}{2})(1 + \frac{1}{N})d = d,$$

which is a contradiction. Similarly $B \cap A'' = \emptyset$.

Let

$$E = \{x : f^*(t'') < f(x) < f^*(t')\}.$$

We will prove that $B \subset E$. By choice of x' and x'' we know that

$$f^*(t'') < f(p) < f^*(t').$$

Suppose there exists a point $q \in B$ where $f(q) < f^*(t'')$. Since f has the intermediate value property there exists a point r along the line segment from p to q where $f(r) = f^*(t'')$. Thus $r \in B \cap A''$, which is a contradiction. In the same way the assumption that $f(x) > f^*(t')$ for some $x \in B$ leads to a contradiction. This proves that $B \subset E$. By our observations (2.10) and (2.11) we then obtain

$$\mathrm{mes}_n B \leq \mathrm{mes}_n E \leq t'' - t'.$$

This gives inequality (2.21). Now

$$f^*(t') - f^*(t'') = f(x') - f(x'') \leq \omega\big(f; 2\frac{N+1}{N-1} v_n^{-1/n} (t'' - t')^{1/n}\big).$$

Since N is arbitrary, we obtain

$$f^*(t') - f^*(t'') \le \omega\big(f; 2v_n^{-1/n}(t''-t')^{1/n}\big). \tag{2.22}$$

By (2.19), this implies (2.20). □

Remark 2.8. Let $n = 1$. Then we have $c = 2$ in (2.20). However, in this case (2.22) gives

$$\omega(f^*;\delta) \le \omega(f;\delta), \tag{2.23}$$

that is, (2.20) holds with $c = 1$. The following shorter proof of inequality (2.23) is already known. Let $0 < t < t+h$. Assume that $f^*(t) > f^*(t+h)$. By Lemma 2.5, there exists $x', x'' \in \mathbb{R}$ such that $|f(x')| = f^*(t)$, $|f(x'')| = f^*(t+h)$, and $f^*(t+h) < |f(x)| < f^*(t)$ for all x between x' and x''. It is clear that $|x'-x''| \le h$ since otherwise we would have

$$\mathrm{mes}_1\{x : f^*(t+h) < |f(x)| < f^*(t)\} > h,$$

which is a contradiction (by (2.10) and (2.11), this set has measure at most h). Thus,

$$f^*(t) - f^*(t+h) = |f(x')| - |f(x'')| \le \omega(f;h).$$

This implies inequality (2.23).

2.3. **Lorentz spaces.** The Lorentz spaces $L^{q,p}$ form a two parameter family of spaces that contains the Lebesgue spaces L^p. We give here the definition and some basic properties.

We observe first that the rearrangement preserves the L^p-norm. Indeed it holds that (see [44, p. 191-192])

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_0^\infty [f^*(t)]^p dt, \tag{2.24}$$

for all $0 < p < \infty$, and

$$\|f\|_\infty = \|f^*\|_\infty. \tag{2.25}$$

It follows from Lemma 3.17 on page 201 in [44] that given $f \in S_0(\mathbb{R}^n)$ and $t > 0$, there exists a measurable set $E_t \subset \mathbb{R}^n$ having measure t such that

$$\int_{E_t} |f(x)|dx = \sup_{|E|=t} \int_E |f(x)|dx = \int_0^t f^*(u)du, \tag{2.26}$$

where $|E|$ denotes the measure of E and the supremum is over all measurable sets $E \subset \mathbb{R}^n$ having measure t.

In what follows we set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)du. \tag{2.27}$$

It follows from (2.26) that the operator $f \mapsto f^{**}$ is subadditive, that is,

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t). \tag{2.28}$$

By the property (2.15) of the rearrangement, we have

$$\lim_{t\to 0+} f^{**}(t) = \|f\|_\infty. \tag{2.29}$$

If $f \in S_0(\mathbb{R}^n)$ and $f \in L^1(E)$, for each measurable set $E \subset \mathbb{R}^n$ with $\mathrm{mes}_n E < \infty$, then $f^{**}(t) < \infty$ for all $t > 0$. Thus, in this case, we have, applying (2.14),

$$\lim_{t\to\infty} f^{**}(t) = 0. \tag{2.30}$$

As was already mentioned in Section 1, when $0 < q, p < \infty$, the space $L^{q,p}(\mathbb{R}^n)$ is defined as the class of all $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{q,p} \equiv \Big(\int_0^\infty [t^{1/q} f^*(t)]^p \frac{dt}{t}\Big)^{1/p} < \infty.$$

By (2.24) we have that $L^{p,p}$ coincides with the space L^p, $0 < p < \infty$. For $0 < q < \infty$ we let $L^{q,\infty}(\mathbb{R}^n)$ be the space of all $f \in S_0(\mathbb{R}^n)$ for which

$$\|f\|_{q,\infty} \equiv \sup_{t>0} t^{1/q} f^*(t) < \infty.$$

We also set $L^{\infty,\infty}(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n)$. When $0 < p \le s \le \infty$, $0 < q < \infty$, there holds the inequality (see [44, Theorem 3.11, p. 192])

$$\Big(\frac{s}{q}\Big)^{1/s} \|f\|_{q,s} \le \Big(\frac{p}{q}\Big)^{1/p} \|f\|_{q,p}. \tag{2.31}$$

Note that we get equality in (2.31) for $f = \chi_E$, $\mathrm{mes}_n E < \infty$, so the constants can not be improved.

The last range of the parameters for which we define the Lorentz space is when $q = \infty$, $0 < p < \infty$. Then we let $L^{\infty,p}(\mathbb{R}^n)$ consist of all $f \in S_0(\mathbb{R}^n)$ such that (see [3], [35])

$$\|f\|_{\infty,p} \equiv \Big(\int_0^\infty \Big[f^{**}(t) - f^*(t)\Big]^p \frac{dt}{t}\Big)^{1/p} < \infty.$$

Observe that

$$L^{\infty,1}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n), \tag{2.32}$$

and the norms coincide. Indeed, note that

$$\frac{d}{dt} f^{**}(t) = -\frac{1}{t}(f^{**}(t) - f^*(t)), \quad \text{a.e. } t > 0.$$

Hence, for every $\varepsilon > 0$,

$$\int_\varepsilon^{1/\varepsilon} \Big[f^{**}(t) - f^*(t)\Big] \frac{dt}{t} = f^{**}(\varepsilon) - f^{**}(1/\varepsilon).$$

Let $\varepsilon \to 0+$. Then the left-hand side tends to $\|f\|_{\infty,1}$, and the right-hand side tends to $\|f\|_\infty$ by (2.29) and (2.30). This proves (2.32).

If $1 \le q, p < \infty$ and $f \in L^{q,p}(\mathbb{R}^n)$, then by (2.31)

$$f^*(t) = O(t^{-1/q}), \tag{2.33}$$

as $t \to 0+$ and as $t \to \infty$.

For any function $f \in S_0(\mathbb{R}^n)$, we will use the notation

$$\Delta_f(t) \equiv f^*(t) - f^*(2t),$$

for $t > 0$. This difference will play an important role in the sequel. We now define the modified Lorentz norm, denoted $\|\cdot\|^*_{q,p}$, which will be equivalent to the Lorentz norm of f but which is defined in terms of Δ_f. This modified Lorentz norm was introduced in [29]. When $1 \le q < \infty$ we set

$$\|f\|^*_{q,p} = \begin{cases} \left(\int_0^\infty [t^{1/q} \Delta_f(t)]^p \frac{dt}{t} \right)^{1/p}, & 1 \le p < \infty \\ \sup\limits_{t>0} t^{1/q} \Delta_f(t), & p = \infty. \end{cases}$$

Clearly, $\|f\|^*_{q,p} \le \|f\|_{q,p}$. To show that $\|f\|_{q,p} \le c\|f\|^*_{q,p}$ for some constant c, we use the inequality:

$$f^*(2t) \le \frac{1}{\ln 2} \int_t^\infty \Delta_f(u) \frac{du}{u}. \tag{2.34}$$

To verify that (2.34) holds, fix $t > 0$ and take $N > 2t$. Then

$$\int_t^N \Delta_f(u) \frac{du}{u} = \int_t^{2t} f^*(u) \frac{du}{u} - \int_N^{2N} f^*(u) \frac{du}{u} \ge f^*(2t) \ln 2 - f^*(N).$$

Now (2.34) follows if we let N tend to ∞ and use (2.14). By (2.34), Hardy's inequality (2.1), and (2.3) we obtain that

$$\|f\|_{q,p} \le \frac{2^{1/q} q}{\ln 2} \|f\|^*_{q,p}, \quad 1 \le q < \infty, 1 \le p \le \infty. \tag{2.35}$$

We define the modified Lorentz norm also when $q = \infty$ and $1 \le p < \infty$. In this case we set

$$\|f\|^*_{\infty,p} \equiv \left(\int_0^\infty (\Delta_f(t))^p \frac{dt}{t} \right)^{1/p}.$$

To prove the equivalence between $\|\cdot\|_{\infty,p}$ and $\|\cdot\|^*_{\infty,p}$ we will use the following inequalities

$$\frac{1}{2} \Delta_f\left(\frac{t}{2}\right) \le f^{**}(t) - f^*(t) \le \frac{2}{t} \int_0^t \Delta_f(u) du. \tag{2.36}$$

The left inequality in (2.36) is immediate,

$$f^{**}(t) - f^*(t) \ge \frac{1}{t} \int_0^{t/2} [f^*(u) - f^*(t)] du \ge \frac{1}{2} [f^*(t/2) - f^*(t)].$$

To prove the right inequality in (2.36) we take $0 < \varepsilon < t/2$ and observe that

$$2\int_\varepsilon^t \Delta_f(u)du \geq \int_\varepsilon^t f^*(u)du - \int_t^{2t} f^*(u)du \geq \int_\varepsilon^t f^*(u)du - tf^*(t).$$

The left inequality in (2.36) immediately implies that $\|f\|^*_{\infty,p} \leq 2\|f\|_{\infty,p}$. By the right inequality in (2.36) and Hardy's inequality (2.2) we have that

$$\|f\|_{\infty,p} \leq 2\|f\|^*_{\infty,p}. \tag{2.37}$$

2.4. **Iterative rearrangements.** We will consider rearrangements with respect to specific variables. Let $f \in S_0(\mathbb{R}^n)$ and $1 \leq k \leq n$. Fix $\hat{x}_k \in \mathbb{R}^{n-1}$, and consider the function $f_{\hat{x}_k}(x_k) = f(\hat{x}_k, x_k)$. By Fubini's theorem, $f_{\hat{x}_k} \in S_0(\mathbb{R})$ for almost all $\hat{x}_k \in \mathbb{R}^{n-1}$. We denote the rearrangement of f with respect to x_k by $\mathcal{R}_k f$. That is, we set

$$\mathcal{R}_k f(t, \hat{x}_k) = f^*_{\hat{x}_k}(t).$$

This function is defined almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. Moreover, $\mathcal{R}_k f$ is a measurable function equimeasurable with f (see [9]). Let $\mathcal{P}_n$ denote the set of all permutations $\sigma = (k_1, \ldots, k_n)$ of the numbers $1, 2, \ldots, n$. For all $\sigma \in \mathcal{P}_n$ and $f \in S_0(\mathbb{R}^n)$, we define the $\mathcal{R}_\sigma$-rearrangement of f as the function

$$\mathcal{R}_\sigma f(t) = \mathcal{R}_{k_n} \cdots \mathcal{R}_{k_1} f(t), \quad t \in \mathbb{R}^n_+.$$

That is, we obtain $\mathcal{R}_\sigma f$ from f by "rearranging" f successively with respect to the variables $x_{k_1}, \ldots, x_{k_n}$, starting with x_{k_1}. By the above, the function $\mathcal{R}_\sigma f$ is equimeasurable with f. As we observed in Section 2.2, this means that their rearrangements coincide, that is $\mathcal{R}_\sigma f \in S_0(\mathbb{R}^n_+)$ and

$$(\mathcal{R}_\sigma f)^*(\tau) = f^*(\tau), \quad \tau > 0. \tag{2.38}$$

Moreover, $\mathcal{R}_\sigma f$ is non-negative on $\mathbb{R}^n_+$, and non-increasing in each variable. In what follows we set

$$\pi(t) = \prod_{k=1}^n t_k, \quad t \in \mathbb{R}^n_+.$$

There holds the inequality

$$\mathcal{R}_\sigma f(t) \leq f^*(\pi(t)), \quad t \in \mathbb{R}^n_+. \tag{2.39}$$

Indeed, since $\mathcal{R}_\sigma f$ is non-increasing in each variable, we have

$$(\mathcal{R}_\sigma f)^*(\pi(t)) \geq \inf_{s \in Q_t} \mathcal{R}_\sigma f(s) \geq \mathcal{R}_\sigma f(t), \tag{2.40}$$

where $Q_t = (0, t_1) \times \cdots \times (0, t_n)$. Apply (2.38), with $\tau = \pi(t)$. This gives (2.39).

Using (2.24) n times, we obtain that for all $f \in S_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n_+} [R_\sigma f(t)]^p dt = \int_{\mathbb{R}^n} |f(x)|^p dx, \tag{2.41}$$

for $0 < p < \infty$. By (2.25), we also have

$$\|\mathcal{R}_\sigma f\|_\infty = \|f\|_\infty.$$

For $0 < p, s < \infty$ and $\sigma \in \mathcal{P}_n$, we define the space $\mathcal{L}^{p,s}_\sigma(\mathbb{R}^n)$ as the class of all functions $f \in S_0(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}^{p,s}_\sigma} \equiv \Big(\int_{\mathbb{R}^n_+} \Big[\pi(t)^{1/p} \mathcal{R}_\sigma f(t) \Big]^s \frac{dt}{\pi(t)} \Big)^{1/s} < \infty$$

(see [9]). We also set

$$\mathcal{L}^{p,s}(\mathbb{R}^n) = \bigcap_{\sigma \in \mathcal{P}_n} \mathcal{L}^{p,s}_\sigma(\mathbb{R}^n).$$

It was proved in [48] that

$$\|f\|_{p,s} \le 2^{1/s-1/p} \|f\|_{\mathcal{L}^{p,s}_\sigma},$$

for $0 < s \le p < \infty$ and $\sigma \in \mathcal{P}_n$. Theorem 2.10 below improves the constant in this inequality. The proof of this theorem is very similar to the proof of the preceding inequality given in [30, pp. 54–55].

Let $\mathcal{M}(\mathbb{R}^n_+)$ denote the class of all measurable non-negative functions on $\mathbb{R}^n_+$, which are non-increasing in each variable.

Lemma 2.9. *Let $\sigma \in \mathcal{P}_n$, $0 < s, p < \infty$, and $f \in \mathcal{M}(\mathbb{R}^n_+) \cap \mathcal{L}^{p,s}_\sigma(\mathbb{R}^n_+)$. There exists a sequence $\{f_k\}$ in $\mathcal{M}(\mathbb{R}^n_+) \cap \mathcal{L}^{p,s}_\sigma(\mathbb{R}^n_+)$, such that:*

(i) *for each k, f_k is positive and strictly decreasing in each variable;*
(ii) $\|f_k\|_{p,s} \to \|f\|_{p,s}$;
(iii) $\|f_k\|_{\mathcal{L}^{p,s}_\sigma} \to \|f\|_{\mathcal{L}^{p,s}_\sigma}$.

Proof. Fix $\alpha > 1/p$ and set

$$g(t) = \Big[\prod_{k=1}^n (1 + t_k) \Big]^{-\alpha}, \quad t \in \mathbb{R}^n_+.$$

We have

$$\|g\|_{\mathcal{L}^{p,s}_\sigma} = \Big(\int_0^\infty u^{s/p-1} (1+u)^{-s\alpha} du \Big)^{n/s}.$$

By our choice of α, this integral converges so that $g \in \mathcal{L}^{p,s}_\sigma(\mathbb{R}^n_+)$. Set

$$f_k = f + \frac{1}{k} g, \quad k \in \mathbb{N}.$$

Clearly $f_k \in \mathcal{M}(\mathbb{R}^n_+) \cap \mathcal{L}^{p,s}_\sigma(\mathbb{R}^n_+)$ and statement (i) holds. Note that

$$|f - f_k| \leq \frac{1}{k}, \tag{2.42}$$

on $\mathbb{R}^n_+$. Further, the sequence $\{f_k\}$ is decreasing and the function

$$t \mapsto \Big(\prod_{k=1}^n t_k^{s/p-1}\Big) f_1(t)^s, \quad t \in \mathbb{R}^n_+,$$

belongs to $L^1(\mathbb{R}^n_+)$. Hence, by the dominated convergence theorem, statement (iii) holds. According to (2.42), $f_k \to f$ in measure on $\mathbb{R}^n_+$. This implies that $f_k^* \to f^*$ a.e. on $\mathbb{R}_+$ by Proposition 2.4. The sequence $\{f_k^*\}$ is decreasing and the function

$$u \mapsto u^{s/p-1} f_1^*(u)^s, \quad u > 0,$$

belongs to $L^1(\mathbb{R}_+)$. To obtain statement (ii), we again use the dominated convergence theorem. □

Theorem 2.10. *Let $f \in S_0(\mathbb{R}^n)$ and $\sigma \in \mathcal{P}_n$. For all $0 < s \leq p < \infty$,*

$$\|f\|_{p,s} \leq \|f\|_{\mathcal{L}^{p,s}_\sigma}. \tag{2.43}$$

Proof. Set $F(t) = \mathcal{R}_\sigma f(t)$. By Lemma 2.9 we may suppose that

$$\mathrm{mes}_n\{t \in \mathbb{R}^n_+ : F(t) = y\} = 0, \tag{2.44}$$

for all $y \geq 0$. Fix $a > 1$ and set

$$A_\nu = \{t \in \mathbb{R}^n_+ : f^*(a^{-\nu+1}) \leq F(t) < f^*(a^{-\nu})\},$$

for $\nu \in \mathbb{Z}$. Let $t \in A_\nu$. Then $f^*(a^{-\nu+1}) \leq F(t) \leq F(s)$ for all $s \in \mathbb{R}^n_+$ such that $0 < s_k \leq t_k$, $k = 1, \ldots, n$. Thus,

$$\pi(t) \leq \mathrm{mes}_n\{s \in \mathbb{R}^n_+ : F(s) \geq f^*(a^{-\nu+1})\}.$$

By our assumption (2.44), this gives

$$\pi(t) \leq \mathrm{mes}_1\{u > 0 : f^*(u) \geq f^*(a^{-\nu+1})\} = a^{-\nu+1},$$

for all $t \in A_\nu$. This inequality implies

$$\begin{aligned}
\|f\|^s_{\mathcal{L}^{p,s}_\sigma} &= \int_{\mathbb{R}^n_+} \pi(t)^{s/p-1} F(t)^s dt = \sum_{\nu\in\mathbb{Z}} \int_{A_\nu} \pi(t)^{s/p-1} F(t)^s dt \geq \\
&\geq \sum_{\nu\in\mathbb{Z}} a^{(s/p-1)(-\nu+1)} \int_{A_\nu} F(t)^s dt = \\
&= a^{s/p-1} \sum_{\nu\in\mathbb{Z}} a^{-\nu(s/p-1)} \int_{a^{-\nu}}^{a^{-\nu+1}} f^*(u)^s du \geq \\
&\geq a^{s/p-1} \sum_{\nu\in\mathbb{Z}} \int_{a^{-\nu}}^{a^{-\nu+1}} u^{s/p-1} f^*(u)^s du = a^{s/p-1} \|f\|^s_{p,s}.
\end{aligned}$$

Since $a > 1$ was arbitrary, inequality (2.43) follows. $\square$

3. Some geometric results

In this section, we will examine some of the properties of the spaces

$$V_k = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})], \quad k = 1, \ldots, n,$$

where, as above, $\hat{x}_k$ denotes the point in $\mathbb{R}^{n-1}$ which is obtained from a given point $x \in \mathbb{R}^n$ by removal of its kth coordinate. Recall from Section 1, that the corresponding norms where used by Gagliardo, and later also by Fournier, to prove embeddings of Sobolev spaces. More complicated mixed norms related to embeddings of Sobolev type spaces will be studied in latter sections of this thesis.

In Section 7 below we will study embeddings of the space $\cap_{k=1}^n V_k$, and in particular we prove a refinement of Fournier's inequality (1.6) for the V_k-norms (see Theorem 7.10 and Remark 7.11 below).

For $E \subset \mathbb{R}^n$ and $1 \le k \le n$, we let $\Pi_k E \subset \mathbb{R}^{n-1}$ be the orthogonal projection of E onto the coordinate hyperplane $x_k = 0$. Throughout this work we study geometric properties of sets, in particular the measures of the projections. An important role is played by the following inequality proved by Loomis and Whitney [34].

Theorem 3.1. *For any F_σ-set $E \subset \mathbb{R}^n$ there holds the inequality*

$$(\mathrm{mes}_n E)^{n-1} \le \prod_{k=1}^n \mathrm{mes}_{n-1} \Pi_k E. \tag{3.1}$$

We shall also use the following elementary lemma.

Lemma 3.2. *Let $n \ge 2$ and $1 \le k \le n$. Assume that $E \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^{n-1}$ are measurable in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ respectively. Then the set*

$$E' = \{x \in E : \hat{x}_k \in D\}$$

is measurable in $\mathbb{R}^n$.

Proof. It is sufficient to consider the case $k = n$. In this case

$$E' = E \cap (D \times \mathbb{R}).$$

Since the Cartesian product of two measurable sets is measurable, the measurability of E' follows. □

For a point $\hat{x}_k \in \mathbb{R}^{n-1}$, denote by $E(\hat{x}_k)$ the $\hat{x}_k$-section of the set E:

$$E(\hat{x}_k) = \{x_k \in \mathbb{R} : (x_k, \hat{x}_k) \in E\}.$$

Assume that $E \subset \mathbb{R}^n$ is measurable. By Fubini's theorem, for any $1 \le k \le n$ and almost all $\hat{x}_k \in \mathbb{R}^{n-1}$, the sections $E(\hat{x}_k)$ are measurable in $\mathbb{R}$, and the functions

$$m_k(\hat{x}_k) = \mathrm{mes}_1 E(\hat{x}_k) \quad (k = 1, \ldots, n)$$

defined a.e. on $\mathbb{R}^{n-1}$ are measurable.

The *essential* projection of a measurable set E onto the hyperplane $x_k = 0$ is defined to be the set $\Pi_k^* E$ of all points $\hat{x}_k \in \mathbb{R}^{n-1}$ such that $E(\hat{x}_k)$ is measurable and $m_k(\hat{x}_k) > 0$. Since the function m_k is measurable, the essential projection $\Pi_k^* E$ is measurable.

Observe that inequality (3.1) holds also if $\Pi_k E$ is replaced by the essential projection $\Pi_k^* E$. That is, for all measurable sets $E \subset \mathbb{R}^n$, there holds the inequality

$$(\mathrm{mes}_n E)^{n-1} \leq \prod_{k=1}^{n} \mathrm{mes}_{n-1} \Pi_k^* E. \tag{3.2}$$

Indeed, for any $\hat{x}_k \in \mathbb{R}^{n-1}$ the section $E(\hat{x}_k)$ is an F_σ-set in $\mathbb{R}$ and therefore it is measurable. Thus, $\Pi_k^* E$ consists exactly of all points $\hat{x}_k$ for which $m_k(\hat{x}_k) > 0$. Put

$$E' = \bigcap_{k=1}^{n} \{x \in E : \hat{x}_k \in \Pi_k^* E\}.$$

Then E' is measurable by Lemma 3.2. Note also that $\Pi_k E' \subset \Pi_k^* E$. Further, we have $\mathrm{mes}_n E' = \mathrm{mes}_n E$. Namely,

$$E \setminus E' = \bigcup_{k=1}^{n} \{x \in E : \hat{x}_k \notin \Pi_k^* E\} = \bigcup_{k=1}^{n} \{x \in E : m_k(\hat{x}_k) = 0\},$$

and each of the sets in the last union have measure 0. Let E'' be an F_σ-subset of E' with $\mathrm{mes}_n E'' = \mathrm{mes}_n E$. Then $\Pi_k E'' \subset \Pi_k^* E$. Using (3.1) we then get

$$(\mathrm{mes}_n E)^{n-1} = (\mathrm{mes}_n E'')^{n-1} \leq \prod_{k=1}^{n} \mathrm{mes}_{n-1} \Pi_k E'' \leq \prod_{k=1}^{n} \mathrm{mes}_{n-1} \Pi_k^* E,$$

so (3.2) is proved.

Let f be a measurable function on $\mathbb{R}^n$ and let $1 \leq k \leq n$. By Fubini's theorem, for almost all $\hat{x}_k \in \mathbb{R}^{n-1}$ the sections $f_{\hat{x}_k}$ (see Section 2.4) are measurable functions on $\mathbb{R}$. Moreover, as we observed above, the function

$$\Psi_k(\hat{x}_k) = \|f_{\hat{x}_k}\|_{L^\infty(\mathbb{R})} = \mathrm{ess\ sup}_{x_k \in \mathbb{R}} |f(x_k, \hat{x}_k)|$$

(defined on $\mathbb{R}^{n-1}$) is measurable. It is sufficient to show this in the case when f is a bounded function with compact support. In this case we have

$$\Psi_k(\hat{x}_k) = \lim_{m \to \infty} \|f_{\hat{x}_k}\|_{L^m(\mathbb{R})},$$

and the functions

$$\hat{x}_k \mapsto \|f_{\hat{x}_k}\|_{L^m(\mathbb{R})}$$

are measurable by Fubini's theorem. Thus, the definition of the spaces V_k (see Section 1) is correct. Now we shall show that the norms in V_k have a simple geometric interpretation.

Theorem 3.3. *Let f be a measurable function on $\mathbb{R}^n$, and put*

$$E = \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ 0 \le y \le |f(x)|\}.$$

For $k \in \{1, \dots, n\}$, we let E_k denote the essential projection of E onto the hyperplane $x_k = 0$. Then $f \in V_k$ if and only if E_k is measurable and $\mathrm{mes}_n E_k < \infty$. Moreover, in this case

$$\mathrm{mes}_n E_k = \|f\|_{V_k}. \tag{3.3}$$

Proof. It is enough to give the proof for $k = n$. Denote

$$M_n(\hat{x}_n) = \mathrm{ess\ sup}_{x_n \in \mathbb{R}} |f(\hat{x}_n, x_n)| \tag{3.4}$$

for all $\hat{x}_n$ where this essential supremum is defined, and put $M_n(\hat{x}_n) = 0$ otherwise. We have

$$E_n(\hat{x}_n) \subset [0, M_n(\hat{x}_n)], \tag{3.5}$$

for all $\hat{x}_n \in \mathbb{R}^{n-1}$ (with the obvious interpretation if $M_n(\hat{x}_n)$ equals 0 or ∞). Indeed, suppose $(\hat{x}_n, y) \in E_n$. We have

$$E(\hat{x}_n, y) = \{x_n \in \mathbb{R} : (\hat{x}_n, x_n, y) \in E\}.$$

By definition of E_n, the set $E(\hat{x}_n, y) \subset \mathbb{R}$ is measurable and $\mathrm{mes}_1 E(\hat{x}_n, y) > 0$. But for all $x_n \in E(\hat{x}_n, y)$, we have $0 \le y \le |f(\hat{x}_n, x_n)|$, and thus $y \le M_n(\hat{x}_n)$. This proves (3.5).

Further, if $M_n(\hat{x}_n) > 0$, then we have

$$[0, M_n(\hat{x}_n)) \subset E_n(\hat{x}_n). \tag{3.6}$$

Indeed, suppose $M_n(\hat{x}_n) > 0$ and let $y \in [0, M_n(\hat{x}_n))$. Note that

$$E(\hat{x}_n, y) = \{x_n \in \mathbb{R} : 0 \le y \le |f(\hat{x}_n, x_n)|\}.$$

Since $y < M_n(\hat{x}_n)$, this set has positive linear measure, so by definition $(\hat{x}_n, y) \in E_n$. That is, (3.6) holds.

By (3.5) and (3.6), $E_n(\hat{x}_n)$ is an interval with the length

$$\mathrm{mes}_1 E_n(\hat{x}_n) = M_n(\hat{x}_n) \tag{3.7}$$

for all $\hat{x}_n$ where $M_n(\hat{x}_n) > 0$. Suppose that E_n is measurable and $\mathrm{mes}_n E_n < \infty$. By Fubini's theorem and (3.7), we have

$$\mathrm{mes}_n E_n = \int_{\mathbb{R}^{n-1}} \mathrm{mes}_1 E_n(\hat{x}_n) d\hat{x}_n = \int_{\mathbb{R}^{n-1}} M_n(\hat{x}_n) d\hat{x}_n = \|f\|_{V_n}.$$

So we have proved (3.3), and thus $f \in V_n$. To prove the converse, suppose that $f \in V_n$. Then $M_n \in L^1(\mathbb{R}^{n-1})$, and $\|M_n\|_{L^1(\mathbb{R}^{n-1})} = \|f\|_{V_n}$. We are done if we prove (3.3). To this end, we consider the sets

$$E'_n = \{(\hat{x}_n, y) \in \mathbb{R}^n : 0 \leq y < M_n(\hat{x}_n)\}$$

and

$$E''_n = \{(\hat{x}_n, y) \in \mathbb{R}^n : 0 \leq y \leq M_n(\hat{x}_n)\}.$$

These sets are measurable and

$$\mathrm{mes}_n E'_n = \mathrm{mes}_n E''_n = \|M_n\|_{L^1(\mathbb{R}^{n-1})} = \|f\|_{V_n}.$$

Note also that

$$E'_n \subset E_n \subset E''_n.$$

These observations imply that E_n is measurable, and that (3.3) holds. $\square$

Next, we let $n = 2$ and show that the space $V_1 \cap V_2$ is *not* invariant under rotation. To make clear what we mean, suppose that $f \in V_1 \cap V_2$, and set

$$g(x, y) = f\Big(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\Big). \tag{3.8}$$

The vectors (x, y) and $((x-y)/\sqrt{2}, (x+y)/\sqrt{2})$ have the same length, and the angle between them is $\pi/4$. Thus, the graph of g is obtained by rotating the graph of f the angle $\pi/4$ around the origin. We prove the following proposition by finding $f \in V_1 \cap V_2$ such that the function g defined by (3.8) does not belong to $V_1 \cap V_2$.

Proposition 3.4. *The space $V_1 \cap V_2$ is not invariant under rotation.*

Proof. Put

$$E = (-1, 1) \times \bigcup_{k=1}^{\infty} (k, k + 2^{-k})$$

and set

$$f(x, y) = \chi_E(x, y),$$

on $\mathbb{R}^2$. We have

$$\|f(\cdot, y)\|_\infty = \sum_{k=1}^{\infty} \chi_{(k,k+2^{-k})}(y), \quad \text{and} \quad \|f(x, \cdot)\|_\infty = \chi_{(-1,1)}(x).$$

Thus,

$$\|f\|_{V_1} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \chi_{(k,k+2^{-k})}(y) dy = 1, \quad \text{and} \quad \|f\|_{V_2} = \int_{\mathbb{R}} \chi_{(-1,1)}(x) dx = 2.$$

So $f \in V_1 \cap V_2$. For this function f, the function g from (3.8) is given by

$$g(x,y) = \chi_{(-1,1)}\Big(\frac{x-y}{\sqrt{2}}\Big) \sum_{k=1}^{\infty} \chi_{(k,k+2^{-k})}\Big(\frac{x+y}{\sqrt{2}}\Big),$$

and as we noted above, it is obtained by rotating f. The proof is complete if we check that g does not belong to $V_1 \cap V_2$.

Fix $x_0 \geq 1/\sqrt{2}$ and let $k \geq 1$ be the integer contained in

$$(x_0\sqrt{2} - 1, x_0\sqrt{2}].$$

Then

$$g(x_0, y) = 1, \qquad \text{for all} \qquad y \in (k\sqrt{2} - x_0, k\sqrt{2} - x_0 + 2^{-k+1/2}). \tag{3.9}$$

Indeed, for y in this interval it holds that

$$\frac{x_0 + y}{\sqrt{2}} \in (k, k + 2^{-k})$$

and

$$\frac{x_0 - y}{\sqrt{2}} \in (x_0\sqrt{2} - k - 2^{-k}, x_0\sqrt{2} - k) \subset (-2^{-k}, 1),$$

where the last inclusion holds since $k \in (x_0\sqrt{2} - 1, x_0\sqrt{2}]$. This proves (3.9). It follows that

$$\|g(x, \cdot)\|_\infty = 1, \quad x \geq 1/\sqrt{2},$$

which implies that $\|g\|_{V_2} = \infty$. (In fact, a similar argument shows that also $\|g\|_{V_1} = \infty$.) The proof is complete. $\square$

Remark 3.5. With the above proposition in mind, we point out that $\cap_{k=1}^n V_k$ contains $W_1^1(\mathbb{R}^n)$ (according to (1.4)), and this Sobolev space is clearly invariant under rotation. Indeed, there holds the obvious inequality

$$\|D_u f\|_1 \leq \sum_{k=1}^{n} \|\frac{\partial f}{\partial x_k}\|_1, \quad D_u f(x) = u \cdot \nabla f(x),$$

for $f \in W_1^1(\mathbb{R}^n)$, where $D_u f$ is the weak directional derivative of f in the direction of a given unit vector u. This shows that if $f \in W_1^1(\mathbb{R}^n)$, and if g is obtained by rotating f a given angle around the origin, then g belongs to the same Sobolev space.

Let X be a linear space of measurable real-valued functions on $\mathbb{R}^n$, which is generated by a norm $\|\cdot\|_X$. We say that such a space is *rearrangement invariant* (r.i. for short) if, whenever $f \in X$ and g is equimeasurable with f, then $g \in X$ and

$$\|f\|_X = \|g\|_X$$

(see [5, p. 59]).

Remark 3.6. It follows immediately from Proposition 3.4 that $V_1 \cap V_2$ is not a r.i. space.

Given any set $W \subset S_0(\mathbb{R}^n)$, we define the *r.i. hull* of W as the smallest r.i. space that contains W.

Remark 3.7. It is well known that $L^{n',1}(\mathbb{R}^n)$ is the r.i. hull of $W_1^1(\mathbb{R}^n)$. Since

$$W_1^1(\mathbb{R}^n) \subset \bigcap_{k=1}^{n} V_k \subset L^{n',1}(\mathbb{R}^n),$$

it follows that $L^{n',1}(\mathbb{R}^n)$ is the r.i. hull also of $\cap_{k=1}^n V_k$. The direct proof of this fact was given in [16, Theorems 3.1 and 4.1].

4. The spaces Λ^σ

In this section we consider a one parameter family of spaces denoted Λ^σ. These spaces were introduced in [29]. Let $\sigma \in \mathbb{R}$. Recall from Section 1 that a function $f \in S_0(\mathbb{R})$ belongs to Λ^σ if

$$\|f\|_{\Lambda^\sigma} \equiv \sup_{t>0} t^\sigma \Delta_f(t) < \infty.$$

In Section 4.1 we see how Λ^σ relates to known spaces. We also give an equivalent definition of $\|\cdot\|_{\Lambda^\sigma}$ for $\sigma < 0$, and show that in this case all functions in this space belong to $L^\infty(\mathbb{R})$. The main results in Section 4.2 are Theorems 4.8 and 4.10, which show how functions in Λ^σ can be approximated by simple functions (defined below) and by continuous functions with compact support.

4.1. **Some general properties of the spaces Λ^σ.**

Propositions 4.1, 4.2, and 4.3 below, state embeddings of Λ^σ for different values of σ. These results were obtained in [29]; for the sake of completeness, we give the proofs.

First we determine to what extent $\|\cdot\|_{\Lambda^\sigma}$ satisfies the properties of a norm. We have $\|f\|_{\Lambda^\sigma} \geq 0$ for all $f \in \Lambda^\sigma$ since Δ_f is non-negative. Moreover, $\Delta_f = 0$ on $\mathbb{R}_+$ if and only if $f^* = 0$ on $\mathbb{R}_+$. Therefore $\|f\|_{\Lambda^\sigma} = 0$ if and only if $f = 0$ a.e. on $\mathbb{R}^n$. Furthermore, by (2.12), we have $\|\lambda f\|_{\Lambda^\sigma} = |\lambda|\|f\|_{\Lambda^\sigma}$, for all $\lambda \in \mathbb{R}$. However, we will show that if $\sigma \leq 0$ then there is no constant c such that the "triangle inequality",

$$\|f+g\|_{\Lambda^\sigma} \leq c(\|f\|_{\Lambda^\sigma} + \|g\|_{\Lambda^\sigma}), \tag{4.1}$$

holds for all $f, g \in \Lambda^\sigma$. Set $f_n = n\chi_{(0,1]}$, $h_2 = \chi_{(1,2]}$, and $h_{n+1} = h_n + \chi_{(1,2^n]}$, $n \geq 2$. Using induction we prove that

$$\Delta_{f_n} = n\chi_{(1/2,1]},$$

$$\Delta_{f_n+h_n} = \chi_{(1/2,2^{n-1}]},$$

and

$$\Delta_{h_n} = \sum_{k=1}^{n-1} \chi_{((2^k-1)/2,2^k-1]}.$$

So, if $\alpha \geq 0$, then $\|f_n\|_{\Lambda^{-\alpha}} = n2^\alpha$, $\|f_n + h_n\|_{\Lambda^{-\alpha}} = 2^\alpha$, and $\|h_n\|_{\Lambda^{-\alpha}} = 2^\alpha$. Clearly there is no constant c for which

$$\|f_n\|_{\Lambda^{-\alpha}} \leq c(\|f_n + h_n\|_{\Lambda^{-\alpha}} + \|h_n\|_{\Lambda^{-\alpha}}),$$

for all $n \geq 2$, so (4.1) is not satisfied when $\sigma \leq 0$. For $\sigma > 0$, (4.1) holds with $c = 4^\sigma/(\sigma \ln 2)$. To prove this, we will use the fact that for $\sigma > 0$, the space Λ^σ coincides with the Marcinkiewicz space, as stated in the following proposition.

Proposition 4.1. *Let $\sigma > 0$ and set $r = 1/\sigma$. Then $\Lambda^\sigma = L^{r,\infty}(\mathbb{R})$ and*

$$\|f\|_{\Lambda^\sigma} \le \|f\|_{r,\infty} \le \frac{2^\sigma}{\sigma \ln 2} \|f\|_{\Lambda^\sigma}. \tag{4.2}$$

Proof. The first inequality in (4.2) is immediate for all $f \in L^{r,\infty}(\mathbb{R})$. Let $f \in \Lambda^\sigma(\mathbb{R})$. By (2.34),

$$\|f\|_{r,\infty} \le \frac{2^\sigma}{\ln 2} \sup_{t>0} t^\sigma \int_t^\infty \Delta_f(u) \frac{du}{u}.$$

The second inequality in (4.2) now follows by inequality (2.3). □

Let $\sigma > 0$ and set $r = 1/\sigma$. Suppose $f, g \in L^{r,\infty}(\mathbb{R})$. By (2.13) we have

$$\|f+g\|_{r,\infty} \le \sup_{t>0} t^{1/r}(f^*(t/2) + g^*(t/2)) \le 2^{1/r}(\|f\|_{r,\infty} + \|g\|_{r,\infty}).$$

This inequality and Proposition 4.1 now give

$$\|f+g\|_{\Lambda^\sigma} \le \frac{4^\sigma}{\sigma \ln 2}(\|f\|_{\Lambda^\sigma} + \|g\|_{\Lambda^\sigma}), \tag{4.3}$$

for all $f, g \in \Lambda^\sigma$, i.e. (4.1) holds when $\sigma > 0$.

Define the space W, called weak-L^∞, as the class of all $f \in S_0(\mathbb{R})$ such that

$$\|f\|_W = \sup_{t>0} [f^{**}(t) - f^*(t)] < \infty. \tag{4.4}$$

This space was introduced in [4] by Bennett, DeVore, and Sharpely.

Proposition 4.2. *The spaces Λ^0 and W coincide and*

$$\frac{1}{2}\|f\|_{\Lambda^0} \le \|f\|_W \le 2\|f\|_{\Lambda^0}.$$

Proof. Let $f \in W$. The first inequality follows immediately from the first inequality in (2.36). Therefore $W \subset \Lambda^0$. Suppose $f \in \Lambda^0$. Fix $t > 0$. By the second inequality in (2.36) we have

$$f^{**}(t) - f^*(t) \le \frac{2}{t} \int_0^t \Delta_f(u) du \le 2\|f\|_{\Lambda^0}.$$

The second inequality now follows. This gives $\Lambda^0 \subset W$. □

As above, $C(\mathbb{R})$ denotes the class of all bounded continuous functions on $\mathbb{R}$. For $0 < \alpha \le 1$ we define $\operatorname{Lip} \alpha$ to be the space of all functions $f \in C(\mathbb{R})$ for which (recall (2.18))

$$\|f\|_{\operatorname{Lip} \alpha} \equiv \sup_{\delta>0} \frac{\omega(f;\delta)}{\delta^\alpha} < \infty. \tag{4.5}$$

Proposition 4.3. *Let $0 < \alpha \le 1$. If $f \in S_0(\mathbb{R}) \cap \operatorname{Lip} \alpha$ then $f \in \Lambda^{-\alpha}$ and*

$$\|f\|_{\Lambda^{-\alpha}} \le \|f\|_{\operatorname{Lip} \alpha},$$

Proof. Fix $t > 0$. By inequality (2.23) in Remark 2.8 we have

$$\Delta_f(t) \leq \omega(f^*; t) \leq \omega(f; t)$$

and then

$$t^{-\alpha}\Delta_f(t) \leq \|f\|_{\mathrm{Lip}\,\alpha}.$$

Taking supremum over all $t > 0$ we obtain the inequality stated in the proposition. □

The next proposition gives an equivalent definition of the space Λ^σ, when $\sigma < 0$.

Proposition 4.4. *Let $\sigma < 0$. Then $f \in \Lambda^\sigma$ if and only if there exists a constant A such that for all $t > 0$*

$$\|f\|_\infty \leq f^*(t) + At^{-\sigma}. \tag{4.6}$$

Moreover, if $A_0 \geq 0$ is the smallest constant such that inequality (4.6) holds for all $t > 0$, then

$$(2^{-\sigma} - 1)A_0 \leq \|f\|_{\Lambda^\sigma} \leq 2^{-\sigma}A_0. \tag{4.7}$$

Proof. Suppose (4.6) holds. Then

$$\Delta_f(t) \leq \|f\|_\infty - f^*(2t) \leq (2t)^{-\sigma}A,$$

and thus

$$\|f\|_{\Lambda^\sigma} \leq 2^{-\sigma}A.$$

So, $f \in \Lambda^\sigma$ and the right-hand side inequality in (4.7) follows. Let now $f \in \Lambda^\sigma$. For any $N > 0$,

$$f^*(2^{-N}t) - f^*(t) = \sum_{k=1}^{N} \Delta_f(2^{-k}t) \leq t^{-\sigma}\|f\|_{\Lambda^\sigma} \sum_{k=1}^{N} 2^{k\sigma}.$$

Let $N \to \infty$. By (2.15) we obtain

$$\|f\|_\infty \leq f^*(t) + t^{-\sigma}\frac{\|f\|_{\Lambda^\sigma}}{1 - 2^\sigma}.$$

Thus, (4.6) holds.

If $A_0 = 0$, then (4.7) follows immediately. Suppose $A_0 > 0$ and fix $\varepsilon \in (0, A_0)$. By definition of A_0 there exists $t_0 > 0$ such that

$$\|f\|_\infty > f^*(t_0) + (A_0 - \varepsilon)t_0^{-\sigma}.$$

Take $N > 0$ such that $f^*(2^{-N}t_0) > \|f\|_\infty - \varepsilon$. We then have

$$A_0 - \varepsilon < t_0^\sigma\big(f^*(2^{-N}t_0) - f^*(t_0) + \varepsilon\big) = \varepsilon t_0^\sigma + \|f\|_{\Lambda^\sigma} \sum_{k=1}^{N} 2^{k\sigma}.$$

Since $\varepsilon \in (0, A_0)$ was arbitrary, it follows that

$$A_0 \leq \|f\|_{\Lambda^\sigma} \sum_{k=1}^{N} 2^{k\sigma}$$

which implies the left-hand side inequality in (4.7). $\square$

Corollary 4.5. *Let $\sigma < 0$. Then $\Lambda^\sigma \subset L^\infty(\mathbb{R})$.*

Proof. Let $f \in \Lambda^\sigma$. By Proposition 4.4, there exists a constant $A > 0$ for which inequality (4.6) holds. This implies that $f \in L^\infty(\mathbb{R})$. $\square$

4.2. **Approximation in Λ^σ.** The main results in this section are Theorem 4.8 and Theorem 4.10 below. They show in particular how functions in Λ^σ can be approximated by simple functions, and by continuous functions with compact support. We first give a negative result on the separability of Λ^σ.

Proposition 4.6. *Let $\sigma > 0$. Then the space Λ^σ is not separable.*

Proof. We need only find an uncountable subfamily $S \subset \Lambda^\sigma$ with the property that

$$\|f - g\|_{\Lambda^\sigma} \geq 1, \tag{4.8}$$

for all functions $f, g \in S$ such that f is not equivalent to g. Indeed, suppose $S = \{f_\xi\}_{\xi \in I}$ is such a family and let $\{g_n\}_{n=1}^\infty$ be any countable sequence in Λ^σ. Set

$$r = \frac{\sigma \ln 2}{2^{2\sigma+1}}.$$

Then the balls $B_\xi = B(f_\xi, r)$ are pairwise disjoint. Indeed, suppose $g \in B_\xi \cap B_\eta$ for some $\xi, \eta \in I$, $\xi \neq \eta$. By inequality (4.3) we would then have

$$\|f_\xi - f_\eta\|_{\Lambda^\sigma} \leq \frac{4^\sigma}{\sigma \ln 2}(\|f_\xi - g\|_{\Lambda^\sigma} + \|g - f_\eta\|_{\Lambda^\sigma}) < 1,$$

which is a contradiction. Since S is uncountable, there must then exist balls B_ξ which does not contain any of the functions g_n. Therefore the sequence $\{g_n\}_{n=1}^\infty$ can not be dense in Λ^σ.

To construct such a family $S \subset \Lambda^\sigma$ we set

$$f_\xi(t) = (t - \xi)^{-\sigma} \chi_{(\xi,1]}(t),$$

for $0 < \xi < 1$ and $t \neq \xi$. Then

$$f_\xi^*(t) = t^{-\sigma} \chi_{(0,1-\xi]}$$

so that

$$\sup_{t>0} t^\sigma \Delta_{f_\xi}(t) \leq \sup_{t>0} t^\sigma f_\xi^*(t) = 1,$$

and therefore $f_\xi \in \Lambda^\sigma$. Let $0 < \xi < \eta < 1$. By (4.2)

$$\frac{2^{\sigma+1}}{\sigma}\|f_\xi - f_\eta\|_{\Lambda^\sigma} \geq \sup_{t>0} t^\sigma (f_\xi - f_\eta)^*(t) \geq \sup_{t>0} t^\sigma \big((f_\xi - f_\eta)\chi_{(0,\eta]}\big)^*(t) = 1,$$

so we can let S consist of the functions $2^{\sigma+1}\sigma^{-1}f_\xi$, $\xi \in (0,1)$. □

Observe that if $\sigma \leq 0$, it makes no sense to speak about approximation "in the norm" $\|\cdot\|_{\Lambda^\sigma}$. Indeed, if $\sigma < 0$ set $\alpha \equiv -\sigma$ and $f_n = \chi_{(0,n]}$. Then

$$\|f_n\|_{\Lambda^{-\alpha}} = \sup_{t>0} t^{-\alpha}\chi_{(n/2,n]}(t) = \Big(\frac{2}{n}\Big)^\alpha.$$

So, $f_n \in \Lambda^{-\alpha}$ and $\|f_n\|_{\Lambda^{-\alpha}} \to 0$, as $n \to \infty$. If $\sigma = 0$ we set

$$g_n(x) = \Big(1 - \frac{\log_2(1+x)}{n}\Big)\chi_{(0,2^n-1]}(x).$$

For $0 < t \leq (2^n - 1)/2$ we have

$$\Delta_{g_n}(t) = \frac{1}{n}\log_2\Big(\frac{1+2t}{1+t}\Big) < \frac{1}{n}$$

and for $((2^n-1)/2) < t \leq 2^n - 1$ we have

$$\Delta_{g_n}(t) = g_n^*(t) \leq g_n((2^n-1)/2) < 1 - \frac{1}{n}\log_2(2^{n-1}) = \frac{1}{n}.$$

So, $\|g_n\|_{\Lambda^0} < 1/n$. Thus $g_n \in \Lambda^0$ and $\|g_n\|_{\Lambda^0} \to 0$, as $n \to \infty$. These examples shows that even if $\|f\|_{\Lambda^\sigma}$ is "small", it can still happen that f is "big".

Let w be a positive continuous function on $\mathbb{R}_+$. We say that a function $f \in S_0(\mathbb{R})$ belongs to the space $\Lambda(w)$ if

$$\|f\|_{\Lambda(w)} \equiv \sup_{t>0} w(t)\Delta_f(t) < \infty.$$

If $w(t) = t^\sigma$, then $\|\cdot\|_{\Lambda(w)} = \|\cdot\|_{\Lambda^\sigma}$. We will give two theorems on how a function $f \in \Lambda(w)$ can be approximated by a function g. The approximation will not be in the sense that $\|f - g\|_{\Lambda(w)}$ is small. As the above example shows, this does not imply that g is "close" to f. Instead we will ensure that g approximates f in measure and at the same time that $\|g\|_{\Lambda(w)}$ approximates $\|f\|_{\Lambda(w)}$. Observe that these results are similar to those obtained for functions of bounded variation (see [49, p. 225]). There is no additional complication of the proofs resulting from the replacement of Λ^σ by $\Lambda(w)$.

By a simple function we mean a real-valued, measurable and everywhere finite function f on $\mathbb{R}$ which takes only finitely many values and which has the property that for every $c \neq 0$, the level set $\{x \in \mathbb{R} : f(x) = c\}$ has finite measure. It is well known that bounded measurable functions can be uniformly approximated by simple functions. We will use this property in the following form.

Lemma 4.7. *Let $f \in S_0(\mathbb{R})$. Suppose that $|f(x)| \le M$ for all $x \in \mathbb{R}$, and*

$$|\{x : f(x) \neq 0\}| < \infty. \tag{4.9}$$

Then for every $\varepsilon > 0$ there exists a simple function g such that:

(i) $|g(x)| \le M$, *for all* $x \in \mathbb{R}$;
(ii) $\{x : |g(x)| = M\} = \{x : |f(x)| = M\}$;
(iii) $\{x : g(x) \neq 0\} = \{x : f(x) \neq 0\}$;
(iv) $|f(x) - g(x)| < \varepsilon$, *for all* $x \in \mathbb{R}$.

Proof. Fix $\varepsilon > 0$. We can assume that $M/\varepsilon \in \mathbb{N}$. Set $g(x) = f(x)$ if $f(x) = 0$ or $|f(x)| = M$. Let $E = \{x : 0 < |f(x)| < M\}$ and set

$$g(x) = \Big(\Big[\frac{f(x)}{\varepsilon}\Big] + \frac{1}{2}\Big)\varepsilon,$$

for all $x \in E$ (here $[a]$ denotes the integral part of a number a). Then for all $x \in E$

$$f(x) - \frac{\varepsilon}{2} < g(x) \le f(x) + \frac{\varepsilon}{2}.$$

This implies statement (iv). Furthermore, $-M < f(x) < M$ on E and therefore

$$-\frac{M}{\varepsilon} \le \Big[\frac{f(x)}{\varepsilon}\Big] \le \frac{M}{\varepsilon} - 1,$$

for all $x \in E$. It follows that

$$-M + \frac{\varepsilon}{2} \le g(x) \le M - \frac{\varepsilon}{2}$$

on E. Thus $|g(x)| < M$ on E, and statements (i) and (ii) hold. We also have that $g(x) \neq 0$ on E, which implies statement (iii). Finally, g satisfies our definition of a simple function. Indeed, clearly g is measurable and everywhere finite. Moreover, by (iii) and (4.9),

$$|\{x : g(x) = c\}| \le |\{x : f(x) \neq 0\}| < \infty,$$

for all $c \neq 0$. $\square$

Our first main result in this section reads:

Theorem 4.8. *Let $f \in \Lambda(w)$. For every $\varepsilon > 0$ there exists a simple function g on $\mathbb{R}$ which satisfies:*

(i) $|\{x \in \mathbb{R} : |f(x) - g(x)| > \varepsilon\}| < \varepsilon$;
(ii) $\big|\|f\|_{\Lambda(w)} - \|g\|_{\Lambda(w)}\big| < \varepsilon$.

Proof. We can assume that $\|f\|_\infty > 0$. Then we have $\|f\|_{\Lambda(w)} > 0$. Fix $0 < \varepsilon < \min(\|f\|_{\Lambda(w)}, \|f\|_\infty)$. We will construct a function f_1 that approximates f and which has certain good properties that allow us to approximate it with a simple function g. To construct f_1 we first define the function f_0 as follows. Take $t_* > 0$ such that

$$|w(t_*)\Delta_f(t_*) - \|f\|_{\Lambda(w)}| < \frac{\varepsilon}{4}. \tag{4.10}$$

Take $t_0 \in (0, \min(t_*, \varepsilon/2))$ and define f_0 as

$$f_0(x) = \begin{cases} f^*(t_0), & f(x) > f^*(t_0) \\ f(x), & -f^*(t_0) \le f(x) \le f^*(t_0) \\ -f^*(t_0), & f(x) < -f^*(t_0). \end{cases}$$

By (2.14), there exists $t_1 > 2t_*$ such that $\lambda \equiv f_0^*(t_1) < \min(\varepsilon/2, f^*(t_0))$. Define f_1 as

$$f_1(x) = \begin{cases} f_0(x) - \lambda, & f_0(x) > \lambda \\ 0, & -\lambda \le f_0(x) \le \lambda \\ f_0(x) + \lambda, & f_0(x) < -\lambda. \end{cases}$$

We will show that f_1 approximates f. If $f(x) = f_0(x)$ then $|f(x) - f_1(x)| = |f_0(x) - f_1(x)| \le \lambda < \varepsilon/2$, so

$$|\{x : |f(x) - f_1(x)| > \frac{\varepsilon}{2}\}| \le |\{x : f(x) \ne f_0(x)\}| =$$

$$= |\{x : |f(x)| > f^*(t_0)\}| \le t_0 \le \frac{\varepsilon}{2}, \tag{4.11}$$

where the second inequality holds by (2.10). By considering the three cases $t \in (0, t_0/2]$, $t \in (t_0/2, t_0]$, and $t \in (t_0, \infty)$ one can verify that $\Delta_{f_0}(t) \le \Delta_f(t)$, for all $t > 0$. Moreover, by considering the three cases $t \in (0, t_1/2]$, $t \in (t_1/2, t_1]$, and $t \in (t_1, \infty)$ one can also verify that $\Delta_{f_1}(t) \le \Delta_{f_0}(t)$ for all $t > 0$. Thus

$$\|f_1\|_{\Lambda(w)} \le \|f\|_{\Lambda(w)}. \tag{4.12}$$

Observe that $f_0^*(t) = \min(f^*(t), f^*(t_0))$. Since $t_0 \le t_*$ we then have

$$f_0^*(t_*) = f^*(t_*) \text{ and } f_0^*(2t_*) = f^*(2t_*). \tag{4.13}$$

We also note that $f_1^*(t) = \max(0, f_0^*(t) - \lambda)$. Since $t_1 \ge 2t_*$ we have $f_0^*(2t_*) \ge f_0^*(t_1) = \lambda$ and then

$$f_1^*(t_*) = f_0(t_*) - \lambda \text{ and } f_1^*(2t_*) = f_0(2t_*) - \lambda.$$

By these two equalities and (4.13) we see that

$$\Delta_{f_1}(t_*) = \Delta_f(t_*). \tag{4.14}$$

By (4.14) and (4.10) we obtain

$$\|f\|_{\Lambda(w)} \le \frac{\varepsilon}{4} + \|f_1\|_{\Lambda(w)}. \tag{4.15}$$

It remains only to approximate f_1 by a simple function. First we observe that $\|f_1\|_\infty < \infty$. Moreover,

$$m \equiv |\{x : f_1(x) = \|f_1\|_\infty\}| > 0. \tag{4.16}$$

Indeed, since $\lambda < f^*(t_0)$, we see that

$$\{x : |f_1(x)| = \|f_1\|_\infty\} = \{x : |f_0(x)| = f^*(t_0)\} = \{x : |f(x)| \ge f^*(t_0)\}.$$

So (4.16) holds by (2.11). We also note that by (2.10),

$$M \equiv |\{x : f_1(x) \ne 0\}| = |\{x : |f_0(x)| > \lambda\}| \le t_1 < \infty. \tag{4.17}$$

Fix $\varepsilon_1 \in (0, \varepsilon/2)$ such that for all $t \in [m/2, M]$,

$$8\varepsilon_1 w(t) < \varepsilon. \tag{4.18}$$

By Lemma 4.7 there exists a simple function g on $\mathbb{R}$ such that

$$|f_1(x) - g(x)| \le \varepsilon_1 \tag{4.19}$$

for all $x \in \mathbb{R}$,

$$\{x : |g(x)| = \|g\|_\infty\} = \{x : |f_1(x)| = \|f_1\|_\infty\} \tag{4.20}$$

and

$$\{x : g(x) \ne 0\} = \{x : f_1(x) \ne 0\}. \tag{4.21}$$

By the triangle inequality

$$|\{x : |f(x) - g(x)| > \varepsilon\}| \le$$

$$\le |\{x : |f(x) - f_1(x)| > \frac{\varepsilon}{2}\}| + |\{x : |f_1(x) - g(x)| > \frac{\varepsilon}{2}\}| \le \frac{\varepsilon}{2},$$

where the last inequality holds by (4.11) and (4.19). Thus, statement (i) is true. According to (4.19) it holds that

$$g(x) - \varepsilon_1 \le f_1(x) \le g(x) + \varepsilon_1,$$

for all $x \in \mathbb{R}$. It follows that

$$g^*(t) - \varepsilon_1 \le f_1^*(t) \le g^*(t) + \varepsilon_1,$$

which in turn implies that

$$|\Delta_{f_1}(t) - \Delta_g(t)| \le 2\varepsilon_1, \tag{4.22}$$

for all $t > 0$.

By (4.12), (4.10), and (4.14) we have

$$\|f_1\|_{\Lambda(w)} \le \|f\|_{\Lambda(w)} \le \frac{\varepsilon}{4} + w(t_*)\Delta_{f_1}(t_*).$$

Applying (4.22) gives

$$\|f_1\|_{\Lambda(w)} \le \frac{\varepsilon}{4} + 2\varepsilon_1 w(t_*) + w(t_*)\Delta_g(t_*) \le \frac{\varepsilon}{4} + 2\varepsilon_1 w(t_*) + \|g\|_{\Lambda(w)}. \quad (4.23)$$

We want to apply (4.18) to estimate $2\varepsilon_1 w(t_*)$, so we must check that $t* \in [m/2, M]$. It is clear that $\Delta_f(t_*) > 0$, indeed if $\Delta_f(t_*) = 0$ then by (4.10) we would have $\|f\|_{\Lambda(w)} < \varepsilon/4$ which contradicts our choice of ε. So by (4.14) we know that $\Delta_{f_1}(t_*) > 0$. However, by (4.16) and (4.17) it holds that $\Delta_{f_1}(t) = 0$ for all $t \in (0, m/2) \cup (M, \infty)$. Thus we conclude that $t_* \in [m/2, M]$, so (4.18) holds for $t = t_*$, i.e. we have

$$8\varepsilon_1 w(t_*) \le \varepsilon$$

This inequality and (4.23) gives

$$\|f_1\|_{\Lambda(w)} \le \frac{\varepsilon}{2} + \|g\|_{\Lambda(w)}.$$

By (4.20) and (4.16) we have $\Delta_g = 0$ on $(0, m/2)$, and by (4.21) and (4.17) we know that $\Delta_g = 0$ on (M, ∞). Therefore

$$\|g\|_{\Lambda(w)} = \sup\{w(t)\Delta_g(t) : \frac{m}{2} \le t \le M\} \le \frac{\varepsilon}{4} + \|f_1\|_{\Lambda(w)},$$

where the inequality holds by (4.22) and (4.18). By (4.12), (4.15), and the two preceding inequalities, we obtain (ii). □

We let $C_0(\mathbb{R})$ denote the class of all continuous functions on $\mathbb{R}$ with compact support.

Lemma 4.9. *Let f be a simple function on $\mathbb{R}$. For every $\delta > 0$ there exists a function $g \in C_0(\mathbb{R})$ such that:*

(i) $|\{x \in \mathbb{R} : f(x) \ne g(x)\}| < \delta$;
(ii) $\|g\|_\infty = \|f\|_\infty$.

Proof. Since f is a simple function, we know that $\|f\|_\infty < \infty$ and

$$M \equiv |\{x \in \mathbb{R} : f(x) \ne 0\}| < \infty. \quad (4.24)$$

We can assume that f is not equivalent to 0, and then

$$m \equiv |\{x \in \mathbb{R} : |f(x)| = \|f\|_\infty\}| > 0. \quad (4.25)$$

Fix $\delta \in (0, m)$. By (4.24) there exists $N > 0$ such that

$$|\{x \in \mathbb{R} : f(x) \ne 0,\ |x| > N\}| < \frac{\delta}{4}. \quad (4.26)$$

Simple functions are finite and measurable. Lusin's theorem then ensures the existence of a closed set $F \subset [-N, N]$ such that f is continuous relative to F and

$$|[-N, N] \setminus F| < \frac{\delta}{4}. \quad (4.27)$$

So, by the extension theorem there exists a function $g \in C_0(\mathbb{R})$ such that

$$g(x) = f(x), \tag{4.28}$$

for all $x \in F$,

$$g(x) = 0, \tag{4.29}$$

if $|x| > N + \delta/4$, and

$$\|g\|_\infty \leq \|f\|_\infty. \tag{4.30}$$

By (4.28) and (4.29) we have the inclusion

$$\{x \in \mathbb{R} : f(x) \neq g(x)\} \subset$$

$$([-N, N] \setminus F) \cup [-N - \frac{\delta}{4}, -N] \cup [N, N + \frac{\delta}{4}] \cup \{x \in \mathbb{R} : f(x) \neq 0,\ |x| > N\}.$$

By this inclusion and inequalities (4.27) and (4.26) we obtain statement (i). Since $\delta < m$, statement (i) and (4.25) implies that g attains the value $\|f\|_\infty$ on some set of positive measure. Thus, $\|f\|_\infty \leq \|g\|_\infty$ which together with (4.30) give statement (ii). $\square$

Our next theorem reads:

Theorem 4.10. *Let $f \in \Lambda(w)$. For every $\varepsilon > 0$ there exists a function $g \in C_0(\mathbb{R})$ such that:*

(i) $|\{x \in \mathbb{R} : |f(x) - g(x)| > \varepsilon\}| < \varepsilon$;
(ii) $\big|\|f\|_{\Lambda(w)} - \|g\|_{\Lambda(w)}\big| < \varepsilon$.

Proof. Fix $\varepsilon > 0$. By Theorem 4.8 we can assume that f is a simple function. Let $c_1 > \cdots > c_N > 0$ be the positive values of $|f|$. We may assume that $c_1 = 1$. For $k = 1, \ldots, N$ we put

$$E_k = \{x \in \mathbb{R} : |f(x)| = c_k\}.$$

We can assume that $|E_k| > 0$ for all $k = 1, \ldots, N$. Indeed, if there is some k for which $|E_k| = 0$ then we replace the value of f by 0 on E_k. This does not change the value of f^* at any point so $\|f\|_{\Lambda(w)}$ remains the same. Put $t_0 = 0$ and

$$t_k = \sum_{i=1}^{k} |E_i|,$$

for all $k = 1, \ldots, N$. Then $0 < t_1 < t_2 < \cdots < t_N$. Choose $\delta_1 \in (0, \varepsilon)$ such that

$$8\delta_1 < \min\{t_k - t_{k-1} : 1 \leq k \leq N\} \tag{4.31}$$

and the condition

$$4|w(t') - w(t'')| < \varepsilon, \tag{4.32}$$

holds for all $t', t'' \in [t_1/8, 2t_N]$ such that $|t' - t''| < \delta_1$ (this is possible since w is uniformly continuous on $[t_1/8, 2t_N]$).

First we will show that we can assume that $2t_k \neq t_l$ for all $1 \leq k < l \leq N$. We prove this by constructing a simple function h which has this property and which approximates f. Define

$$\eta' \equiv \frac{1}{2} \min\{|2t_k - t_l| : 1 \leq k, l \leq N,\ 2t_k \neq t_l\},$$

and set $\eta \equiv \min(\delta_1, \eta')$. Choose in E_1 any measurable subset of measure η and replace the value of f by 0 on this subset. Denote the new function by h. We then have

$$h^*(t) = f^*(t + \eta), \tag{4.33}$$

for all $t > 0$. Let $t'_0 \equiv 0$ and $t'_k \equiv t_k - \eta$, $k = 1, \ldots, N$. The intervals of constancy of h^* are $(t'_{k-1}, t'_k]$, $k = 1, \ldots, N$. Furthermore, for all $1 \leq k, l \leq N$ the numbers t'_k and t'_l satisfy

$$|2t'_k - t'_l| \geq \eta. \tag{4.34}$$

Indeed, fix $1 \leq k, l \leq N$. By the definition of t'_k and t'_l we have

$$2t'_k - t'_l = 2t_k - t_l - \eta, \tag{4.35}$$

so if $2t_k = t_l$ then (4.34) holds. On the other hand, if $2t_k \neq t_l$ then by the definition of η

$$0 < \eta \leq \frac{1}{2}|2t_k - t_l|.$$

From this and (4.35) we get (4.34).

Next we will show that

$$\|f\|_{\Lambda(w)} - \varepsilon \leq \|h\|_{\Lambda(w)} \leq \|f\|_{\Lambda(w)} + \varepsilon. \tag{4.36}$$

We start with the proof of the right-hand side inequality of (4.36). Fix $t \in [t_1/4, t_N]$. By (4.33) it holds that

$$\Delta_h(t) = f^*(t + \eta) - f^*(2t + \eta) \leq \Delta_f(t + \eta).$$

From this and the fact that $\Delta_h = 0$ on $(0, t_1/4) \cup (t_N - \eta, \infty)$, we see that

$$\|h\|_{\Lambda(w)} \leq \sup\{w(s)\Delta_f(s + \eta) : \frac{t_1}{4} \leq s \leq t_N - \eta\}. \tag{4.37}$$

Since $\eta \leq \delta_1$, we know from (4.32) that

$$w(s)\Delta_f(s + \eta) \leq$$

$$\leq w(s + \eta)\Delta_f(s + \eta) + \frac{\varepsilon}{4}\Delta_f(s + \eta) \leq \|f\|_{\Lambda(w)} + \frac{\varepsilon}{4}\|f\|_\infty,$$

for all $s \in [t_1/4, t_N - \eta]$. By this, (4.37), and the assumption that $\|f\|_\infty = 1$ we now obtain the right-hand side inequality in (4.36).

To obtain the left-hand side inequality in (4.36) we will first show that

$$\Delta_f(t) \leq \max(\Delta_h(t - \eta), \Delta_h(t - \eta/2)), \tag{4.38}$$

for all $t \in [t_1/4, t_N]$. To prove this estimate we will consider the three cases $t \in [t_1/4, t_N/2]$, $t \in (t_N/2, t_N/2 + \eta/2]$, and $t \in (t_N/2 + \eta/2, t_N]$. Suppose first that $t \in (t_N/2 + \eta/2, t_N]$. Then $f^*(2t) = 0$, and using (4.33) we see that also $h^*(2t - 2\eta) = 0$. Thus

$$\Delta_f(t) = f^*(t) = h^*(t - \eta) = \Delta_h(t - \eta),$$

where the second equality is (4.33). So, (4.38) holds in this case. Next we suppose that $t \in (t_N/2, t_N/2 + \eta/2]$. Take $k \in \{1, \dots, N\}$ such that $t \in (t_{k-1}, t_k]$. Then $t \in (t_{k-1}, t_k - \eta/2]$. Indeed, if $t \in (t_k - \eta/2, t_k]$ then

$$2t \in (t_N, t_N + \eta] \cap (2t_k - \eta, 2t_k]. \tag{4.39}$$

So we would have $|2t_k - t_N| < 2\eta$, but this contradicts the definition of η (to see this, note that $2t_k \neq t_N$ by (4.39) so by definition $\eta \leq |2t_k - t_N|/2$). Thus, $t \in (t_{k-1}, t_k - \eta/2]$ and then $f^*(t) = f^*(t+\eta/2)$. From this and (4.33)

$$\Delta_f(t) = f^*(t + \eta/2) - f^*(2t) = h^*(t - \eta/2) - h^*(2t - \eta) = \Delta_h(t - \eta/2),$$

and thus (4.38) holds also in this case. The last case in the proof of (4.38) is when $t \in [t_1/4, t_N/2]$. Then there exist $k, l \in \{1, 2, \dots, N\}$ such that $t \in (t_{k-1}, t_k]$ and $2t \in (t_{l-1}, t_l]$. We then have either

$$t \in (t_{k-1}, t_k - \frac{\eta}{2}], \tag{4.40}$$

or

$$2t \in (t_{l-1} + \eta, t_l]. \tag{4.41}$$

Indeed, suppose neither (4.40) nor (4.41) holds. Then we have

$$2t \in (2t_k - \eta, 2t_k] \cap (t_{l-1}, t_{l-1} + \eta]. \tag{4.42}$$

Therefore,

$$|2t_k - t_{l-1}| < 2\eta, \tag{4.43}$$

which contradicts the definition of η (to see this, observe that $2t_k \neq t_{l-1}$ by (4.42), so by definition $\eta \leq |2t_k - t_{l-1}|/2$). If (4.40) holds, then $f^*(t) = f^*(t + \eta/2)$. Using (4.33) then gives

$$\Delta_f(t) = \Delta_h(t - \frac{\eta}{2}), \tag{4.44}$$

so (4.38) is satisfied. In the case (4.41), we have $f^*(2t) = f^*(2t - \eta)$. Applying again (4.33), we obtain

$$\Delta_f(t) = \Delta_h(t - \eta), \tag{4.45}$$

and thus (4.38) holds. The proof of (4.38) is now complete.

Since $\Delta_f = 0$ on $(0, t_1/4) \cup (t_N, \infty)$, we have

$$\|f\|_{\Lambda(w)} = \sup\{w(t)\Delta_f(t) : \frac{t_1}{4} \leq t \leq t_N\}.$$

Applying (4.38), we get

$$\|f\|_{\Lambda(w)} \le \sup\left\{w(t)\max\left(\Delta_h(t-\eta/2),\Delta_h(t-\eta)\right) : \frac{t_1}{4} \le t \le t_N\right\}. \quad (4.46)$$

But according to (4.32),

$$w(t)\Delta_h(t-\eta) \le \frac{\varepsilon}{4}\|h\|_\infty + w(t-\eta)\Delta_h(t-\eta) \le \frac{\varepsilon}{4} + \|h\|_{\Lambda(w)},$$

and similarly, $w(t)\Delta_h(t-\eta/2) \le \varepsilon/4 + \|h\|_{\Lambda(w)}$, for all $t \in [t_1/4, t_N]$, so (4.46) implies the left hand side of inequality (4.36). The proof of (4.36) is then complete. We have now proved that we can assume that

$$2t_k \neq t_l,$$

for all $1 \le k < l \le N$.

We now choose $\delta \in (0,\delta_1)$ such that

$$8\delta < \min\{|2t_k - t_l| : 1 \le k \le l \le N\}. \quad (4.47)$$

Since f is a simple function on $\mathbb{R}$, Lemma 4.9 ensures the existence of a function $g \in C_0(\mathbb{R})$ such that

$$\|g\|_\infty = \|f\|_\infty \quad (4.48)$$

and

$$|\{x \in \mathbb{R} : f(x) \neq g(x)\}| < \delta. \quad (4.49)$$

By this inequality we have statement (i) and the equality

$$(f-g)^*(\delta) = 0. \quad (4.50)$$

It only remains to check that also statement (ii) holds. First we will verify that

$$\|f\|_{\Lambda(w)} \le \varepsilon + \|g\|_{\Lambda(w)}. \quad (4.51)$$

From (4.50), and the subadditivity (2.13) of the rearrangement, we get that

$$f^*(t) \le g^*(t-\delta) \text{ and } g^*(2t-2\delta) \le f^*(2t-3\delta),$$

for all $t > 3\delta/2$. Set

$$\Psi(t) = f^*(t) - f^*(2t-3\delta), \quad t > 3\delta/2.$$

By the two preceding inequalities and (4.32) we obtain

$$w(t)\Psi(t) \le w(t)\Delta_g(t-\delta) \le \frac{\varepsilon}{4}\|g\|_\infty + \|g\|_{\Lambda(w)} = \frac{\varepsilon}{4} + \|g\|_{\Lambda(w)}, \quad (4.52)$$

for all $t > 3\delta/2$ (we use here that $\|g\|_\infty = 1$ by (4.48)). To obtain (4.51) we only need to show that

$$w(t)\Delta_f(t) \le \varepsilon + \|g\|_{\Lambda(w)}, \quad (4.53)$$

for all $t \in [t_1/4, t_N]$, since $\Delta_f = 0$ outside this interval. To prove (4.53) we will consider the three cases $t \in [t_1/4, t_N/2]$, $t \in [t_N/2, t_N/2 + 3\delta/2]$, and

$t \in [t_N/2 + 3\delta/2, t_N]$. Suppose first that $t \in [t_1/4, t_N/2]$. In this case there exists $k, l \in \{1, \dots, N\}$ such that $t \in (t_{k-1}, t_k]$ and $2t \in (t_{l-1}, t_l]$. By choice of δ we have that either

$$t \in (t_{k-1}, t_k - 2\delta] \tag{4.54}$$

or

$$2t \in (t_{l-1} + 3\delta, t_l]. \tag{4.55}$$

Indeed, if neither (4.54) nor (4.55) holds then

$$2t \in (2t_k - 4\delta, 2t_k] \cap (t_{l-1}, t_{l-1} + 3\delta]$$

and then we would have

$$|2t_k - t_{l-1}| \leq |2t_k - 2t| + |2t - t_{l-1}| < 7\delta,$$

which contradicts the definition of δ. In the case of (4.54), we have

$$\Delta_f(t) = f^*(t + 2\delta) - f^*(2t) \leq f^*(t + 2\delta) - f^*(2t + \delta) = \Psi(t + 2\delta).$$

By (4.32) and (4.52) we then get

$$w(t)\Delta_f(t) \leq \Big(w(t + 2\delta) + \frac{\varepsilon}{4}\Big)\Psi(t + 2\delta) \leq \frac{\varepsilon}{4}\Psi(t + 2\delta) + \frac{\varepsilon}{4} + \|g\|_{\Lambda(w)}.$$

Since Ψ is bounded by $\|f\|_\infty = 1$, inequality (4.53) follows in this case. If instead (4.55) holds, then $f^*(2t) = f^*(2t - 3\delta)$ so

$$\Delta_f(t) = \Psi(t).$$

In this case we immediately get inequality (4.53) from (4.52). Thus (4.53) holds when $t \in [t_1/4, t_N/2]$. Next we suppose that $t \in (t_N/2, t_N/2 + 3\delta/2]$. Then there exists $k \in \{1, \dots, N\}$ such that $t \in (t_{k-1}, t_k]$. As above, the definition of δ implies that $t \in (t_{k-1}, t_k - 2\delta]$. As in the case (4.54) above, we obtain (4.53). The last case is when $t \in (t_N/2 + 3\delta/2, t_N]$. Then $f^*(2t) = f^*(2t - 3\delta) = 0$, so again

$$\Delta_f(t) = \Psi(t),$$

and then (4.53) follows directly from (4.52). We have now proved inequality (4.53) for all $t \in [t_1/4, t_N]$. This implies (4.51).

To obtain statement (ii) we must also show that

$$\|g\|_{\Lambda(w)} \leq \varepsilon + \|f\|_{\Lambda(w)}. \tag{4.56}$$

Since $\delta < t_1/8$, by (4.49) and (4.48) we see that $\Delta_g = 0$ outside the interval $[t_1/2 - \delta/2, t_N + \delta]$, so (4.56) follows if we prove

$$w(t)\Delta_g(t) \leq \varepsilon + \|f\|_{\Lambda(w)}, \tag{4.57}$$

for all $t \in [t_1/2 - \delta/2, t_N + \delta]$.

Fix $t \in [t_1/2 - \delta/2, t_N + \delta]$ and prove (4.57). By (4.50) and the subadditivity (2.13) of the rearrangement we have

$$g^*(t) \leq f^*(t - \delta) \text{ and } g^*(2t) \geq f^*(2t + \delta),$$

Set

$$\Phi(t) = f^*(t) - f^*(2t + 3\delta).$$

By the two preceding inequalities

$$\Delta_g(t) \leq f^*(t - \delta) - f^*(2t + \delta) = \Phi(t - \delta).$$

By (4.32) it holds that $w(t) \leq \varepsilon/4 + w(t - \delta)$, so the above estimate gives

$$w(t)\Delta_g(t) \leq \frac{\varepsilon}{4} + w(t - \delta)\Phi(t - \delta)$$

(we use here that $\Phi \leq \|f\|_\infty = 1$). Thus, for all $t \in [t_1/2 - \delta/2, t_N + \delta]$ it holds that

$$w(t)\Delta_g(t) \leq \frac{\varepsilon}{4} + \sup\{w(s)\Phi(s) : s \in [t_1/4, t_N]\}. \tag{4.58}$$

So, (4.57) follows from (4.58) if we prove that

$$w(s)\Phi(s) \leq \frac{\varepsilon}{2} + \|f\|_{\Lambda(w)}, \tag{4.59}$$

for all $s \in [t_1/4, t_N]$. Fix $s \in [t_1/4, t_N]$ and prove (4.59). Suppose first that $s \in (t_N/2, t_N]$. Then $\Phi(s) = \Delta_f(s)$ and so (4.59) follows immediately. Next we suppose that $s \in [t_1/4, t_N/2]$. Take $k, l \in \{1, \ldots, N\}$ such that $s \in (t_{k-1}, t_k]$ and $2s \in (t_{l-1}, t_l]$. As above, the definition of δ gives that either

$$s \in (t_{k-1}, t_k - \frac{3\delta}{2}], \tag{4.60}$$

or

$$2s \in (t_{l-1}, t_l - 3\delta]. \tag{4.61}$$

Suppose that (4.60) is true. Then $f^*(s) = f^*(s + 3\delta/2)$, so

$$\Phi(s) = f^*(s + \frac{3\delta}{2}) - f^*(2s + 3\delta) = \Delta_f(s + \frac{3\delta}{2}).$$

By (4.32) we then have

$$w(s)\Phi(s) = \frac{\varepsilon}{4} + w(s + \frac{3\delta}{2})\Delta_f(s + \frac{3\delta}{2})$$

which implies (4.59). In the case (4.61) we have $f^*(2s) = f^*(2s+3\delta)$ so that we again obtain $\Phi(s) = \Delta_f(s)$. So in this case (4.59) follows immediately. The proof of (4.59) is now complete. As we noted above, (4.59) together with (4.58) implies (4.56). Thus, statement (ii) holds. $\square$

Remark 4.11. Theorems 4.8 and 4.10 fail if one replaces statement (ii) in their formulations by the statement $\|f - g\|_{\Lambda(w)} < \varepsilon$. Indeed, let $w(t) = t^\sigma$ with $\sigma > 0$ and set

$$f(x) = \begin{cases} x^{-\sigma} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then $\|f\|_{\Lambda^\sigma} = 1 - 2^{-\sigma}$, so $f \in \Lambda^\sigma(\mathbb{R})$. Let $g \in L^\infty(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$ and set $M = \|g\|_\infty$. Then $f(x) - g(x) \geq x^{-\sigma} - M > 0$ for $0 < x < M^{-1/\sigma}$. Thus $(f-g)^*(t) \geq t^{-\sigma} - M$ for $0 < t < M^{-1/\sigma}$. By Proposition 4.1 we then have

$$\|f - g\|_{\Lambda^\sigma} \geq \frac{\sigma \ln 2}{2^\sigma}.$$

So there is no function $g \in L^\infty(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$ such that $\|f - g\|_{\Lambda^\sigma} < (\sigma \ln 2)/2^\sigma$.

Applying Theorem 4.10 we obtain the following result.

Theorem 4.12. *Let $f \in \Lambda(w)$. Then there exists a sequence $\{f_n\}$, $f_n \in C_0(\mathbb{R})$, such that $\{f_n\}$ converges to f in measure and $\|f_n\|_{\Lambda(w)} \to \|f\|_{\Lambda(w)}$.*

Observe that by Riesz's theorem there exists a subsequence $\{f_{n_k}\}$ converging to f a.e.

As it was pointed out above, there is an analogy between Theorems 4.8 and 4.10 and the results concerning the so called approximation in variation [49], [24, Section 9.1].

5. Mixed norm spaces

The main result in this section is a theorem on embedding of anisotropic mixed norm spaces into Lorentz spaces. As it was already pointed out in the introduction, the first results in this direction where obtained by Gagliardo [17] and Fournier [16] (see also [8]). These results were extended by Kolyada [29] to more general mixed norm spaces. Our main theorem is a follow-up of the work [29]. We consider fully anisotropic mixed norm spaces. Our study is based on the methods developed in the works by Kolyada [29] and Kolyada and Pérez [32].

In Section 5.1 we give the lemmas that we will use, and in Section 5.2 we state and prove Theorem 5.4.

5.1. **Some lemmas.**

The following lemma was proved in [26].

Lemma 5.1. *Let ψ be a measurable non-negative function on $\mathbb{R}^n$ and let $P \subset \mathbb{R}^n$ be a measurable set with $\mathrm{mes}_n P = \mu > 0$. Then for any $0 < \tau < \mu$ the set P can be decomposed into measurable disjoint subsets E' and E'' such that $\mathrm{mes}_n E' = \tau$,*

$$\sup_{x \in E''} \psi(x) \leq \inf_{x \in E'} \psi(x),$$

and

$$\int_{E''} \psi(x) dx \leq \int_\tau^\mu \psi^*(t) dt.$$

The next lemma was proved in [32]. It it is similar to a regularization lemma proved using other methods in [41, Lemma 3.3]. We include the proof in order to get an explicit value of the constant in statement (iii) in this lemma.

Lemma 5.2. *Let $\phi \in L^{p,s}(\mathbb{R}_+)$ $(1 \leq p, s < \infty)$ be a non-negative non-increasing function on $\mathbb{R}_+$. Then for any $\delta \in (0, 1/p)$ there exists a continuously differentiable function ψ on $\mathbb{R}_+$ such that:*

(i) *$\phi(t) \leq \psi(t)$, $t \in \mathbb{R}_+$;*
(ii) *$\psi(t) t^{1/p-\delta}$ decreases and $\psi(t) t^{1/p+\delta}$ increases on $\mathbb{R}_+$;*
(iii) *$\|\psi\|_{p,s} \leq \dfrac{8}{\delta^2} \|\phi\|_{p,s}$.*

Proof. Define

$$\phi_1(t) \equiv 2t^{\delta-1/p} \int_{t/2}^\infty u^{1/p-\delta-1} \phi(u) du,$$

for $t > 0$. Since $\phi \in L^{p,s}(\mathbb{R}_+)$ and $\phi^* = \phi$, we have by (2.33) that

$$\phi(u) = O(u^{-1/p}),$$

as $u \to \infty$. Therefore the integral in the definition of ϕ_1 converges, so ϕ_1 is well defined. Moreover, since ϕ is non-increasing on $\mathbb{R}_+$ it is easy to see that

$$\phi_1(t) \geq 2t^{\delta-1/p}\phi(t) \int_{t/2}^{t} u^{1/p-\delta-1}du \geq \phi(t), \tag{5.1}$$

for all $t > 0$. Since $\delta < 1/p$, then ϕ_1 is decreasing on $\mathbb{R}_+$ and thus $\phi_1^* = \phi_1$. By this observation and Hardy's inequality (2.1), we have

$$\|\phi_1\|_{p,s} = 2^{1+\delta}\Big(\int_0^\infty t^{\delta s-1}\Big(\int_t^\infty u^{1/p-\delta-1}\phi(u)du\Big)^s dt\Big)^{1/s} \leq \frac{4}{\delta}\|\phi\|_{p,s}. \tag{5.2}$$

Thus, $\phi_1 \in L^{p,s}(\mathbb{R}_+)$, so by (2.33) we obtain that

$$\phi_1(u) = O(u^{-1/p}),$$

as $u \to 0+$ (here we again use that $\phi_1^* = \phi_1$). Therefore the function

$$\psi(t) \equiv \Big(\delta + \frac{1}{p}\Big)t^{-1/p-\delta}\int_0^t \phi_1(u)u^{1/p+\delta-1}du$$

is well defined on $\mathbb{R}_+$, since the integral converges. The function ψ is continuously differentiable on $\mathbb{R}_+$ since ϕ_1 is continuous on $\mathbb{R}_+$. Since ϕ_1 decreases on $\mathbb{R}_+$ it holds that

$$\psi(t) \geq \Big(\delta + \frac{1}{p}\Big)t^{-1/p-\delta}\phi_1(t)\int_0^t u^{1/p+\delta-1}du = \phi_1(t).$$

This estimate and (5.1) gives statement (i).

Clearly, $\psi(t)t^{1/p+\delta}$ increases on $\mathbb{R}_+$. To obtain statement (ii) we must also show that $\psi(t)t^{1/p-\delta}$ decreases on $\mathbb{R}_+$. We make the change of variables $u \mapsto u^{2\delta}$ to see that

$$\psi(t)t^{1/p-\delta} = \frac{\delta p+1}{2\delta p}t^{-2\delta}\int_0^{t^{2\delta}} \eta(v^{1/(2\delta)})dv$$

for all $t > 0$, where $\eta(u) \equiv u^{1/p-\delta}\phi_1(u)$. Differentiating with respect to t in the preceding equality gives

$$\frac{d}{dt}(\psi(t)t^{1/p-\delta}) = \frac{1+\delta p}{p}\,t^{-1}\Big(\eta(t) - t^{-2\delta}\int_0^{t^{2\delta}} \eta(v^{1/(2\delta)})dv\Big),$$

for all $t > 0$. Clearly η is non-increasing on $\mathbb{R}_+$, so by the preceding equality

$$\frac{d}{dt}(\psi(t)t^{1/p-\delta}) \leq 0,$$

and thus the function $\psi(t)t^{1/p-\delta}$ is non-increasing on $\mathbb{R}_+$. So, statement (ii) holds.

The function ψ is decreasing on $\mathbb{R}_+$ since $\psi(t)t^{1/p-\delta}$ is non-increasing and $\delta < 1/p$. Therefore $\psi^* = \psi$. By this observation and Hardy's inequality (2.2) we have

$$\|\psi\|_{p,s} = \Big(\delta + \frac{1}{p}\Big)\Big(\int_0^\infty t^{-\delta s-1}\Big(\int_0^t u^{1/p+\delta-1}\phi_1(u)du\Big)^s dt\Big)^{1/s} \leq$$

$$\leq \Big(1 + \frac{1}{\delta p}\Big)\|\phi_1\|_{p,s} \leq \frac{2}{\delta}\|\phi_1\|_{p,s}$$

(here we again use that $\phi_1^* = \phi_1$ and that $\delta < 1/p$). From this inequality and (5.2) we obtain statement (iii). $\square$

The following lemma is similar to Lemma 2.2 in [32] and the proof is based on the same reasonings.

Lemma 5.3. *Let $n \geq 2$, $1 \leq p_1, \dots, p_n, s_1, \dots, s_n < \infty$, and $\alpha_1, \dots, \alpha_n > 0$. Put*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1}, \ p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}, \ \textit{and}\ s = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k s_k}\Big)^{-1}.$$

Assume that $p \leq n/\alpha$. Let

$$q = \begin{cases} np/(n-\alpha p), & \alpha p < n \\ \infty, & \alpha p = n. \end{cases}$$

For all $k = 1, \dots, n$ we set $\sigma_k = 1/p_k - \alpha_k$ and assume that

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \sigma_k > 0. \tag{5.3}$$

Denote

$$R = \max_{k=1,\dots,n} \frac{r_k}{\alpha_k} \max_{k=1,\dots,n} \frac{1}{r_k} \tag{5.4}$$

and

$$c_k = \frac{\alpha_k}{r_k}, \tag{5.5}$$

$k = 1, \dots, n$. For each $k = 1, \dots, n$ we let $\phi_k \in L^{p_k,s_k}(\mathbb{R}_+)$ be a non-increasing and non-negative function on $\mathbb{R}_+$ and define

$$\eta_k(z,t) = \Big(\frac{z}{t}\Big)^{\sigma_k}\phi_k(z), \quad z, t > 0.$$

Set also

$$w(t) = \inf\{\max_{k=1,\dots,n} \eta_k(z_k,t) : \prod_{k=1}^n z_k = t^{n-1}, z_k > 0\},$$

for $t > 0$. Then there holds the inequality

$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt\Big)^{1/s} \le c \prod_{k=1}^n \|\phi_k\|_{p_k,s_k}^{\alpha/(n\alpha_k)}, \tag{5.6}$$

where

$$c = K_n \prod_{k=1}^n (c_k^{1/s_k} \max(R^2, p_k^2))^{\alpha/(n\alpha_k)}, \tag{5.7}$$

and K_n only depends on n.

Proof. Fix $t > 0$. By (5.3) we see that $R > 0$. Set $\delta = 1/(2R)$. For each k we set $\delta_k = \min(\delta, 1/(2p_k))$ and apply Lemma 5.2 to the function ϕ_k. In this way we obtain continuously differentiable functions ψ_k, $k = 1, \dots, n$, on $\mathbb{R}_+$ such that:

$$\phi_k(z) \le \psi_k(z), \text{ for all } z \in \mathbb{R}_+; \tag{5.8}$$

$$\psi_k(z) z^{1/p_k - \delta} \text{ decreases on } \mathbb{R}_+; \tag{5.9}$$

$$\psi_k(z) z^{1/p_k + \delta} \text{ increases on } \mathbb{R}_+; \tag{5.10}$$

$$\|\psi_k\|_{p_k,s_k} \le \frac{8}{\delta_k^2} \|\phi_k\|_{p_k,s_k} = 32 \max(R^2, p_k^2) \|\phi_k\|_{p_k,s_k}. \tag{5.11}$$

For $z > 0$ we define

$$G_k(z) = z^{\sigma_k} \psi_k(z)$$

and

$$\xi_k(z,t) = t^{-\sigma_k} G_k(z),$$

$k = 1, \dots, n$. Observe that

$$\delta < \alpha_k, \tag{5.12}$$

for all k. Indeed,

$$\delta < \frac{1}{R} \le \frac{\alpha_l}{r_l} \min_{k=1,\dots,n} r_k,$$

for all $l = 1, \dots, n$ which implies (5.12). Write G_k as

$$G_k(z) = \frac{\psi_k(z) z^{1/p_k - \delta}}{z^{\alpha_k - \delta}}. \tag{5.13}$$

It follows from (5.13), (5.9), and (5.12) that

$$\lim_{z \to 0+} G_k(z) = \infty \text{ and } \lim_{z \to \infty} G_k(z) = 0. \tag{5.14}$$

Define

$$\mu_t(z_1, \dots, z_n) = \max_{k=1,\dots,n} \xi_k(z_k, t).$$

Set also

$$v(t) = \inf\{\max_{k=1,\dots,n} \xi_k(z_k, t) : \prod_{k=1}^n z_k = t^{n-1},\ z_k > 0\}.$$

The function μ_t is continuous on $\mathbb{R}_+^n$ and $v(t)$ is the infimum of μ_t over the set

$$E_t = \{(z_1, \dots, z_n) \in \mathbb{R}_+^n : \prod_{k=1}^{n} z_k = t^{n-1}\}.$$

From the definition of E_t we see that by choosing $z = (z_1, \dots, z_n) \in E_t$ so that $|z|$ is sufficiently big, we can make $\min_{k=1,\dots,n} z_k$ arbitrarily small, i.e. there holds the relation

$$\lim_{|z|\to\infty,\ z\in E_t} \big(\min_{k=1,\dots,n} z_k \big) = 0.$$

Furthermore, (5.14) implies that for each $k = 1, \dots, n$

$$\lim_{z_k\to 0+} \xi_k(z_k, t) = \infty.$$

By the two preceding equalities we see that

$$\lim_{|z|\to\infty,\ z\in E_t} \mu_t(z) = \infty,$$

and therefore the infimum in the definition of v need only be taken over some compact subset of E (we use here that E_t is closed in $\mathbb{R}^n$). This infimum is then attained at some point, i.e. there is a point $(u_1^*, \dots, u_n^*) \in E_t$ where

$$\mu_t(u_1^*, \dots, u_n^*) = v(t). \tag{5.15}$$

Differentiate in (5.13) to get

$$G_k'(z) = \frac{d}{dz}(\psi_k(z) z^{1/p_k - \delta}) z^{\delta - \alpha_k} - (\alpha_k - \delta) z^{\sigma_k - 1} \psi_k(z),$$

for all $z > 0$. The first term on the right-hand side of this equality is non-positive by (5.9), so we have

$$G_k'(z) \le -(\alpha_k - \delta) z^{\sigma_k - 1} \psi_k(z). \tag{5.16}$$

We may assume that each of the functions ϕ_k is positive at some point. Indeed, suppose that $\phi_1(z) = 0$ for all $z > 0$. Then $\eta_1(z,t) = 0$, $z > 0$, so for $N > 0$ and $(z_1, \dots, z_n) = ((t/N)^{n-1}, N, \dots, N)$, we have

$$w(t) \le \max_{k=1,\dots,n} \eta_k(z_1, \dots, z_n) = \max_{k=2,\dots,n} \eta_k(N, t).$$

By (5.8) and (5.14), $\eta_k(N,t) \to 0$ as $N \to \infty$. Thus $w(t) = 0$ and (5.6) holds.

Since ϕ_k is non-increasing it follows from (5.8) and (5.10) that $\psi_k(z) > 0$ for all $z \in \mathbb{R}_+$. Using this observation and (5.12) in the estimate (5.16) gives

$$G_k'(z) < 0, \tag{5.17}$$

for all $z > 0$. So by (5.14) and (5.17) each G_k is a bijection of $\mathbb{R}_+$ onto $\mathbb{R}_+$ and G_k^{-1} is continuously differentiable. Since $v(t) \in \mathbb{R}_+$ we then have that for each $k = 1, \dots, n$ there exists a unique number $u_k = u_k(t) > 0$ such that $G_k(u_k) = t^{\sigma_k} v(t)$, and then

$$\xi_k(u_k, t) = v(t). \tag{5.18}$$

We will now show that $u_k = u_k^*$ for all k. Observe that by (5.17),

$$\frac{\partial \xi_k}{\partial z}(z, t) = t^{-\sigma_k} G_k'(z) < 0 \tag{5.19}$$

so ξ_k is strictly decreasing with respect to z. So if $u_k > u_k^*$ for some k, then (5.18) gives

$$v(t) = \xi_k(u_k, t) < \xi_k(u_k^*, t) \le \mu_t(u_1^*, \dots, u_n^*),$$

but this contradicts (5.15). Thus $u_k \le u_k^*$ for all k. Fix $k \in \{1, \dots, n\}$ and suppose that $u_k < u_k^*$. Since $(u_1^*, \dots, u_n^*) \in E_t$, we know that

$$\prod_{k=1}^{n} u_k^* = t^{n-1}. \tag{5.20}$$

By (5.20) and the assumption $u_k < u_k^*$ there are positive numbers d_l, $l = 1, \dots, n$ such that $0 < u_k < d_k < u_k^*$ and $0 < u_l \le u_l^* < d_l$ for $l \ne k$, which satisfies

$$\prod_{l=1}^{n} d_l = t^{n-1}.$$

Therefore $(d_1, \dots, d_n) \in E_t$ so that

$$v(t) \le \mu_t(d_1, \dots, d_n). \tag{5.21}$$

As we observed above, the functions $z \mapsto \xi_l(z, t)$, $l = 1, \dots, n$, are strictly decreasing on $\mathbb{R}_+$, and thus $\xi_l(d_l, t) < \xi_l(u_l, t)$ for all l. Therefore

$$\mu_t(d_1, \dots, d_n) < \mu_t(u_1, \dots, u_n).$$

This inequality and (5.21) gives

$$v(t) < \mu_t(u_1, \dots, u_n) = \max_{k=1,\dots,n} \xi_k(u_k, t).$$

which is a contradiction according to (5.18). Thus, $u_k = u_k^*$ for all $k = 1, \dots, n$, so (5.20) becomes

$$\prod_{k=1}^{n} u_k(t) = t^{n-1}. \tag{5.22}$$

We will now show that $u_k \in C^1(\mathbb{R}_+)$, for all k. By (5.18), $v(t) = t^{-\sigma_k} G_k(u_k(t))$, and therefore

$$u_k(t) = G_k^{-1}(t^{\sigma_k} v(t)) \tag{5.23}$$

for all $t > 0$, $k = 1, \ldots, n$. Define

$$\Psi(z,t) = \prod_{k=1}^{n} G_k^{-1}(zt^{\sigma_k}),$$

for $z, t > 0$. Then, by (5.23) and (5.22), $\Psi(v(t), t) = t^{n-1}$. By (5.17) we also have that $(G_k^{-1})' < 0$ on $\mathbb{R}_+$. Therefore

$$\frac{\partial}{\partial z}\Psi(z,t) = \sum_{k=1}^{n} \frac{\Psi(z,t)}{G_k^{-1}(zt^{\sigma_k})}(G_k^{-1})'(zt^{\sigma_k})t^{\sigma_k} < 0,$$

for all $z > 0$. By the implicit function theorem we obtain that $v \in C^1(\mathbb{R}_+)$. From (5.23) we then see that $u_k \in C^1(\mathbb{R}_+)$ for all $k = 1, \ldots, n$.

Next we will show that

$$\frac{u_k(t)}{t} \leq 4c_k u_k'(t), \tag{5.24}$$

for all $t > 0$, where c_k is the constant defined in (5.5). Write (5.18) as $v(t) = t^{-\sigma_k} G_k(u_k(t))$ and differentiate to get

$$-\frac{v'(t)}{v(t)} = \frac{\sigma_k}{t} - u_k'(t)\frac{G_k'(u_k(t))}{G_k(u_k(t))}. \tag{5.25}$$

By (5.9) and (5.10) we know that $G_k(z)z^{\alpha_k - \delta}$ is decreasing and $G_k(z)z^{\alpha_k+\delta}$ is increasing on $\mathbb{R}_+$. Therefore

$$\frac{\alpha_k - \delta}{z} \leq -\frac{G_k'(z)}{G_k(z)} \leq \frac{\alpha_k + \delta}{z}. \tag{5.26}$$

By (5.25) and (5.26) we get

$$\frac{\sigma_k}{t} + (\alpha_k - \delta)\frac{u_k'(t)}{u_k(t)} \leq -\frac{v'(t)}{v(t)} \leq \frac{\sigma_k}{t} + (\alpha_k + \delta)\frac{u_k'(t)}{u_k(t)}, \tag{5.27}$$

for $k = 1, \ldots, n$. Differentiate (5.22) with respect to t to obtain

$$\sum_{k=1}^{n} \frac{u_k'(t)}{u_k(t)} = \frac{n-1}{t}. \tag{5.28}$$

Observe that

$$\sum_{k=1}^{n} \frac{r_k}{\alpha_k} = n - 1.$$

Since $r_k, \alpha_k > 0$ (see (5.3)), this equality and (5.28) implies that there exists a number $m \in \{1, \ldots, n\}$ such that

$$\frac{r_m}{\alpha_m t} \leq \frac{u_m'(t)}{u_m(t)}. \tag{5.29}$$

Take $k = m$ in the left-hand side inequality in (5.27) and apply (5.29). We then get

$$-\frac{v'(t)}{v(t)} \geq \frac{\sigma_m}{t} + (\alpha_m - \delta)\frac{u_m'(t)}{u_m(t)} \geq \frac{1}{t}(\sigma_m + (\alpha_m - \delta)\frac{r_m}{\alpha_m}).$$

Set

$$\gamma \equiv \max_{k=1,\dots,n} \frac{r_k}{\alpha_k}.$$

Using the latter inequality and the definition (5.3) of r_m, we get

$$-\frac{v'(t)}{v(t)} \geq \frac{1}{t}(\frac{1}{p} - \frac{\alpha}{n} - \delta\gamma).$$

The right-hand side inequality in (5.27) implies

$$(\alpha_k + \delta)\frac{u_k'(t)}{u_k(t)} \geq (r_k - \delta\gamma)\frac{1}{t}.$$

Thus,

$$\frac{u_k(t)}{t} \leq \frac{\alpha_k + \delta}{r_k - \gamma\delta}\, u_k'(t).$$

By (5.12) and by observing that $\gamma\delta \leq r_k/2$, we see that the constant in this inequality is smaller than $4c_k$, so (5.24) holds.

We are now ready to prove inequality (5.6). First we observe that by (5.8), $\eta_k(z,t) \leq \xi_k(z,t)$ for all $k = 1,\dots,n$ and all $z > 0$, and thus

$$w(t) \leq v(t).$$

This inequality together with (5.18), and the fact that $\sum_{k=1}^{n} \alpha/(n\alpha_k) = 1$ gives

$$w(t) \leq \prod_{k=1}^{n} \xi_k(u_k(t),t)^{\alpha/(n\alpha_k)}.$$

It follows that

$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt\Big)^{1/s} \leq$$

$$\leq \Big(\int_0^\infty t^{s/q-1} \prod_{k=1}^{n} \Big(\Big(\frac{u_k(t)}{t}\Big)^{\sigma_k} \psi_k(u_k(t))\Big)^{s\alpha/(n\alpha_k)} dt\Big)^{1/s} =$$

$$= \Big(\int_0^\infty \prod_{k=1}^{n} \Big(u_k(t)^{s_k/p_k-1}\frac{u_k(t)}{t}\psi_k(u_k(t))^{s_k}\Big)^{s\alpha/(n\alpha_k s_k)} dt\Big)^{1/s}. \qquad (5.30)$$

Indeed, the equality in (5.30) can be proved by checking that

$$t^{s/q-1}\prod_{k=1}^{n}\Big(\frac{u_k(t)}{t}\Big)^{s\alpha\sigma_k/(n\alpha_k)} = \prod_{k=1}^{n}\Big(u_k(t)^{s_k/p_k-1}\frac{u_k(t)}{t}\Big)^{s\alpha/(n\alpha_k s_k)},$$

which is equivalent to

$$t^a = \prod_{k=1}^{n} u_k(t)^{b_k}, \tag{5.31}$$

where

$$a = \frac{s}{q} - 1 + \frac{s\alpha}{n} \sum_{k=1}^{n} \Big(\frac{1}{s_k \alpha_k} - \frac{\sigma_k}{\alpha_k} \Big)$$

and

$$b_k = \frac{s\alpha}{n\alpha_k p_k} - \frac{s\alpha\sigma_k}{n\alpha_k}.$$

But,

$$\frac{\sigma_k}{\alpha_k} = \frac{1}{p_k \alpha_k} - 1.$$

Thus,

$$a = \frac{s}{q} - 1 + \frac{s\alpha}{n} \sum_{k=1}^{n} \frac{1}{s_k \alpha_k} - \frac{s\alpha}{n} \Big(\sum_{k=1}^{n} \frac{1}{p_k \alpha_k} - n \Big) =$$
$$= \frac{s}{q} - \frac{s}{p} + s\alpha = \frac{s\alpha}{n}(n-1)$$

and

$$b_k = \frac{s\alpha}{n},$$

$k = 1, \dots, n$. Thus, (5.31) reduces to (5.22).

Observe that $\sum_{k=1}^{n} s\alpha/(n\alpha_k s_k) = 1$. We can then apply Hölder's inequality with the parameters $n\alpha_k s_k/(s\alpha)$, $k = 1, \dots, n$, in the last integral in (5.30) to get

$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt \Big)^{1/s} \leq$$
$$\leq \prod_{k=1}^{n} \Big(\int_0^\infty u_k(t)^{s_k/p_k - 1} \frac{u_k(t)}{t} \psi_k(u_k(t))^{s_k} dt \Big)^{\alpha/(n\alpha_k s_k)} \leq$$
$$\leq \prod_{k=1}^{n} \Big(4c_k \int_0^\infty u_k(t)^{s_k/p_k - 1} u_k'(t) \psi_k(u_k(t))^{s_k} dt \Big)^{\alpha/(n\alpha_k s_k)},$$

where the last inequality holds by (5.24). Make the change of variables $z = u_k(t)$. By (5.24), u_k increases on $\mathbb{R}_+$, so we obtain

$$\Big(\int_0^\infty t^{s/q-1} w(t)^s dt \Big)^{1/s} \leq 4^n \prod_{k=1}^{n} (c_k^{1/s_k} \|\psi_k\|_{p_k, s_k})^{\alpha/(n\alpha_k)}.$$

Applying (5.11) we get inequality (5.6). The lemma is proved. □

5.2. **The main theorem.** In the proof of the next theorem we will derive inequalities involving sections of sets. As in Section 3, for $E \subset \mathbb{R}^n$ and $\hat{x}_k \in \mathbb{R}^{n-1}$, we define the $\hat{x}_k$-section of E as the set

$$E(\hat{x}_k) = \{x_k \in \mathbb{R} : (\hat{x}_k, x_k) \in E\},$$

where $(\hat{x}_k, x_k) \equiv (x_1, \ldots, x_n)$.

Theorem 5.4. *Let $n \geq 2$, $1 \leq p_1, \ldots, p_n, s_1, \ldots, s_n < \infty$, and $\alpha_1, \ldots, \alpha_n > 0$. Put*

$$\alpha = n\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k}\Big)^{-1}, \; p = \frac{n}{\alpha}\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1}, \; \textit{and } s = \frac{n}{\alpha}\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k s_k}\Big)^{-1}.$$

Assume that $p \leq n/\alpha$ and put

$$q = \begin{cases} np/(n - \alpha p), & \alpha p < n \\ \infty, & \alpha p = n. \end{cases}$$

Set

$$\sigma_k = \frac{1}{p_k} - \alpha_k, \; \textit{and } V_k = L^{p_k, s_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{\sigma_k}_{x_k}(\mathbb{R})],$$

and assume that

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \sigma_k > 0, \tag{5.32}$$

for $k = 1, \ldots, n$. Suppose that

$$f \in S_0(\mathbb{R}^n) \textit{ and } f \in \bigcap_{k=1}^{n} V_k.$$

Then $f \in L^{q,s}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,s} \leq c \prod_{k=1}^{n} \|f\|_{V_k}^{\alpha/(n\alpha_k)}, \tag{5.33}$$

where

$$c = K_n c' \max_{k=1,\ldots,n} 4^{n\alpha_k} \prod_{k=1}^{n} \Big(1 + \frac{1}{\alpha_k}\Big)^{\alpha/(n\alpha_k)}, \tag{5.34}$$

K_n only depends on n, and c' is the constant from Lemma 5.3 defined in (5.7).

Proof. We may assume that $f \geq 0$. Fix $t > 0$. We will give a non-negative upper bound on $\Delta_f(t)$ and can therefore assume that $\Delta_f(t) > 0$. Let

$$E_1 = \{x : f(x) \geq f^*(t)\} \quad \text{and} \quad E_2 = \{x : f(x) > f^*(2t)\}.$$

By (2.11), $\mathrm{mes}_n E_1 \geq t$ so there exists an F_σ-set $A \subset E_1$ such that $\mathrm{mes}_n A = t$. Moreover, by (2.10), $\mathrm{mes}_n E_2 \leq 2t$ so there exists a G_δ-set $B \subset \mathbb{R}^n$ such that $E_2 \subset B$ and $\mathrm{mes}_n B = 2t$. Since F_σ-sets and G_δ-sets have measurable sections, the functions

$$a_k(\hat{x}_k) \equiv \mathrm{mes}_1 A(\hat{x}_k) \quad \text{and} \quad b_k(\hat{x}_k) \equiv \mathrm{mes}_1 B(\hat{x}_k),$$

$k = 1, \ldots, n$, are defined for all $\hat{x}_k \in \mathbb{R}^{n-1}$. By Fubini's theorem, these functions are measurable on $\mathbb{R}^{n-1}$. Since $f(x) \geq f^*(t)$ for all $x \in A$ we have (recall the definition of the partial rearrangement, $\mathcal{R}_k f$, from Section 2.4)

$$f^*(t) \leq \inf_{x_k \in A(\hat{x}_k)} f(x_k, \hat{x}_k) \leq \mathcal{R}_k f(a_k(\hat{x}_k), \hat{x}_k), \tag{5.35}$$

for all $k = 1, \ldots, n$ and all $\hat{x}_k$ such that $0 < a_k(\hat{x}_k) < \infty$. Moreover, if $0 < b_k(\hat{x}_k) < \infty$ then

$$\mathcal{R}_k f(2b_k(\hat{x}_k), \hat{x}_k) \leq f^*(2t). \tag{5.36}$$

Indeed, suppose $0 < b_k(\hat{x}_k) < \infty$ and let $E \subset \mathbb{R}$ be a measurable set of measure $2b_k(\hat{x}_k)$. Then there exists a point $x_k \in E$ such that $(x_k, \hat{x}_k) \notin B$. But $E_2 \subset B$ so then $f(x_k, \hat{x}_k) \leq f^*(2t)$ and thus

$$\inf_{x_k \in E} f(x_k, \hat{x}_k) \leq f^*(2t).$$

Since E was an arbitrary measurable set of measure $2b_k(\hat{x}_k)$, (5.36) follows.

Observe that by our assumption $\Delta_f(t) > 0$, it holds that

$$A \subset E_1 \subset E_2 \subset B. \tag{5.37}$$

For each k we put (the projection $\Pi_k A$ was defined in Section 3)

$$P_k = \{\hat{x}_k \in \Pi_k A : 0 < b_k(\hat{x}_k) \leq 2^{n+1} a_k(\hat{x}_k) < \infty\}.$$

The sets P_k are measurable since $\Pi_k A$ is measurable (A is an F_σ-set) and since the functions a_k and b_k are measurable. For all $\hat{x}_k \in P_k$ we have the inequalities (5.35) and (5.36), and from these we obtain

$$\begin{aligned}
\Delta_f(t) &\leq \mathcal{R}_k f(a_k(\hat{x}_k), \hat{x}_k) - \mathcal{R}_k f(2b_k(\hat{x}_k), \hat{x}_k) \\
&\leq \mathcal{R}_k f(a_k(\hat{x}_k), \hat{x}_k) - \mathcal{R}_k f(2^{n+2} a_k(\hat{x}_k), \hat{x}_k) \\
&= \sum_{l=0}^{n+1} \big(\mathcal{R}_k f(2^l a_k(\hat{x}_k), \hat{x}_k) - \mathcal{R}_k f(2^{l+1} a_k(\hat{x}_k), \hat{x}_k)\big) \\
&\leq (a_k(\hat{x}_k))^{-\sigma_k} \Psi_k(\hat{x}_k) \sum_{l=0}^{n+1} 2^{-l\,\sigma_k},
\end{aligned}$$

where $\Psi_k(\hat{x}_k) = \|f(\hat{x}_k, \cdot)\|_{\Lambda^{\sigma_k}}$. So for all $\hat{x}_k \in P_k$ and every $k = 1, \ldots, n$ it holds that

$$\Delta_f(t) \leq c(a_k(\hat{x}_k))^{-\sigma_k} \Psi_k(\hat{x}_k), \tag{5.38}$$

where

$$c = \sum_{l=0}^{n+1} 2^{-l\sigma_k} \le 2n \max(1, 2^{-(n+1)\sigma_k}).$$

For all $\hat{x}_k \in (\Pi_k A) \setminus P_k$ we have

$$2^{n+1} a_k(\hat{x}_k) \le b_k(\hat{x}_k). \tag{5.39}$$

Indeed, take $\hat{x}_k \in (\Pi_k A) \setminus P_k$. By the definition of P_k, we have either $b_k(\hat{x}_k) = 0$, $2^{n+1} a_k(\hat{x}_k) < b_k(\hat{x}_k)$, or $a_k(\hat{x}_k) = \infty$. However, if $b_k(\hat{x}_k) = 0$ or $a_k(\hat{x}_k) = \infty$ then (5.39) holds by (5.37).

For $k = 1, \dots, n$ we put

$$A_k = \{x \in A : \hat{x}_k \in P_k\}.$$

These sets are measurable by Lemma 3.2. Moreover,

$$\mathrm{mes}_n A_k \ge t(1 - 2^{-n}), \tag{5.40}$$

for all k. Indeed, $\Pi_k A_k = P_k$ and for all $\hat{x}_k \in P_k$ we have $\mathrm{mes}_1(A_k(\hat{x}_k)) = a_k(\hat{x}_k)$, so

$$\mathrm{mes}_n A_k = \int_{P_k} a_k(\hat{x}_k) d\hat{x}_k.$$

By (5.39) and this equality we get

$$2t = \int_{\Pi_k B} b_k(\hat{x}_k) d\hat{x}_k \ge 2^{n+1} \int_{(\Pi_k A)\setminus P_k} a_k(\hat{x}_k) d\hat{x}_k =$$

$$= 2^{n+1}(\mathrm{mes}_n A - \mathrm{mes}_n A_k) = 2^{n+1}(t - \mathrm{mes}_n A_k),$$

and then (5.40) follows.

Let A^* be an F_σ-subset of $\cap_{k=1}^n A_k$ such that $\mathrm{mes}_n A^* = \mathrm{mes}_n(\cap_{k=1}^n A_k)$. By (5.40) we have $\mathrm{mes}_n(A \setminus A^*) \le n 2^{-n} t$. This implies that

$$\mathrm{mes}_n A^* \ge \frac{t}{2}. \tag{5.41}$$

Let u_k, $k = 1, \dots, n$, be positive numbers such that

$$\prod_{k=1}^n u_k = t^{n-1}. \tag{5.42}$$

Put

$$\Omega = \{k \in \{1, \dots, n\} : \mathrm{mes}_{n-1} P_k \ge \frac{u_k}{2}\}.$$

Then $\Omega \ne \emptyset$. Indeed, suppose $\Omega = \emptyset$. By (5.41) and the Loomis-Whitney inequality (3.1),

$$\left(\frac{t}{2}\right)^{n-1} \le (\mathrm{mes}_n A^*)^{n-1} \le \prod_{k=1}^n \mathrm{mes}_{n-1} \Pi_k A^*. \tag{5.43}$$

Since $\Pi_k A^* \subset P_k$ and $\Omega = \emptyset$, it follows that

$$\left(\frac{t}{2}\right)^{n-1} < \frac{1}{2^n} \prod_{k=1}^{n} u_k,$$

but this is false by (5.42). Thus, $\Omega \neq \emptyset$.

Fix k in Ω. Assume that $\sigma_k \leq 0$ and define

$$\tilde{P}_k = \{\hat{x}_k \in P_k : a_k(\hat{x}_k) \leq \frac{4t}{u_k}\}.$$

Then $\tilde{P}_k$ is measurable since P_k and the function a_k are measurable. Since $a_k(\hat{x}_k) > 4t/u_k$ for all $\hat{x}_k \in P_k \setminus \tilde{P}_k$, we have

$$t = \mathrm{mes}_n A = \int_{\Pi_k A} a_k(\hat{x}_k) d\hat{x}_k \geq \int_{P_k \setminus \tilde{P}_k} a_k(\hat{x}_k) d\hat{x}_k \geq$$

$$\geq \frac{4t}{u_k} (\mathrm{mes}_{n-1} P_k - \mathrm{mes}_{n-1} \tilde{P}_k).$$

Since $k \in \Omega$, we know that $\mathrm{mes}_{n-1} P_k \geq u_k/2$, so by the preceding inequality

$$\mathrm{mes}_{n-1} \tilde{P}_k \geq \frac{u_k}{4}. \tag{5.44}$$

Since $\sigma_k \leq 0$, (5.38) gives that

$$\Delta_f(t) \leq 2n 4^{n\alpha_k} \left(\frac{u_k}{t}\right)^{\sigma_k} \Psi_k(\hat{x}_k),$$

for all $\hat{x}_k \in \tilde{P}_k$. Taking infimum over the set $\tilde{P}_k$ and using (5.44), we get

$$\Delta_f(t) \leq 2n 4^{n\alpha_k} \left(\frac{u_k}{t}\right)^{\sigma_k} \Psi_k^*(\frac{u_k}{4}). \tag{5.45}$$

From here on we assume that $\sigma_k > 0$ for each k in Ω. We now partition the sets P_k, $k \in \Omega$, as follows. If $\mathrm{mes}_{n-1} P_k > u_k/2$, then we apply Lemma 5.1 to obtain disjoint measurable sets P_k' and P_k'' such that $P_k = P_k' \cup P_k''$,

$$\mathrm{mes}_{n-1} P_k' = \frac{u_k}{2}, \tag{5.46}$$

and

$$\int_{P_k''} \Psi_k(\hat{x}_k)^{1/\sigma_k} d\hat{x}_k \leq \int_{u_k/2}^{\infty} \Psi_k^*(z)^{1/\sigma_k} dz. \tag{5.47}$$

On the other hand, if $\mathrm{mes}_{n-1} P_k = u_k/2$ then we put $P_k' = P_k$ and $P_k'' = \emptyset$. Clearly (5.46) and (5.47) are satisfied also in this case.

For each $k \in \Omega$ we put

$$A_k'' = \{x \in A^* : \hat{x}_k \in P_k''\}.$$

These sets are measurable by Lemma 3.2. We will consider two cases. First we assume that there exists $k \in \Omega$ for which $\mathrm{mes}_n A_k'' \geq t/(4n)$. Fix such an index k. Since $\mathrm{mes}_1(A_k''(\hat{x}_k)) \leq a_k(\hat{x}_k)$ and $\Pi_k A_k'' \subset P_k''$, it holds that

$$\frac{t}{4n} \leq \mathrm{mes}_n A_k'' \leq \int_{P_k''} a_k(\hat{x}_k) d\hat{x}_k \tag{5.48}$$

(the first inequality is by our assumption on k). Since $\sigma_k > 0$, (5.38) gives

$$a_k(\hat{x}_k)\Delta_f(t)^{1/\sigma_k} \leq (2n\Psi_k(\hat{x}_k))^{1/\sigma_k},$$

for all $\hat{x}_k \in P_k$. Integrating over P_k'' and applying (5.47) we get

$$\Delta_f(t)^{1/\sigma_k} \int_{P_k''} a_k(\hat{x}_k) d\hat{x}_k \leq (2n)^{1/\sigma_k} \int_{u_k/2}^{\infty} \Psi_k^*(z)^{1/\sigma_k} dz. \tag{5.49}$$

By this inequality and (5.48),

$$\Delta_f(t) \leq 4n^2 t^{-\sigma_k} \Big(\int_{u_k}^{\infty} \Psi_k^*\big(\frac{z}{2}\big)^{1/\sigma_k} dz \Big)^{\sigma_k}. \tag{5.50}$$

Now we turn to the remaining case when

$$\mathrm{mes}_n A_k'' < \frac{t}{4n}, \tag{5.51}$$

for all $k \in \Omega$. Put

$$D = A^* \setminus \bigcup_{k\in\Omega} A_k''.$$

By (5.41) and (5.51),

$$\mathrm{mes}_n D \geq \mathrm{mes}_n A^* - \sum_{k\in\Omega} \mathrm{mes}_n A_k'' \geq \frac{t}{2} - \sum_{k=1}^{n} \frac{t}{4n} = \frac{t}{4}. \tag{5.52}$$

Fix $k \in \Omega$. Let the set S be defined by

$$S = \{x \in D : a_k(\hat{x}_k) \geq \frac{t}{4u_k}\}.$$

This set is measurable by Lemma 3.2. Let Q be an F_σ-subset of $D \setminus S$ such that

$$\mathrm{mes}_n Q = \mathrm{mes}_n(D \setminus S).$$

Now,

$$\mathrm{mes}_n Q \leq \int_{\Pi_k Q} \mathrm{mes}_1 D(\hat{x}_k) d\hat{x}_k \leq \int_{\Pi_k Q} a_k(\hat{x}_k) d\hat{x}_k \leq \frac{t}{4u_k} \mathrm{mes}_{n-1} \Pi_k Q.$$

But $\Pi_k Q \subset \Pi_k D \subset \Pi_k(A^* \setminus A_k'') \subset P_k'$ so we have by (5.46) and the preceding inequality that

$$\mathrm{mes}_n(D \setminus S) = \mathrm{mes}_n Q \leq \frac{t}{8}.$$

By this and (5.52),

$$\mathrm{mes}_n S = \mathrm{mes}_n D - \mathrm{mes}_n (D \setminus S) \geq \frac{t}{8}. \tag{5.53}$$

Let $\tilde{S}$ be an F_σ-subset of S such that $\mathrm{mes}_n \tilde{S} = \mathrm{mes}_n S$. Then

$$\mathrm{mes}_{n-1} \Pi_l \tilde{S} \leq \frac{u_l}{2}, \tag{5.54}$$

for all $l = 1, \ldots, n$. Indeed, if $l \in \Omega$, then we have

$$\tilde{S} \subset S \subset D \subset A^* \setminus A_l'',$$

so that

$$\Pi_l \tilde{S} \subset \Pi_l (A^* \setminus A_l'') \subset P_l'.$$

By (5.46) we then have (5.54), for all $l \in \Omega$. Suppose $\Omega \neq \{1, \ldots, n\}$ and fix $l \in \{1, \ldots, n\} \setminus \Omega$. Then $\mathrm{mes}_{n-1} P_l < u_l/2$. But

$$\tilde{S} \subset S \subset D \subset A^* \subset A_l$$

and then

$$\Pi_l \tilde{S} \subset \Pi_l A_l = P_l,$$

so we again obtain (5.54).

By (5.53) and the Loomis-Whitney inequality (3.1),

$$\Big(\frac{t}{8}\Big)^{n-1} \leq (\mathrm{mes}_n S)^{n-1} = (\mathrm{mes}_n \tilde{S})^{n-1} \leq \prod_{l=1}^{n} \mathrm{mes}_{n-1} \Pi_l \tilde{S}.$$

Applying (5.54) for each $l \in \{1, \ldots, n\}$, except for $l = k$ (recall that $k \in \Omega$ is fixed), we obtain

$$\big(\frac{t}{8}\big)^{n-1} \leq \frac{2^{-n+1}}{u_k} \mathrm{mes}_{n-1} \Pi_k \tilde{S} \prod_{l=1}^{n} u_l.$$

By (5.42), this implies

$$\frac{u_k}{4^{n-1}} \leq \mathrm{mes}_{n-1} \Pi_k \tilde{S}. \tag{5.55}$$

Let $\hat{x}_k \in \Pi_k \tilde{S}$. Then inequality (5.38) holds and $a_k(\hat{x}_k) \geq t/(4u_k)$, so we have

$$\Delta_f(t) \leq 8n \Big(\frac{u_k}{t}\Big)^{\sigma_k} \Psi_k(\hat{x}_k)$$

(here we used that $0 < \sigma_k < 1$ to estimate the constant). Taking infimum over all $\hat{x}_k \in \Pi_k \tilde{S}$ in the preceding inequality and using (5.55), we obtain

$$\Delta_f(t) \leq 8n \Big(\frac{u_k}{t}\Big)^{\sigma_k} \Psi_k^* \Big(\frac{u_k}{4^{n-1}}\Big).$$

Since Ψ_k^* is non-increasing, it follows that

$$\Delta_f(t) \le 8nt^{-\sigma_k}\Big(\int_{u_k}^{\infty} \Psi_k^*(z/4^n)^{1/\sigma_k}dz\Big)^{\sigma_k}. \tag{5.56}$$

For each $k \in \{1,\dots,n\}$ and $z > 0$ we define the function

$$\phi_k(z) = \begin{cases} \Psi_k^*(z/4), & \sigma_k \le 0 \\ z^{-\sigma_k}\Big(\int_z^{\infty} \Psi_k^*(\tau/4^n)^{1/\sigma_k}d\tau\Big)^{\sigma_k}, & \sigma_k > 0, \end{cases}$$

and set $\eta_k(z,t) \equiv (z/t)^{\sigma_k}\phi_k(z)$. It holds that

$$\|\phi_k\|_{p_k,s_k} \le 4^{n+1}\Big(1 + \frac{1}{\alpha_k}\Big)\|\Psi_k\|_{p_k,s_k}. \tag{5.57}$$

Indeed, fix $k \in \{1,\dots,n\}$. Assume first that $\sigma_k \le 0$. Then

$$\|\phi_k\|_{p_k,s_k} = \Big(\int_0^{\infty} z^{s_k/p_k-1}\Psi_k^*(z/4)^{s_k}dz\Big)^{1/s_k}.$$

Making the change of variables $z \mapsto z/4$ we obtain (5.57), with 4 as the constant. Now we suppose that $\sigma_k > 0$. Then

$$\|\phi_k\|_{p_k,s_k} = \Big(\int_0^{\infty} z^{\alpha_k s_k-1}\Big(\int_z^{\infty} \Psi_k^*(\tau/4^n)^{1/\sigma_k}d\tau\Big)^{s_k\sigma_k}dz\Big)^{1/s_k}.$$

Make the change of variables $\tau \mapsto 4^{-n}\tau$ and $z \mapsto 4^{-n}z$ to get

$$\|\phi_k\|_{p_k,s_k} = 4^{n/p_k}\Big(\int_0^{\infty} z^{\alpha_k s_k-1}\Big(\int_z^{\infty} \Psi_k^*(\tau)^{1/\sigma_k}d\tau\Big)^{s_k\sigma_k}dz\Big)^{1/s_k}.$$

Assume first that $1 \le s_k\sigma_k$. By Hardy's inequality (2.1),

$$\|\phi_k\|_{p_k,s_k} \le 4^{n/p_k}\Big(\frac{\sigma_k}{\alpha_k}\Big)^{\sigma_k}\Big(\int_0^{\infty} t^{\alpha_k s_k+s_k\sigma_k-1}\Psi_k^*(t)^{s_k}dt\Big)^{1/s_k} \le$$

$$\le \frac{4^n}{\alpha_k}\|\Psi_k\|_{p_k,s_k},$$

where we used that $0 < \alpha_k$, $\sigma_k < 1$ to estimate the constant. Suppose now that $0 < s_k\sigma_k < 1$. We apply inequality (2.6), but with the constant from Remark 2.3 (this constant is sufficient for our purposes and it has a simpler form that the optimal constant - see Theorem 2.2). We now obtain inequality (5.57) with the constant $4^{n/p_k}(e/(\alpha_k p_k))^{1/s_k}$ (this estimate is similar to the previous case, so we omit the details). This constant is less then $4^{n+1}/\alpha_k$, since $0 < \alpha_k < 1$ and $1 \le p_k, s_k$. In each of these three cases, the constant we get is less than $4^{n+1}(1 + 1/\alpha_k)$, so (5.57) holds.

By (5.45), (5.50), and (5.56) there exists $k \in \{1,\dots,n\}$ such that

$$\Delta_f(t) \le 8n^2 4^{n\alpha_k}\eta_k(u_k,t). \tag{5.58}$$

Set

$$w(t) \equiv \inf\{\max_{k=1,\dots,n} \eta_k(z_k,t) : \prod_{k=1}^{n} z_k = t^{n-1},\ 0 < z_1,\dots,z_n\}.$$

The numbers u_k, $k = 1,\dots,n$, are arbitrary positive numbers satisfying (5.42), so it follows from (5.58) that

$$\Delta_f(t) \le dw(t),$$

where

$$d \equiv 8n^2 \max_{k=1,\dots,n} 4^{n\alpha_k}.$$

Since $t > 0$ was arbitrary, we get

$$\|f\|^*_{q,s} = \Big(\int_0^\infty t^{s/q-1}\Delta_f(t)^s dt\Big)^{1/s} \le d\Big(\int_0^\infty t^{s/q-1}w(t)^s dt\Big)^{1/s}.$$

Each of the functions ϕ_k is non-negative and non-increasing. Furthermore, by (5.57) we have that $\phi_k \in L^{p_k,s_k}(\mathbb{R}_+)$. Since we also assumed (5.32), we can apply Lemma 5.3. This gives

$$\|f\|^*_{q,s} \le dc' \prod_{k=1}^{n} \|\phi_k\|_{p_k,s_k}^{\alpha/(n\alpha_k)},$$

where c' is the constant from Lemma 5.3, given by (5.7). By (5.57) we then get

$$\|f\|^*_{q,s} \le 4^{n+1}dc' \prod_{k=1}^{n} \Big[\Big(1+\frac{1}{\alpha_k}\Big)\|f\|_{V_k}\Big]^{\alpha/(n\alpha_k)},$$

so we have proved (5.33). It follows from (5.33) that $f \in L^{q,p}(\mathbb{R}^n)$. Indeed, when $q = \infty$, we apply (2.37) and when $q < \infty$ we apply (2.35). $\square$

Remark 5.5. As was mentioned above, for $p_k = s_k = p$, $k = 1,\dots,n$, Theorem 5.4 was proved in [29]. Note that in this case the condition (5.32) reduces to the inequality $\alpha_k > \alpha/n$ which is certainly true for any $k = 1,\dots,n$. Indeed,

$$\frac{\alpha}{n} = \Big(\sum_{k=1}^{n}\frac{1}{\alpha_k}\Big)^{-1} < \alpha_k.$$

6. Applications

It was shown in [29] that estimates in terms of mixed norms provide a unified approach to embeddings of Sobolev spaces and Besov spaces, and enable us to obtain optimal embedding constants. In this section we apply Theorem 5.4 to get similar results for anisotropic Sobolev-Liouville spaces and anisotropic Sobolev-Besov spaces.

6.1. **Anisotropic Sobolev-Liouville spaces.** Let $1 \le p < \infty$ and $\alpha \in \mathbb{N}$. A function $f \in L^p(\mathbb{R}^n)$ is said to belong to the *partial* Sobolev space $W^\alpha_{p;k}(\mathbb{R}^n)$ $(1 \le k \le n)$ if f has a usual (weak) derivative $D^\alpha_k f \in L^p(\mathbb{R}^n)$. The norm in this space is defined as

$$\|f\|_{W^\alpha_{p;k}} = \|f\|_p + \|D^\alpha_k f\|_p.$$

Let $1 \le p_1, \dots, p_n < \infty$ and $\alpha_1, \dots, \alpha_n \in \mathbb{N}$. The anisotropic Sobolev space $W^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$ is defined as the intersection

$$W^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n) = \bigcap_{k=1}^n W^{\alpha_k}_{p_k;k}(\mathbb{R}^n),$$

with the norm

$$\|f\|_{W^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}} = \sum_{k=1}^n \|f\|_{W^{\alpha_k}_{p_k;k}}.$$

The Sobolev spaces can be extended to fractional orders of smoothness. To this end, we consider the Bessel kernel G_α. It is defined via its Fourier transform by

$$\hat{G}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}, \quad \xi \in \mathbb{R}. \tag{6.1}$$

It holds that (see e.g. [43, p. 132])

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi x^2/t} e^{-t/(4\pi)} t^{(\alpha-1)/2} \frac{dt}{t}, \quad x \in \mathbb{R}, \tag{6.2}$$

for $\alpha > 0$.

For $1 \le s \le \infty$ and $\alpha > 1/s'$, we have

$$\|G_\alpha\|_{L^s(\mathbb{R})} < \infty. \tag{6.3}$$

Indeed, suppose first that $1 \le s < \infty$. Using Minkowski's inequality for integrals (see e.g. [15, p. 194]), we get

$$\|G_\alpha\|_{L^s(\mathbb{R})} \le c \int_0^\infty e^{-t/(4\pi)} t^{(\alpha-1)/2} \Big(\int_{\mathbb{R}} e^{-\pi s x^2/t} dx \Big)^{1/s} \frac{dt}{t} =$$

$$= c' \int_0^\infty e^{-t/(4\pi)} t^{(\alpha-1)/2} t^{1/(2s)} \frac{dt}{t}.$$

The last integral converges since $(\alpha-1)/2+1/(2s)>0$, that is, since $\alpha > 1/s'$. So, (6.3) holds for $1 \le s < \infty$. It remains to show that $G_\alpha \in L^\infty(\mathbb{R})$ for $\alpha > 1$. This is true since

$$\|G_\alpha\|_{L^\infty(\mathbb{R})} \le c\int_0^\infty e^{-t/(4\pi)} t^{(\alpha-1)/2}\frac{dt}{t},$$

and this integral converges for $\alpha > 1$. The proof of (6.3) is complete.

Since $G_\alpha > 0$ on $\mathbb{R}$, we obtain from (6.1) that

$$\|G_\alpha\|_{L^1(\mathbb{R})} = \hat{G}_\alpha(0) = 1, \quad \alpha > 0. \tag{6.4}$$

From (6.1) it also follows that

$$G_{\alpha+\beta}(x) = G_\alpha * G_\beta(x), \tag{6.5}$$

for $x \in \mathbb{R}$ and α, $\beta > 0$.

Let f be a measurable function on $\mathbb{R}^n$ and let $\alpha > 0$. A function $g_k \in L^p(\mathbb{R}^n)$ is said to be the Bessel derivative of order α of f with respect to x_k, if

$$f(x) = \int_{\mathbb{R}} G_\alpha(x_k - u) g_k(u, \hat{x}_k)du, \quad \text{a.e. } x \in \mathbb{R}^n.$$

For all $\alpha > 0$ we let $J_k^\alpha f$ denote the Bessel derivative g_k. Moreover, for $\alpha \notin \mathbb{N}$ we also use the notation $D_k^\alpha f$ for the Bessel derivative. However, if $\alpha \in \mathbb{N}$, then $D_k^\alpha f$ will always denote only the usual (weak) derivative. We will sometimes refer to Bessel derivatives of non-integer order as fractional derivatives (see [27, pp. 132–133]).

The Bessel derivative is uniquely defined. Namely, if $g, h \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ and if (for some $1 \le k \le n$ and $\alpha > 0$) it holds that

$$\int_{\mathbb{R}} G_\alpha(x_k - u) g(u, \hat{x}_k)du = \int_{\mathbb{R}} G_\alpha(x_k - u) h(u, \hat{x}_k)du \tag{6.6}$$

for a.e. $x \in \mathbb{R}^n$, then g and h are equivalent. Indeed, in the case $n = 1$ this is proved in [43, p. 135]. Further, if $n \ge 2$ we let A_k denote the set of all $\hat{x}_k \in \mathbb{R}^{n-1}$ such that (6.6) holds for a.e. $x_k \in \mathbb{R}$, and the functions $g(\hat{x}_k, \cdot)$ and $h(\hat{x}_k, \cdot)$ belong to $L^p(\mathbb{R})$. By Fubini's theorem

$$\mathrm{mes}_{n-1}\big(\mathbb{R}^{n-1} \setminus A_k\big) = 0.$$

Moreover, for $\hat{x}_k \in A_k$ we have $g(\hat{x}_k, x_k) = h(\hat{x}_k, x_k)$ for a.e. $x_k \in \mathbb{R}$, according to (6.6) with $n = 1$. This proves that $g = h$ a.e. on $\mathbb{R}^n$.

The partial Sobolev-Liouville space $L^\alpha_{p;k}(\mathbb{R}^n)$ (see [27, Section 6.2]), $1 \le k \le n$, is defined in one way for $\alpha \in \mathbb{N}$, and in another way for $\alpha \notin \mathbb{N}$ $(\alpha > 0)$. In the case $\alpha \in \mathbb{N}$, we let $L^\alpha_{p;k}(\mathbb{R}^n)$ denote the partial Sobolev space $W^\alpha_{p;k}(\mathbb{R}^n)$, and set $\|f\|_{L^\alpha_{p;k}} = \|f\|_{W^\alpha_{p;k}}$. Further, for $\alpha > 0$, $\alpha \notin \mathbb{N}$, we say that a measurable function f on $\mathbb{R}^n$ belongs to $L^\alpha_{p;k}(\mathbb{R}^n)$ if it has a Bessel derivative $D_k^\alpha f \in L^p(\mathbb{R}^n)$, and we then set $\|f\|_{L^\alpha_{p;k}} = \|D_k^\alpha f\|_p$. The anisotropic Sobolev-Liouville space is defined as the intersection of these partial spaces:

$$L^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n) = \bigcap_{k=1}^{n} L^{\alpha_k}_{p_k;k}(\mathbb{R}^n)$$

for $1 \le p_1, \dots, p_n < \infty$ and $\alpha_1, \dots, \alpha_n > 0$, with the norm

$$\|f\|_{L^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}} = \sum_{k=1}^{n} \|f\|_{L^{\alpha_k}_{p_k;k}}.$$

We also write $L^{\alpha_1,\dots,\alpha_n}_{p}(\mathbb{R}^n)$ for the space $L^{\alpha_1,\dots,\alpha_n}_{p,\dots,p}(\mathbb{R}^n)$.

If $\alpha > 0$ and $\alpha \notin \mathbb{N}$, then by Minkowski's inequality for integrals, and by (6.4), we easily obtain the following proposition (this result is immediate if $\alpha \in \mathbb{N}$).

Proposition 6.1. *Let $1 \le p < \infty$ and $\alpha > 0$. Suppose that $f \in L^{\alpha}_{p;k}(\mathbb{R}^n)$ for some $k \in \{1, \dots, n\}$. Then $f \in L^p(\mathbb{R}^n)$, and*

$$\|f\|_p \le \|f\|_{L^{\alpha}_{p;k}}.$$

Let φ be a function on $\mathbb{R}$. We consider its difference of order k

$$\Delta^k(h)\varphi(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \varphi(x+ih). \tag{6.7}$$

We also set $\Delta(h) = \Delta^1(h)$. For functions on $\mathbb{R}^n$, we define $\Delta^k_j(h)f(x) = \Delta^k(h)f_{\hat{x}_j}(x_j)$. Note that if $f \in L^1_{loc}(\mathbb{R}^n)$ has a usual (weak) derivative $D^k_j f \in L^1_{loc}(\mathbb{R}^n)$, then it holds that

$$\Delta^k_j(h)f(x) = \int_{(0,h)^k} D^k_j f(\hat{x}_j, x_j + u_1 + \dots + u_k)du \tag{6.8}$$

for almost all $x \in \mathbb{R}^n$ [7, Section 16, (8)]. By (6.8),

$$|\Delta^k_j(h)f(x)| \le h^{k-1} \int_0^{kh} |D^k_j f(\hat{x}_j, x_j + w)|dw. \tag{6.9}$$

It follows from inequality (7) on page 292 in [38] that

$$|G^{(j)}_{\alpha}(x)| \le c|x|^{\alpha-j-1}, \quad 0 < \alpha < j+1,\ x \in \mathbb{R}. \tag{6.10}$$

Apparently the following lemma is known, but we could not find a reference, and therefore we give a sketch of the proof.

Lemma 6.2. *Let $1 \le r \le \infty$ and $1/r < \alpha < \infty$. For $r = \infty$ we assume that $\alpha \notin \mathbb{N}$. Suppose k is the least integer such that $k > \alpha - 1/r$. Then*

$$\|\Delta^k(h)G_{\alpha}\|_{r'} \le ch^{\alpha-1/r}, \quad h \ge 0, \tag{6.11}$$

where c depends only on p, k, and α.

Proof. Note that $k - 1/r' \le \alpha < k + 1/r$ (and $\alpha \notin \mathbb{N}$ for $r = \infty$). We will first prove (6.11) under the assumption $\alpha \neq k$.

If $\alpha > k$, then by (6.9) and (6.10) (with $j = k$), we have for $|x| \le 2kh$ that

$$|\Delta^k(h)G_\alpha(x)| \le ch^{k-1} \int_0^{3kh} |G_\alpha^{(k)}(y)|dy \le$$
$$\le c'h^{k-1} \int_0^{3kh} y^{\alpha-k-1} dy = c''h^{\alpha-1}. \tag{6.12}$$

From (6.12) we obtain

$$\operatorname{ess\,sup}_{|x|\le 2kh} |\Delta^k(h)G_\alpha(x)| \le ch^{\alpha-1} \tag{6.13}$$

for $\alpha > k$.

Next, if $\alpha < k$ and $k \ge 2$, then by (6.9) and (6.10) (with $j = k-1$), we have for $|x| \le 2kh$ that

$$|\Delta^k(h)G_\alpha(x)| \le ch^{k-2} \int_0^{3kh} |G_\alpha^{(k-1)}(y)|dy \le$$
$$\le c'h^{k-2} \int_0^{3kh} y^{\alpha-k} dy = c''h^{\alpha-1}, \tag{6.14}$$

where the last integral converges since $\alpha \ge k - 1/r' \ge k - 1$ and $\alpha \notin \mathbb{N}$ for $r = \infty$ (and thus $\alpha > k - 1$).

For $1 < r \le \infty$, we will show that

$$\Big(\int_{|x|\le 2kh} |\Delta^k(h)G_\alpha(y)|^{r'} dy\Big)^{1/r'} \le ch^{\alpha-1/r} \tag{6.15}$$

(recall that we consider the case $\alpha \neq k$). Indeed, (6.15) follows directly from (6.12) for $\alpha > k$, $k \ge 1$, and from (6.14) for $\alpha < k$, $k \ge 2$. In the remaining case $k = 1$, $1/r < \alpha < 1$ ($1 < r \le \infty$), we have

$$\Big(\int_{|x|\le 2h} |\Delta(h)G_\alpha(y)|^{r'} dy\Big)^{1/r'} \le c\Big(\int_0^{3h} \big(G_\alpha(y)\big)^{r'} dy\Big)^{1/r'} \le$$
$$\le c'\Big(\int_0^{3h} y^{(\alpha-1)r'} dy\Big)^{1/r'} = c''h^{\alpha-1/r},$$

as required, where the second inequality holds by (6.10) with $j = 0$. This completes the proof of (6.15).

By (6.9) and (6.10) (we can apply (6.10) with $j = k$ since $\alpha < k + 1/r \le k + 1$),

$$|\Delta^k(h)G_\alpha(x)| \le ch^{k-1} \int_0^{kh} |G_\alpha^{(k)}(x+w)|dw \le$$
$$\le ch^{k-1} \int_0^{kh} |x+w|^{\alpha-k-1} dw.$$

For $|x| \geq 2kh$, this implies (since if $|x| \geq 2kh$ and $0 < w < kh$, then $|x+w| \geq |x|/2$)

$$|\Delta^k(h)G_\alpha(x)| \leq ch^k|x|^{\alpha-k-1}, \tag{6.16}$$

since $\alpha < k + 1/r$. So, for $1 < r \leq \infty$ (recall that $k - 1/r' \leq \alpha < k + 1/r$), we have

$$\Big(\int_{|x|\geq 2kh} |\Delta^k(h)G_\alpha(x)|^{r'}dx\Big)^{1/r'} \leq$$
$$\leq ch^k\Big(\int_{2kh}^{\infty} x^{(\alpha-k-1)r'}dx\Big)^{1/r'} = c'h^{\alpha-1/r}, \tag{6.17}$$

where the last integral converges since $\alpha < k + 1/r$. Furthermore, for $k < \alpha < k+1$ (corresponding to $r = 1$), (6.16) gives

$$\text{ess sup}_{|x|\geq 2kh} |\Delta^k(h)G_\alpha(x)| \leq ch^{\alpha-1}.$$

From (6.13) and the preceding estimate, we obtain (6.11) for $r = 1$ and $\alpha \neq k$ (in this case $k < \alpha < k+1$). Moreover, by (6.15) and (6.17), we obtain (6.11) for $1 < r \leq \infty$ and $\alpha \neq k$. This completes the proof of (6.11) in the case $\alpha \neq k$.

Assume now that $\alpha = k$. Then we have $1 \leq r < \infty$. Let $1/r' < \delta < 1$. By (6.5), it holds that

$$\Delta^k(h)G_k(x) = \Delta^k(h)G_{k-\delta} * G_\delta(x) = \int_{\mathbb{R}} \Delta^{k-1}(h)G_{k-\delta}(x-u)\Delta(h)G_\delta(u)du.$$

By Minkowski's inequality, we then have

$$\|\Delta^k(h)G_k\|_{r'} \leq \|\Delta^{k-1}(h)G_{k-\delta}\|_{r'}\|\Delta(h)G_\delta\|_1 \leq ch^{k-1/r}, \tag{6.18}$$

where the last inequality holds by inequality (6.11). Indeed, by the above we have that $\|\Delta(h)G_\delta\|_1 \leq ch^\delta$ if $0 < \delta < 1$, and $\|\Delta^{k-1}(h)G_{k-\delta}\|_{r'} \leq ch^{k-\delta-1/r}$ if $k-2 \leq k-\delta-1/r < k-1$ and $k-\delta \neq k-1$, i.e. for $1/r' < \delta \leq 1+1/r'$ and $\delta \neq 1$. Thus, (6.11) holds also for $\alpha = k$. $\square$

The following lemma was proved in [27, p. 148].

Lemma 6.3. *Let $\varphi \in L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$ and assume that $0 < t < \infty$. Then, for each $x \in \mathbb{R}$*

$$|\varphi(x)| \leq \varphi^*(t) + \frac{1}{t}\int_0^{2t} |\Delta(h)\varphi(x)|dh.$$

The next lemma states the embedding from anisotropic Sobolev-Liouville spaces to the mixed norm spaces considered in Theorem 5.4. In the case when $\alpha = 1$, it was proved in [29]. The proof in the case $0 < \alpha < 1$ is quite different from the arguments that apply in the case $\alpha = 1$. Recall that $D_k^\alpha f$ denotes the Bessel derivative if $0 < \alpha < 1$ and the usual (weak) derivative if $\alpha = 1$.

Lemma 6.4. *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $n \ge 2$, and $k \in \{1, \dots, n\}$. If $f \in L^{\alpha}_{p;k}(\mathbb{R}^n)$, then*

$$f \in V_k \equiv L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p-\alpha}_{x_k}(\mathbb{R})] \tag{6.19}$$

and

$$\|f\|_{V_k} \le c\|D^{\alpha}_k f\|_p, \tag{6.20}$$

where c depends only on p, α, and n.

Proof. Fix $t > 0$. Write $f_{\hat{x}_k}(x_k) = f(\hat{x}_k, x_k)$. Then $f_{\hat{x}_k} \in L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$, for a.e. $\hat{x}_k \in \mathbb{R}^{n-1}$. Indeed, $L^{\alpha}_{p,k}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ by Proposition 6.1. By Fubini's theorem it follows that $f_{\hat{x}_k} \in L^p(\mathbb{R}) \subset L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$, for a.e. $\hat{x}_k \in \mathbb{R}^{n-1}$. Fix such $\hat{x}_k$. By Lemma 6.3,

$$|f_{\hat{x}_k}(x_k)| \le f^*_{\hat{x}_k}(2t) + \frac{1}{2t}\Phi(x_k) \tag{6.21}$$

for all $x_k \in \mathbb{R}$, where

$$\Phi(x_k) = \int_0^{4t} |\Delta(h) f_{\hat{x}_k}(x_k)| dh.$$

We have $\Phi \in S_0(\mathbb{R})$. Indeed, by Minkowski's inequality

$$\|\Phi\|_p \le \int_0^{4t} \|\Delta(h) f_{\hat{x}_k}\|_p dh \le 8t\|f_{\hat{x}_k}\|_p.$$

As we noted above, $f_{\hat{x}_k} \in L^p(\mathbb{R})$ and thus $\Phi \in L^p(\mathbb{R}) \subset S_0(\mathbb{R})$.

It follows from (6.21) that

$$f^*_{\hat{x}_k}(t) - f^*_{\hat{x}_k}(2t) \le \frac{1}{t}\Phi^*(t). \tag{6.22}$$

Set $g \equiv D^{\alpha}_k f$, that is, if $0 < \alpha < 1$ then g denotes the Bessel derivative, and if $\alpha = 1$ then g is the usual (weak) derivative. Write $g_{\hat{x}_k}(x_k) = g(\hat{x}_k, x_k)$. In the case $\alpha = 1$,

$$\Delta(h) f_{\hat{x}_k}(x_k) = \int_0^h g_{\hat{x}_k}(x_k + u) du.$$

So

$$\Phi(x_k) \le 4t \int_0^{4t} |g_{\hat{x}_k}(x_k + u)| du \le 16t \int_0^t g^*_{\hat{x}_k}(s) ds,$$

and thus

$$\Phi^*(t) \le 16t \int_0^t g^*_{\hat{x}_k}(s) ds. \tag{6.23}$$

Assume now that $0 < \alpha < 1$. By definition of the Bessel derivative,

$$f_{\hat{x}_k}(x_k) = \int_{\mathbb{R}} G_{\alpha}(x_k - u) g_{\hat{x}_k}(u) du.$$

It follows that

$$\Delta(h)f_{\hat{x}_k}(x_k) = \int_{\mathbb{R}} g_{\hat{x}_k}(u)\Delta(h)G_\alpha(x_k - u)du.$$

Changing variables we get

$$\Delta(h)f_{\hat{x}_k}(x_k) = \int_{\mathbb{R}} g_{\hat{x}_k}(x_k - u)\Delta(h)G_\alpha(u)du.$$

So by Fubini's theorem,

$$\Phi(x_k) \le \int_{\mathbb{R}} |g_{\hat{x}_k}(x_k - u)|\varphi(u)du,$$

where

$$\varphi(u) = \int_0^{4t} |\Delta(h)G_\alpha(u)|dh.$$

Let $E \subset \mathbb{R}$ be a measurable set having measure t. Integrating over E in the preceding inequality and using Fubini's theorem and (2.26), we obtain

$$\int_E \Phi(x_k)dx_k \le \int_{\mathbb{R}} \varphi(u)\Big(\int_E |g_{\hat{x}_k}(x_k - u)|dx_k\Big)du \le \|\varphi\|_1 \int_0^t g^*_{\hat{x}_k}(s)ds.$$

Since E was an arbitrary set of measure t, (2.26) then implies

$$\Phi^*(t) \le \frac{1}{t}\int_0^t \Phi^*(s)ds = \frac{1}{t}\sup_{|E|=t}\int_E \Phi(x_k)dx_k \le \frac{1}{t}\|\varphi\|_1 \int_0^t g^*_{\hat{x}_k}(s)ds.$$

By Fubini's theorem and Lemma 6.2, we have

$$\|\varphi\|_1 = \int_0^{4t} \|\Delta(h)G_\alpha\|_1 dh \le ct^{\alpha+1}.$$

Thus,

$$\Phi^*(t) \le ct^\alpha \int_0^t g^*_{\hat{x}_k}(s)ds. \tag{6.24}$$

By (6.22), (6.23), (6.24) we obtain

$$f^*_{\hat{x}_k}(t) - f^*_{\hat{x}_k}(2t) \le ct^{\alpha-1}\int_0^t g^*_{\hat{x}_k}(s)ds,$$

for all $0 < \alpha \le 1$. For all $0 < \alpha \le 1$ and $1 \le p < \infty$ we then get (using Hölder's inequality in the case $1 < p < \infty$)

$$\|f_{\hat{x}_k}\|_{\Lambda^{1/p-\alpha}} \le c\|g_{\hat{x}_k}\|_{L^p(\mathbb{R})}.$$

Taking $L^p(\mathbb{R}^{n-1})$-norm with respect to $\hat{x}_k$ and using Fubini's theorem we obtain (6.20), and then also (6.19). $\square$

We will now apply Theorem 5.4 and the preceding lemma to prove the aforementioned embedding of the spaces $L^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$. We remind that $D^\alpha_k f$ denotes the Bessel derivative if $0<\alpha<1$ and the usual (weak) derivative if $\alpha=1$.

Theorem 6.5. *Let $n\ge 2$, $1\le p_1,\dots,p_n<\infty$, and $0<\alpha_1,\dots,\alpha_n\le 1$. Put*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \quad \text{and} \quad p=\frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}.$$

Assume that $p\le n/\alpha$ and that

$$\frac{1}{p}-\frac{\alpha}{n}-\frac{1}{p_k}+\alpha_k>0, \qquad k=1,\dots,n.$$

Put

$$q=\begin{cases} np/(n-\alpha p), & p<n/\alpha \\ \infty, & p=n/\alpha. \end{cases}$$

Suppose $f\in L^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$. Then $f\in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|_{q,p}\le c\prod_{k=1}^n \|D^{\alpha_k}_k f\|^{\alpha/(n\alpha_k)}_{p_k}, \tag{6.25}$$

where c depends only on $\alpha_1,\dots,\alpha_n,p_1,\dots,p_n$, and n.

Proof. As in Theorem 5.4 we set

$$V_k\equiv L^{p_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p_k-\alpha_k}_{x_k}(\mathbb{R})].$$

Since $f\in L^{\alpha_k}_{p_k;\,k}(\mathbb{R}^n)$, for $k=1,\dots,n$, it follows from Lemma 6.4 that $f\in V_k$ and

$$\|f\|_{V_k}\le c\|D^{\alpha_k}_k f\|_{p_k}<\infty. \tag{6.26}$$

Thus $f\in\cap_{k=1}^n V_k$. We also have $f\in S_0(\mathbb{R}^n)$, by Proposition 6.1. So by Theorem 5.4, $f\in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p}\le c\prod_{k=1}^n \|f\|^{\alpha/(n\alpha_k)}_{V_k}. \tag{6.27}$$

Now, (6.25) follows from (6.27), (6.26), (2.35), and (2.37). □

Remark 6.6. Let $p<n/\alpha$. The following special cases of the preceding theorem are known:

If $\alpha_1=\cdots=\alpha_n=1$, but the numbers p_k may be distinct, then the above theorem is a special case of Theorem 13.1 in [27].

If the numbers p_k all coincide, but the numbers α_k may be distinct, then the above theorem is a special case of Theorem 9.3 in [27].

6.2. **Limiting embeddings and anisotropic Sobolev-Besov spaces.** Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$. For $x, h \in \mathbb{R}^n$ we use the notation

$$\Delta(h)f(x) = f(x+h) - f(x),$$

and set

$$I_p(h) = \|\Delta(h)f\|_p. \tag{6.28}$$

The modulus of continuity in $L^p(\mathbb{R}^n)$ of f is the function $t \mapsto \omega(f;t)_p$ defined for $t \ge 0$ by

$$\omega(f;t)_p = \sup_{|h|\le t} I_p(h). \tag{6.29}$$

Further, for $\tau \ge 0$ and $x \in \mathbb{R}^n$, we set

$$I_{p,k}(h) = \|\Delta(\tau e_k)f\|_p$$

(e_k is the kth unit coordinate vector in $\mathbb{R}^n$). We define the *partial* modulus of continuity in $L^p(\mathbb{R}^n)$ of f with respect to x_k as

$$\omega_k(f;t)_p = \sup_{0\le\tau\le t} I_{p,k}(\tau). \tag{6.30}$$

To distinguish, we shall call the modulus of continuity defined in (6.29) the *total modulus of continuity.* Let $0 < \alpha < 1$ and $1 \le p, \theta < \infty$. The following relation between $\omega(f;t)_p$ and $\omega_k(f;t)_p$ is easy to verify

$$\max_{k=1,\dots,n} \omega_k(f;t)_p \le \omega(f;t)_p \le \sum_{k=1}^{n} \omega_k(f;t)_p. \tag{6.31}$$

We define the Besov space $B^\alpha_{p,\theta}(\mathbb{R}^n)$ as consisting of all $f \in L^p(\mathbb{R}^n)$ for which

$$\|f\|_{b^\alpha_{p,\theta}} \equiv \left(\int_0^\infty (t^{-\alpha}\omega(f;t)_p)^\theta \frac{dt}{t}\right)^{1/\theta} < \infty.$$

The partial Besov space $B^\alpha_{p,\theta;k}(\mathbb{R}^n)$ is similarly defined as the class of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{b^\alpha_{p,\theta;k}} \equiv \left(\int_0^\infty (t^{-\alpha}\omega_k(f;t)_p)^\theta \frac{dt}{t}\right)^{1/\theta} < \infty.$$

We set $B^\alpha_p \equiv B^\alpha_{p,p}$ and $B^\alpha_{p;k} \equiv B^\alpha_{p,p;k}$. Let $0 < \alpha_1, \dots, \alpha_n < 1$ and $1 \le p_1, \dots, p_n < \infty$. The anisotropic Besov space $B^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$ is defined as the intersection

$$B^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n) = \bigcap_{k=1}^{n} B^{\alpha_k}_{p_k;k}(\mathbb{R}^n).$$

We set also

$$\|f\|_{b^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}} = \sum_{k=1}^{n} \|f\|_{b^{\alpha_k}_{p_k;k}}.$$

We let $B_p^{\alpha_1,\dots,\alpha_n}$ denote $B_{p,\dots,p}^{\alpha_1,\dots,\alpha_n}$. It follows from (6.31) that if $\alpha_k = \alpha$ and $p_k = p$, $k = 1, \dots, n$, then the spaces $B_{p_1,\dots,p_n}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ and $B_p^\alpha(\mathbb{R}^n)$ coincide and

$$\frac{1}{n}\|f\|_{b_{p,\dots,p}^{\alpha,\dots,\alpha}} \le \|f\|_{b_p^\alpha} \le \|f\|_{b_{p,\dots,p}^{\alpha,\dots,\alpha}} \tag{6.32}$$

The following theorem is well known (see [18]).

Theorem 6.7. *Let* $0 < \alpha_1, \dots, \alpha_n < 1$,

$$\alpha \equiv n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1},$$

and $1 \le p < n/\alpha$. *Set* $q = np/(n - \alpha p)$. *For every* $f \in B_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ *we have* $f \in L^{q,p}(\mathbb{R}^n)$ *and*

$$\|f\|_{q,p} \le c \sum_{k=1}^n \|f\|_{b_{p;k}^{\alpha_k}}, \tag{6.33}$$

where c *depends only on* $\alpha_1, \dots, \alpha_n$, p, *and* n.

Suppose $\alpha_1 = \dots = \alpha_n = \alpha$. By (2.31), (6.33), and (6.32),

$$\|f\|_q \le c\|f\|_{b_p^\alpha}. \tag{6.34}$$

Bourgain, Brezis, and Mironescu [10] proved a limiting relation between the Besov norm and the Sobolev norm. They showed that for any $f \in W_p^1(\mathbb{R}^n)$ $(1 \le p < \infty)$ there holds the equality

$$\lim_{\alpha\to 1-}(1-\alpha)^{1/p}\|f\|_{b_p^\alpha} = \Big(\frac{1}{p}\Big)^{1/p}\|\nabla f\|_p. \tag{6.35}$$

The sharp asymptotic of the best constant in (6.34) as $\alpha \to 1-$ was found by Bourgain, Brezis, and Mironescu in [11]. Namely, they proved that if $1/2 < \alpha < 1$, $1 \le p < n/\alpha$, and $q = np/(n - \alpha p)$, then for any $f \in B_p^\alpha(\mathbb{R}^n)$,

$$\|f\|_q^p \le c_n \frac{1-\alpha}{(n-\alpha p)^{p-1}}\|f\|_{b_p^\alpha}^p. \tag{6.36}$$

They where the first to explicitly observe that embeddings for Sobolev spaces can be derived from embeddings of Besov spaces. Indeed, in view of (6.35), Sobolev's inequality (1.1) can be considered as a limiting case of (6.36).

The next theorem was obtained in [29] as a corollary of estimates via mixed norms (see Theorem 1.6 and Proposition 6.10 below).

Theorem 6.8. *Let* $1 \le p < \infty$, $n \ge 2$, *and* $1/2 < \alpha_1, \dots, \alpha_n < 1$. *Assume that*

$$\alpha \equiv n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \le \frac{n}{p}.$$

Let

$$q = \begin{cases} np/(n-\alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Then, for every $f \in B_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ *we have that* $f \in L^{q,p}(\mathbb{R}^n)$ *and*

$$\|f\|_{q,p}^* \le c_n \prod_{k=1}^n [(1-\alpha_k)^{1/p} \|f\|_{b_{p;k}^{\alpha_k}}]^{\alpha/(n\alpha_k)}, \tag{6.37}$$

where c_n *only depends on* n.

Inequality (6.37) gives the sharp asymptotic behaviour of the best constant in (6.33) as some of the numbers α_k tend to 1.

We shall apply Theorem 5.4 to extend these results to the fully anisotropic space $B_{p_1,\dots,p_n}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$. First we state the following proposition, it gives a version of the relation (6.35) for the partial Besov norm. It is proved as in [31, Proposition 2.5].

Proposition 6.9. *Let* $1 \le p, \theta < \infty$. *If* $f \in W_{p;k}^1(\mathbb{R}^n)$, *then*

$$\lim_{\alpha \to 1-} (1-\alpha)^{1/\theta} \|f\|_{b_{p,\theta;k}^\alpha} = \theta^{-1/\theta} \|D_k f\|_p. \tag{6.38}$$

As it was observed in [29], the constant in (6.37) has the sharp asymptotic behaviour as some of the numbers α_k tend to 1. Indeed, if a function $f \in B_p^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$ has a usual (weak) derivative $D_k f \in L^p(\mathbb{R}^n)$ for some k, then for the corresponding factor in (6.37) we have by (6.38) (with $\theta = p$)

$$(1-\alpha_k)^{1/p} \|f\|_{b_{p;k}^{\alpha_k}} \to \left(\frac{1}{p}\right)^{1/p} \|D_k f\|_p, \quad \alpha_k \to 1- .$$

The next proposition was proved in [29].

Proposition 6.10. *Let* $n \ge 2$, $0 < \alpha < 1$, $1 \le p < \infty$, *and* $1 \le k \le n$. *Set*

$$V_k = L_{\hat{x}_k}^p(\mathbb{R}^{n-1})[\Lambda_{x_k}^{1/p-\alpha}(\mathbb{R})].$$

Assume that $f \in B_{p;k}^\alpha(\mathbb{R}^n)$. *Then* $f \in V_k$ *and*

$$\|f\|_{V_k} \le 100[\alpha(1-\alpha)]^{1/p} \|f\|_{b_{p;k}^\alpha}.$$

Applying Theorem 5.4 and Proposition 6.10, we obtain the following.

Theorem 6.11. *Let* $n \ge 2$, $1/2 < \alpha_1, \dots, \alpha_n < 1$, *and* $1 \le p_1, \dots, p_n < \infty$. *Set*

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} \quad \textit{and} \quad p = \frac{n}{\alpha}\Big(\sum_{k=1}^n \frac{1}{\alpha_k p_k}\Big)^{-1}.$$

Assume that $p \leq n/\alpha$ *and set*

$$q = \begin{cases} np/(n-\alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Assume also that for $k = 1, \dots, n$,

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \frac{1}{p_k} + \alpha_k > 0.$$

If $f \in B^{\alpha_1,\dots,\alpha_n}_{p_1\dots,p_n}(\mathbb{R}^n)$, *then* $f \in L^{q,p}(\mathbb{R}^n)$ *and*

$$\|f\|^*_{q,p} \leq c_n \prod_{k=1}^n \left[d_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}} \right]^{\alpha/(n\alpha_k)}, \tag{6.39}$$

where c_n *depends only on* n,

$$d_k = r_k^{-1/p_k} \max(R, p_k)^2, \quad \textit{and} \quad R = \max_{k=1,\dots,n} \frac{1}{r_k} \max_{k=1,\dots,n} r_k.$$

Proof. By Proposition 6.10,

$$f \in V_k \equiv L^{p_k}_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{1/p_k - \alpha_k}_{x_k}(\mathbb{R})]$$

and (since $\alpha_k < 1$ for all k)

$$\|f\|_{V_k} \leq 100(1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}}.$$

So, $f \in \cap_{k=1}^n V_k$. By assumption, $f \in B^{\alpha_1,\dots,\alpha_n}_{p_1\dots,p_n}(\mathbb{R}^n) \subset S_0(\mathbb{R}^n)$. Hence, by Theorem 5.4, $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \leq c \prod_{k=1}^n \|f\|^{\alpha/(n\alpha_k)}_{V_k},$$

where c is the constant defined in (5.34). We have

$$c = c_n \max_{k=1,\dots,n} 4^{n\alpha_k} \prod_{k=1}^n \left(1 + \frac{1}{\alpha_k}\right)^{\alpha/(n\alpha_k)} \prod_{k=1}^n \left[\left(\frac{\alpha_k}{r_k}\right)^{1/p_k} \max(R', p_k)^2 \right]^{\alpha/(n\alpha_k)},$$

where $R' = \max_{k=1,\dots,n} r_k/\alpha_k \max_{k=1,\dots,n} 1/r_k$. Since $1/2 < \alpha_1, \dots, \alpha_n < 1$ it follows that

$$c \leq c_n \prod_{k=1}^n d_k^{\alpha/(n\alpha_k)}.$$

Now (6.39) follows from the three preceding inequalities. □

We will now define the Sobolev-Besov space $WB^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$. Let $n \geq 2$, $0 \leq m \leq n$, $\alpha_1 = \dots = \alpha_m = 1$, and $0 < \alpha_{m+1}, \dots, \alpha_n < 1$ (with the obvious interpretation if $m = 0$ or $m = n$). Also let $1 \leq p_1, \dots, p_n < \infty$. A measurable function f on $\mathbb{R}^n$ belongs to the space $WB^{\alpha_1,\dots,\alpha_n}_{p_1,\dots,p_n}(\mathbb{R}^n)$ if $f \in W^1_{p_k;k}(\mathbb{R}^n)$

for $k = 1, \ldots, m$ and $f \in B^{\alpha_k}_{p_k;k}(\mathbb{R}^n)$ for $k = m+1, \ldots, n$. A number of embedding theorems have been obtained for these spaces by Gagliardo, Slobodeckii, Uspenskii, and other authors (see [7, Chapter 18.15]).

The next result coincides with the preceding theorem in the case $m = 0$.

Theorem 6.12. *Let $n \geq 2$, $1 \leq p_1, \ldots, p_n < \infty$, and $0 \leq m \leq n$. We also let $\alpha_1 = \cdots = \alpha_m = 1$ and $1/2 < \alpha_{m+1}, \ldots, \alpha_n < 1$. Set*

$$\alpha = n\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k}\Big)^{-1} \quad \textit{and} \quad p = \frac{n}{\alpha}\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1}.$$

Assume that $p \leq n/\alpha$ and set

$$q = \begin{cases} np/(n - \alpha p), & p < n/\alpha \\ \infty, & p = n/\alpha. \end{cases}$$

Assume also that for $k = 1, \ldots, n$,

$$r_k \equiv \frac{1}{p} - \frac{\alpha}{n} - \frac{1}{p_k} + \alpha_k > 0.$$

If $f \in WB^{\alpha_1,\ldots,\alpha_n}_{p_1\ldots,p_n}(\mathbb{R}^n)$, then $f \in L^{q,p}(\mathbb{R}^n)$ and

$$\|f\|^*_{q,p} \leq c_n \prod_{k=1}^{m} \Big[d_k \|D_k f\|_{p_k}\Big]^{\alpha/n} \prod_{k=m+1}^{n} \Big[d_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}}\Big]^{\alpha/(n\alpha_k)},$$

where c_n depends only on n,

$$d_k = r_k^{-1/p_k} \max(R, p_k)^2, \quad \textit{and} \quad R = \max_{k=1,\ldots,n} \frac{1}{r_k} \max_{k=1,\ldots,n} r_k.$$

Theorem 6.12 can be proved in the same way as Theorem 6.11, using Theorem 5.4, Proposition 6.10, and Lemma 6.4. As we will now show, Theorem 6.12 can also be obtained by using Theorem 6.11 and letting $\alpha_1, \ldots, \alpha_m \to 1-$. Assume therefore that the conditions of Theorem 6.12 hold. For simplicity we shall only consider the case $m = 1$. Then $f \in W^1_{p_1;1}(\mathbb{R}^n)$. By Proposition 6.9, it follows that there exists an $\varepsilon \in (0,1)$ such that $f \in B^{1-\varepsilon}_{p_1;1}(\mathbb{R}^n)$. Put $\tilde{\alpha}_1 = 1 - \varepsilon$. Let $\tilde{\alpha}$, $\tilde{p}$, $\tilde{q}$, $\tilde{r}_k$, and $\tilde{d}_k$, $k = 1, \ldots, n$, be defined by replacing α_1 by $\tilde{\alpha}_1$ in α, p, q, r_k, and d_k, respectively. By assumption we know that $r_k > 0$, for all k. Therefore we may assume that ε was chosen so small that also $\tilde{r}_k > 0$, for all k. Moreover, $\tilde{p} < n/\tilde{\alpha}$. Indeed,

$$\tilde{\alpha}\tilde{p} = n\Big(\frac{1}{\tilde{\alpha}_1 p_1} + \sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1} < n\Big(\sum_{k=1}^{n} \frac{1}{\alpha_k p_k}\Big)^{-1} = \alpha p \leq n.$$

Hence, the conditions of Theorem 6.11 are satisfied, and so we have that

$$\|f\|^*_{\tilde{q},\tilde{p}} \leq c_n \Big[\tilde{d}_1 (1-\tilde{\alpha}_1)^{1/p_1} \|f\|_{b^{\tilde{\alpha}_1}_{p_1;1}}\Big]^{\tilde{\alpha}/(n\tilde{\alpha}_1)} \times$$

$$\times \prod_{k=2}^{n} \left[\tilde{d}_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}}\right]^{\tilde{\alpha}/(n\alpha_k)}. \tag{6.40}$$

By Fatou's lemma,

$$\int_0^\infty \left[t^{1/q}\Delta_f(t)\right]^p \frac{dt}{t} \le \liminf_{\varepsilon\to 0+} \int_0^\infty \left[t^{1/\tilde{q}}\Delta_f(t)\right]^{\tilde{p}} \frac{dt}{t}.$$

Let $\varepsilon \to 0+$ in (6.40). By the preceding inequality and Proposition 6.9,

$$\|f\|^*_{q,p} \le c_n \Big[d_1 \Big(\frac{1}{p_1}\Big)^{1/p_1} \|D_1 f\|_{p_1}\Big]^{\alpha/n} \prod_{k=2}^{n} \left[d_k (1-\alpha_k)^{1/p_k} \|f\|_{b^{\alpha_k}_{p_k;k}}\right]^{\alpha/(n\alpha_k)}.$$

This proves Theorem 6.12.

7. On Fournier's theorem

Throughout this section we set

$$V_k = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})],$$

for $n \geq 2$, $k = 1, \ldots, n$. As it was pointed out in Section 1, one of the basic results for mixed norm spaces that we study in this work is the following theorem due to Fournier [16].

Theorem 7.1. *Let $n \geq 2$. If $f \in \cap_{k=1}^n V_k$, then $f \in L^{n',1}(\mathbb{R}^n)$ and*

$$\|f\|_{n',1} \leq n' \prod_{k=1}^n \|f\|_{V_k}^{1/n}. \tag{7.1}$$

It follows from Theorem 1.6 that the preceding inequality remains true if we replace the interior L^∞-norm in V_k by the weaker norm of $weak - L^\infty$. This section is devoted to the further study of embeddings of the spaces V_k.

7.1. **Iterative rearrangement inequalities.**

As we observed in Section 1, Theorem 7.1 implies the Sobolev-type inequality (1.5) for $p = 1$. It was proved in [28] that the $L^{q,p}$-norm on the left-hand side in (1.5) can be replaced by the stronger $\mathcal{L}^{q,p}$-norm (see Section 2.4). Namely, there holds the following strengthening of inequality (1.5).

Theorem 7.2. *Let $n \geq 2$ and $1 \leq p < n$. Set $q = np/(n-p)$. If $f \in W_p^1(\mathbb{R}^n)$, then $f \in \mathcal{L}^{q,p}(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{L}^{q,p}} \leq c \sum_{k=1}^n \|D_k f\|_p,$$

where c depends only on n and p.

This motivates us to study a similar refinement for mixed norm spaces. In particular, we will show that the left-hand side in (7.1) can be replaced by the stronger $\mathcal{L}^{n',1}_\sigma$-norm. In the next lemma we show that the iterative rearrangement does not increase the V_k-norm of a function. Note that (for $f \in S_0(\mathbb{R}^n)$ and $\sigma \in \mathcal{P}_n$) the function $\mathcal{R}_\sigma f$ is defined only on $\mathbb{R}^n_+$, so its V_k-norm is given by

$$\|\mathcal{R}_\sigma f\|_{V_k(\mathbb{R}^n_+)} = \int_{\mathbb{R}^{n-1}_+} \operatorname{ess\,sup}_{t_k>0} \mathcal{R}_\sigma f(t) d\hat{t}_k.$$

Lemma 7.3. *Let $f \in S_0(\mathbb{R}^n)$. Then*

$$\|\mathcal{R}_\sigma f\|_{V_k(\mathbb{R}^n_+)} \leq \|f\|_{V_k(\mathbb{R}^n)} \quad (k = 1, \ldots, n) \tag{7.2}$$

for all $\sigma \in \mathcal{P}_n$.

Proof. Let $\sigma \in \mathcal{P}_n$. We have

$$|f(x)| \le \|f(\hat{x}_k, \cdot)\|_\infty \equiv \alpha_k(\hat{x}_k)$$

for almost all $x \in \mathbb{R}^n$. Hence, $\mathcal{R}_\sigma f(t) \le \mathcal{R}_{\sigma_k}\alpha_k(\hat{t}_k)$, where σ_k is obtained from σ by removing k. This gives

$$\|\mathcal{R}_\sigma f(\hat{t}_k, \cdot)\|_\infty \le \mathcal{R}_{\sigma_k}\alpha_k(\hat{t}_k), \quad \hat{t}_k \in \mathbb{R}^{n-1}_+.$$

Now (7.2) follows from this inequality by taking $L^1(\mathbb{R}^{n-1}_+)$-norm of both sides and applying (2.41). □

Theorem 7.4. *Let $n \ge 2$, $1 \le p_1, \dots, p_n < \infty$, and*

$$\sum_{k=1}^n \frac{1}{p'_k} = 1. \tag{7.3}$$

Assume that $f \in \cap_{k=1}^n V_k$. Then

$$\int_{\mathbb{R}^n_+} \Big(\prod_{k=1}^n t_k^{1/p_k - 1}\Big)\mathcal{R}_\sigma f(t)dt \le \sum_{k=1}^n p_k \|f\|_{V_k}, \tag{7.4}$$

for all $\sigma \in \mathcal{P}_n$.

Proof. [1] Fix $\sigma \in \mathcal{P}_n$ and denote $\varphi = \mathcal{R}_\sigma f$. Set

$$A_j = \{t \in \mathbb{R}^n_+ : t_j^{1/p_j} \le \prod_{k \ne j} t_k^{1/p'_k}\}.$$

Then

$$\bigcup_{j=1}^n A_j = \mathbb{R}^n_+. \tag{7.5}$$

Indeed, suppose that there exists $t \in \mathbb{R}^n_+ \setminus \Big(\cup_{j=1}^n A_j\Big)$. For any j we get

$$\tau \equiv \prod_{k=1}^n t_k^{1/p'_k} < t_j.$$

This implies that

$$\tau^{1/p'_1 + \dots + 1/p'_n} < \tau,$$

which contradicts the assumption (7.3).

Observe that

$$\varphi(t) \le \|\varphi(\hat{t}_j, \cdot)\|_\infty \equiv \alpha_j(\hat{t}_j), \quad t \in \mathbb{R}^n_+, \quad j = 1, \dots, n.$$

[1] In this proof we apply arguments similar to those used in [28].

We now get

$$\int_{A_j}\Big(\prod_{k=1}^{n}t_k^{1/p_k-1}\Big)\varphi(t)dt \le \int_{\mathbb{R}_+^{n-1}}\Big(\prod_{k\ne j}t_k^{1/p_k-1}\Big)\alpha_j(\hat{t}_j)d\hat{t}_j\int_0^{\tau_j}t_j^{1/p_j-1}dt_j,$$

where

$$\tau_j=\Big(\prod_{k\ne j}t_k^{1/p_k'}\Big)^{p_j}.$$

Computing the last integral and using (7.5), we get

$$\int_{\mathbb{R}_+^{n}}\Big(\prod_{k=1}^{n}t_k^{1/p_k-1}\Big)\varphi(t)dt \le \sum_{j=1}^{n}p_j\int_{\mathbb{R}_+^{n-1}}\alpha_j(\hat{t}_j)d\hat{t}_j=\sum_{j=1}^{n}p_j\|\varphi\|_{V_j}.$$

Applying (7.2), we obtain (7.4). □

Remark 7.5. For $n=2$, $1\le p<\infty$, the above theorem states that

$$\int_{\mathbb{R}_+^2}t_1^{1/p-1}t_2^{1/p'-1}\mathcal{R}_\sigma f(t)dt\le p\|f\|_{V_1}+p'\|f\|_{V_2}. \tag{7.6}$$

We stress that the constant coefficients on the left-hand side are the best possible. Indeed, for $f=\chi_{(0,1)^2}$, the right-hand side equals pp', and $\|f\|_{V_1}=\|f\|_{V_2}=1$. Thus, for this function, we have equality in (7.6), and so the constants are sharp. However, for $n\ge 3$, the constants in (7.4) are not optimal. In Proposition 7.7 below, we find the sharp constant in the case $n=3$ and $p_1=p_2=p_3$.

We will use the following inequality due to Chebyshev (see [21]).

Lemma 7.6. *Let f and g be non-negative functions on $\mathbb{R}_+$, and assume that f is non-increasing and g is non-decreasing. For every $a>0$,*

$$\int_0^a f(t)g(t)dt\le\frac{1}{a}\int_0^a f(t)dt\int_0^a g(t)dt. \tag{7.7}$$

Proposition 7.7. *If $f\in S_0(\mathbb{R}^3)$, then*

$$\iiint_{\mathbb{R}_+^3}(xyz)^{-1/3}\mathcal{R}_\sigma f(x,y,z)dxdydz\le\frac{9}{8}\Big(\|f\|_{V_1}+\|f\|_{V_2}+\|f\|_{V_3}\Big), \tag{7.8}$$

for all $\sigma\in\mathcal{P}_3$.

Proof. By Lemma 7.3, we can assume that f is non-negative and non-increasing in each variable on $\mathbb{R}_+^3$, and that f vanishes outside $\mathbb{R}_+^3$. Let

$$A_1=\{(x,y,z)\in\mathbb{R}_+^3 : x\le\min(y,z)\}.$$

We define the sets A_2 and A_3 analogously. Note that

$$\mathbb{R}_+^3=A_1\cup A_2\cup A_3. \tag{7.9}$$

Set

$$I_k = \iiint_{A_k} (xyz)^{-1/3} f(x,y,z)\,dxdydz,$$

for $k = 1,2,3$. We write I_1 as the sum

$$I_1 = I_1' + I_1'', \tag{7.10}$$

where we in I_1' integrate over the part of A_1 where $y \le z$, and in I_1'' we integrate over the part where $y \ge z$. Then we have

$$I_1' = \int_0^\infty z^{-1/3} dz \int_0^z y^{-1/3} dy \int_0^y x^{-1/3} f(x,y,z)\,dx.$$

Note that

$$f(x,y,z) \le \operatorname{ess\,sup}_{x>0} f(x,y,z) \equiv \alpha(y,z).$$

Thus,

$$I_1' \le \frac{3}{2} \int_0^\infty z^{-1/3} dz \int_0^z y^{1/3} \alpha(y,z)\,dy.$$

Since $\alpha(y,z)$ is non-increasing in y, we can apply inequality (7.7) from Lemma 7.6. This gives

$$I_1' \le \frac{3}{2} \int_0^\infty z^{-4/3} \Big(\int_0^z y^{1/3} dy \int_0^z \alpha(y,z)\,dy \Big) dz,$$

and thus

$$I_1' \le \frac{9}{8} \int_0^\infty dz \int_0^z \alpha(y,z)\,dy. \tag{7.11}$$

In I_1'' we integrate over the set where $0 < x \le z \le y$, so we have

$$I_1'' = \int_0^\infty y^{-1/3} dy \int_0^y z^{-1/3} dz \int_0^z x^{-1/3} f(x,y,z)\,dx.$$

Changing the roles of y and z in the proof of (7.11), we obtain

$$I_1'' \le \frac{9}{8} \int_0^\infty dy \int_0^y \alpha(y,z)\,dz.$$

Combining this estimate with (7.10) and (7.11), we get

$$I_1 \le \frac{9}{8} \iint_{\mathbb{R}_+^2} \alpha(y,z)\,dydz = \frac{9}{8} \|f\|_{V_1^1(\mathbb{R}_+^2)}.$$

In the same way we prove that

$$I_k \le \frac{9}{8} \|f\|_{V_k^1(\mathbb{R}_+^2)},$$

also for $k = 2,3$. By (7.9),

$$\iiint_{\mathbb{R}_+^3} (xyz)^{-1/3} f(x,y,z)\,dxdydz = I_1 + I_2 + I_3.$$

Apply the preceding estimates of I_k, $k = 1, 2, 3$. This proves (7.8). $\square$

Remark 7.8. We emphasize that the constant coefficient in (7.8) is the best possible. Indeed, for $f = \chi_{(0,1)^3}$, the left-hand side in this inequality equals $(3/2)^3$, and $\|f\|_{V_k} = 1$, $k = 1, 2, 3$. Thus, for this function we have equality in (7.8), and so the constant is optimal.

In the next theorem we use standard reasonings (see e.g. [38, Chapter 7, 7.2]) to obtain the multiplicative inequality (7.13) from inequality (7.4) in Theorem 7.4.

Theorem 7.9. *Let $n \geq 2$ and $1 \leq p_1, \dots, p_n < \infty$. Assume that*

$$\sum_{k=1}^{n} \frac{1}{p'_k} = 1. \tag{7.12}$$

Suppose that $f \in \cap_{k=1}^{n} V_k$. Then

$$\int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) \mathcal{R}_\sigma f(t) dt \leq \prod_{k=1}^{n} \left(p_k p'_k \|f\|_{V_k} \right)^{1/p'_k}, \tag{7.13}$$

for all $\sigma \in \mathcal{P}_n$.

Proof. Fix $\sigma \in \mathcal{P}_n$ and denote $\varphi = \mathcal{R}_\sigma f$. For $\varepsilon_1, \dots, \varepsilon_n > 0$ we set

$$\varepsilon = \prod_{k=1}^{n} \varepsilon_k \quad \text{and} \quad g(t) = \varphi(\varepsilon_1 t_1, \dots, \varepsilon_n t_n), \quad t \in \mathbb{R}_+^n.$$

Then

$$\|g\|_{V_k(\mathbb{R}_+^n)} = \int_{\mathbb{R}_+^{n-1}} \operatorname{ess\,sup}_{t_k > 0} \varphi(\varepsilon_1 t_1, \dots, \varepsilon_n t_n) d\hat{t}_k.$$

Making the change of variables $t_j \mapsto \varepsilon_j t_j$, $j \neq k$, we obtain

$$\|g\|_{V_k(\mathbb{R}_+^n)} = \frac{\varepsilon_k}{\varepsilon} \|\varphi\|_{V_k(\mathbb{R}_+^n)}. \tag{7.14}$$

We also have

$$\int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) g(t) dt = \Big(\prod_{k=1}^{n} \varepsilon_k^{-1/p_k} \Big) \int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) \varphi(t) dt. \tag{7.15}$$

By Theorem 7.4, we have (here we use that $g = \mathcal{R}_\sigma g$)

$$\int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) g(t) dt \leq \sum_{k=1}^{n} p_k \|g\|_{V_k(\mathbb{R}_+^n)}.$$

Applying (7.14) and (7.15) to this inequality gives

$$\int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) \varphi(t) dt \leq \Big(\prod_{k=1}^{n} \varepsilon_k^{1/p_k} \Big) \sum_{k=1}^{n} \frac{\varepsilon_k p_k}{\varepsilon} \|\varphi\|_{V_k} =$$

$$= \Big(\prod_{k=1}^{n} \varepsilon_k^{-1/p'_k} \Big) \sum_{k=1}^{n} p_k \|\varphi\|_{V_k} \varepsilon_k. \tag{7.16}$$

Set now $\varepsilon_k = (p'_k p_k \|\varphi\|_{V_k(\mathbb{R}^n_+)})^{-1}$, $k = 1, \dots, n$ (it is easy to show that for these values of $\varepsilon_1, \dots, \varepsilon_n$, the right-hand side of (7.16) is minimized). By (7.12), inequality (7.16) then becomes

$$\int_{\mathbb{R}^n_+} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) \varphi(t) dt \le \prod_{k=1}^{n} [p'_k p_k \|\varphi\|_{V_k(\mathbb{R}^n_+)}]^{1/p'_k}.$$

Applying (7.2) yields (7.13). □

Applying Theorem 7.9, we obtain the next result.

Theorem 7.10. *Let $n \ge 2$. If $f \in \cap_{k=1}^{n} V_k$, then $f \in \mathcal{L}_\sigma^{n',1}(\mathbb{R}^n)$, and*

$$\|f\|_{\mathcal{L}_\sigma^{n',1}} \le nn' \prod_{k=1}^{n} \|f\|_{V_k}^{1/n},$$

for every $\sigma \in \mathcal{P}_n$.

Proof. Set $p_1 = \cdots = p_n = n'$ in Theorem 7.9. Then the condition (7.12) is satisfied. Thus, we have

$$\|f\|_{\mathcal{L}_\sigma^{n',1}} = \int_{\mathbb{R}^n_+} \Big(\prod_{k=1}^{n} t_k^{1/n' - 1} \Big) \mathcal{R}_\sigma f(t) dt \le nn' \prod_{k=1}^{n} \|f\|_{V_k}^{1/n}.$$

□

Remark 7.11. Since, by Theorem 2.10,

$$\|f\|_{n',1} \le \|f\|_{\mathcal{L}_\sigma^{n',1}},$$

Theorem 7.10 implies Theorem 7.1 (although with a worse constant coefficient). We emphasize that the norm in $\mathcal{L}_\sigma^{n',1}(\mathbb{R}^n)$ is *stronger* than the norm in $L^{n',1}(\mathbb{R}^n)$. Namely, there exists a function f for which $\|f\|_{n',1} < \infty$ and $\|f\|_{\mathcal{L}_\sigma^{n',1}} = \infty$ (see [30, Proposition 5.4]). Thus, Theorem 7.10 gives a refinement of Theorem 7.1.

7.2. **Intermediate embeddings.** Let $n \ge 2$ and $1 \le p \le (n-1)'$. Set $r = p'/(n-1)$. For $k = 1, \dots, n$ we define the mixed norm space

$$V_k^p = L_{\hat{x}_k}^{p,1}(\mathbb{R}^{n-1})[L_{x_k}^{r,1}(\mathbb{R})]. \tag{7.17}$$

Observe that $V_k^1 = V_k$ and the norms coincide. Indeed, if $p = 1$, then $r = \infty$, and therefore

$$V_k^1 = L_{\hat{x}_k}^{1,1}(\mathbb{R}^{n-1})[L_{x_k}^{\infty,1}(\mathbb{R})].$$

Recall that the norm in $L^{\infty,1}$ is defined by

$$\|f\|_{\infty,1} = \int_0^\infty \Big[f^{**}(t) - f^*(t)\Big]\frac{dt}{t},$$

and that $\|f\|_{\infty,1} = \|f\|_\infty$ (see (2.32)). It follows that

$$V_k^1 = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})] = V_k, \tag{7.18}$$

and

$$\|f\|_{V_k^1} = \|f\|_{V_k} \quad (k = 1, \dots, n).$$

Note that the case $n = 2$ and $p = \infty$ also is included in the definition of V_k^p. Then $r = p' = 1$, so by (2.32)

$$V_k^\infty = L^{\infty,1}_{\hat{x}_k}(\mathbb{R})[L^{1,1}_{x_k}(\mathbb{R})] = L^\infty_{\hat{x}_k}(\mathbb{R})[L^1_{x_k}(\mathbb{R})], \quad k = 1, 2. \tag{7.19}$$

The next theorem states that the space $\cap_{k=1}^n V_k$ is embedded to each of the spaces V_j^p, for $n \geq 2$, $1 < p \leq (n-1)'$, and $j = 1, \dots, n$.

Theorem 7.12. *Let $n \geq 2$, $1 < p \leq (n-1)'$, and $r = p'/(n-1)$. Set*

$$V_k = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})]$$

and

$$V_k^p = L^{p,1}_{\hat{x}_k}(\mathbb{R}^{n-1})[L^{r,1}_{x_k}(\mathbb{R})],$$

$k = 1, \dots, n$. If $f \in \cap_{k=1}^n V_k$, then $f \in \cap_{k=1}^n V_k^p$, and for every $j = 1, \dots, n$ it holds that

$$\|f\|_{V_j^p} \leq c_{n,p} \|f\|_{V_j}^{1/r'} \prod_{k \neq j} \|f\|_{V_k}^{1/p'}, \tag{7.20}$$

where $c_{n,p} = 1$ if $n = 2$ and $p = \infty$, and $c_{n,p} = (rr')^{1/r'}(pp')^{(n-1)/p'}$ otherwise.

Proof. First we consider the case when $n = 2$ and $p = \infty$. Then $p' = 1$, $r = 1$, and $r' = \infty$. We show that

$$\|f\|_{V_1^\infty} \leq \|f\|_{V_2} \quad \text{and} \quad \|f\|_{V_2^\infty} \leq \|f\|_{V_1}. \tag{7.21}$$

As we observed in (7.19),

$$V_1^\infty = L^\infty_{x_2}(\mathbb{R})[L^1_{x_1}(\mathbb{R})]$$

and the norms are equal. Hence, the first inequality in (7.21) says that

$$\operatorname{ess\,sup}_{x_2 \in \mathbb{R}} \int_{\mathbb{R}} |f(x_1, x_2)| dx_1 \leq \int_{\mathbb{R}} \operatorname{ess\,sup}_{x_2 \in \mathbb{R}} |f(x_1, x_2)| dx_1. \tag{7.22}$$

It is obvious that

$$\int_{\mathbb{R}} |f(x_1, x_2)| dx_1 \leq \int_{\mathbb{R}} \operatorname{ess\,sup}_{y \in \mathbb{R}} |f(x_1, y)| dx_1,$$

for almost every $x_2 \in \mathbb{R}$. From this, (7.22) is immediate. The second inequality in (7.21) is obtained in the same way.

Assume now that either $n = 2$ and $1 < p < \infty$, or $n \geq 3$ and $1 < p \leq (n-1)'$. We will prove (7.20) for $j = n$ (this inequality can be proved in the same way for any j). Set $p_1 = \cdots = p_{n-1} = p$ and $p_n = r$. Then

$$\sum_{k=1}^{n} \frac{1}{p_k'} = \frac{n-1}{p'} + \frac{1}{r'} = \frac{1}{r} + \frac{1}{r'} = 1.$$

That is, the condition (7.12) in Theorem 7.9 is satisfied. Let $\sigma = (n, \ldots, 1)$. According to Theorem 7.9, we then have

$$\int_{\mathbb{R}_+^{n-1}} \Big(\prod_{k=1}^{n-1} t_k \Big)^{1/p-1} \Phi(\hat{t}_n) d\hat{t}_n \leq (rr'\|f\|_{V_n})^{1/r'} \prod_{k=1}^{n-1} (pp'\|f\|_{V_k})^{1/p'}, \quad (7.23)$$

where

$$\Phi(\hat{t}_n) = \int_0^\infty t_n^{1/r-1} \mathcal{R}_\sigma f(t) dt_n.$$

Set

$$\Psi(\hat{x}_n) = \int_0^\infty t_n^{1/r-1} \mathcal{R}_n f(\hat{x}_n, t_n) dt_n.$$

By the definition of V_n^p,

$$\|f\|_{V_n^p} = \|\Psi\|_{p,1} = \int_0^\infty s^{1/p-1} \Psi^*(s) ds.$$

Replace $s^{1/p-1}$ in the preceding integral by

$$s^{1/p-1} = \frac{1}{p'} \int_s^\infty u^{1/p-2} du$$

and change the order of integration. This gives

$$\|f\|_{V_n^p} = \frac{1}{p'} \int_0^\infty u^{1/p-2} \Big(\int_0^u \Psi^*(s) ds \Big) du. \quad (7.24)$$

As we noticed in Section 2.3 (see (2.26)), for every $u > 0$ there exists a measurable set $E_u \subset \mathbb{R}^{n-1}$ such that $\mathrm{mes}_{n-1} E_u = u$ and

$$\int_0^u \Psi^*(s) ds = \int_{E_u} \Psi(\hat{x}_n) d\hat{x}_n. \quad (7.25)$$

Set $F_{t_n}(\hat{x}_n) = \mathcal{R}_n f(\hat{x}_n, t_n)$. By the definition of Ψ, we get from (7.25) that

$$\int_0^u \Psi^*(s) ds = \int_0^\infty t_n^{1/r-1} \Big(\int_{E_u} F_{t_n}(\hat{x}_n) d\hat{x}_n \Big) dt_n.$$

Using (2.26), we get

$$\int_0^u \Psi^*(s) ds \leq \int_0^\infty t_n^{1/r-1} \Big(\int_0^u F_{t_n}^*(\tau) d\tau \Big) dt_n.$$

Applying this inequality at the right-hand side in (7.24) and changing the the order of integration, we get

$$\|f\|_{V_n^p} \le \int_0^\infty t_n^{1/r-1} I_p(t_n)dt_n, \tag{7.26}$$

where

$$I_p(t_n) = \frac{1}{p'}\int_0^\infty u^{1/p-2}\Big(\int_0^u F_{t_n}^*(\tau)d\tau\Big)du.$$

Further, the change of the order of integration in $I_p(t_n)$ yields

$$I_p(t_n) = \frac{1}{p'}\int_0^\infty \Big(\int_\tau^\infty u^{1/p-2}du\Big)F_{t_n}^*(\tau)d\tau = \|F_{t_n}\|_{p,1}.$$

Let $t_n > 0$ be fixed and apply Theorem 2.10. We get

$$I_p(t_n) = \|F_{t_n}\|_{p,1} \le \|F_{t_n}\|_{\mathcal{L}^{p,1}_{\widehat{\sigma}_1}} = \int_{\mathbb{R}_+^{n-1}} \Big(\prod_{k=1}^{n-1} t_k\Big)^{1/p-1} \mathcal{R}_{\widehat{\sigma}_1} F_{t_n}(\hat{t}_n)d\hat{t}_n,$$

where $\widehat{\sigma}_1$ denotes the $(n-1)$-dimensional vector obtained from σ by removal of its first coordinate, i.e. $\widehat{\sigma}_1 = (n-1,\dots,1)$ since $\sigma = (n, n-1, \dots, 1)$. Recall that $F_{t_n}(\hat{x}_n) = \mathcal{R}_n f(\hat{x}_n, t_n)$. By the definition of the iterative rearrangement, we then have

$$\mathcal{R}_{\widehat{\sigma}_1} F_{t_n}(\hat{t}_n) = \mathcal{R}_\sigma f(t).$$

So, by the preceding inequality,

$$I_p(t_n) \le \int_{\mathbb{R}_+^{n-1}} \Big(\prod_{k=1}^{n-1} t_k\Big)^{1/p-1} \mathcal{R}_\sigma f(t)d\hat{t}_n.$$

We use this estimate of $I_p(t_n)$ in (7.26), and change the order of integration. By the definition of Φ, we then obtain

$$\|f\|_{V_n^p} \le \int_{\mathbb{R}_+^{n-1}} \Big(\prod_{k=1}^{n-1} t_k\Big)^{1/p-1} \Phi(\hat{t}_n)d\hat{t}_n.$$

This estimate together with (7.23) implies inequality (7.20). $\square$

Remark 7.13. In the preceding theorem there is proved an estimate of the mixed norm by the iterated rearrangement Lorentz norm (inequality (7.26)). This estimate can also be derived from a more general theorem obtained by Blozinski [9, Theorem 4.5 I]. Similar questions have also been studied in [2].

Corollary 7.14. *Let $n \ge 2$ and $1 < p \le (n-1)'$. Put $r = p'/(n-1)$. Set*

$$V_k = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})]$$

and

$$V_k^p = L^{p,1}_{\hat{x}_k}(\mathbb{R}^{n-1})[L^{r,1}_{x_k}(\mathbb{R})],$$

$k = 1, \dots, n$. *If* $f \in \cap_{k=1}^n V_k$, *then* $f \in \cap_{k=1}^n V_k^p$, *and*

$$\prod_{k=1}^n \|f\|_{V_k^p} \le c_{n,p}^n \prod_{k=1}^n \|f\|_{V_k}, \tag{7.27}$$

where $c_{n,p} = 1$ *if* $n = 2$ *and* $p = \infty$, *and* $c_{n,p} = (rr')^{1/r'}(pp')^{(n-1)/p'}$ *otherwise.*

Proof. The conditions of Theorem 7.12 are satisfied, so inequality (7.20) holds. It follows from this inequality that

$$\prod_{j=1}^n \|f\|_{V_j^p} \le c_{n,p}^n \prod_{j=1}^n \left[\|f\|_{V_j}^{1/r'} \prod_{k\neq j} \|f\|_{V_k}^{1/p'}\right] = c_{n,p}^n \prod_{j=1}^n \|f\|_{V_j}$$

(here we use the fact that $1/r' + (n-1)/p' = 1$). Hence, we have obtained inequality (7.27). □

The next corollary shows that inequality (7.20) in Theorem 7.12 implies simpler, additive, inequalities. To prove this result, we will use the following classical convexity inequality: for any $x_1, \dots, x_n \ge 0$ and any $\alpha_1, \dots, \alpha_n > 0$ such that

$$\sum_{k=1}^n \alpha_k = 1,$$

it holds that

$$\prod_{k=1}^n x_k^{\alpha_k} \le \sum_{k=1}^n \alpha_k x_k. \tag{7.28}$$

Corollary 7.15. *Let* $n \ge 2$ *and* $1 < p \le (n-1)'$. *Set* $r = p'/(n-1)$. *Assume that* $f \in \cap_{k=1}^n V_k$. *Then* $f \in \cap_{k=1}^n V_k^p$. *If* $n = 2$ *and* $p = \infty$, *then*

$$\|f\|_{V_1^\infty} \le \|f\|_{V_2} \quad \text{and} \quad \|f\|_{V_2^\infty} \le \|f\|_{V_1}. \tag{7.29}$$

If either $n = 2$ *and* $1 < p < \infty$, *or* $n \ge 3$ *and* $1 < p \le (n-1)'$, *then*

$$\|f\|_{V_j^p} \le r\|f\|_{V_j} + p \sum_{k\neq j} \|f\|_{V_k}, \tag{7.30}$$

for $j = 1, \dots, n$.

Proof. Let $n = 2$ and $p = \infty$. Then $r = p' = 1$ and $r' = \infty$. Hence, (7.29) follows immediately from (7.20).

Assume now that either $n = 2$ and $1 < p < \infty$, or $n \geq 3$ and $1 < p \leq (n-1)'$. Fix j. Set $x_j = rr'\|f\|_{V_j}$ and $x_k = pp'\|f\|_{V_k}$, $k \neq j$. We also put $\alpha_j = 1/r'$ and $\alpha_k = 1/p'$, $k \neq j$. Observe that

$$\sum_{k=1}^{n} \alpha_k = \frac{1}{r'} + (n-1)\frac{1}{p'} = \frac{1}{r'} + \frac{1}{r} = 1.$$

Hence, (7.28) holds for these values of $x_1, \ldots, x_n$ and $\alpha_1, \ldots, \alpha_n$. By inequalities (7.20) and (7.28), we get

$$\|f\|_{V_j^p} \leq \prod_{k=1}^{n} x_k^{\alpha_k} \leq \sum_{k=1}^{n} \alpha_k x_k = r\|f\|_{V_j} + p\sum_{k\neq j} \|f\|_{V_k}.$$

Thus, we have proved (7.30). $\square$

Remark 7.16. Let $n = 2$ and $f \in V_1 \cap V_2$. Taking $j = 1$ in (7.20), we get

$$\|f\|_{V_1^p} \leq pp'\|f\|_{V_1}^{1/p}\|f\|_{V_2}^{1/p'}, \tag{7.31}$$

for $1 < p < \infty$ and

$$\|f\|_{V_1^\infty} \leq \|f\|_{V_2}. \tag{7.32}$$

Furthermore, for $j = 1$, inequality (7.30) says that

$$\|f\|_{V_1^p} \leq p'\|f\|_{V_1} + p\|f\|_{V_2}. \tag{7.33}$$

Note that there hold corresponding inequalities for $j = 2$. If $f = \chi_{(0,1)^2}$, then $\|f\|_{V_1^p} = pp'$ for $1 < p < \infty$, $\|f\|_{V_1^\infty} = 1$, and $\|f\|_{V_k} = 1$, $k = 1, 2$. So, for this function we get equality in (7.31), (7.32), and (7.33) (we use here that $pp' = p + p'$). Hence, the constants in these inequalities are optimal. We observe that the constant in (7.31) tends to ∞ as $p \to \infty$, but for $p = \infty$ we have inequality (7.32) with the constant 1. It follows directly from (7.31) that

$$\limsup_{p\to\infty} \frac{1}{p}\|f\|_{V_1^p} \leq \|f\|_{V_2} \tag{7.34}$$

Moreover, as we will prove in Theorem 7.28 below, it is also true that

$$\|f\|_{V_1^\infty} \leq \liminf_{p\to\infty} \frac{1}{p}\|f\|_{V_1^p}.$$

These two relations show that (7.32) follows, as a limiting case, from (7.31).

In (7.18) we observed that when $p = 1$, then V_j^p coincides with V_j and the norms are equal. In Theorem 7.12, we showed that if $1 < p \leq (n-1)'$, then the space $\cap_{k=1}^n V_k$ is embedded to each V_j^p, $j = 1, \ldots, n$, $(n \geq 2)$. The remaining case is when $n \geq 3$ and $(n-1)' < p < \infty$. The following proposition shows that the embedding in Theorem 7.12 does not hold for any p in this range.

Proposition 7.17. *For $n \geq 3$ and $k = 1\ldots,n$ we set*

$$V_k = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})].$$

Let $r > 0$ and $(n-1)' < p < \infty$. There exists a function $f \in S_0(\mathbb{R}^n)$ such that $f \in \cap_{k=1}^n V_k$ and

$$\|f\|_{L^p_{\hat{x}_j}(\mathbb{R}^{n-1})[L^r_{x_j}(\mathbb{R})]} = \infty, \tag{7.35}$$

for each $j = 1,\ldots,n$.

Proof. We have $p > (n-1)'$, or equivalently $(n-1)/p < n-2$. Fix $(n-1)/p \leq \alpha < n-2$. For $j = 1,\ldots,n$ we define

$$f_j(x) = |\hat{x}_j|^{-\alpha}\chi_{I_n}(x), \quad x \in \mathbb{R}^n,$$

where $I_n = [0,1]^n$. Fix j. We have

$$\operatorname{ess\,sup}_{x_j \in \mathbb{R}} f_j(x) = |\hat{x}_j|^{-\alpha}\chi_{I_{n-1}}(\hat{x}_j),$$

and thus

$$\|f_j\|_{V_j} = \int_{I_{n-1}} |\hat{x}_j|^{-\alpha} d\hat{x}_j.$$

Since $\alpha < n-2$, we have that $\|f_j\|_{V_j} < \infty$. Further, for $k \neq j$ it holds that

$$\operatorname{ess\,sup}_{x_k \in \mathbb{R}} f_j(x) = |\hat{x}_{j,k}|^{-\alpha}\chi_{I_{n-1}}(\hat{x}_k),$$

where $\hat{x}_{j,k}$ denotes the $(n-2)$-dimensional vector, obtained from a given vector $x \in \mathbb{R}^n$ by removal of its jth and kth coordinates $(k \neq j)$. We get

$$\|f_j\|_{V_k} = \int_{I_{n-1}} |\hat{x}_{j,k}|^{-\alpha} d\hat{x}_k = \int_{I_{n-2}} |\hat{x}_{j,k}|^{-\alpha} d\hat{x}_{j,k}.$$

Since $\alpha < n-2$, this integral converges. We have now proved that

$$\|f_j\|_{V_k} < \infty, \quad k = 1,\ldots,n \tag{7.36}$$

(j is fixed). Next we observe that

$$\|f_j(\hat{x}_j,\cdot)\|_r = |\hat{x}_j|^{-\alpha}\chi_{I_{n-1}}(\hat{x}_j),$$

and since $\alpha p \geq n-1$,

$$\|f_j\|_{L^p_{\hat{x}_j}[L^r_{x_j}]} = \left(\int_{I_{n-1}} |\hat{x}_j|^{-\alpha p} d\hat{x}_j\right)^{1/p} = \infty. \tag{7.37}$$

Set now

$$f = \sum_{j=1}^n f_j.$$

By (7.36), it holds that $f \in \cap_{k=1}^n V_k$. Moreover, for every j, $f \geq f_j$ on $\mathbb{R}^n$. Hence, (7.37) implies (7.35) for all j. $\square$

It is clear that embeddings of the space $\cap_{k=1}^n V_k^p$ are closely connected with Theorem 5.4. In particular, the next proposition follows from Theorem 7.1 when $p=1$, and from Theorem 5.4 when $1<p<n'$.

Proposition 7.18. *Let $n \ge 2$ and $1 \le p < n'$. Set $r = p'/(n-1)$. Suppose $f \in \cap_{k=1}^n V_k^p$ and $f \in S_0(\mathbb{R}^n)$. Then $f \in L^{n',1}(\mathbb{R}^n)$ and*

$$\|f\|_{n',1} \le c \prod_{k=1}^n \|f\|_{V_k^p}^{1/n}, \tag{7.38}$$

where c depends only on p and n.

Proof. Let $p=1$. As we observed in (7.18), $\|f\|_{V_k^1} = \|f\|_{V_k}$, $k=1,\dots,n$. Thus, inequality (7.38) holds with $c=n'$, according to Theorem 7.1.

Assume now that $1<p<n'$. Then $p'>n$ and $r=p'/(n-1)>n'$. Thus, $r>p$. Set $p_k=p$, $s_k=1$, and $\alpha_k = 1/p-1/r$, for $k=1,\dots,n$. We will check that the conditions of Theorem 5.4 are satisfied. First we note that $\alpha_k>0$. Moreover, since all α_k coincide,

$$\alpha = n\Big(\sum_{k=1}^n \frac{1}{\alpha_k}\Big)^{-1} = \frac{1}{p} - \frac{1}{r},$$

and thus $\alpha p = 1 - p/r < 1$. It follows that $p < n/\alpha$, as required. We also have $\sigma_k = 1/p_k - \alpha_k = 1/p - \alpha = 1/r$ and

$$\frac{1}{p} - \frac{\alpha}{n} - \sigma_k = \alpha\Big(1 - \frac{1}{n}\Big) > 0,$$

for all $k=1,\dots,n$. Set $(k=1,\dots,n)$

$$\tilde V_k^p \equiv L^{p,1}_{\hat x_k}(\mathbb{R}^{n-1})[\Lambda^{1/r}_{x_k}(\mathbb{R})].$$

By Proposition 4.1 and inequality (2.31), we get

$$\|f\|_{\tilde V_k^p} \le c\|f\|_{V_k^p}$$

for all k. So $f \in \cap_{k=1}^n \tilde V_k^p$. Thus, all the conditions of Theorem 5.4 hold.

Since $\alpha = 1 - n/p'$, we see that

$$q = \frac{np}{n-\alpha p} = n\Big(\frac{n}{p} - \alpha\Big)^{-1} = n\Big(\frac{n}{p} + \frac{n}{p'} - 1\Big)^{-1} = n'.$$

By Theorem 5.4, we now have $f \in L^{q,s}(\mathbb{R}^n) = L^{n',1}(\mathbb{R}^n)$ and

$$\|f\|_{n',1}^* \le c \prod_{k=1}^n \|f\|_{\tilde V_k^p}^{1/n}.$$

The two latter inequalities together with (2.35) imply (7.38). $\square$

Remark 7.19. By Corollary 7.14 and Proposition 7.18, there holds the intermediate embedding

$$\bigcap_{k=1}^{n} V_k \subset \bigcap_{k=1}^{n} V_k^p \subset L^{n',1}(\mathbb{R}^n),$$

for $n \geq 2$ and $1 < p < n'$.

7.3. **Sobolev spaces.** Applying our previous results, we obtain some new embedding theorems for Sobolev spaces. The following result is a corollary of Theorem 7.9.

Corollary 7.20. *Let $n \geq 2$ and $1 \leq p_1, \dots, p_n < \infty$. Assume that*

$$\sum_{k=1}^{n} \frac{1}{p_k'} = 1. \tag{7.39}$$

If $f \in W_1^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/p_k - 1} \Big) \mathcal{R}_\sigma f(t) dt \leq \prod_{k=1}^{n} \big(p_k p_k' \|D_k f\|_1 \big)^{1/p_k'}, \tag{7.40}$$

for all $\sigma \in \mathcal{P}_n$.

Proof. By Theorem 7.9 we have the inequality (7.13). Further, (7.13) and (1.4) imply (7.40). □

Remark 7.21. Corollary 7.20 implies Theorem 7.2, in the case $p = 1$. Indeed, set $p_1 = \cdots = p_n = n'$ in this corollary. Then condition (7.39) holds, and inequality (7.40) becomes

$$\|f\|_{\mathcal{L}_\sigma^{n',1}} = \int_{\mathbb{R}_+^n} \Big(\prod_{k=1}^{n} t_k^{1/n' - 1} \Big) \mathcal{R}_\sigma f(t) dt \leq nn' \prod_{k=1}^{n} \|D_k f\|_1^{1/n}.$$

This estimate implies the conclusion of Theorem 7.2, for $p = 1$.

By inequality (1.4), for $k = 1, \dots, n$,

$$\|f\|_{V_k} \leq \frac{1}{2} \|D_k f\|_1.$$

Applying this estimate and Corollary 7.14, we immediately get

Corollary 7.22. *Let $n \geq 2$ and $1 < p \leq (n-1)'$. Set $r = p'/(n-1)$ and*

$$V_k^p = L_{\hat{x}_k}^{p,1}(\mathbb{R}^{n-1})[L_{x_k}^{r,1}(\mathbb{R})],$$

$k = 1, \dots, n$. If $f \in W_1^1(\mathbb{R}^n)$, then $f \in \cap_{k=1}^n V_k^p$, and

$$\prod_{k=1}^{n} \|f\|_{V_k^p} \leq c \prod_{k=1}^{n} \|D_k f\|_1, \tag{7.41}$$

where c depends only on p and n.

Remark 7.23. Corollary 7.22 gives relevant information on the behaviour of sections of Sobolev functions. In particular, for $p = n'$, this corollary states that

$$W_1^1(\mathbb{R}^n) \subset L_{\hat{x}_k}^{n',1}(\mathbb{R}^{n-1})[L_{x_k}^{n',1}(\mathbb{R})], \quad k = 1, \dots, n,$$

so for $p = n = 2$ we have that

$$W_1^1(\mathbb{R}^2) \subset L_{\hat{x}_k}^{2,1}(\mathbb{R})[L_{x_k}^{2,1}(\mathbb{R})], \quad k = 1, 2.$$

This inclusion does not follow from the strong type Sobolev inequality (1.5), which states that

$$W_1^1(\mathbb{R}^2) \subset L^{2,1}(\mathbb{R}^2).$$

Indeed, it was proved by Cwikel [13] that

$$L^{2,1}(\mathbb{R}^2) \not\subset L_{\hat{x}_k}^{2,1}(\mathbb{R})[L_{x_k}^{2,1}(\mathbb{R})], \quad k = 1, 2.$$

We point out that the known results in terms of iterative rearrangements are stronger. In particular, by Theorem 7.2 and [9, Theorem 4.5 I],

$$W_1^1(\mathbb{R}^2) \subset \mathcal{L}^{2,1}(\mathbb{R}^2) \subset L_{\hat{x}_k}^{2,1}(\mathbb{R})[L_{x_k}^{2,1}(\mathbb{R})], \quad k = 1, 2.$$

7.4. **Limiting relations.** It was proved in Theorem 7.12 that the the space $\cap_{k=1}^n V_k$ is embedded to V_j^p, for $n \geq 2$, $1 < p \leq (n-1)'$, and $j = 1, \dots, n$. We also observed that $\|\cdot\|_{V_j^1} = \|\cdot\|_{V_j}$, $j = 1, \dots, n$ (see (7.18)). In Theorem 7.26 below, we will clarify the limiting behaviour of $\|f\|_{V_k^p}$, as $p \to 1+$. We shall use the following lemma.

Lemma 7.24. *Let $r_0 \geq 1$. Suppose that $\phi \in L^{r_0,1}(\mathbb{R})$. Then*

$$\lim_{r\to\infty} \frac{1}{r}\|\phi\|_{r,1} = \|\phi\|_\infty.$$

Proof. We have

$$\frac{1}{r}\|\phi\|_{r,1} \geq \frac{1}{r}\int_0^{1/r} t^{1/r-1}\phi^*(t)dt \geq r^{-1/r}\phi^*(1/r).$$

So by (2.15),

$$\liminf_{r\to\infty} \frac{1}{r}\|\phi\|_{r,1} \geq \lim_{r\to\infty} r^{-1/r}\phi^*(1/r) = \|\phi\|_\infty. \tag{7.42}$$

On the other hand,

$$\frac{1}{r}\|\phi\|_{r,1} = \frac{1}{r}\int_0^1 t^{1/r-1}\phi^*(t)dt + \frac{1}{r}I_r \leq \|\phi\|_\infty + \frac{1}{r}I_r, \tag{7.43}$$

where

$$I_r \equiv \int_1^\infty t^{1/r-1}\phi^*(t)dt.$$

For $t > 1$, $t^{1/r}$ is decreasing with respect to r, and therefore I_r is decreasing in r. Moreover, since $\phi \in L^{r_0,1}(\mathbb{R})$ we have that $I_{r_0} < \infty$. These observations imply that

$$\lim_{r\to\infty} \frac{1}{r} I_r = 0,$$

so by (7.43), we then have

$$\limsup_{r\to\infty} \frac{1}{r} \|\phi\|_{r,1} \leq \|\phi\|_\infty.$$

According to (7.42), the proof is now complete. □

Remark 7.25. In (2.32) we observed that $\|\cdot\|_\infty = \|\cdot\|_{\infty,1}$, so the conclusion of the above lemma is that

$$\lim_{r\to\infty} \frac{1}{r} \|\phi\|_{r,1} = \|\phi\|_{\infty,1}.$$

Theorem 7.26. *Let $n \geq 2$ and $p > 1$. Put $r = p'/(n-1)$. Set*

$$V_k = L^1_{\hat{x}_k}(\mathbb{R}^{n-1})[L^\infty_{x_k}(\mathbb{R})]$$

and

$$V_k^p = L^{p,1}_{\hat{x}_k}(\mathbb{R}^{n-1})[L^{r,1}_{x_k}(\mathbb{R})],$$

for $k = 1, \ldots, n$. Let $k \in \{1, \ldots, n\}$. If $f \in S_0(\mathbb{R}^n)$ and $f \in \cap_{j\neq k} V_j$, then

$$\lim_{p\to 1+} \frac{n-1}{p'} \|f\|_{V_k^p} = \|f\|_{V_k}. \tag{7.44}$$

Proof. We can assume that $f \geq 0$. Fix $k \in \{1, \ldots, n\}$. Let $\{p_\nu\}_{\nu=1}^\infty$ be a decreasing sequence such that $p_\nu > 1$ and $p_\nu \to 1$ as $\nu \to \infty$. Denote $r_\nu = p'_\nu/(n-1)$. Observe that $r_\nu \to \infty$, as $\nu \to \infty$. To prove (7.44), it is enough to show that

$$\lim_{\nu\to\infty} \frac{1}{r_\nu} \|f\|_{V_k^{p_\nu}} = \|f\|_{V_k}. \tag{7.45}$$

For $M > 1$ and $x \in \mathbb{R}$ we set

$$f_M(x) = \min(M, f(x))\chi_{(-M,M)^n}(x),$$

$$\Phi_M^{(\nu)}(\hat{x}_k) = \|f_M(\hat{x}_k, \cdot)\|_{r_\nu,1},$$

and

$$\alpha_M(\hat{x}_k) = \|f_M(\hat{x}_k, \cdot)\|_\infty.$$

Since $r_\nu \to \infty$ as $\nu \to \infty$, Lemma 7.24 shows that

$$\lim_{\nu\to\infty} \frac{1}{r_\nu} \Phi_M^{(\nu)}(\hat{x}_k) = \alpha_M(\hat{x}_k),$$

for all $\hat{x}_k \in \mathbb{R}^{n-1}$. Since all $\Phi_M^{(\nu)}$, $\nu \in \mathbb{N}$, have support in $(-M, M)^{n-1}$, it follows that (see e.g. [47, Theorem 4.21])

$$\frac{1}{r_\nu}\Phi_M^{(\nu)} \to \alpha_M$$

in measure on $\mathbb{R}^{n-1}$, as $\nu \to \infty$. Hence, by Proposition 2.4

$$\frac{1}{r_\nu}(\Phi_M^{(\nu)})^*(t) \to \alpha_M^*(t) \quad \text{as} \quad \nu \to \infty, \tag{7.46}$$

for almost every $t > 0$. As we observed, supp $\Phi_M^{(\nu)} \subset (-M, M)^{n-1}$ for all $\nu \in \mathbb{N}$. Also, for all $\hat{x}_k \in \mathbb{R}^{n-1}$ we have

$$\Phi_M^{(\nu)}(\hat{x}_k) \le \int_0^{2M} t^{1/r_\nu - 1}\mathcal{R}_k f_M(\hat{x}_k, t)dt \le r_\nu M(2M)^{1/r_\nu} \le 2r_\nu M^2.$$

Therefore

$$\frac{1}{r_\nu}(\Phi_M^{(\nu)})^*(t) \le 2M^2 \chi_{(0,(2M)^{n-1}]}(t),$$

for all $t > 0$ and $\nu \in \mathbb{N}$. By this estimate and (7.46), the dominated convergence theorem gives

$$\lim_{\nu\to\infty} \frac{1}{r_\nu}\|f_M\|_{V_k^{p_\nu}} = \lim_{\nu\to\infty} \frac{1}{r_\nu}\|\Phi_M^{(\nu)}\|_{p_\nu,1} =$$

$$= \lim_{\nu\to\infty} \frac{1}{r_\nu}\int_0^\infty t^{1/p_\nu - 1}(\Phi_M^{(\nu)})^*(t)dt = \int_0^\infty \alpha_M^*(t)dt = \|f_M\|_{V_k}, \tag{7.47}$$

where the last equality holds by (2.24). We have $f_M(x) \le f(x)$ on $\mathbb{R}^n$. Hence, $\|f_M(\hat{x}_k, \cdot)\|_{r_\nu,1} \le \|f(\hat{x}_k, \cdot)\|_{r_\nu,1}$, and then also $\|f_M\|_{V_k^{p_\nu}} \le \|f\|_{V_k^{p_\nu}}$. From this and (7.47) we get

$$\|f_M\|_{V_k} \le \liminf_{\nu\to\infty} \frac{1}{r_\nu}\|f\|_{V_k^{p_\nu}}, \tag{7.48}$$

for all $M > 1$. Observe that $\alpha_M(\hat{x}_k)$ is increasing with respect to M, and that

$$\lim_{M\to\infty} \alpha_M(\hat{x}_k) = \|f(\hat{x}_k, \cdot)\|_\infty \equiv \alpha(\hat{x}_k),$$

for all $\hat{x}_k$. Then, by the monotone convergence theorem

$$\|f\|_{V_k} = \|\alpha\|_1 = \lim_{M\to\infty} \|\alpha_M\|_1 = \lim_{M\to\infty} \|f_M\|_{V_k}.$$

By this and (7.48), we see that

$$\|f\|_{V_k} \le \liminf_{\nu\to\infty} \frac{1}{r_\nu}\|f\|_{V_k^{p_\nu}}. \tag{7.49}$$

First, this implies (7.45) if $\|f\|_{V_k} = \infty$. Suppose now that $f \in V_k$. Then $f \in \cap_{j=1}^n V_j$, and by inequality (7.30), it holds that

$$\frac{1}{r_\nu} \|f\|_{V_k^{p_\nu}} \leq \|f\|_{V_k} + \frac{p_\nu}{r_\nu} \sum_{j \neq k} \|f\|_{V_j}.$$

Since $p_\nu / r_\nu \to 0$, as $\nu \to \infty$, we conclude that

$$\limsup_{\nu \to \infty} \frac{1}{r_\nu} \|f\|_{V_k^{p_\nu}} \leq \|f\|_{V_k}.$$

This, together with (7.49), gives (7.45). $\square$

Remark 7.27. In Theorem 7.26 we assumed that $f \in \cap_{j \neq k} V_j$. This assumption can not be omitted. Namely, there exists a function $f \in S_0(\mathbb{R}^2)$ such that $\|f\|_{V_1} < \infty$ and $\|f\|_{V_1^p} = \infty$ for $1 < p < \infty$. An example of such a function is

$$f(x, y) = \phi(x) \chi_{(0,1)}(y),$$

where

$$\phi(x) = \frac{1}{\ln(e + |x|)}.$$

Indeed,

$$\text{ess sup}_{x \in \mathbb{R}} f(x, y) = \chi_{(0,1)}(y),$$

so that $\|f\|_{V_1} = 1$. Further, the function ϕ is even and decreasing on $\mathbb{R}_+$. Therefore (using (2.8)),

$$\phi^*(t) = \sup_{|E|=t} \inf_{x \in E} \phi(x) = \inf_{|x| \leq t/2} \phi(x) = \phi(t/2),$$

and then

$$\mathcal{R}_1 f(t, y) = \phi^*(t) \chi_{(0,1)}(y) = \phi(t/2) \chi_{(0,1)}(y).$$

So, for $0 < y < 1$ and $1 < p < \infty$,

$$\|f(\cdot, y)\|_{p',1} = \|\phi\|_{p',1} = \int_0^\infty \frac{t^{1/p'-1}}{\ln(e + t/2)} \, dt = \infty.$$

It follows that $\|f\|_{V_1^p} = \infty$.

As we observed in Remark 7.16 above, for $n = 2$ and $j = 1$, Theorem 7.12 states that

$$\|f\|_{V_1^p} \leq pp' \|f\|_{V_1}^{1/p} \|f\|_{V_2}^{1/p'}, \quad 1 < p < \infty,$$

and thus

$$\limsup_{p \to \infty} \frac{1}{p} \|f\|_{V_1^p} \leq \|f\|_{V_2}$$

(note that there holds a corresponding statement for $j = 2$). The next theorem gives a further result on the limiting behaviour of $\|f\|_{V_k^p}$, $k = 1, 2$, as p approaches ∞.

Theorem 7.28. *For $f \in S_0(\mathbb{R}^2)$, there holds the relation*

$$\|f\|_{V_k^\infty} \le \liminf_{p\to\infty} \frac{1}{p}\|f\|_{V_k^p}, \quad k = 1, 2. \tag{7.50}$$

Proof. We consider the case $k = 1$. The proof is carried out in the same way for $k = 2$. We denote by Ω the set of all $p \ge 1$ such that $\|f\|_{V_1^p} < \infty$. In what follows we shall consider only the values of $p \in \Omega$. Then the function

$$\Psi_{p'}(y) = \|f(\cdot, y)\|_{p',1}$$

is in $S_0(\mathbb{R})$, and for $\tau > 0$,

$$\|f\|_{V_1^p} = \|\Psi_{p'}\|_{p,1} \ge \int_0^\tau t^{1/p-1}\Psi_{p'}^*(t)dt \ge p\tau^{1/p}\Psi_{p'}^*(\tau). \tag{7.51}$$

Let $N \in \mathbb{N}$. Observe that

$$\Psi_{p'}(y) \ge \int_0^N s^{1/p'-1}\mathcal{R}_1 f(s, y)ds \ge N^{1/p'-1}F_N(y),$$

where

$$F_N(y) = \int_0^N \mathcal{R}_1 f(s, y)ds, \quad y \in \mathbb{R}.$$

Since $\Psi_{p'} \in S_0(\mathbb{R})$, we also have $F_N \in S_0(\mathbb{R})$. Rearranging with respect to y in the preceding inequality, we get

$$\Psi_{p'}^*(\tau) \ge N^{1/p'-1}F_N^*(\tau).$$

Combining this inequality with (7.51), we obtain

$$\|f\|_{V_1^p} \ge p\tau^{1/p}N^{1/p'-1}F_N^*(\tau).$$

This implies that

$$\liminf_{q\to\infty} \frac{1}{q}\|f\|_{V_1^q} \ge F_N^*(\tau),$$

for $\tau > 0$. By (2.25), it follows that

$$\liminf_{q\to\infty} \frac{1}{q}\|f\|_{V_1^q} \ge \|F_N\|_\infty. \tag{7.52}$$

Observe that $F_N(y)$ is increasing in N for all $y \in \mathbb{R}$, and by (2.24),

$$\lim_{N\to\infty} F_N(y) = \|f(\cdot, y)\|_1 \equiv F(y).$$

It follows that

$$\lim_{N\to\infty} \|F_N\|_\infty = \|F\|_\infty. \tag{7.53}$$

Indeed, fix $0 < M < \|F\|_\infty$ and set

$$E = \{y \in \mathbb{R} : F(y) > M\} \quad \text{and} \quad E_N = \{y \in \mathbb{R} : F_N(y) > M\},$$

$N \in \mathbb{N}$. Observe that $E_N \subset E_{N+1}$, $N = 1, 2, \dots$, and that

$$E = \cup_{N=1}^{\infty} E_N.$$

Hence $|E| = \lim_{N\to\infty} |E_N|$. But $|E| > 0$, since $M < \|F\|_\infty$, and therefore there exists a number N_0 such that $|E_N| > 0$, for all $N > N_0$. It follows that $\|F_N\|_\infty > M$, for all $N > N_0$, and so

$$M < \lim_{N\to\infty} \|F_N\|_\infty \le \|F\|_\infty.$$

Since this holds for all $M < \|F\|_\infty$, we have now proved (7.53). Letting $N \to \infty$ in (7.52), and applying (7.53), we obtain

$$\|F\|_\infty \le \liminf_{q\to\infty} \frac{1}{q} \|f\|_{V_1^q}.$$

But $\|F\|_\infty = \|f\|_{V_1^\infty}$, so we have obtained (7.50). □

7.5. **On relations between the spaces V^p.** Throughout this section we let $n = 2$. Recall from (7.17) that

$$V_k^p = L_{\hat{x}_k}^{p,1}(\mathbb{R})[L_{x_k}^{p',1}(\mathbb{R})], \quad 1 < p < \infty,\ k = 1, 2.$$

We also remind that for $p = 1$, this space is given by (see (7.18))

$$V_k^1 = L_{\hat{x}_k}^1(\mathbb{R})[L_{x_k}^\infty(\mathbb{R})], \quad k = 1, 2. \tag{7.54}$$

and for $p = \infty$ we have (see (7.19))

$$V_k^\infty = L_{\hat{x}_k}^\infty(\mathbb{R})[L_{x_k}^1(\mathbb{R})], \quad k = 1, 2$$

Denote

$$V^p = V_1^p \cap V_2^p.$$

We have seen that the spaces V^p play an important role in embeddings of the Sobolev space $W_1^1(\mathbb{R}^2)$ and, more generally, in embeddings of the Gagliardo mixed norm space V^1. Recall that according to Corollary 7.22,

$$W_1^1(\mathbb{R}^2) \subset V^p, \quad 1 < p \le \infty. \tag{7.55}$$

It is natural to ask whether the space V^p increases or decreases as p grows. Studying this question, we have obtained the following proposition.

Proposition 7.29. *Let the parameters p and q satisfy one of the conditions:*

(i) $1 < p \le 2$ *and* $p' < q \le \infty$*;*
(ii) $2 < p < \infty$ *and* $p < q \le \infty$*;*
(iii) $1 < p \le \infty$ *and* $1 \le q < p$.

Then it holds that

$$V^p \not\subset V_1^q \cup V_2^q. \tag{7.56}$$

Proof. For $\alpha > 0$, we set

$$f_\alpha(x,y) = \begin{cases} (xy)^{-\alpha}, & 0 < x, y \le 1 \\ 0, & (x,y) \in \mathbb{R}^2 \setminus (0,1]^2. \end{cases}$$

If $1 < r < \infty$, $0 < \alpha < \min(1/r, 1/r')$, then

$$f_\alpha \in V^r, \tag{7.57}$$

and if $1 < r \le \infty$, $\min(1/r, 1/r') < \alpha$, then

$$f_\alpha \notin V_1^r \cup V_2^r. \tag{7.58}$$

Indeed, assume that $1 < r < \infty$ and $0 < \alpha < \min(1/r, 1/r')$. Then

$$\|f_\alpha(x,\cdot)\|_{r',1} = x^{-\alpha} \int_0^1 t^{1/r'-1} t^{-\alpha} dt = cx^{-\alpha},$$

for $0 < x \le 1$. Thus,

$$\|f_\alpha\|_{V_2^r} = \int_0^1 t^{1/r-1} t^{-\alpha} dt < \infty.$$

Similarly, $\|f_\alpha\|_{V_1^r} < \infty$, and so, (7.57) holds. We will now prove (7.58). Assume that $1 < r \le \infty$ and $\min(1/r, 1/r') < \alpha$. If $\alpha \ge 1/r'$, then

$$\|f_\alpha(x,\cdot)\|_{r',1} = x^{-\alpha} \int_0^1 t^{1/r'-1} t^{-\alpha} dt = \infty,$$

for $0 < x \le 1$, which implies that $f_\alpha \notin V_2^r$. On the other hand, if $\alpha < 1/r'$ then $1/r < \alpha$, and then

$$\|f_\alpha(x,\cdot)\|_{r',1} = cx^{-\alpha},$$

for $0 < x \le 1$, and

$$\|f_\alpha\|_{V_2^r} = c \int_0^1 t^{1/r-1} t^{-\alpha} dt = \infty.$$

In the same way $\|f_\alpha\|_{V_1^r} = \infty$. This proves (7.58).

Using (7.57) and (7.58), we will now show that (7.56) holds if one of the conditions (i) and (ii) is satisfied. Assume that (i) is true. Then

$$\min(\frac{1}{p}, \frac{1}{p'}) = \frac{1}{p'} \quad \text{and} \quad \min(\frac{1}{q}, \frac{1}{q'}) = \frac{1}{q}.$$

We also have $1/q < 1/p'$. Let $1/q < \alpha < 1/p'$. By (7.57), $f_\alpha \in V^p$, and by (7.58), $f_\alpha \notin V_1^q \cup V_2^q$. So, (i) implies (7.56). Assume instead that (ii) holds. Then

$$\min(\frac{1}{p}, \frac{1}{p'}) = \frac{1}{p} \quad \text{and} \quad \min(\frac{1}{q}, \frac{1}{q'}) = \frac{1}{q}.$$

Further, $1/q < 1/p$. Take $1/q < \alpha < 1/p$. We again have that by (7.57), $f_\alpha \in V^p$, and by (7.58), $f_\alpha \notin V_1^q \cup V_2^q$. So also (ii) implies (7.56).

It remains to prove that (7.56) holds under the assumption (iii). To see this, we consider the functions

$$\phi_\beta(t) = \begin{cases} t^{-\beta}, & 0 < t \le 1 \\ 0, & t \in \mathbb{R} \setminus (0,1] \end{cases}$$

and

$$g_\beta(x,y) = \begin{cases} \phi_\beta(x-y), & (x,y) \in (0,2]^2 \\ 0, & (x,y) \in \mathbb{R}^2 \setminus (0,2]^2. \end{cases}$$

We will show that if $1 < r \le \infty$ and $0 < \beta < 1/r'$, then

$$g_\beta \in V^r, \tag{7.59}$$

and if $1 \le r < \infty$ and $1/r' < \beta$, then

$$g_\beta \notin V_1^r \cup V_2^r. \tag{7.60}$$

For all $t > 0$, we have

$$\mathcal{R}_1 g_\beta(t,y) = \begin{cases} \phi_\beta(t), & 0 < y \le 1 \\ \phi_\beta(t)\chi_{(0,2-y]}(t), & 1 < y < 2 \\ 0, & y \in \mathbb{R} \setminus (0,2). \end{cases} \tag{7.61}$$

Indeed, it is clear that $\mathcal{R}_1 g_\beta(t,y) = 0$, if $y \in \mathbb{R} \setminus (0,2)$. Moreover, for $0 < y < 2$,

$$g_\beta(x,y) = \begin{cases} (x-y)^{-\beta}, & 0 < x \le 2,\ 0 < x-y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $0 < y \le 1$. Then $g_\beta(x,y) = (x-y)^{-\beta}\chi_{(y,\,y+1]}(x)$, so that $\mathcal{R}_1 g_\beta(t,y) = t^{-\beta}\chi_{(0,1]}(t)$. Assume now that $1 < y < 2$. Then $g_\beta(x,y) = (x-y)^{-\beta}\chi_{(y,2]}(x)$, so that $\mathcal{R}_1 g_\beta(t,y) = t^{-\beta}\chi_{(0,2-y]}(t)$. We have now verified that (7.61) holds.

Assume that $1 < r \le \infty$, $0 < \beta < 1/r'$, and prove (7.59). By (7.61),

$$\|g_\beta(\cdot,y)\|_{r',1} \le \int_0^1 t^{1/r'-1} t^{-\beta} dt = \frac{r'}{1-r'\beta},$$

for $0 < y < 2$. Thus,

$$\|g_\beta\|_{V_1^r} \le c \int_0^2 t^{1/r-1} dt < \infty.$$

Similarly, $\|g_\beta\|_{V_2^r} < \infty$. This proves (7.59). Next we shall prove (7.60). First we consider the case $r = 1$, $\beta > 0$. Observe that

$$\text{ess sup}_{x\in\mathbb{R}}\, g_\beta(x,y) = \infty,$$

for $0 < y < 2$, so $\|g_\beta\|_{V_1^1} = \infty$. The remaining case is when $1 < r < \infty$ and $1/r' < \beta$. By (7.61), we then have

$$\|g_\beta(\cdot, y)\|_{r',1} = \int_0^1 t^{1/r'-1} t^{-\beta} dt = \infty,$$

for $0 < y \le 1$. Hence, $\|g_\beta\|_{V_1^r} = \infty$. Similarly we can verify that $\|g_\beta\|_{V_2^r} = \infty$, for $1 \le r < \infty$, $1/r' < \beta$. The proof of (7.60) is now complete.

We will use (7.59) and (7.60) to show that (iii) implies (7.56). Assume that (iii) holds. Then $p' < q'$, so that $1/q' < 1/p'$. Let $1/q' < \beta < 1/p'$. By (7.59), $g_\beta \in V^p$, and by (7.60), $g_\beta \notin V_1^q \cup V_2^q$. Thus, (7.56) follows from (iii). $\square$

Remark 7.30. Proposition 7.29 shows that the scale of spaces V^p is neither increasing nor decreasing. Indeed, statements (i) and (ii) in this proposition imply that for all $1 < p < \infty$ there exists $q > p$ such that $V^p \not\subset V^q$. Further, statement (iii) shows that for all $1 < p \le \infty$ there exists $q < p$ for which $V^p \not\subset V^q$.

8. Functions with smooth sections

Theorem 1.6 (and its extension - Theorem 5.4) is closely related to embeddings of Sobolev spaces with smoothness not greater than 1. The objective of this section is to obtain an extension of Theorem 1.6 related to Sobolev spaces of higher smoothness. For simplicity, we study here only isotropic mixed norm spaces.

Let f be a real-valued function on $\mathbb{R}$. Recall from (6.7) that

$$\Delta^r(h)f(x) = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}f(x+ih), \quad h \geq 0. \tag{8.1}$$

The rth order modulus of continuity of f is defined for $t \geq 0$ as

$$\omega^r(f;t) = \sup_{0\leq h\leq t}\Big(\sup_{x\in\mathbb{R}}|\Delta^r(h)f(x)|\Big). \tag{8.2}$$

Using induction, it is easy to check that

$$\Delta^s(h)f(x) = \Delta^{s-r}(h)\Delta^r(h)f(x), \quad 1 \leq r < s.$$

It follows that

$$\omega^s(f;t) \leq 2^{s-r}\omega^r(f;t), \quad t \geq 0. \tag{8.3}$$

According to (4.5), p. 332, in [5], we have

$$\omega^r(f;2t) \leq 2^r\omega^r(f;t), \quad t \geq 0. \tag{8.4}$$

We will also use Marchaud's inequality, which states that (see e.g. [5, pp. 332–333])

$$\omega^r(f;t) \leq ct^r\int_t^\infty \frac{\omega^s(f;u)}{u^r}\frac{du}{u}, \tag{8.5}$$

for all $f \in L^\infty(\mathbb{R})$, $1 \leq r < s$, and $t \geq 0$.

We say that f belongs to the class $\mathcal{C}^\lambda(\mathbb{R})$ $(\lambda > 0)$ given that

$$\|f\|^*_{\mathcal{C}^\lambda} \equiv \sup_{t>0}\frac{\omega^r(f;t)}{t^\lambda} < \infty,$$

where $r = r(\lambda)$ is the least integer such that $\lambda < r$. Observe that $\|\cdot\|^*_{\mathcal{C}^\lambda}$ is not a norm but only a seminorm. Indeed, $\|f\|^*_{\mathcal{C}^\lambda} = 0$ for all constant functions f. We also define the space $C^\lambda(\mathbb{R}) = \mathcal{C}^\lambda(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with the norm

$$\|f\|_{C^\lambda} = \|f\|_\infty + \|f\|^*_{\mathcal{C}^\lambda}.$$

In [46, Chapter 1.2.2], these spaces are referred to as Hölder-Zygmund classes.

We note here that $\|f\|^*_{\mathcal{C}^\lambda}$ coincides with $\|f\|_{\mathrm{Lip}\,\lambda}$ (defined by (4.5)) for $0 < \lambda < 1$. However, for $\lambda = 1$ we only have $\|f\|^*_{\mathcal{C}^1} \leq 2\|f\|_{\mathrm{Lip}\,1}$.

In this section we study embeddings of the mixed norm spaces

$$U^{p,\lambda} = \bigcap_{k=1}^{n} U_k^{p,\lambda},$$

where

$$U_k^{p,\lambda} = L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\mathcal{C}^\lambda_{x_k}(\mathbb{R})], \quad (k = 1, \dots, n),$$

for $1 \le p < \infty$ and $\lambda > 0$. In Sections 8.1 and 8.2 we consider the cases $0 < \lambda < (n-1)/p$ and $(n-1)/p < \lambda < \infty$ respectively. In the first case we obtain embeddings to Lorentz spaces (Theorem 8.5), and in the second we prove estimates of smoothness (Theorem 8.13).

8.1. **Embeddings to Lorentz spaces.** In this subsection we use the approach developed in [26]. We begin with some auxiliary lemmas. The following result is Lemma 5 from [26].

Lemma 8.1. *Let $E \subset \mathbb{R}$ be a measurable set of measure $0 < t < \infty$, $x_0 \in \mathbb{R}$, and $r \in \mathbb{N}$. Then there exists a number $h \in (0, (r+1)t)$ such that $x_0 + ih$ does not belong to E for any $i = 1, \dots, r$.*

The next lemma ensures boundedness and uniform continuity of the rearrangable functions in $\mathcal{C}^\lambda(\mathbb{R})$.

Lemma 8.2. *Let $\lambda > 0$. Suppose $f \in S_0(\mathbb{R}) \cap \mathcal{C}^\lambda(\mathbb{R})$. Then f is bounded and uniformly continuous on $\mathbb{R}$. Moreover, for all $s > 0$, there holds the inequality*

$$\|f\|_\infty \le c\big(f^*(s) + s^\lambda \|f\|^*_{\mathcal{C}^\lambda}\big), \tag{8.6}$$

where c depends only on λ.

Proof. Fix $s > 0$. By (2.10), there exists a measurable set $E_s \subset \mathbb{R}$, such that $\mathrm{mes}_1 E_s = s$ and

$$\{x \in \mathbb{R} : |f(x)| > f^*(s)\} \subset E_s.$$

Fix $x_0 \in \mathbb{R}$ and let $r = r(\lambda)$, i.e. $r \in \mathbb{N}$ and $r - 1 \le \lambda < r$. By Lemma 8.1, there exists a number $h \in (0, (r+1)s)$, such that $x_0 + ih \notin E_s$ for $i = 1, \dots, r$. By (8.1), we have

$$|f(x_0)| \le \Big|\sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} f(x_0 + ih)\Big| + |\Delta^r(h) f(x_0)|. \tag{8.7}$$

Since $x_0 + ih \in \mathbb{R} \setminus E_s$, it follows that

$$|f(x_0 + ih)| \le f^*(s), \quad i = 1, \dots, r.$$

Use this in (8.7). We get

$$|f(x_0)| \le 2^r f^*(s) + \omega^r(f; h) \le 2^r f^*(s) + h^\lambda \|f\|^*_{\mathcal{C}^\lambda}. \tag{8.8}$$

Since $0 < h < (r+1)s$, this implies (8.6) and proves that f is bounded.

It remains to show that f is uniformly continuous on $\mathbb{R}$. If $0 < \lambda < 1$, then $r = r(\lambda) = 1$ and we have

$$\omega(f;\delta) \le \delta^{\lambda} \|f\|^*_{\mathcal{C}^{\lambda}}$$

for $\delta > 0$. Hence, f is uniformly continuous on $\mathbb{R}$. Suppose now that $\lambda \ge 1$. Then $r = r(\lambda) \ge 2$. Since f is bounded, Marchaud's inequality (8.5) ensures that

$$\omega(f;\delta) \le c\delta \int_{\delta}^{\infty} \frac{\omega^r(f;u)}{u^2}\, du$$

for $\delta \ge 0$. Let $0 < \delta < 1/2$. Observe that

$$\int_{\delta}^{1} \frac{\omega^r(f;u)}{u^2}\, du \le \|f\|^*_{\mathcal{C}^{\lambda}} \int_{\delta}^{1} u^{\lambda-2} du \le c\|f\|^*_{\mathcal{C}^{\lambda}} \ln\frac{1}{\delta}.$$

Further, from the definitions (8.1) and (8.2), we see that $\omega^r(f;u) \le 2^r\|f\|_{\infty}$, for all $u \ge 0$. Therefore,

$$\int_{1}^{\infty} \frac{\omega^r(f;u)}{u^2}\, du \le 2^r\|f\|_{\infty}.$$

Combining these three estimates, we get

$$\omega(f;\delta) \le c(\|f\|_{\infty} + \|f\|^*_{\mathcal{C}^{\lambda}})\delta \ln\frac{1}{\delta}$$

for $0 < \delta < 1/2$. This shows that f is uniformly continuous on $\mathbb{R}$. □

Recall from Section 2.4 that $\mathcal{R}_k f$ denotes the rearrangement of f with respect to x_k.

Lemma 8.3. *Let $n \ge 2$ and $\lambda > 0$. Suppose $f \in S_0(\mathbb{R}^n)$ and $f_{\hat{x}_k} \in \mathcal{C}^{\lambda}(\mathbb{R})$ for a.e. $\hat{x}_k \in \mathbb{R}^{n-1}$, $k = 1, \dots, n$. Set*

$$N_k(\hat{x}_k) = \|f_{\hat{x}_k}\|^*_{\mathcal{C}^{\lambda}},$$

and assume that $N_k \in S_0(\mathbb{R}^{n-1})$, $k = 1, \dots, n$. For all $0 < t < \tau < \infty$, it holds that

$$f^*(t) \le c\Big(f^*(\tau) + \Big(\frac{\tau}{t^{1-1/n}}\Big)^{\lambda} \sum_{k=1}^{n} N_k^*\Big(\frac{1}{2}t^{1-1/n}\Big)\Big), \tag{8.9}$$

where c depends only on n and λ.

Proof. Let $0 < t < \tau < \infty$ and assume that $f^*(t) > f^*(\tau)$, and thereby $f^*(t) > 0$. Denote

$$E = \{x : |f(x)| \ge f^*(t)\}.$$

Take E_k $(k = 1, \dots, n)$ to be the set of all $\hat{x}_k$ in the essential projection $\Pi_k^* E$ (see Section 3) such that $f_{\hat{x}_k} \in \mathcal{C}^\lambda(\mathbb{R}) \cap S_0(\mathbb{R})$. By assumption, this is true for a.e. $\hat{x}_k$, so E_k is measurable and

$$\mu_k \equiv \mathrm{mes}_{n-1} \Pi_k^* E = \mathrm{mes}_{n-1} E_k.$$

Hence, by (2.11) and Loomis-Whitney's inequality for essential projections (see (3.2)), we get

$$t^{n-1} \leq (\mathrm{mes}_n E)^{n-1} \leq \prod_{k=1}^{n} \mu_k. \tag{8.10}$$

For $\hat{x}_k \in E_k \subset \Pi_k^* E$, the set $E(\hat{x}_k)$ is measurable in $\mathbb{R}$ and

$$\mathrm{mes}_1 E(\hat{x}_k) > 0 \tag{8.11}$$

(by the definition of the essential projection). Further, for $x \in E$, $f^*(t) \leq |f(x)|$. This and (8.11) imply that

$$f^*(t) \leq \|f_{\hat{x}_k}\|_\infty, \tag{8.12}$$

for all $\hat{x}_k \in E_k$.

Let $P \subset \mathbb{R}^n$ be a G_δ-set which contains $\{x : |f(x)| > f^*(\tau)\}$, and has the same measure as this set. Since G_δ-sets have measurable sections, the functions

$$p_k(\hat{x}_k) = \mathrm{mes}_1 P(\hat{x}_k), \quad k = 1, \dots, n,$$

are defined on $\Pi_k P$. Further, by Fubini's theorem, each p_k is a measurable function. Since $E \subset P$ (we use here that $f^*(\tau) < f^*(t)$), it follows from (8.11) that

$$p_k(\hat{x}_k) > 0 \quad \text{for } \hat{x}_k \in E_k. \tag{8.13}$$

Let A_k, $k = 1, \dots, n$, be the set of all $\hat{x}_k \in E_k$ such that

$$p_k(\hat{x}_k) \leq \frac{2\tau}{\mu_k}. \tag{8.14}$$

Since p_k is a measurable function, the set A_k is measurable. Further,

$$\mathrm{mes}_{n-1} A_k \geq \frac{\mu_k}{2}, \quad k = 1, \dots, n. \tag{8.15}$$

Indeed, by (2.10), and since $E_k \setminus A_k \subset \Pi_k P$, we have

$$\tau \geq \mathrm{mes}_n P \geq \int_{E_k \setminus A_k} p_k(\hat{x}_k) d\hat{x}_k.$$

On $E_k \setminus A_k$, it holds that $p_k(\hat{x}_k) > 2\tau/\mu_k$. Thus,

$$\tau \geq \frac{2\tau}{\mu_k} \mathrm{mes}_{n-1}(E_k \setminus A_k),$$

which implies that $\mathrm{mes}_{n-1}(E_k \setminus A_k) \leq \mu_k/2$. Since $\mathrm{mes}_{n-1} E_k = \mu_k$, we obtain (8.15).

For $\hat{x}_k \in A_k$ we apply Lemma 8.2 to the function $f_{\hat{x}_k}$, with $s = 2p_k(\hat{x}_k)$. Using (8.12), we then obtain

$$f^*(t) \le c\Big(\mathcal{R}_k f(\hat{x}_k, 2p_k(\hat{x}_k)) + (p_k(\hat{x}_k))^\lambda N_k(\hat{x}_k)\Big) \tag{8.16}$$

for $\hat{x}_k \in A_k$ and $k = 1, \dots, n$. Observe that

$$\mathcal{R}_k f(\hat{x}_k, 2p_k(\hat{x}_k)) \le f^*(\tau) \tag{8.17}$$

for $\hat{x}_k \in A_k$. Indeed, let $U \subset \mathbb{R}$ be a measurable set of measure $2p_k(\hat{x}_k)$. Then there exists a point $x_k \in U$ such that $(x_k, \hat{x}_k) \notin P$ (we use here that $0 < p_k(\hat{x}_k) < \infty$ for $\hat{x}_k \in A_k$, by (8.13) and (8.14)). Therefore $f(x_k, \hat{x}_k) \le f^*(\tau)$, and then

$$\inf_{z \in U} f(\hat{x}_k, z) \le f^*(\tau).$$

Since U was an arbitrary measurable set of measure $2p_k(\hat{x}_k)$, (8.17) follows.

Apply (8.17) and (8.14) in (8.16). We obtain

$$f^*(t) \le c\Big(f^*(\tau) + \Big(\frac{\tau}{\mu_k}\Big)^\lambda N_k(\hat{x}_k)\Big)$$

for $\hat{x}_k \in A_k$ and $k = 1, \dots, n$. By (8.10), there exists $j \in \{1, \dots, n\}$ such that

$$\mu_j \ge t^{1/n'}.$$

For such j and $\hat{x}_j \in A_j$, we have

$$f^*(t) \le c\Big(f^*(\tau) + \Big(\frac{\tau}{t^{1/n'}}\Big)^\lambda N_j(\hat{x}_j)\Big).$$

By (8.15), we now have $\mathrm{mes}_{n-1} A_j \ge \mu_j/2 \ge t^{1/n'}/2$. It follows that

$$f^*(t) \le c\Big(f^*(\tau) + \Big(\frac{\tau}{t^{1/n'}}\Big)^\lambda N_j^*\Big(\frac{1}{2}t^{1/n'}\Big)\Big).$$

This implies (8.9). □

Remark 8.4. To prove the above lemma, we have followed the method in the proof of Lemma 8 in [26]. We also used similar reasonings for proving Theorem 5.4.

We shall now prove the main result of this section.

Theorem 8.5. *Let $n \ge 2$, $1 \le p < \infty$, $0 < \lambda < (n-1)/p$, and $q = np/(n-1-\lambda p)$. Set*

$$U_k^{p,\lambda} = L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\mathcal{C}^\lambda_{x_k}(\mathbb{R})], \quad k = 1, \dots, n.$$

Suppose that $f \in S_0(\mathbb{R}^n)$, $f \in \cap_{k=1}^n U_k^{p,\lambda}$, and that

$$f^*(t) = O(t^{-1/q}), \quad t \to \infty. \tag{8.18}$$

Then $f \in L^{q,p}(\mathbb{R}^n)$, *and*

$$\|f\|_{q,p} \le c \sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}}, \tag{8.19}$$

where c *depends only on* p, λ, *and* n.

Proof. Fix $\varepsilon > 0$. By (8.18), there exist numbers $B, M_0 > 0$, such that

$$t^{1/q} f^*(t) \le B, \quad \text{for all } t > M_0. \tag{8.20}$$

Let c_0 be the constant from inequality (8.9) in Lemma 8.3. Fix $\sigma > \max(2, 1/c_0)$ such that

$$\frac{B}{\sigma}(\ln K)^{1/p} < \frac{\varepsilon}{2}, \tag{8.21}$$

where $K = (c_0\sigma)^q$. According to Lemma 8.3 (with $\tau = Kt$) it holds that

$$f^*(t) \le c_0 f^*(Kt) + c t^{\lambda/n} \sum_{k=1}^{n} N_k^*\Big(\frac{1}{2} t^{1/n'}\Big),$$

for $t > 0$, where $N_k(\hat{x}_k) = \|f_{\hat{x}_k}\|^*_{\mathcal{C}^\lambda}$ ($N_k \in S_0(\mathbb{R}^{n-1})$ since $N_k \in L^p(\mathbb{R}^{n-1})$). For $M \ge M_0$, we denote

$$I_M = \Big(\int_{1/M}^{M} t^{p/q-1} f^*(t)^p dt\Big)^{1/p}.$$

By the preceding inequality,

$$I_M \le c_0 \Big(\int_{1/M}^{M} t^{p/q-1} f^*(Kt)^p dt\Big)^{1/p} +$$

$$+ c \sum_{k=1}^{n} \Big(\int_{1/M}^{M} t^{-1/n} N_k^*\Big(\frac{1}{2} t^{1/n'}\Big)^p dt\Big)^{1/p}. \tag{8.22}$$

Making the change of variables $t = Kt$, we obtain

$$c_0 \Big(\int_{1/M}^{M} t^{p/q-1} f^*(Kt)^p dt\Big)^{1/p} = \frac{1}{\sigma} \Big(\int_{K/M}^{KM} t^{p/q-1} f^*(t)^p dt\Big)^{1/p} \le$$

$$\le \frac{1}{\sigma} I_M + \frac{1}{\sigma} \Big(\int_{M}^{KM} t^{p/q-1} f^*(t)^p dt\Big)^{1/p} \le$$

$$\le \frac{1}{\sigma} I_M + \frac{B}{\sigma}(\ln K)^{1/p} \le \frac{1}{\sigma} I_M + \frac{\varepsilon}{2}, \tag{8.23}$$

where in the first inequality we use that $K > 1$ since $\sigma > 1/c_0$, the second estimate holds by (8.20), and the last holds by (8.21).

Next we estimate the remaining integrals on the right-hand side in (8.22). Making the change of variables $t = \frac{1}{2}t^{1/n'}$, we get

$$\int_{1/M}^{M} t^{-1/n} N_k^*\Big(\frac{1}{2}t^{1/n'}\Big)^p dt \le$$

$$\le c \int_0^\infty t^{-n'/n} t^{n'-1} N_k^*(t)^p dt = c\|N_k\|_p^p = c\|f\|_{U_k^{p,\lambda}}^p, \qquad (8.24)$$

where the first equality holds by (2.24) and the fact that $-n'/n + n' - 1 = 0$. Using (8.23) and (8.24) in (8.22), we obtain

$$I_M \le \frac{\sigma}{\sigma - 1}\Big(\frac{\varepsilon}{2} + c\sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}}\Big) \le \varepsilon + 2c \sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}},$$

for all $M \ge M_0$ (in the last step we use that $\sigma > 2$). This implies (8.19) since $I_M \to \|f\|_{q,p}$, as $M \to \infty$, and since ε was chosen arbitrarily. □

Remark 8.6. In the case when $1 \le p < \infty$ and $0 < \lambda < \min(1, (n-1)/p)$, Theorem 8.5 follows from Proposition 4.3 and Theorem 1.6 (the special case of Theorem 1.6 that we will use is stated by Proposition 8.7 below). Further, the assumption (8.18) in Theorem 8.5 can be omitted for these values of p and λ. Indeed, suppose that $f \in S_0(\mathbb{R}^n)$ and $f \in \cap_{k=1}^n U_k^{p,\lambda}$. Proposition 4.3 ensures that (recall that $\|\cdot\|_{\mathrm{Lip}\,\lambda}$ and $\|\cdot\|_{\mathcal{C}^\lambda}^*$ coincide for $0 < \lambda < 1$)

$$\|f_{\hat{x}_k}\|_{\Lambda^{-\lambda}} \le \|f_{\hat{x}_k}\|_{\mathcal{C}^\lambda}^*, \qquad (8.25)$$

for a.e. $\hat{x}_k$ and $k = 1, \dots, n$. Hence,

$$\|f\|_{V_k} \equiv \|f\|_{L^p_{\hat{x}_k}[\Lambda^{-\lambda}_{x_k}]} \le \|f\|_{U_k^{p,\lambda}}, \qquad (8.26)$$

$k = 1, \dots, n$, and so $f \in \cap_{k=1}^n V_k$. Put $q = np/(n - 1 - \lambda p)$. By Proposition 8.7 below, $f \in L^{q,p}(\mathbb{R}^n)$ and there holds the inequality

$$\|f\|_{q,p} \le c \sum_{k=1}^{n} \|f\|_{V_k}, \qquad (8.27)$$

where c depends only on n, p, and λ. Inequality (8.19) in Theorem 8.5 now follows from (8.27) and inequality (8.26). We stress that the assumption $f^*(t) = O(t^{-1/q})$, $t \to \infty$, in Theorem 8.5 was not used in this case. We also emphasize that the condition

$$g^*(t) - g^*(2t) \le ct^\lambda, \quad t \ge 0, \qquad (8.28)$$

is weaker then the condition

$$\omega(g; t) \le ct^\lambda, \quad t \ge 0. \qquad (8.29)$$

Indeed, by (8.25) we see that (8.29) implies (8.28). On the other hand, clearly there exists a function g such that (8.28) holds but (8.29) does not hold. In fact, every (non-zero) function can be rearranged in such a way that it does not satisfy (8.29).

In the preceding remark we used the following proposition.

Proposition 8.7. *Let* $n \geq 2$, $1 \leq p < \infty$, *and* $0 < \lambda < \min(1, (n-1)/p)$. *Set* $q = np/(n-1-\lambda p)$ *and*

$$V_k = L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\Lambda^{-\lambda}_{x_k}(\mathbb{R})], \quad k = 1, \ldots, n.$$

Suppose $f \in S_0(\mathbb{R}^n)$ *and* $f \in \cap_{k=1}^n V_k$. *Then* $f \in L^{q,p}(\mathbb{R}^n)$ *and inequality* (8.27) *holds.*

This result follows from Theorem 1.6. Indeed, let the conditions in Proposition 8.7 be satisfied. Set $\alpha_k = 1/p + \lambda$. Then, the parameters α and σ_k ($k = 1, \ldots, n$) in Theorem 1.6 are equal to $1/p + \lambda$ and $-\lambda$ respectively. Thus, the mixed norm spaces considered in Theorem 1.6 are the same as the spaces V_k, $k = 1, \ldots, n$, in Proposition 8.7. Since $\lambda < (n-1)/p$, we see that the condition $\alpha < n/p$ in Theorem 1.6 holds. Further, the parameter q in Theorem 1.6 satisfies

$$q = \frac{np}{n - \alpha p} = \frac{np}{n - 1 - \lambda p},$$

so it is the same as in Proposition 8.7. Since the conditions of Theorem 1.6 are fulfilled, it follows that $f \in L^{q,p}(\mathbb{R}^n)$ and inequality (1.10) holds, that is

$$\|f\|_{q,p} \leq c \prod_{k=1}^n \|f\|_{V_k}^{1/n}.$$

This implies (8.27), so Proposition 8.7 holds.

8.2. **Embeddings to classes of smooth functions.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. For $r \in \mathbb{N}$, $k = 1, \ldots, n$, and $h > 0$, we define the generalized Steklov averages

$$V^r_{h,k} f(x) = |Q_h|^{-1} \sum_{i=1}^r (-1)^{i-1} \binom{r}{i} \int_{Q_h} f(x_k + i\sigma(u), \hat{x}_k) du, \tag{8.30}$$

where $Q_h = (0,h)^r$ and $\sigma(u) = \sum_{k=1}^r u_k$ for $u \in Q_h$. We have

$$\int_{Q_h} |f_{\hat{x}_k}(x_k + i\sigma(u))| du =$$

$$= \frac{1}{i^r} \int_0^{ih} \cdots \int_0^{ih} |f_{\hat{x}_k}(x_k + v_1 + \cdots + v_r)| dv_1 \ldots dv_r =$$

$$= \frac{1}{i^r} \int_{(0,ih)^{r-1}} d\hat{v}_k \int_{\sigma(\hat{v}_k)}^{ih + \sigma(\hat{v}_k)} |f_{\hat{x}_k}(x_k + w)| dw,$$

where $\sigma(\hat{v}_k) = v_1 + \cdots + v_{k-1} + v_{k+1} + \cdots + v_r$ for $\hat{v}_k \in (0, ih)^{r-1}$. Estimate the last integral by integrating over the bigger interval $(0, r^2h)$ instead. This gives

$$\int_{Q_h} |f_{\hat{x}_k}(x_k + i\sigma(u))|du \leq \frac{h^{r-1}}{i} \int_0^{r^2h} |f_{\hat{x}_k}(x_k + \xi_k)|d\xi_k.$$

Thus,

$$|V^r_{h,k} f(x)| \leq \frac{2^r}{h} \int_0^{r^2h} |f(x_k + \xi_k, \hat{x}_k)|d\xi_k. \tag{8.31}$$

Further, denote

$$V^r_\mu f(x) = V^r_{\mu_1,1} \cdots V^r_{\mu_n,n} f(x), \tag{8.32}$$

for $\mu \in \mathbb{R}^n_+$. Applying successively (8.31), we obtain the following lemma.

Lemma 8.8. *Let $r \in \mathbb{N}$ and $\mu \in \mathbb{R}^n_+$. If $f \in L^1_{loc}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$*

$$|V^r_\mu f(x)| \leq \frac{c}{|R_\mu|} \int_{R_\mu} |f(x + \xi)|d\xi, \tag{8.33}$$

where $R_\mu = (0, r^2\mu_1) \times \cdots \times (0, r^2\mu_n)$ and c only depends on r and n.

Lemma 8.9. *Let $r \in \mathbb{N}$ and $f \in L^1_{loc}(\mathbb{R}^n)$. Set*

$$E = \{\mu \in \mathbb{R}^n_+ : \mu_k \leq 2\mu_l,\ 1 \leq k, l \leq n\}.$$

Then

$$\lim_{\mu \in E, \mu \to 0} V^r_\mu f(x) = f(x) \tag{8.34}$$

for a.e $x \in \mathbb{R}^n$.

Proof. For each $\mu \in E$ we denote $\mu^* = \max(\mu_1, \ldots, \mu_n)$ and set $P_\mu = (0, r^2\mu^*)^n$. It is clear that if f is constant, $f(x) = A$ say, then $V^r_\mu f(x) = A$. Thus, by Lemma 8.8, for any $x \in \mathbb{R}^n$ and $\mu \in E$, we have

$$|V^r_\mu f(x) - f(x)| \leq \frac{c}{|R_\mu|} \int_{R_\mu} |f(x + \xi) - f(x)|d\xi \leq$$

$$\leq \frac{c'}{|P_\mu|} \int_{P_\mu} |f(x + \xi) - f(x)|d\xi,$$

(recall that $R_\mu = (0, r^2\mu_1) \times \cdots \times (0, r^2\mu_n)$) where the second inequality holds because $R_\mu \subset P_\mu$ and (since $\mu_k \geq \mu^*/2$, $k = 1, \ldots, n$)

$$\mathrm{mes}_n R_\mu = r^{2n} \prod_{k=1}^n \mu_k \geq \Big(\frac{\mu^* r^2}{2}\Big)^n = 2^{-n} \mathrm{mes}_n P_\mu.$$

For every Lebesgue point x of f, Lebesgue's differentiation theorem (see e.g. Theorem 7.16 on p. 108–109 in [47]) ensures that the right-hand side of this inequality vanishes, as $\mu \to 0$ in E. This implies (8.34). □

Lemma 8.10. *Let $r \in \mathbb{N}$ and $\mu \in \mathbb{R}^n_+$. If $f \in L^1_{loc}(\mathbb{R}^n)$, then $V^r_\mu f$ is continuous on $\mathbb{R}^n$. Further, if $f \in L^\infty(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$, then $V^r_\mu f$ is uniformly continuous on $\mathbb{R}^n$.*

Proof. Fix $\mu \in \mathbb{R}^n_+$. For $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, we have by Lemma 8.8,

$$\begin{aligned}\big|V^r_\mu f(x) - V^r_\mu f(x+h)\big| &= \big|V^r_\mu\big(f(x) - f(x+h)\big)\big| \le \\ &\le \frac{c}{|R_\mu|}\int_{R_\mu}\big|f(x+\xi) - f(x+h+\xi)\big|d\xi = \\ &= \frac{c}{|R_\mu|}\int_{R_\mu + x}|f(y) - f(y+h)|dy,\end{aligned}$$

where $R_\mu + x = (x_1, x_1 + r^2\mu_1) \times \cdots \times (x_n, x_n + r^2\mu_n)$. Since f is locally integrable on $\mathbb{R}^n$, the last integral tends to 0 as $h \to 0$. This proves that $V^r_\mu f$ is continuous.

Assume now that $f \in L^\infty(\mathbb{R}^n) \cap S_0(\mathbb{R}^n)$. We will show that

$$\lim_{|x|\to\infty} V^r_\mu f(x) = 0. \tag{8.35}$$

Fix $\varepsilon > 0$. For $k \in \mathbb{N}$ we put

$$E_k = \{x \in \mathbb{R}^n : |f(x)| > \varepsilon,\ |x| > k\}.$$

Since $f \in S_0(\mathbb{R}^n)$, we have $\mathrm{mes}_n\{x \in \mathbb{R}^n : |f(x)| > \varepsilon\} < \infty$. Furthermore, $E_k \supset E_{k+1}$ for all k and $\cap_{k=1}^\infty E_k = \emptyset$. Thus $\lim_{k\to\infty} \mathrm{mes}_n E_k = 0$ (see e.g. [47, Theorem 3.26, p. 41]). Fix k_1 such that $\mathrm{mes}_n E_{k_1} < \varepsilon$. By Lemma 8.8,

$$|V^r_\mu f(x)| \le \frac{c}{|R_\mu|}\int_{R_\mu + x}|f(\xi)|d\xi \tag{8.36}$$

for all x. Let $|x| > k_1 + \sqrt{n}r^2\mu^*$, where $\mu^* = \max(\mu_1, \ldots, \mu_n)$. Take $y \in R_\mu + x$. Then $|y| > k_1$. Indeed, $y - x \in R_\mu$ so $|y - x| \le \sqrt{n}r^2\mu^*$, and thus

$$|y| \ge |x| - |y - x| \ge |x| - \sqrt{n}r^2\mu^* > k_1.$$

Thus, if $|x| > k_1 + \sqrt{n}r^2\mu^*$, then

$$|f(y)| \le \varepsilon \quad \text{for all } y \in (R_\mu + x) \setminus E_{k_1}. \tag{8.37}$$

Inequality (8.36) implies

$$|V^r_\mu f(x)| \le \frac{c}{|R_\mu|}\Big(\int_{(R_\mu+x)\cap E_{k_1}}|f(\xi)|d\xi + \int_{(R_\mu+x)\setminus E_{k_1}}|f(\xi)|d\xi\Big)$$

for all x. For $|x| > k_1 + \sqrt{n}r^2\mu^*$ we apply (8.37) and use the fact that $\mathrm{mes}_n E_{k_1} < \varepsilon$. We get

$$|V^r_\mu f(x)| \le \frac{c}{|R_\mu|}\Big(\|f\|_\infty|E_{k_1}| + \varepsilon|R_\mu + x|\Big) \le c\,\varepsilon(\|f\|_\infty|R_\mu|^{-1} + 1).$$

This proves (8.35). Since $V_h^r f$ is continuous on $\mathbb{R}^n$, it follows from (8.35) that $V_h^r f$ is actually uniformly continuous. $\square$

Remark 8.11. The assumption that $f \in L^\infty(\mathbb{R})$ in Lemma 8.10 can not be omitted. Namely, we shall construct an unbounded function $f \in L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$, for which the Steklov average

$$f_h(x) \equiv V_h^r f(x)$$

is *not* a uniformly continuous function on $\mathbb{R}$ for any $h > 0$. For simplicity we take $r = 1$. Set

$$f(x) = \sum_{k=1}^{\infty} k^2 \chi_{(2^k, 2^k + 1/k^2)}(x).$$

Then f is unbounded and belongs to $L^1_{loc}(\mathbb{R}) \cap S_0(\mathbb{R})$. Fix $h > 0$ and $\delta \in (0, h)$. Take k such that $1/k^2 < \delta$ and

$$h < 2^{k-1} - \frac{1}{(k-1)^2}. \tag{8.38}$$

Set $x_0 = 2^k - h$. Then

$$f_h(x_0) = \frac{1}{h} \int_0^h f(x_0 + u)du = \frac{1}{h} \int_{2^k - h}^{2^k} f(u)du.$$

By (8.38), $2^k - h > 2^{k-1} + 1/(k-1)^2$. Since f vanishes on the interval $(2^{k-1} + 1/(k-1)^2, 2^k)$, it follows that $f_h(x_0) = 0$. Further,

$$f_h(x_0 + \delta) = \frac{1}{h} \int_{x_0+\delta}^{2^k+\delta} f(u)du \geq \frac{1}{h} \int_{2^k}^{2^k+1/k^2} f(u)du = \frac{1}{h},$$

where the inequality holds since $x_0 + \delta = 2^k - h + \delta < 2^k$ and $1/k^2 < \delta$. We have now proved that for every $\delta \in (0, h)$, there exists $x_0 \in \mathbb{R}$ such that $f_h(x_0) = 0$ and $f_h(x_0 + \delta) \geq 1/h$. Thus, f_h is not uniformly continuous on $\mathbb{R}$.

Our next lemma uses the following well known property: For all sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, and for all real valued functions f defined on $A \times B$, it holds that

$$\inf_{(x,y) \in A \times B} f(x, y) = \inf_{x \in A} \Big(\inf_{y \in B} f(x, y) \Big). \tag{8.39}$$

To see this we observe that for all $(x, y) \in A \times B$,

$$f(x, y) \geq \inf_{y \in B} f(x, y) \geq \inf_{x \in A} \Big(\inf_{y \in B} f(x, y) \Big).$$

This implies that

$$\inf_{(x,y) \in A \times B} f(x, y) \geq \inf_{x \in A} \Big(\inf_{y \in B} f(x, y) \Big). \tag{8.40}$$

On the other hand,

$$\inf_{(x,y)\in A\times B} f(x,y) \le f(x,y)$$

for all $(x,y)\in A\times B$, and thus

$$\inf_{(x,y)\in A\times B} f(x,y) \le \inf_{y\in B} f(x,y)$$

for all $x\in A$. From this we see that (8.40) holds also when the inequality is reversed. This proves (8.39).

Lemma 8.12. *Let $\lambda>0$, $1\le p\le\infty$, and $f\in S_0(\mathbb{R}^n)$. Suppose that the functions $N_k(\hat{x}_k)=\|f(\cdot,\hat{x}_k)\|^*_{\mathcal{C}^\lambda}$, $k=1,\dots,n$, belong to $L^p(\mathbb{R}^{n-1})$. Then $f\in L^1_{loc}(\mathbb{R}^n)$, and for every cube $Q\subset\mathbb{R}^n$ with side length* 1, *there holds the inequality*

$$\int_Q |f(x)|dx \le c\Big(f^*(2^{-n})+\sum_{k=1}^n \|N_k\|_p\Big), \tag{8.41}$$

where c depends only on p, λ, and n.

Proof. We give the proof for $Q=(0,1)^n$. The same reasonings can be used for general Q. To prove this lemma, we will use induction on n. First we consider the case $n=2$. By Lemma 8.2,

$$|f(x,y)|\le c\big(\mathcal{R}_1 f(1,y)+N_1(y)\big) \tag{8.42}$$

and

$$|f(x,y)|\le c\big(\mathcal{R}_2 f(x,1)+N_2(x)\big)$$

for a.e. $x,y\in\mathbb{R}$. Rearrange with respect to x in the last inequality. By (2.13) and (2.39),

$$\mathcal{R}_1 f(1,y)\le c\big(\mathcal{R}_{2,1} f(1/2,1)+N_2^*(1/2)\big)\le c\Big(f^*(1/2)+2\int_0^1 N_2^*(t)dt\Big).$$

Apply Hölder's inequality in the case $1<p<\infty$, and use (2.24). We obtain

$$\mathcal{R}_1 f(1,y)\le c\big(f^*(1/2)+\|N_2\|_p\big).$$

Use this estimate in (8.42), and integrate over $(0,1)^2$. This gives

$$\iint_{(0,1)^2} |f(x,y)|dxdy \le c\Big(f^*(1/2)+\int_0^1 N_1(y)dy+\|N_2\|_p\Big).$$

Now (8.41) follows (for $n=2$ and $Q=(0,1)^2$) by applying Hölder's inequality to the integral on the right-hand side.

Fix $n\ge 3$ ($n\in\mathbb{N}$). Our induction hypothesis is that this lemma holds for $n-1$. Suppose that $f\in S_0(\mathbb{R}^n)$ satisfies the conditions of this lemma. For $y\in\mathbb{R}$ we let f_y denote the function $x\mapsto f(x,y)$, where

$(x,y) = (x_1, \dots, x_{n-1}, y) \in \mathbb{R}^n$. Note that $f_y \in S_0(\mathbb{R}^{n-1})$ for a.e. $y \in \mathbb{R}$. For such y, our induction hypothesis states that

$$\int_{(0,1)^{n-1}} |f(x,y)|dx \le c\Big(f_y^*(2^{-n+1}) + \sum_{k=1}^{n-1} \|N_k(\cdot,y)\|_{L^p(\mathbb{R}^{n-2})}\Big). \tag{8.43}$$

By Lemma 8.2,

$$|f(x,y)| \le c\big(\mathcal{R}_n f(x,1) + N_n(x)\big)$$

for a.e. $(x,y) \in \mathbb{R}^n$. Rearrange with respect to x. By (2.13),

$$f_y^*(2^{-n+1}) \le c\big(\varphi^*(2^{-n}) + N_n^*(2^{-n})\big), \tag{8.44}$$

where $\varphi(x) = \mathcal{R}_n f(x,1)$. Recall from Section 2.4 that f and $\mathcal{R}_n f$ are equimeasurable. Therefore

$$f^*(\tau) = (\mathcal{R}_n f)^*(\tau), \quad \tau > 0. \tag{8.45}$$

Further, we have

$$(\mathcal{R}_n f)^*(\tau) = \sup_A \Big(\inf_{(x,t)\in A} \mathcal{R}_n f(x,t)\Big),$$

where the supremum is taken over all F_σ-sets $A \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$ with measure τ. If we instead take supremum only over sets of the special form $A = E \times (0,1)$, where $E \subset \mathbb{R}^{n-1}$ is of type F_σ and $\text{mes}_{n-1} E = \tau$, then we get

$$(\mathcal{R}_n f)^*(\tau) \ge \sup_E \Big(\inf_{(x,t)\in E\times(0,1)} \mathcal{R}_n f(x,t)\Big). \tag{8.46}$$

According to the property (8.39), for every set $E \subset \mathbb{R}^{n-1}$ it holds that

$$\inf_{(x,t)\in E\times(0,1)} \mathcal{R}_n f(x,t) = \inf_{x\in E}\Big(\inf_{0<t<1} \mathcal{R}_n f(x,t)\Big) = \inf_{x\in E} \varphi(x)$$

(recall $\varphi(x) = \mathcal{R}_n f(x,1)$). Using this in (8.46), we get

$$(\mathcal{R}_n f)^*(\tau) \ge \sup_E \Big(\inf_{x\in E} \varphi(x)\Big) = \varphi^*(\tau).$$

This inequality and (8.45) implies that

$$\varphi^*(\tau) \le f^*(\tau)$$

for all $\tau > 0$. We apply this estimate with $\tau = 2^{-n}$ in (8.44). This gives

$$f_y^*(2^{-n+1}) \le c\big(f^*(2^{-n}) + N_n^*(2^{-n})\big) \le$$

$$\le c\Big(f^*(2^{-n}) + 2^n \int_0^1 N_n^*(t)dt\Big).$$

So, by Hölder's inequality, we get

$$f_y^*(2^{-n+1}) \le c\Big(f^*(2^{-n}) + \|N_n^*\|_{L^p(\mathbb{R}_+)}\Big) =$$

$$= c\Big(f^*(2^{-n}) + \|N_n\|_{L^p(\mathbb{R}^{n-1})}\Big),$$

where the equality holds by (2.24). Use this estimate in (8.43), and integrate with respect to y over $(0,1)$. This gives

$$\int_{(0,1)^n} |f(z)|dz \le c\Big(f^*(2^{-n}) + \|N_n\|_{L^p(\mathbb{R}^{n-1})} + \sum_{k=1}^{n-1} \int_0^1 \|N_k(\cdot, y)\|_{L^p(\mathbb{R}^{n-2})}\, dy\Big).$$

We have by Hölder's inequality that

$$\int_0^1 \|N_k(\cdot, y)\|_{L^p(\mathbb{R}^{n-2})}\, dy \le \|N_k\|_{L^p(\mathbb{R}^{n-1})}$$

for all k. By the two preceding estimates, we obtain (8.41) for $Q = (0,1)^n$, so the proof is complete. □

In (8.2) we defined the modulus of continuity of order $r \in \mathbb{N}$ for functions of one variable. Now we extend this definition to functions of several variables. Let f be a real-valued function on $\mathbb{R}^n$. For h, $x \in \mathbb{R}^n$, and $r \in \mathbb{N}$, the difference of order r is given by

$$\Delta^r(h)f(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih).$$

The *total* modulus of continuity of the order r of f is defined as

$$\omega^r(f;t) = \sup_{|h| \le t} \Big(\sup_{x \in \mathbb{R}^n} |\Delta^r(h)f(x)|\Big).$$

We shall also consider *partial* modulus of continuity. First we denote

$$\Delta_j^r(\tau)f(x) = \Delta^r(\tau e_j)f(x), \tag{8.47}$$

where e_j is the jth unit coordinate vector. The *partial* modulus of continuity of the order r of f, in the direction of the variable x_j, is defined by

$$\omega_j^r(f;t) = \sup_{0 \le \tau \le t} \Big(\sup_{x \in \mathbb{R}^n} |\Delta_j^r(\tau)f(x)|\Big).$$

We set $\omega(f;t) = \omega^1(f;t)$ and $\omega_j(f;t) = \omega_j^1(f;t)$. Let us remind that both the *total* and the *partial* moduli of continuity in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, of the first order were defined in Section 6.2. For $r = 1$, it is easy to verify the relation

$$\max_{j=1,\dots,n} \omega_j(f;t) \le \omega(f;t) \le \sum_{j=1}^{n} \omega_j(f;t)$$

(the corresponding relations for modulus of continuity in $L^p(\mathbb{R}^n)$ were given in (6.31)). The second of these inequalities fails for modulus of continuity of higher order, however for $r \geq 1$ we still have

$$\max_{j=1,\dots,n} \omega_j^r(f;t) \leq \omega^r(f;t).$$

We also point out the relation ([45, Chapter 2.5.13])

$$\omega^{nr}(f;t) \leq c\sum_{j=1}^{n} \omega_j^r(f;t), \quad t \geq 0,\ r \geq 1. \tag{8.48}$$

Marchaud's inequality for functions of one variable is given by (8.5). The corresponding estimate holds also for functions on $\mathbb{R}^n$. Namely, Marchaud's inequality states that if r and s are integers satisfying $1 \leq r < s$, and if $f \in L^\infty(\mathbb{R}^n)$, then for all $t > 0$,

$$\omega^r(f;t) \leq ct^r \int_t^\infty \frac{\omega^s(f;u)}{u^r}\frac{du}{u} \tag{8.49}$$

and, for $j = 1, \dots, n$,

$$\omega_j^r(f;t) \leq ct^r \int_t^\infty \frac{\omega_j^s(f;u)}{u^r}\frac{du}{u} \tag{8.50}$$

((8.49) is proved in [5, Theorem 4.4, pp. 332–333], and (8.50) can be proved in the same way).

Theorem 8.13. *Let $n \geq 2$, $1 \leq p \leq \infty$, and $(n-1)/p < \lambda < \infty$. Set*

$$U_k^{p,\lambda} = L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\mathcal{C}^\lambda_{x_k}(\mathbb{R})],$$

$k = 1, \dots, n$. Suppose that $f \in S_0(\mathbb{R}^n)$ and that $f \in \cap_{k=1}^n U_k^{p,\lambda}$. Then there exists a bounded and uniformly continuous function $g \in \cap_{k=1}^n U_k^{p,\lambda}$, such that $f = g$ a.e. and $\|g\|_{U_k^{p,\lambda}} = \|f\|_{U_k^{p,\lambda}}$, $k = 1, \dots, n$. Moreover, if $\beta \equiv \lambda - (n-1)/p$ and $s > \beta$, $s \in \mathbb{N}$, then

$$\omega^s(g;\delta) \leq c\delta^\beta \sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}} \tag{8.51}$$

for $\delta \geq 0$, where c depends only on p, λ, and n.

Proof. Denote $r = r(\lambda)$ (recall that $r(\lambda)$ is the smallest integer such that $\lambda < r(\lambda)$). By Lemma 8.12, $f \in L^1_{loc}(\mathbb{R}^n)$. So the iterated Steklov average $V_\mu^r f$ is defined on $\mathbb{R}^n$ for every $\mu \in \mathbb{R}^n_+$. Fix $h > 0$ and denote $(k = 0, 1, \dots, n)$

$$\mu_k(h) = (h, \dots, h, 2h, \dots, 2h) \in \mathbb{R}^n_+,$$

where the first k entries are equal to h. Set also

$$f_{\mu_k(h)}(x) = V^r_{\mu_k(h)} f(x).$$

We will prove that

$$|f_{\mu_n(h)}(x) - f_{\mu_0(h)}(x)| \le ch^{\beta} \sum_{k=1}^{n} \|f\|_{U_k^{p,\lambda}} \tag{8.52}$$

for all $x \in \mathbb{R}^n$. Observe that

$$|f_{\mu_n(h)}(x) - f_{\mu_0(h)}(x)| \le \sum_{k=1}^{n} |f_{\mu_k(h)}(x) - f_{\mu_{k-1}(h)}(x)|. \tag{8.53}$$

Hence, (8.52) follows from (8.53) if we prove that

$$|f_{\mu_k(h)}(x) - f_{\mu_{k-1}(h)}(x)| \le ch^{\beta} \|f\|_{U_k^{p,\lambda}} \tag{8.54}$$

for $k = 1, \ldots, n$ and all $x \in \mathbb{R}^n$. For simplicity of notation, we give the proof of (8.54) in the case $k = n$ and $x = 0$. Note that

$$f_{\mu_n(h)}(0) - f_{\mu_{n-1}(h)}(0) = V_{h,1}^r \cdots V_{h,n-1}^r I_h(0),$$

where

$$I_h(y) = (V_{h,n}^r - V_{2h,n}^r) f(y,0), \quad y \in \mathbb{R}^{n-1}.$$

Applying successively (8.31) (with $k = 1, \ldots, n-1$), we obtain

$$|f_{\mu_n(h)}(0) - f_{\mu_{n-1}(h)}(0)| \le \frac{c}{h^{n-1}} \int_{T_h} |I_h(y)| dy, \tag{8.55}$$

where $T_h = (0, r^2 h)^{n-1}$. Changing variables, we see that

$$V_{2h,n}^r f(y,0) = h^{-r} \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} \int_{(0,h)^r} f(y, 2i\sigma(v)) dv$$

(recall that $\sigma(v) = \sum_{j=1}^r v_j$). Thus,

$$I_h(y) = |Q_h|^{-1} \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} \int_{Q_h} \Big(f(y, i\sigma(v)) - f(y, 2i\sigma(v)) \Big) dv,$$

where $Q_h = (0,h)^r$. Adding and subtracting $f(y,0)$ to the integrand, and using the definition (8.47), we get

$$I_h(y) = (-1)^{r-1} h^{-r} \int_{Q_h} \Big(\Delta_n^r(\sigma(v)) - \Delta_n^r(2\sigma(v)) \Big) f(y,0) dv.$$

By (8.4), we then have

$$|I_h(y)| \le h^{-r} \int_{Q_h} \Big(\omega^r(f_y; \sigma(v)) + \omega^r(f_y; 2\sigma(v)) \Big) dv \le$$

$$\le ch^{-r} \int_{Q_h} \omega^r(f_y; \sigma(v)) dv.$$

Let $N_n(y) = \|f(y,\cdot)\|^*_{\mathcal{C}^\lambda}$. Since $r-1 \le \lambda < r$, we get

$$|I_h(y)| \le ch^{-r} N_n(y) \int_{Q_h} \sigma(v)^\lambda dv \le$$

$$\le ch^{-1} N_n(y) \int_0^{rh} z^\lambda dz = c' h^\lambda N_n(y).$$

Use the preceding estimate in (8.55). We get

$$|f_{\mu_n(h)}(0) - f_{\mu_{n-1}(h)}(0)| \le ch^{\lambda-n+1} \int_{T_h} N_n(y) dy \le ch^\beta \|N_n\|_p,$$

where the last estimate holds by Hölder's inequality and the fact that

$$\lambda - n + 1 + (n-1)/p' = \lambda - (n-1)/p = \beta.$$

This completes the proof of (8.54) in the case $k = n$ and $x = 0$. One proves (8.54) in the same way if $k \in \{1, \ldots, n-1\}$ or $x \ne 0$. As we mentioned above, combining (8.54) and (8.53), we immediately get (8.52).

For $\nu \in \mathbb{Z}$, we set $h_\nu = 2^{-\nu}$ and

$$g_\nu = f_{h_\nu,\ldots,h_\nu} = V^r_{h_\nu,1} \cdots V^r_{h_\nu,n} f.$$

As we observed before, $f \in L^1_{loc}(\mathbb{R}^n)$. Thus, by Lemma 8.9, for a.e. $x \in \mathbb{R}^n$ there exists the limit

$$g(x) \equiv \lim_{k\to\infty} g_k(x),$$

and $g(x) = f(x)$ a.e. on $\mathbb{R}^n$. We have

$$g(x) = g_\nu(x) + \sum_{k=\nu}^\infty \Big(g_{k+1}(x) - g_k(x)\Big)$$

for any $\nu \in \mathbb{Z}$. To see that g is bounded and uniformly continuous on $\mathbb{R}^n$, we first observe that

$$|g(x) - g_\nu(x)| \le \sum_{k=\nu}^\infty |g_{k+1}(x) - g_k(x)|.$$

Applying (8.52) to each term in this series, we find that

$$|g(x) - g_\nu(x)| \le c \sum_{k=\nu}^\infty 2^{-k\beta} \sum_{j=1}^n \|f\|_{U_j^{p,\lambda}} \le c' 2^{-\nu\beta} \sum_{j=1}^n \|f\|_{U_j^{p,\lambda}} \tag{8.56}$$

for all $x \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}$. Thus, the sequence $\{g_\nu\}$ converges uniformly to g on $\mathbb{R}^n$ as $\nu \to \infty$.

Let ν be the greatest integer satisfying $2^\nu \le r^2$. Then

$$\frac{1}{r^2} \le 2^{-\nu} < \frac{2}{r^2}.$$

Lemma 8.8 with $\mu = (2^{-\nu}, \dots, 2^{-\nu})$ states that

$$|g_\nu(x)| = |V_\mu^r f(x)| \le c|P_\nu(x)|^{-1} \int_{P_\nu(x)} |f(z)|dz,$$

for a.e. x, where $P_\nu(x) = (x_1, x_1 + r^2 2^{-\nu}) \times \dots \times (x_n, x_n + r^2 2^{-\nu})$. Subdivide P_ν into subcubes having side length less than one, split the last integral into the corresponding sum, and apply Lemma 8.12 to each of the terms. This gives

$$|g_\nu(x)| \le c\Big(f^*(2^{-n}) + \sum_{j=1}^n \|f\|_{U_j^{p,\lambda}}\Big).$$

Whence, $g_\nu \in L^\infty(\mathbb{R}^n)$. From (8.56) and the boundedness of g_ν, we conclude that $g \in L^\infty(\mathbb{R}^n)$. But $f = g$ a.e., and thus $f \in L^\infty(\mathbb{R}^n)$. By assumption we also have $f \in S_0(\mathbb{R}^n)$, so Lemma 8.10 then ensures that g_k is uniformly continuous on $\mathbb{R}^n$ for each k. Since the convergence of the sequence $\{g_k\}$ is uniform, this proves that the limit function g also is uniformly continuous on $\mathbb{R}^n$.

Next we will prove (8.51). We first estimate $\omega_j^r(g;\delta)$. Fix $\delta > 0$. Take $\nu \in \mathbb{Z}$ such that

$$2^{-\nu-1} < \delta \le 2^{-\nu}. \tag{8.57}$$

For $j = 1, \dots, n$ and $x \in \mathbb{R}^n$,

$$|\Delta_j^r(\delta)g(x)| \le |\Delta_j^r(\delta)(g - g_\nu)(x)| + |\Delta_j^r(\delta)g_\nu(x)|. \tag{8.58}$$

From the definition of $\Delta_j^r(\delta)$, we see that

$$|\Delta_j^r(\delta)(g - g_\nu)(x)| \le 2^r\|g - g_\nu\|_\infty \le c2^{-\nu\beta} \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}},$$

where the second inequality holds by (8.56). By (8.57), $2^{-\nu\beta} < (2\delta)^\beta$. Hence,

$$\big|\Delta_j^r(\delta)\big(g(x) - g_\nu(x)\big)\big| \le c\delta^\beta \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}}, \tag{8.59}$$

$j = 1, \dots, n$, $x \in \mathbb{R}^n$. We will prove the following estimate for the second term on the right-hand side in (8.58):

$$|\Delta_j^r(\delta)g_\nu(x)| \le c\delta^\beta \|f\|_{U_j^{p,\lambda}}, \tag{8.60}$$

for $j = 1, \dots, n$ and all $x \in \mathbb{R}^n$. To avoid some notational inconvenience, we give the proof of (8.60) in the case $x = 0$. Set $h = 2^{-\nu}$. Observe that

$$\begin{aligned}|\Delta_j^r(\delta)g_\nu(0)| = \big|\Delta_j^r(\delta)V_{(h,\dots,h)}^r f(0)\big| = \big|V_{(h,\dots,h)}^r \Delta_j^r(\delta) f(0)\big| \le \\ \le ch^{-n} \int_{(0,r^2h)^n} |\Delta_j^r(\delta) f(x)|dx, \qquad (8.61)\end{aligned}$$

where the inequality holds by Lemma 8.8. Observe that

$$|\Delta_j^r(\delta)f(x)| \le \omega^r(f_{\hat{x}_j};\delta) \le \delta^\lambda N_j(\hat{x}_j),$$

where $N_j(\hat{x}_j) = \|f(\cdot,\hat{x}_j)\|^*_{\mathcal{C}^\lambda}$. Use the preceding estimate in (8.61), and apply Hölder's inequality. We obtain

$$|\Delta_j^r(\delta)g_\nu(0)| \le c\delta^\lambda h^{-n+1}\int_{(0,r^2h)^{n-1}} N_j(\hat{x}_j)d\hat{x}_j \le c'\delta^\lambda h^{-(n-1)/p}\|N_j\|_p.$$

By (8.57), $h^{-(n-1)/p} = (2^{-\nu})^{-(n-1)/p} \le \delta^{-(n-1)/p}$. Use this inequality in the preceding estimate. This gives (8.60) for $x = 0$. The proof of (8.60) for $x \ne 0$ is obtained in the same way. Combining (8.58), (8.59), and (8.60) yields the inequality

$$|\Delta_j^r(\delta)g(x)| \le c\delta^\beta \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}}$$

for all $x \in \mathbb{R}^n$, which implies

$$\omega_j^r(g;\delta) \le c\delta^\beta \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}}, \tag{8.62}$$

for $j = 1,\dots,n$. We will prove that $(j = 1,\dots,n)$

$$\omega_j^s(g;\delta) \le c\delta^\beta \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}}. \tag{8.63}$$

Indeed, if $r \le s$, then inequality (8.63) follows from (8.62) by (8.3). Suppose $\beta < s < r$. Since g is bounded, Marchaud's inequality (8.50) together with (8.62) imply that

$$\omega_j^s(g;\delta) \le c\delta^s \int_\delta^\infty \frac{\omega_j^r(g;u)}{u^s}\frac{du}{u} \le$$

$$\le c'\delta^s \int_\delta^\infty u^{\beta-s-1}du \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}} = c''\delta^\beta \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}}.$$

So (8.63) is proved.

Combine (8.63) with (8.48) (taking $r = s$). We obtain

$$\omega^{ns}(g;\delta) \le c\delta^\beta \sum_{k=1}^n \|f\|_{U_k^{p,\lambda}}.$$

Marchaud's inequality (8.49) ensures that

$$\omega^s(g;\delta) \le c\delta^s \int_\delta^\infty \frac{\omega^{ns}(g;u)}{u^s}\frac{du}{u}.$$

Combine the two preceding estimates, and compute the integral in u. We then obtain (8.51).

By assumption, $f_{\hat{x}_k} \in \mathcal{C}^\lambda(\mathbb{R}) \cap S_0(\mathbb{R})$ for a.e. $\hat{x}_k$, $k = 1, \ldots, n$. For such $\hat{x}_k$, $f_{\hat{x}_k}$ is continuous on $\mathbb{R}$. But g is continuous on $\mathbb{R}^n$ and $f = g$ a.e. It follows that for a.e. $\hat{x}_k$, $f_{\hat{x}_k}(y) = g_{\hat{x}_k}(y)$ for all $y \in \mathbb{R}$. Thus, $\|f\|_{U_k^{p,\lambda}} = \|g\|_{U_k^{p,\lambda}}$, $k = 1, \ldots, n$. This completes the proof. $\square$

Remark 8.14. Theorem 8.13 does not hold for $s = \beta$, $\beta \in \mathbb{N}$. Actually, this fact can be derived from a similar statement about Sobolev spaces. A corresponding example is given in [1, Example 4.44, pp. 110–111]. However, there is inaccuracy in this example (in fact, it is incorrect and needs a slight modification). For simplicity, we consider a special case. Let $n = p = 2$, $\lambda = 3/2$. Then $\beta = \lambda - (n-1)/p = 1$. Denote

$$U^{2,3/2} \equiv U_1^{2,3/2} \bigcap U_2^{2,3/2}.$$

Suppose that Theorem 8.13 holds for these parameters and for $s = 1$. As usual, we denote by $\mathrm{Lip}\, 1 = \mathrm{Lip}\,(1; \mathbb{R}^2)$ the class of all $\varphi \in C(\mathbb{R}^2)$ such that $\omega(\varphi; t) \le ct$, $t \ge 0$. Then, by Theorem 8.13,

$$U^{2,3/2} \subset \mathrm{Lip}\, 1. \tag{8.64}$$

It is easy to show that $W_2^2(\mathbb{R}^2) \subset U^{2,3/2}$ (see Theorem 8.21 below). We shall obtain a contradiction to (8.64) by finding a function $f \in W_2^2(\mathbb{R}^2)$, such that $f \notin \mathrm{Lip}\, 1$.

We define $v(0,0) = 0$ and

$$v(x,y) = x\varphi(x,y), \quad 0 < x^2 + y^2 \le 4,$$

where

$$\varphi(x,y) = \left(\ln \frac{2}{r}\right)^{1/4}, \quad r = r(x,y) = \sqrt{x^2 + y^2}.$$

Let $u \in C_0^\infty(\mathbb{R}^2)$ satisfy that $u = 1$ on the ball $B_{1/2} \equiv B(0, 1/2)$, and $u = 0$ on $\mathbb{R}^2 \setminus B(0,1)$ (the existence of such a function u is ensured e.g. by Theorem 1.4.1 on page 25 in [23]). Set $f(x,y) = u(x,y)v(x,y)$ on $\mathbb{R}^2$. Clearly f belongs to $C_0^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$, and $f = v$ on $B_{1/2}$. So, to see that $f \in W_2^2(\mathbb{R}^2)$ it is enough to check that $v \in W_2^2(B_{1/2})$. For $(x,y) \in B_{1/2}$, it holds that

$$|v(x,y)| \le \varphi(x,y).$$

Further,

$$v'_x(x,y) = \varphi(x,y) - \frac{x^2}{4r^2}\big(\varphi(x,y)\big)^{-3},$$

so on $B_{1/2}$, we have

$$|v'_x(x,y)| \le \varphi(x,y) + \big(\varphi(x,y)\big)^{-3} \le 2\varphi(x,y),$$

since $\varphi(x,y) > 1$ on this set. In the same way, we obtain

$$|v'_y(x,y)| \le \frac{1}{r}\big(\varphi(x,y)\big)^{-3}$$

on $B_{1/2}$. We also have

$$v''_{xx}(x,y) = -\frac{x(x^2+3y^2)}{4r^4}\big(\varphi(x,y)\big)^{-3} - \frac{3x^3}{16r^4}\big(\varphi(x,y)\big)^{-7}.$$

Using again that $\varphi(x,y) > 1$ on $B_{1/2}$, we find that

$$|v''_{xx}(x,y)| \le \frac{1}{r}\big(\varphi(x,y)\big)^{-3} + \frac{1}{r}\big(\varphi(x,y)\big)^{-7} \le \frac{2}{r}\big(\varphi(x,y)\big)^{-3},$$

on this ball. Similarly, we obtain the same estimate for v''_{xy} and v''_{yy}. So, to verify that $v \in W_2^2(B_{1/2})$, we only need to check that $\varphi(x,y)$ and $r^{-1}(\varphi(x,y))^{-3}$ belong to $L^2(B_{1/2})$. Changing to polar coordinates, we obtain

$$\iint_{B_{1/2}} \big(\varphi(x,y)\big)^2 dxdy = 2\pi \int_0^{1/2} \Big(\ln\frac{2}{r}\Big)^{1/2} r dr < \infty$$

and

$$\iint_{B_{1/2}} \frac{(\varphi(x,y))^{-6}}{x^2+y^2}\, dxdy = 2\pi \int_0^{1/2} \Big(\ln\frac{2}{r}\Big)^{-3/2} \frac{dr}{r} < \infty.$$

We have now proved that $v \in W_2^2(B_{1/2})$, and thus $f \in W_2^2(\mathbb{R}^2)$. Further, for $0 < x \le 1/2$ we have $f(x,0) = v(x,0)$, and so

$$f(x,0) - f(0,0) = x\Big(\ln\frac{2}{x}\Big)^{1/4}.$$

Thus $f \notin \mathrm{Lip}\,1$, as required.

In the following remark we consider functions that satisfy the conditions of Theorem 8.13 and for which *all* sections are continuous. We note that this is not true for an arbitrary rearrangable functions in $\cap_{k=1}^n U_k^{p,\lambda}$. Take for example $\chi_{\{0\}}$ - the characteristic function of the set $\{0\}$. Clearly it belongs to $S_0(\mathbb{R}^n)$ and $\cap_{k=1}^n U_k^{p,\lambda}$, but at the same time, n of its sections are discontinuous.

Remark 8.15. In Theorem 8.13, a.e. section of f is continuous, and $f = g$ a.e. However, if *all* sections of f are continuous, then in fact $f(x) = g(x)$ for *all* $x \in \mathbb{R}^n$. Namely, if f and g are any two measurable equivalent functions on $\mathbb{R}^n$, and all sections of f and g are continuous on $\mathbb{R}$, then $f(x) = g(x)$ for all $x \in \mathbb{R}^n$. We will prove this using induction on n. Observe that this statement is trivial for $n = 1$, and assume that it holds with $n = m$ for some $m \ge 1$. We will prove that the statement in question is true also for $n = m + 1$.

Let f and g be measurable equivalent functions on $\mathbb{R}^{m+1}$, and assume that all sections of f and g are continuous on $\mathbb{R}$. Show that $f = g$ everywhere on $\mathbb{R}^{m+1}$. Let $A = \{z \in \mathbb{R}^{m+1} : f(z) \neq g(z)\}$. Then $\text{mes}_{m+1} A = 0$. So for a.e. $x \in \mathbb{R}$, the set

$$A_x = \{y \in \mathbb{R}^m : (x, y) \in A\} = \{y \in \mathbb{R}^m : f(x, y) \neq g(x, y)\}$$

is measurable (in $\mathbb{R}^m$) and has measure 0. Suppose $f(x^*, y^*) \neq g(x^*, y^*)$ for some $x^* \in \mathbb{R}$, $y^* \in \mathbb{R}^m$. Then, since f_{y^*} and g_{y^*} are continuous on $\mathbb{R}$, there exists $\varepsilon > 0$ such that

$$f(x, y^*) \neq g(x, y^*), \quad \text{for all } x \in I \equiv (x^* - \varepsilon, x^* + \varepsilon).$$

For a.e. $x \in \mathbb{R}$, the functions $f(x, \cdot)$ and $g(x, \cdot)$ are measurable on $\mathbb{R}^m$. So for such $x \in I$, our induction hypothesis implies that $f(x, \cdot)$ and $g(x, \cdot)$ are not equivalent on $\mathbb{R}^m$(since if they were, then we would have $f(x, y) = g(x, y)$ for all $y \in \mathbb{R}^m$). That is, for a.e. $x \in I$ we have $\text{mes}_m A_x > 0$, which is a contradiction.

Remark 8.16. Let $\lambda > 0$ and set $r = r(\lambda)$, i.e. $r - 1 \leq \lambda < r$, $r \in \mathbb{N}$. For any integer $s > r$, we have equivalence between the semi norm

$$\|f\|_{\mathcal{C}^{\lambda,s}} \equiv \sup_{t>0} \frac{\omega^s(f; t)}{t^\lambda}$$

and $\|f\|_{\mathcal{C}^{\lambda,r}} = \|f\|_{\mathcal{C}^\lambda}$.

Indeed, suppose $f \in \mathcal{C}^{\lambda,s}(\mathbb{R})$. Arguing as in the proof of inequality (8.8), we obtain that

$$|f(x)| \leq 2^s f^*(t) + h^\lambda \|f\|_{\mathcal{C}^{\lambda,s}},$$

for $x \in \mathbb{R}$, $t > 0$, and $0 < h < (s + 1)t$. This shows that f is bounded, and thus Marchaud's inequality (8.5) ensures that

$$\omega^r(f; t) \leq ct^r \int_t^\infty \frac{\omega^s(f; u)}{u^r} \frac{du}{u} \leq ct^r \int_t^\infty u^{\lambda - r - 1} du \|f\|_{\mathcal{C}^{\lambda,s}}.$$

Thus,

$$\omega^r(f; t) \leq ct^\lambda \|f\|_{\mathcal{C}^{\lambda,s}},$$

so that $\|f\|_{\mathcal{C}^\lambda} \leq c\|f\|_{\mathcal{C}^{\lambda,s}}$ and thus $f \in \mathcal{C}^\lambda(\mathbb{R})$. Suppose now that $f \in \mathcal{C}^\lambda(\mathbb{R})$. By (8.3) we have

$$\|f\|_{\mathcal{C}^{\lambda,s}} \leq 2^{s-r} \|f\|_{\mathcal{C}^\lambda},$$

and then $f \in \mathcal{C}^{\lambda,s}(\mathbb{R})$. These estimates show that $\|f\|_{\mathcal{C}^\lambda}$ and $\|f\|_{\mathcal{C}^{\lambda,s}}$ are equivalent for $s > r$.

8.3. **Sobolev-Liouville spaces.** The main result in this section is Theorem 8.21, which gives an embedding from Sobolev-Liouville spaces (defined in Section 6) to the mixed norm spaces considered above:

$$U_k^{p,\lambda} = L^p_{\hat{x}_k}(\mathbb{R}^{n-1})[\mathcal{C}^\lambda_{x_k}(\mathbb{R})], \quad k = 1, \dots, n.$$

We will also illustrate that the results obtained in Section 8.1 and Section 8.2 (in particular Theorem 8.5 and Theorem 8.13) are consistent with known theorems about Sobolev-Liouville spaces (see Theorem 8.22 and Theorem 8.23 below).

First we recall the notion of the ε-regularization of a function (see e.g. [49, Chapter 1.6]). Let η be a non-negative function in $C^\infty(\mathbb{R}^n)$ which vanishes outside the unit ball $B(0,1)$, and satisfies the condition

$$\int_{\mathbb{R}^n} \eta(x)dx = 1 \tag{8.65}$$

(as usual, $C^\infty(\mathbb{R}^n)$ denotes the class of functions that have continuous derivatives of all orders). An example of such a function is

$$\eta(x) = \begin{cases} c\exp(1/(|x|^2-1)), & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$$

where c is chosen so that (8.65) is fulfilled. Let $\varepsilon > 0$. The function $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$ belongs to $C^\infty(\mathbb{R}^n)$ and vanishes outside the ball $B(0,\varepsilon)$. For $f \in L^1_{loc}(\mathbb{R}^n)$, the convolution

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)f(y)dy \tag{8.66}$$

is called the ε-regularization of the function f. Important properties of regularizations are contained in the next theorem (see e.g. Theorem 1.6.1 on page 22 in [49]).

Theorem 8.17. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then:*

(i) *For all $\varepsilon > 0$, $f_\varepsilon \in C^\infty(\mathbb{R}^n)$;*
(ii) $\lim_{\varepsilon\to 0+} f_\varepsilon(x) = f(x)$, *at every Lebesgue point of f;*
(iii) *If $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then $\|f_\varepsilon\|_p \le \|f\|_p$ $(\varepsilon > 0)$ and*

$$\lim_{\varepsilon\to 0+} \|f_\varepsilon - f\|_p = 0.$$

For $f \in L^1_{loc}(\mathbb{R}^n)$ that have a usual (weak) derivative $D_k^r f \in L^1_{loc}(\mathbb{R}^n)$ ($r \in \mathbb{N}$, $1 \le k \le n$), it holds that

$$(D_k^r f)_\varepsilon(x) = D_k^r f_\varepsilon(x), \quad a.e.\ x \in \mathbb{R}^n$$

(actually this is proved in [49], page 44). The following lemma shows that this property is true also for functions with Bessel derivatives.

Lemma 8.18. *Let $1 \le p < \infty$ and $\alpha > 0$. Suppose that $f \in L^p(\mathbb{R}^n)$ has Bessel derivatives $J_k^\alpha f \in L^p(\mathbb{R}^n)$, $k = 1, \dots, n$. Then for all $\varepsilon > 0$ and $k = 1, \dots, n$, it holds that*

$$(J_k^\alpha f)_\varepsilon(x) = J_k^\alpha f_\varepsilon(x), \quad a.e.\ x \in \mathbb{R}^n. \tag{8.67}$$

Proof. We have

$$\int_{\mathbb{R}} G_\alpha(u)(J_k^\alpha f)_\varepsilon(x_k - u, \hat{x}_k)du =$$
$$= \int_{\mathbb{R}} G_\alpha(u)du \int_{\mathbb{R}^n} \eta_\varepsilon(x_k - u - y_k, \hat{x}_k - \hat{y}_k) J_k^\alpha f(y)dy \equiv I(x) \tag{8.68}$$

for all $x \in \mathbb{R}^n$. The integral $I(x)$ is absolutely convergent for all x. Indeed, consider the iterated integral

$$\int_{\mathbb{R}} G_\alpha(u)du \int_{\mathbb{R}^n} \eta_\varepsilon(x_k - u - y_k, \hat{x}_k - \hat{y}_k)\big|J_k^\alpha f(y)\big|dy,$$

and apply Hölder's inequality to the interior integral. We then obtain that the preceding integral converges and is bounded by

$$\|G_\alpha\|_{L^1(\mathbb{R})}\|\eta_\varepsilon\|_{L^{p'}(\mathbb{R}^n)}\|J_k^\alpha f\|_{L^p(\mathbb{R}^n)} < \infty.$$

Thus, by Tonelli's theorem the integrand $G_\alpha(u)\eta_\varepsilon(x_k - u - y_k, \hat{x}_k - \hat{y}_k)J_k^\alpha f(y)$ in $I(x)$ belongs to $L^1(\mathbb{R}^{n+1})$ and the order of integration can be interchanged. We then get

$$I(x) = \int_{\mathbb{R}^n} J_k^\alpha f(y)dy \int_{\mathbb{R}} G_\alpha(u)\eta_\varepsilon(x_k - u - y_k, \hat{x}_k - \hat{y}_k)du =$$
$$= \int_{\mathbb{R}^n} \eta_\varepsilon(x - z)dz \int_{\mathbb{R}} G_\alpha(z_k - v)J_k^\alpha f(v, \hat{z}_k)dv =$$
$$= \int_{\mathbb{R}^n} \eta_\varepsilon(x - z)f(z)dz = f_\varepsilon(x)$$

for all $x \in \mathbb{R}^n$, where in the second equality we have made the change of variables $v = y_k$, $\hat{z}_k = \hat{y}_k$, and $z_k = y_k + u$. From this, (8.68), and the uniqueness of the Bessel derivative, we obtain (8.67). □

If a function $f \in L^p(\mathbb{R}^n)$ has usual (weak) derivatives $D_k f \in L^p(\mathbb{R}^n)$, $k = 1, \dots, n$, then f can be modified on a set of measure 0 so that almost every section becomes locally absolutely continuous on $\mathbb{R}$ (see e.g. [49, Theorem 2.1.4, p. 44]). The next lemma states a similar result for functions with Bessel derivatives.

Lemma 8.19. *Let $1 \le p < \infty$ and $\alpha > 1/p$. Suppose that $f \in L^p(\mathbb{R}^n)$ has Bessel derivatives $J_k^\alpha f \in L^p(\mathbb{R}^n)$, $k = 1, \dots, n$. Then f is equivalent to a function f_0 such that $x_k \mapsto f_0(x_k, \hat{x}_k)$ is continuous on $\mathbb{R}$ for a.e. $\hat{x}_k \in \mathbb{R}^{n-1}$.*

Proof. Define f_ε as in (8.66). For all $x \in \mathbb{R}^n$, $\varepsilon > 0$, and $1 \le k \le n$, we have

$$f_\varepsilon(x) = \int_{\mathbb{R}} G_\alpha(u)(J_k^\alpha f)_\varepsilon(x_k - u, \hat{x}_k)du \equiv I_{k,\varepsilon}(x). \tag{8.69}$$

Indeed, note first that (8.69) holds for a.e. $x \in \mathbb{R}^n$ by Lemma 8.18 and the definition of the Bessel derivative. Further, f_ε is continuous on $\mathbb{R}^n$ by Theorem 8.17. Since two continuous and equivalent functions on $\mathbb{R}^n$ coincide, it is enough to show that the right-hand side in (8.69) (denoted by $I_{k,\varepsilon}$) also is continuous on $\mathbb{R}^n$. For all $x, h \in \mathbb{R}^n$ we have

$$|\Delta(h)I_{k,\varepsilon}(x)| \le \int_{\mathbb{R}} G_\alpha(u)\Big|\Delta(h)(J_k^\alpha f)_\varepsilon(x_k - u, \hat{x}_k)\Big|du \le$$

$$\le \int_{\mathbb{R}} G_\alpha(u)du \int_{\mathbb{R}^n} \Big|J_k^\alpha f(y)\Delta(h)\eta_\varepsilon(x_k - u - y_k, \hat{x}_k - \hat{y}_k)\Big|dy \le$$

$$\le \|G_\alpha\|_{L^1(\mathbb{R})}\|J_k^\alpha f\|_{L^p(\mathbb{R}^n)}\|\Delta(h)\eta_\varepsilon\|_{L^{p'}(\mathbb{R}^n)},$$

where η_ε is the function from the definition of the ε-regularization and where the last estimate holds by Hölder's inequality. Note that $\Delta(h)\eta_\varepsilon \to 0$ uniformly on $\mathbb{R}^n$ as $h \to 0$ since η_ε is continuous and has compact support, and thus $\|\Delta(h)\eta_\varepsilon\|_{L^{p'}(\mathbb{R}^n)} \to 0$. Also, $\|G_\alpha\|_{L^1(\mathbb{R})} < \infty$ and $\|J_k^\alpha f\|_{L^p(\mathbb{R}^n)} < \infty$, so the preceding estimates imply that $I_{k,\varepsilon}$ is continuous on $\mathbb{R}^n$. We have now proved that (8.69) holds for *all* $x \in \mathbb{R}^n$.

From Theorem 8.17, we have that

$$\lim_{\varepsilon \to 0+} \|(J_k^\alpha f)_\varepsilon - J_k^\alpha f\|_p = 0. \tag{8.70}$$

By Fubini's theorem, $\|(J_k^\alpha f)_\varepsilon - J_k^\alpha f\|_{L^p(\mathbb{R}^n)}$ is the $L^p(\mathbb{R}^{n-1})$-norm of the function

$$\hat{x}_k \mapsto \|(J_k^\alpha f)_\varepsilon(\cdot, \hat{x}_k) - J_k^\alpha f(\cdot, \hat{x}_k)\|_{L^p(\mathbb{R})}.$$

That is, (8.70) says that the latter norm tends to 0 in $L^p(\mathbb{R}^{n-1})$ as $\varepsilon \to 0+$. This implies the existence of a sequence $\{\varepsilon_j\}_{j=1}^\infty$ in $\mathbb{R}_+$ such that

$$\lim_{j \to \infty} \|(J_k^\alpha f)_{\varepsilon_j}(\cdot, \hat{x}_k) - J_k^\alpha f(\cdot, \hat{x}_k)\|_{L^p(\mathbb{R})} = 0 \tag{8.71}$$

for all $k = 1, \dots, n$ and almost every $\hat{x}_k \in \mathbb{R}^{n-1}$.

Define

$$f_0(x) \equiv \lim_{j \to \infty} f_{\varepsilon_j}(x)$$

at all points $x \in \mathbb{R}^n$ where the limit exists. Then f and f_0 are equivalent on $\mathbb{R}^n$ by Theorem 8.17.

For $1 \le k \le n$ we let E_k be the set of all $\hat{x}_k \in \mathbb{R}^{n-1}$ such that (8.71) is true and $J_k^\alpha f(\cdot, \hat{x}_k)$ belongs to $L^p(\mathbb{R})$. Then $\mathrm{mes}_{n-1}(\mathbb{R}^{n-1} \setminus E_k) = 0$ by Fubini's theorem.

For $\hat{x}_k \in E_k$ and $x_k \in \mathbb{R}$ we define

$$g_k(x_k, \hat{x}_k) = \int_{\mathbb{R}} G_\alpha(x_k - u) J_k^\alpha f(u, \hat{x}_k) du.$$

We have that $J_k^\alpha f(\cdot, \hat{x}_k)$ belongs to $L^p(\mathbb{R})$ (since $\hat{x}_k \in E_k$) and $G_\alpha \in L^{p'}(\mathbb{R})$ (by (6.3) since $\alpha > 1/p$), and therefore the preceding integral converges for all $x_k \in \mathbb{R}$ by Hölder's inequality.

By (8.69) with $\varepsilon = \varepsilon_j$ and by Hölder's inequality, we have for all $1 \le k \le n$, $\hat{x}_k \in E_k$, $x_k \in \mathbb{R}$, and $j \in \mathbb{N}$ that

$$|f_{\varepsilon_j}(x_k, \hat{x}_k) - g_k(x_k, \hat{x}_k)| \le \int_{\mathbb{R}} G_\alpha(x_k - u) \Big| (J_k^\alpha f)_{\varepsilon_j}(u, \hat{x}_k) - J_k^\alpha f(u, \hat{x}_k) \Big| du \le$$
$$\le \|G_\alpha\|_{L^{p'}(\mathbb{R})} \|(J_k^\alpha f)_{\varepsilon_j}(\cdot, \hat{x}_k) - J_k^\alpha f(\cdot, \hat{x}_k)\|_{L^p(\mathbb{R})}.$$

Since (8.71) holds for $\hat{x}_k \in E_k$ (and since $G_\alpha \in L^{p'}(\mathbb{R})$), it follows that $f_{\varepsilon_j}(x_k, \hat{x}_k)$ converges uniformly to $g_k(x_k, \hat{x}_k)$ on $\mathbb{R}$ with respect to x_k as $j \to \infty$. Thus, $g_k(x_k, \hat{x}_k)$ is continuous in x_k on $\mathbb{R}$ and coincides with the pointwise limit, i.e. $g_k(x_k, \hat{x}_k) = f_0(x_k, \hat{x}_k)$ for all $x_k \in \mathbb{R}$. This proves that almost all sections of f_0 are continuous. $\square$

The following result goes back to Hardy and Littlewood [19] (see also [50, Theorem 9.1]). Even stronger theorems are known (see [22, Theorem 4′, p. 318]).

Theorem 8.20. *Let* $1 \le p < \infty$ *and* $1/p < \alpha < \infty$. *Put* $\lambda = \alpha - 1/p$. *Let* $\phi \in L^p(\mathbb{R})$ *and set*

$$g(x) = \int_{\mathbb{R}} G_\alpha(x - u)\phi(u) du. \tag{8.72}$$

Then the integral (8.72) *converges for all* $x \in \mathbb{R}$. *Moreover,* $g \in \mathcal{C}^\lambda(\mathbb{R})$ *and*

$$\|g\|^*_{\mathcal{C}^\lambda} \le c\|\phi\|_{L^p}, \tag{8.73}$$

where c *only depends on* p *and* α.

Theorem 8.20 can be derived from Lemma 6.2. Indeed, let the assumptions of the preceding theorem hold. The integral (8.72) converges for all $x \in \mathbb{R}$ by Hölder's inequality, since $\phi \in L^p(\mathbb{R})$ and $G_\alpha \in L^{p'}(\mathbb{R})$ for $\alpha > 1/p$ (according to (6.3)).

Let k be the least integer such that $\alpha - 1/p < k$. By Hölder's inequality,

$$|\Delta^k(h)g(x)| \le \int_{\mathbb{R}} |\Delta^k(h)G_\alpha(x - u)\phi(u)| du \le \|\Delta^k(h)G_\alpha\|_{p'} \|\phi\|_p \tag{8.74}$$

for all $x \in \mathbb{R}$ (recall the definition of $\Delta^k(h)$ from (8.1)). Apply Lemma 6.2 with $r = p$. This gives

$$|\Delta^k(h)g(x)| \le ch^{\alpha - 1/p}\|\phi\|_p$$

for all $x \in \mathbb{R}$. Thus $\omega^k(g;h) \le ch^{\alpha-1/p}\|\phi\|_p$, which implies (8.73). We have now derived Theorem 8.20 from Lemma 6.2.

The next theorem states that every function in the Sobolev-Liouville space $L_p^\alpha(\mathbb{R}^n)$ ($n \ge 2$, $1 \le p < \infty$, $1/p < \alpha$) is equivalent to some function in the mixed norm space $\cap_{j=1}^n U_j^{p,\lambda}$ ($\lambda = \alpha - 1/p$). This theorem is used in the proofs of Theorem 8.22 and Theorem 8.23 below. We recall that $D_j^\alpha f$ denotes the usual (weak) derivative for $\alpha \in \mathbb{N}$ and the Bessel derivative for $\alpha \notin \mathbb{N}$, $\alpha > 0$.

Theorem 8.21. *Let $1 \le p < \infty$ and $\alpha > 1/p$. Set $\lambda = \alpha - 1/p$ and*

$$U_j^{p,\lambda} = L_{\hat{x}_j}^p(\mathbb{R}^{n-1})[\mathcal{C}_{x_j}^\lambda(\mathbb{R})], \quad j = 1, \dots, n.$$

Suppose that $f \in L_p^\alpha(\mathbb{R}^n)$. Then there exists a function $f_0 \in \cap_{j=1}^n U_j^{p,\lambda}$ which is equivalent to f (and then $f_0 \in L_p^\alpha(\mathbb{R}^n)$ and $D_j^\alpha f_0 = D_j^\alpha f$ a.e., $j = 1, \dots, n$). Moreover,

$$\|f_0\|_{U_j^{p,\lambda}} \le c\|D_j^\alpha f_0\|_p, \quad j = 1, \dots, n, \tag{8.75}$$

where c depends only on n, p, and α.

Proof. The construction of the function f_0 will depend on whether or not α is an integer. First we consider the case when $\alpha \in \mathbb{N}$. Then f has usual (weak) derivatives $D_j^\alpha f \in L^p(\mathbb{R}^n)$, $j = 1, \dots, n$. So there exists a function f_0 which is equivalent to f, such that $f_0(x_j, \hat{x}_j)$ is continuous on $\mathbb{R}$ with respect to x_j for a.e. $\hat{x}_j$ (see e.g. Theorem 2.1.4 on page 44 in [49]). By (6.9) and Hölder's inequality, we have

$$\begin{aligned}|\Delta_j^\alpha(h) f_0(x)| \le h^{\alpha-1}\int_0^{\alpha h} |D_j^\alpha f_0(x_j + w, \hat{x}_j)| dw \le \\ \le ch^{\alpha-1/p}\|D_j^\alpha f_0(\cdot, \hat{x}_j)\|_p\end{aligned} \tag{8.76}$$

for a.e. $x \in \mathbb{R}^n$, where the second estimate is immediate if $p = 1$. Since $f_0(x_j, \hat{x}_j)$ is continuous on $\mathbb{R}$ in x_j for a.e. $\hat{x}_j$, and (8.76) holds for a.e. x, it follows that (8.76) holds along the whole line $\{(\hat{x}_j, y) : y \in \mathbb{R}\}$ for a.e. $\hat{x}_j$. For such $\hat{x}_j$, (8.76) implies that

$$\omega^\alpha(f_0(\cdot, \hat{x}_j); \delta) \le c\delta^\lambda \|D_j^\alpha f_0(\cdot, \hat{x}_j)\|_p, \quad \delta \ge 0.$$

Since $1 \le p < \infty$ and $\lambda = \alpha - 1/p$, we have $\alpha - 1 \le \lambda < \alpha$. That is, α is the smallest integer such that $\lambda < \alpha$. Thus, from the preceding inequality we get

$$\|f_0(\cdot, \hat{x}_j)\|_{\mathcal{C}^\lambda}^* \le c\|D_j^\alpha f_0(\cdot, \hat{x}_j)\|_p$$

for a.e. $\hat{x}_j$, $j = 1, \dots, n$. Taking $L^p(\mathbb{R}^{n-1})$-norm and using Fubini's theorem, we obtain (8.75) for all $\alpha \in \mathbb{N}$.

We now turn to the case when $\alpha \notin \mathbb{N}$. Then f has Bessel derivatives $D_j^\alpha f \in L^p(\mathbb{R}^n)$ for $j = 1, \dots, n$. By Lemma 8.19, there exists a function f_0 which is equivalent to f, and for which a.e. section is continuous. Fix $j \in \{1, \dots, n\}$. By Fubini's theorem, $D_j^\alpha f_0(\cdot, \hat{x}_j)$ belongs to $L^p(\mathbb{R})$ for a.e. $\hat{x}_j$. For such $\hat{x}_j$, Theorem 8.20 shows that the integral

$$g_j(x_j, \hat{x}_j) = \int_{\mathbb{R}} G_\alpha(x_j - u) D_j^\alpha f_0(u, \hat{x}_j) du$$

converges for all $x_j \in \mathbb{R}$, and that

$$\|g_j(\cdot, \hat{x}_j)\|^*_{\mathcal{C}^\lambda} \le c \|D_j^\alpha f_0(\cdot, \hat{x}_j)\|_{L^p(\mathbb{R})}. \tag{8.77}$$

Since f_0 and g_j are equivalent on $\mathbb{R}^n$ (by definition of the Bessel derivative), and since a.e. $\hat{x}_j$-section of f_0 and g_j is continuous on $\mathbb{R}$, it follows that $f_0(\cdot, \hat{x}_j)$ coincides with $g_j(\cdot, \hat{x}_j)$ on $\mathbb{R}$ for a.e. $\hat{x}_j$. Thus, (8.77) holds for a.e. $\hat{x}_j$ with g_j replaced by f_0, and from this inequality we obtain (8.75) by applying the $L^p(\mathbb{R}^{n-1})$-norm and using Fubini's theorem. □

The embedding stated in the next theorem is known (see Theorem 9.3 on page 159 in [27]). For $0 < \alpha \le 1/p$, we here check that it is contained in Theorem 6.5. Further, for $1/p < \alpha < n/p$ we shall prove it by combining Theorem 8.5 and Theorem 8.21, thereby showing that these results (for the mixed norm space $\cap_{j=1}^n U_j^{p,\lambda}$) are in agreement with known properties of Sobolev-Liouville spaces. We again recall that $D_j^\alpha f$ denotes the usual (weak) derivative for $\alpha \in \mathbb{N}$ and the Bessel derivative for $\alpha \notin \mathbb{N}$, $\alpha > 0$.

Theorem 8.22. *Let $n \ge 2$, $1 \le p < \infty$, and $0 < \alpha < n/p$. Denote $q = np/(n - \alpha p)$. Suppose that $f \in L_p^\alpha(\mathbb{R}^n)$. Then $f \in L^{q,p}(\mathbb{R}^n)$, and*

$$\|f\|_{q,p} \le c \sum_{j=1}^n \|D_j^\alpha f\|_p, \tag{8.78}$$

where c depends only on n, p, and α.

Proof. Assume first that $\alpha \le 1/p$. Then the conditions of Theorem 6.5 are satisfied with $\alpha_k = \alpha$ and $p_k = p$, $k = 1, \dots, n$. Indeed, we have $f \in L_p^\alpha(\mathbb{R}^n) = L_{p_1,\dots,p_n}^{\alpha_1,\dots,\alpha_n}(\mathbb{R}^n)$, $p < n/\alpha$, and for all $k = 1, \dots, n$ it holds that $\alpha_k \le 1$ and $1/p - \alpha/n - 1/p_k + \alpha_k = \alpha(1 - 1/n) > 0$. So by Theorem 6.5, $f \in L^{q,p}(\mathbb{R}^n)$ and inequality (8.78) holds.

Suppose now that $1/p < \alpha$. Put $\lambda = \alpha - 1/p$. Then $0 < \lambda < (n-1)/p$. By Theorem 8.21, there exists a function $f_0 \in \cap_{j=1}^n U_j^{\lambda,p}$, which is equivalent to f (and thus $f_0 \in L_p^\alpha(\mathbb{R}^n)$). Moreover,

$$\|f_0\|_{U_j^{p,\lambda}} \le c \|D_j^\alpha f_0\|_p = c \|D_j^\alpha f\|_p, \quad j = 1, \dots, n.$$

By Proposition 6.1, we have that $f_0 \in L^p(\mathbb{R}^n)$. So, according to (2.31) and (2.24), it holds that $\|f_0\|_{p,\infty} \le c\|f_0\|_{p,p} = c\|f_0\|_p < \infty$. Thus, $f_0^*(t) = O(t^{-1/p})$, as $t \to \infty$. Since $p < q$, it follows that

$$f_0^*(t) = O(t^{-1/q}), \quad \text{as } t \to \infty.$$

From Theorem 8.5 we then obtain that $f_0 \in L^{q,p}(\mathbb{R}^n)$ and that

$$\|f_0\|_{q,p} \le c \sum_{j=1}^{n} \|f_0\|_{U_j^{p,\lambda}}$$

(we have here the same q as in Theorem 8.5 since $q = np/(n - \alpha p) = np/(n - 1 - \lambda p)$). Combine the two preceding inequalities, and use that $\|f_0\|_{q,p} = \|f\|_{q,p}$. This gives (8.78). $\square$

We prove the following known fact about $L_p^\alpha(\mathbb{R}^n)$, by using Theorem 8.13 and Theorem 8.21. As for the preceding theorem, this illustrates how the properties of the mixed norm spaces $\cap_{j=1}^n U_j^{p,\lambda}$ are consistent with known results for $L_p^\alpha(\mathbb{R}^n)$. We again remind that $D_j^\alpha f$ denotes the usual (weak) derivative for $\alpha \in \mathbb{N}$ and the Bessel derivative for $\alpha \notin \mathbb{N}$, $\alpha > 0$.

Theorem 8.23. *Let $n \ge 2$, $1 \le p < \infty$, and $n/p < \alpha < \infty$. Suppose $f \in L_p^\alpha(\mathbb{R}^n)$. Then there exists a bounded and uniformly continuous function $g \in L_p^\alpha(\mathbb{R}^n)$ which is equivalent to f. Moreover, if $\beta \equiv \alpha - n/p$ and $s > \beta$, $s \in \mathbb{N}$, then*

$$\omega^s(g;t) \le ct^\beta \sum_{j=1}^{n} \|D_j^\alpha f\|_p \tag{8.79}$$

for $t \ge 0$, where c depends only on n, p, and α.

Proof. Set $\lambda = \alpha - 1/p$. Then $(n-1)/p < \lambda < \infty$. By Theorem 8.21, f is equivalent to a function $f_0 \in \cap_{j=1}^n U_j^{p,\lambda}$. Moreover,

$$\|f_0\|_{U_j^{p,\lambda}} \le c\|D_j^\alpha f_0\|_p = c\|D_j^\alpha f\|_p \tag{8.80}$$

for $j = 1, \ldots, n$. Then, by Theorem 8.13, f_0 is equivalent to a bounded and uniformly continuous function $g \in \cap_{j=1}^n U_j^{p,\lambda}$. Thus, g and f are equivalent. Therefore, $g \in L_p^\alpha(\mathbb{R}^n)$ and $D_j^\alpha g = D_j^\alpha f$ almost everywhere for $j = 1, \ldots, n$. Furthermore, Theorem 8.13 ensures that for all $s > \beta$, $s \in \mathbb{N}$, it holds that

$$\omega^s(g;t) \le ct^\beta \sum_{j=1}^{n} \|f_0\|_{U_j^{p,\lambda}}, \quad t \ge 0, \quad k = 1, \ldots, n.$$

Apply (8.80). This proves (8.79). $\square$

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