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Baire category theorem

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Abstract

In this thesis we give an exposition of the notion of *category* and the *Baire category theorem* as a set theoretical method for proving existence. The category method was introduced by René Baire to describe the functions that can be represented by a limit of a sequence of continuous real functions. Baire used the term *functions of the first class* to denote these functions.

The usage of the Baire category theorem and the category method will be illustrated by some of its numerous applications in real and functional analysis. Since the usefulness, and generality, of the category method becomes fully apparent in Banach spaces, the applications provided have been restricted to these spaces.

To some extent, basic concepts of metric topology will be revised, as the Baire category theorem is formulated and proved by these concepts. In addition to the Baire category theorem, we will give proof of equivalence between different versions of the theorem.

Explicit examples, of first class functions will be presented, and we shall state a theorem, due to Baire, providing a necessary condition on the set of points of continuity for any function of the first class.

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Chapter 1

Introduction

The notion of countability as a method of comparing sets with the set of natural numbers, is often introduced at an early stage in undergraduate studies of real analysis. We know that the set of integers, the set of odd integers and the set of rationals, are all examples of countable sets. Sets that are not countable, fall under the definition of uncountable sets, as for example the set of all irrationals. A set is either countable or uncountable, depending on whether there exists a one-to-one relation between the set and the natural numbers.

The notion of *category*, as presented in the doctoral thesis of René Baire in 1899, is based on that of countability. The subsets of metric spaces are divided into two categories: *first category* and *second category*. The former subsets can be seen as small and the latter subsets can be seen as large, since in most cases, any first category set is a subset of some second category set, but the reversed inclusion never holds.

A metric space is by definition a set with a distance function. Since there are no other requirements on the set, the notion of category can be applied to many different metric spaces, as for example Euclidean spaces, function spaces and sequence spaces.

Thus, theorems developed from the notion of categories are in this sense general, and have been proved a useful tool in real analysis and functional analysis. At the heart of these theorems, we often find the *Baire category theorem* as a method for proving existence.

The intention of this thesis is to give an exposition of topics in analysis that relate to the notion of category. This includes both proof of different theorems originating from the definition of categories as well as applications of the category method in analysis.

We begin chapter 2 by stating some well known set theoretical concepts and theorems, and then proceed by presenting the Baire category theorem. The Cantor set will be used as an example in a comparison of the notion of category and that of countability. Although first category sets are in some sense defined in terms of countability, a set of the first category is not necessarily countable. We shall introduce the Borel sets, F_σ-sets and G_δ-sets, since these classes of sets relate to the category method. At the end of chapter 2, the Uniform boundedness theorem is presented as an application of the Baire category theorem.

We devote chapter 3 to the topics of real valued functions and sequences of real valued functions, and in this context we introduce *functions of the first class*, another term defined by René Baire. To illustrate these concepts, some explicit examples of functions of the first class are provided.

Finally, in chapter 4, we present Baire category theorem on general metric spaces, together with several examples of using the category method on well known metric spaces. We shall also include some applications of the Baire category theorem in the field of functional analysis.

The main results of this thesis are the following:

In section 2.2 we prove the equivalence between five different versions of the Baire category theorem on the real line.

In section 3.2 we give explicit examples of some well known functions of the first class.

In section 4.2 we show that the set of piecewise linear functions is of the first category, and a dense subset of the set of all continuous functions on the unit interval.

In section 4.4 we give a set theoretical proof of the statement that the unit interval can not be expressed as a countable sequence of non-empty disjoint closed sets.

In section 4.5 prove certain property of functions with derivatives of all orders on the unit interval.

In section 5.4 we examine subspaces of well known Banach spaces with respect to category.

Chapter 2

Baire category theorem on the real line

The main theme of this chapter is to present the fundamental definitions of the category method, to give a proof of the Baire category theorem, and to prove equivalence between five versions of the theorem. However, we shall first revise some basic concepts from topology and prove some statements, needed throughout this thesis. Section 2.3 contains a comparison between the notion of category and that of countability, and section 2.5 provides a proof of the Uniform boundedness theorem to illustrate an application of the Baire category theorem.

2.1 Basic topology

Definition 2.1 (Neighborhood)**.** For any point $x_0 \in \mathbb{R}$ and any real number $\varepsilon > 0$, the set of point

$$\{x : |x_0 - x| < \varepsilon\}$$

is called a *neighborhood* of x_0. This is often written as $N_\varepsilon(x_0)$ or $B(x_0, r)$. On the real line, a neighborhood corresponds to an open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, centered about x_0.

Definition 2.2. All sets and all points mentioned below are understood to be subsets and points of $\mathbb{R}$.

(i) A point $x \in E$ is said to be an *interior point* of E if there exists a neighborhood $N_\varepsilon(x)$, such that $N_\varepsilon(x) \subset E$.

(ii) A point $x \in E$ is said to be an *isolated point* of E if there exists a neighborhood $N_\varepsilon(x)$, such that $N_\varepsilon(x) \cap E = \{x\}$.

(iii) A point x is said to be an *accumulation point* x of E, if every neighborhood of x contains at least one point, not equal to x, in E. That is, $(N_\varepsilon(x) \setminus \{x\}) \cap E \neq \emptyset$.

(iv) E is said to be *closed* if all accumulation points of E are points in E.

(v) A closure of a set E, denoted by $\overline{E}$, is the union of E and all accumulation points of E.

(vi) E is said to be *open* if every point in E is an interior point of E.

(vii) The *interior* of a set E, denoted by E°, is the set containing all interior points of E.

($viii$) The *complement* of a set E, denoted by E^c, is the set of all points not contained in E.

(ix) A point x is said to be a *boundary point* x of E, if every neighborhood of x contains points both in E and in E^c. That is, $N_\varepsilon(x) \cap E \neq \emptyset$ and $N_\varepsilon(x) \cap E^c \neq \emptyset$.

(x) The *boundary* of a set E, denoted by ∂E, is the set containing all boundary points of E.

(xi) E is said to be *perfect*, if E is closed and every point in E is an accumulation point of E.

Definition 2.3 (Dense)**.** Given two sets $A, B \subset \mathbb{R}$. We say that A is *dense* in B, if $A \subset B$ and every open interval I that intersects B also intersects A.

Definition 2.4 (Nowhere dense)**.** Given a set $A \subset \mathbb{R}$. We say that A is *nowhere dense*, provided that every open interval I contains an open subinterval $J \subset I$, such that $J \cap A = \emptyset$.

Definition 2.5. The following three definitions are fundamental concepts of the Baire category theorem.

(i) S is said to be of the *first category* if it can be represented as a countable union of nowhere dense sets.

(ii) S is said to be of the *second category* if it is not of the first category.

(iii) S is said to be *residual* if its complement, S^c, is of the first category.

Theorem 2.6. Let $A \subset \mathbb{R}$, then A is open if and only if A^c is closed.

Proof. Suppose that A is open. Prove that A^c is closed. Let p be an accumulation point of A^c. Then every neighborhood of p intersects A^c, and therefore p can not be an interior point of A. Since A is open, p is in A^c. Hence A^c is closed.

Suppose that A^c is closed. Prove that A is open. Choose $p \in A$, since p is not an accumulation point of A^c, there exists a neighborhood, N, about p such that $N \cap A^c = \emptyset$. But this means that $N \subset A$, so p must be an interior point of A. Hence A is open. □

Theorem 2.7. Let $A \subset B \subset \mathbb{R}$, then A is dense in B if and only if $\overline{A} \supset B$.

Proof. Assume that A is dense in B. Prove that $\overline{A} \supset B$. Let $x \in B$ and assume $x \notin A$. By assumption, A is dense in B, so every neighborhood of x contains points from both A and B. Thus by definition, x is an accumulation point of A. So $\overline{A} \supset B$.

Conversely, assume that $\overline{A} \supset B$. Prove that A is dense in B. Given an arbitrary open interval I such that $B \cap I \neq \emptyset$, we must show that $A \cap I \neq \emptyset$. By assumption, $I \cap \overline{A} \neq \emptyset$. Let x be any point in $I \cap \overline{A}$. Suppose also that $x \notin A$. Then x is an interior point of I and an accumulation point of A. Let N be a neighborhood of x such that $N \subset I$. Since $N \cap A \neq \emptyset$ we have that $I \cap A \neq \emptyset$. So A is dense in B □

Theorem 2.8. Let $A \subset \mathbb{R}$ and $B = \mathbb{R} \setminus A$. Then A is a closed nowhere dense set in $\mathbb{R}$ if and only if B is an open dense set in $\mathbb{R}$.

Proof. Assume that A is closed and nowhere dense in $\mathbb{R}$. Prove that B is open and dense in $\mathbb{R}$. B is open since A is closed (by theorem 2.6). By assumption, we know that for every open interval I there exists an open subinterval $J \subset I$ such that $A \cap J = \emptyset$. This means that $J \subset B$, hence $J \subset I \cap B$, thus I intersects B. So B is open and dense in $\mathbb{R}$.

Now, assume that B is open dense in $\mathbb{R}$. Prove that A is closed and nowhere dense in $\mathbb{R}$. A is closed, since B is open. By assumption every open interval I intersects B. Now let $J = I \cap B$. J is open since both I and B are open. Moreover, $J \subset B$ and therefore can not intersect A. Thus for every open interval I there exists an open subset J, such that $A \cap J = \emptyset$. So A is closed and nowhere dense in $\mathbb{R}$. ☐

Theorem 2.9. Let F be a closed set. Then the set of boundary points has an empty interior.

Proof. Assume F is closed. Prove that the ∂F contains no open interval. Since F is closed we have

$$\partial F \subset F.$$

Thus, if ∂F contains an open interval I, then $I \subset F$, thus I must be contained in the interior of F, and therefore I can not be in ∂F. ☐

Theorem 2.10. Let I be an open interval and A be a set, in $\mathbb{R}$. Then I intersects both A and A^c if and only if I contains a boundary point of A.

Proof. Assume I has a non-empty intersection both with A and A^c. Prove that I contains a boundary point of A. Let $a, b \in I$, such that $a \in A$ and $b \in A^c$. We may assume that $a < b$ (see remark 2.11). Let

$$z = \sup\{x : a \leq x \leq b \text{ and } x \in A\}.$$

Since $\mathbb{R}$ has the least upper bound property, and supremum is taken from a set bounded above by b, we have that z exists. From $a \leq z \leq b$ it follows that $z \in I$. We shall continue by proving that z is a boundary point of A.

If $z \in A$, then $z \neq b$. For any $\delta > 0$, such that $z < z + \delta < b$, we have that the interval $J = [z, z + \delta]$ is a subset of I. Also, J contains points from A^c, since otherwise $z + \delta$ is a larger candidate for the supremum. So every J (and hence I) contains points from A and A^c. If $z \in A^c$, then z must be an accumulation point of A. Thus every open interval about z contains points from A and A^c. It follows that z is a boundary point of A.

Conversely, assume that I contains a boundary point of A, show that I intersects A and A^c. Let z be a boundary point of A in I. Since I is open, z is an interior point. Thus there exists an open interval J about z in I. Now, J contains points from A and A^c, since z is a boundary point of A. Thus I intersects both A and A^c. ☐

Remark 2.11. The assumption $a < b$ in the proof of theorem 2.10 impose no restriction on either A or A^c. Since if no such a and b can be found, we could substitute $B = A^c$ and continue the proof using B instead of A. From here on we can find $a \in B$ and $b \in B^c$, such that $a < b$. Since any boundary point from B, by definition is a boundary point of A, the theorem holds.

Theorem 2.12. A is a nowhere dense subset of $\mathbb{R}$ if and only if $\overline{A}$ contains no interval. That is, $(\overline{A})^\circ = \emptyset$

Proof. Suppose that A is nowhere dense. Prove that $\overline{A}$ contains no interval. By using an indirect proof, we assume that $\overline{A}$ contains an interval I. But this is an immediate contradiction to A being nowhere dense. Thus $\overline{A}$ contains no interval.

Suppose that $\overline{A}$ contains no interval. Prove that A is nowhere dense. Assume, that A fails to be nowhere dense in $\mathbb{R}$, then there exits an open interval I such that the set $A \cap I$ is dense in I.

By theorem 2.7 we know that

$$\overline{A} \supset \overline{(A \cap I)} \supset I.$$

So, $\overline{A}$ contains the interval I. We have reached a contradiction. Hence A is nowhere dense. □

Theorem 2.13 (Cantor's lemma)**.** If $\{F_n\}$ is a nested sequence of non-empty closed sets in $\mathbb{R}$, such that $F_n \supset F_{n+1}$ for all n, then

$$\bigcap_{n=0}^{\infty} F_n \neq \emptyset.$$

Proof. To show that the intersection is non-empty, we need to find a point $x \in \mathbb{R}$ such that $x \in F_n$ for all n.

Let $a_n = \inf(F_n)$ and $b_n = \sup(F_n)$. Since all F_n are closed we know that $a_n, b_n \in F_n$. Define

$$I_n = [a_n, b_n]$$

By the construction of I_n we know that $I_j \subset I_i$ if $j > i$ and therefore

$$I_j \cap I_i \neq \emptyset. \tag{1}$$

Also we have that $F_n \subset I_n$. The sequence $\{a_n\}$ is infinite and non-decreasing with an upper bound (b_1). Then

$$x = \sup(a_n)$$

exists. We know that $x \geq a_n$ for all $n \in \mathbb{N}$. We must show that $x \leq b_n$ for all $n \in \mathbb{N}$. Assume that there exists b_i such that $b_i < x$. Since x is a supremum of $\{a_n\}$ there exists some a_j such that $b_i < a_j \leq x$. But this would mean that

$$a_i \leq b_i < a_j \leq b_j.$$

We have found two intervals I_i and I_j that shares no points. This contradicts assertion (1), hence there exists $x \in \mathbb{R}$ such that $a_n \leq x \leq b_n$ for all n, which implies that $x \in I_n$ for all n.

Finally, we show that x is in every F_n. If $x = a_i$ for some i, then it follows that x is in every F_n. If $x \notin \{a_n\}$ then x must be an accumulation point of $\{a_n\}$, and since all F_n are closed, x is a point in every F_n. □

Corollary 2.14. If $\{I_n\}$ is a nested sequence of non-empty closed intervals in $\mathbb{R}$, such that $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$ and $|I_n| \to 0$, then

$$\bigcap_{n=0}^{\infty} I_n = \{x\}$$

for some point $x \in \mathbb{R}$.

Proof. By Cantor's lemma (theorem 2.13) we know that the intersection is non-empty. Suppose that it contains at least two points $x_1 < x_2$. This means that the closed interval $[x_1, x_2]$ is a subset of every open interval I_n. But since $|I_n| \to 0$ we can find $N > 0$ such that $|[x_1, x_2]| > |I_n|$ for $n > N$ which is impossible. So, x_1 and x_2 can not both be points in the countable intersection of I_n. □

Theorem 2.15. A prefect set is uncountable.

Proof. Suppose that S is a perfect set. Prove that S is uncountable.

Assume that S is countable and defined as

$$S = \{s_1, s_2, \dots\}.$$

To arrive at a contradiction, we construct a nested sequence of non-empty closed sets in S, such that the intersection contains no element from S.

By assumption S is perfect, so every point in S is an accumulation point of S. Let s_1 be a point in S and I_1 be any neighborhood of s_1. Since s_1 is an accumulation point we now that I_1 contains infinitely many other points from S. Now, assume that I_n is an open interval such that $I_n \cap S$ is infinite. Proceed inductively, by defining an interval I_{n+1}, such that:

(*i*) I_{n+1} is an open neighborhood of some point in $I_n \cap S$

(*ii*) $\overline{I_{n+1}} \subset I_n$

(*iii*) $|I_{n+1}| < \frac{1}{2}|I_n|$

(*iv*) $s_n \notin \overline{I_{n+1}}$

Since I_{n+1} is a neighborhood of some point in S, it satisfies our induction hypothesis that $I_{n+1} \cap S$ should be infinite. So we can proceed with our construction of a countable sequence of I_n. Now define

$$A_n = \overline{I_n} \cap S,$$

where both $\overline{I_n}$ and S are closed, thus A_n is closed. Also $s_m \notin A_n$ if $m < n$. Let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Then $A \subset S$ since each A_n is a subset of S. We know that that A is non-empty. So a point in A must be a point in S but that contradicts our construction of A, since no point in S is in every A_n. The assumption that S is countable, must be false. ☐

2.2 Baire category theorem

The Baire category theorem can be expressed in different forms. We present five versions of this theorem and give proof of their equivalence.

(1) Every interval $[a, b]$ is a set of the second category.

(2) $\mathbb{R}$ is of the second category.

(3) Every residual subset of $\mathbb{R}$ is dense.

(4) Any countable union of closed sets with empty interior has an empty interior.

(5) Any countable intersection of open dense sets is dense.

First, we choose to prove the third version of the theorem. Secondly, we present how the other versions can be derived.

Theorem 2.16 (Baire category theorem)**.** Every residual subset of $\mathbb{R}$ is dense.

Proof. Let A be residual in $\mathbb{R}$. We must show that an arbitrary interval $I \subset \mathbb{R}$ intersects A. Since A is residual, its complement $B = \mathbb{R} \setminus A$ is of the first category, and can be expressed as

$$B = \bigcup_{i=1}^{\infty} B_i,$$

where B_i are nowhere dense sets in $\mathbb{R}$. We now inductively define a sequence of closed non-degenerate intervals $\{I_n\}$, as follows

(*i*) $I_1 \subset I$

(*ii*) $B_1 \cap I_1 = \emptyset$.

We know that such I_1 exists since B_1 is nowhere dense. As induction hypothesis, we assume I_n has an empty intersection with B_n. We now define I_{n+1}, such that

(*i*) $I_{n+1} \subset I_n$

(*ii*) $B_{n+1} \cap I_{n+1} = \emptyset$.

Such interval I_{n+1} exists, since B_{n+1} is a nowhere dense set. By this, I_{n+1} satisfies our induction hypothesis. The construction can proceed. The countable intersection of these intervals is non-empty. Let

$$I' = \bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

By the construction of I' we know $I' \cap B = \emptyset$. It follows that $I' \subset A$, and since $I' \subset I$, we have that the interval I intersects A. So, A is dense in $\mathbb{R}$. □

Remark 2.17. In the doctoral thesis of René Baire [1] we find the theorem presented as "the continuum constitutes a set of the second category". The outline of the proof of Baire is very similar to the proof given in this thesis. First Baire assumes that P is a set of the first category in $\mathbb{R}$, hence P is the countable union of nowhere dense sets P_n. Then he constructs a nested sequence of closed intervals, such that I_n does not intersect P_n. By assigning the point M to be the intersection of all I_n, M can not be in P. Thus, Baire conclude that $\mathbb{R}$ is not of the first category.

Now, we shall prove equivalence between the five versions of the theorem.

Proof. **(3) ⇒ (1)** Assume that every residual set in $\mathbb{R}$ is dense. Prove that every interval is of the second category. If an interval $I = [a,b]$ were of the first category, then $\mathbb{R} \setminus I$ would be residual in $\mathbb{R}$. Hence by (3) dense in $\mathbb{R}$. But the set $\mathbb{R} \setminus I$ can not be dense, since $(\mathbb{R} \setminus I) \cap I = \emptyset$. Hence we have a contradiction, thus an interval can not be of the first category. □

Proof. **(1) ⇒ (2)** Assume that every interval is of the second category. Prove that $\mathbb{R}$ is of the second category. Suppose that (2) were false. Then we can express $\mathbb{R}$ as a countable union

$$\mathbb{R} = \bigcup_{i=0}^{\infty} R_i,$$

where R_i are nowhere dense sets. Then any arbitrary interval $I = [a,b]$ can be expressed as

$$I = \bigcup_{i=0}^{\infty} R_i \cap I.$$

But the set $R_i \cap I$ is again nowhere dense for every i. Thus I is of the first category, which contradicts our assumption that (1) holds. So, $\mathbb{R}$ is of the second category. □

Proof. **(2) ⇒ (3)** Assume that $\mathbb{R}$ is of the second category. Prove that every residual set in $\mathbb{R}$ is dense. Suppose that (3) were false, then there exists a residual subset A, in $\mathbb{R}$, not dense in $\mathbb{R}$. This means that there exist an interval I such that $A \cap I = \emptyset$. Since A is residual the complement set, $\mathbb{R} \setminus A$, is of the first category. This would mean that the interval I also is of the first category, since $I \subset \mathbb{R} \setminus A$. We will now proceed by proving that if an interval is of the first category then the whole real line must be of the first category.

First, we show that if I is of the first category then any closed interval is of the first category. Let $\Phi_{a,b}$ be a linear bijective mapping between any interval $[a,b]$ of non-zero length to the unit interval $[0,1]$.

$$\Phi_{a,b}(x) = \frac{x-a}{b-a} \qquad \Phi_{a,b} : [a,b] \mapsto [0,1]$$
$$\Phi_{a,b}^{-1}(x) = x(b-a)+a \qquad \Phi_{a,b}^{-1} : [0,1] \mapsto [a,b]$$

Note that the function Φ is a dilation from $\mathbb{R}$ into $\mathbb{R}$, and in it simplest form, when the length of the interval is not changed, a pure translation.

Since I is of the first category, we can find chained mappings that map I to any other bounded interval. For example $\Phi_{c,d}^{-1}(\Phi_{a,b}(x))$ maps $[a,b] \mapsto [c,d]$. Hence any bounded interval is of the first category (see remark 2.18).

Finally, since $\mathbb{R}$ can be expressed as

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n,n],$$

and every interval is of the first category, then $\mathbb{R}$ must be of the first category. But that contradicts our assumption that (2) holds. □

Remark 2.18. In the previous proof we used the fact that the mappings Φ and Φ^{-1} preserve category, and since I was assumed to be an interval of the first category, the image of I under any combination of Φ and Φ^{-1} is again of the first category. The proof of this statement is deferred until theorem 4.11.

So far we have proved the Baire category theorem expressed in form (3) and also showed that

$$(3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3),$$

which means that we have equivalence between the first three versions of the Baire category theorem. Next, we continue by showing that version (4) and (5) are equivalent and (3) ⇒ (5) and (4) ⇒ (3).

Proof. **(5) ⇔ (4)** To show that (5) and (4) are equivalent, we shall first define two sets A and B, such that

$$A = \bigcap_{n=0}^{\infty} A_n,$$

and the complement of A, as

$$B = \mathbb{R} \setminus A = \mathbb{R} \setminus \bigcap_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \mathbb{R} \setminus A_n.$$

Assume that (5) is valid. Let A_n be open and dense sets in $\mathbb{R}$. Then by (5) we know that A is dense. Since A_n are open dense sets we know (by theorem 2.8) that the sets B_n are closed and nowhere dense sets. It follows that B satisfies the assumptions in (4). Suppose now that B contains some interval I. Then $A \cap I = \emptyset$, in contradiction to (5). Thus, B has an empty interior.

Conversely, assume that (4) is valid. Let B_n be closed sets with empty interiors (i.e. nowhere dense). By (4) we know that B has an empty interior. We also know (by theorem 2.8) that A_n are open and dense, and thus satisfy the assumptions in (5). Suppose now that A is not dense. Then there exists an interval I such that $A \cap I = \emptyset$ and thus $I \subset B$, in contradiction to B being nowhere dense. Thus A must be dense. □

Proof. **(3) ⇒ (5)** As in version (5) we assume that A_n are open dense sets, and we define

$$A = \bigcap_{n=0}^{\infty} A_n.$$

We shall prove that A is dense. Since the complement of A_n is nowhere dense, and

$$A^c = \bigcup_{n=0}^{\infty} A_n^c,$$

it follows that A^c is of the first category. Thus A is residual. By (3) it follows that the residual set A, is dense. □

Proof. **(4) ⇒ (3)** As in version (3) we assume that A is a residual set in $\mathbb{R}$. We need to prove that A is dense. Since A is residual, we know that $B = A^c$ is of the first category, thus there exists a sequence of nowhere dense sets B_n, such that

$$B = A^c = \bigcup_{n=0}^{\infty} B_n.$$

By theorem 2.12 we know that $\overline{B}_n$ has an empty interior. Thus by (4)

$$B' = \bigcup_{n=0}^{\infty} \overline{B}_n,$$

we have that B' has an empty interior. Since B is a subset of B', B has an empty interior. It follows that arbitrary interval intersects A, thus A is dense. □

2.3 Cantor set

The sets of the first category are in some sense defined in terms of countability, since they can be expressed as the countable union of nowhere dense sets. A natural question to ask, is whether there is an immediate relation between the property of first category and the property of countability. In this section we will find that there exist both countable and uncountable sets of the first category on the real line.

Theorem 2.19. Let E be a set of the first category. Then for any interval $I = (a,b)$, the set $I \setminus E$ is uncountable.

Proof. Since E is of the first category E can be expressed as a countable union of nowhere dense sets E_i. Suppose that the set $I \setminus E$ is countable, then this set can be expressed as a union of singleton sets e_i.

$$I \setminus E = \bigcup_{i=0}^{\infty} e_i$$

The interval I can be defined as

$$I = (I \setminus E) \cup (I \cap E) = \Big(\bigcup_{i=0}^{\infty} e_i\Big) \cup \Big(I \cap \bigcup_{i=0}^{\infty} E_i\Big) =$$
$$= \Big(\bigcup_{i=0}^{\infty} e_i\Big) \cup \Big(\bigcup_{i=0}^{\infty} I \cap E_i\Big) = \bigcup_{i=0}^{\infty} e_i \cup (I \cap E_i)$$

Since $E_i \cap I$ are nowhere dense sets and since e_i are nowhere dense sets. Then I is of the first category. But this contradicts the Baire category theorem. So, $I \setminus E$ is uncountable. ☐

Theorem 2.20. Let A be a countable set in $\mathbb{R}$. Then A is of the first category in $\mathbb{R}$.

Proof. This theorem follows immediately from the fact that a set containing a single point in $\mathbb{R}$, is nowhere dense, and any countable set in $\mathbb{R}$ can be expressed as a countable union of single points. Thus, a countable set is of the first category. ☐

We can easily construct countable sets of the first category, for example $\mathbb{N}$ and $\mathbb{Q}$ are examples of such sets. It is however harder to picture an uncountable set being of the first category.

The Cantor set, which we in this section will construct, have these properties: uncountable and of the first category. The construction is done stepwise, starting with the unit interval

$$K_0 = [0,1].$$

Divide this interval in three equal parts and remove the middle part from K_0 and let K_1 be

$$K_1 = \left[\frac{0}{3}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{3}\right].$$

Continue by removing the middle third from all closed intervals in K_1 to obtain K_2

$$K_2 = \left[\frac{0}{9}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{9}{9}\right].$$

In this manner we construct a sequence of sets K_n, such that

(i) $K_1 \supset K_2 \supset K_3 \supset \dots$

(ii) K_n is the union of 2^n closed, pairwise disjoint, intervals, each of length 3^{-n}.

We will now call the countable intersection the *Cantor set*,

$$K = \bigcap_{n=0}^{\infty} K_n.$$

Theorem 2.21. The Cantor set K is a prefect set.

Proof. Since $K \subset [0,1]$, it is bounded. K is also closed since it is a countable intersection of closed sets.

It now remains to show that every point p in K is an accumulation point of K. Let $N_\varepsilon(p)$ be a neighborhood, and let n be a number such that $1/3^n < \varepsilon$. We must show that this neighborhood contains some other point from K. Since p is a point in K_n it must be a point in one of the closed disjoint interval components of K_n, say L. With respect to K_{n+1} the interval L will be divided into two closed intervals, L_0 and L_1. The point p must be in one of these intervals, say L_0, pick any point $q \in K \cap L_1$. We have now found another point $q \in K$ such that $|p-q| \leq 1/3^n < \varepsilon$. This proves that K is perfect. ☐

Theorem 2.22. The Cantor set K is nowhere dense.

Proof. Assume that there exists an interval $I \subset K$. Let λ be the length of the interval, and let n be a number such that $1/3^n < \lambda$. If I were a subset of K then I must also be contained in K_n, but all closed intervals that constitutes K_n all have length $1/3^n$, i.e none of these intervals could contain the interval I. Since I was arbitrary, we have proved that K contains no interval, and thus K is nowhere dense. ☐

By theorem 2.15 it follows that the Cantor set is uncountable, and by definition, it is a set of the first category. We conclude this section by an observation on the structure of K.

Definition 2.23. Let K^{1st} be the set of all endpoint of every interval component in K. We say that these points are of the *first kind* in K. Let $K^{2nd} = K \setminus K^{1st}$. We say that these points are of the *second kind* in K.

In each step of the Cantor set construction we have 2^{n+1} points of the first kind in K_n. Thus there are countably many points of the first kind in K. Since K is a uncountable set, the points of the second kind must be uncountable.

Suppose that I is an open interval, such that $K \cap I \neq \emptyset$. Then there exists some interval component K_{n_i} such that $K_{n_i} \subset I$. This means that

$$K \cap I \neq \emptyset \quad \Leftrightarrow \quad K^{1st} \cap I \neq \emptyset \quad \Leftrightarrow \quad K^{2nd} \cap I \neq \emptyset.$$

Thus, K^{1st} and K^{2nd} are dense in K.

2.4 F_σ- and G_δ-sets

The Borel sets, F_σ and G_δ, form two classes of sets. It is immediate from theorem 2.26 that these classes have a significance for the category method. We shall also see in the proceeding chapter, that the Borel sets are important when describing the set of points of continuity of real functions.

Definition 2.24. S is said to be a F_σ-set if S can be expressed as a countable union of closed sets.

Definition 2.25. S is said to be a G_δ-set if S can be expressed as a countable intersection of open sets.

Theorem 2.26. A dense G_δ-set in $\mathbb{R}$, is a residual set.

Proof. Let A be a dense G_δ-set. Then there exist a sequence of open sets G_n, such that

$$A = \bigcap_{n=0}^{\infty} G_n.$$

Since A is dense it follows that G_n is dense for every $n = 1, 2, \ldots$. By considering the complement of G_n we have that G_n^c is a closed and nowhere dense set. Thus

$$A^c = \bigcup_{n=0}^{\infty} G_n^c,$$

is a set of the first category. It follows that A is a residual set. ☐

Theorem 2.27. This theorem states how the properties G_δ and F_σ are preserved under union and intersection operations.

(i) The intersection of any collection of G_δ-sets is a G_δ-set.

(ii) The union of any collection of F_σ-sets is again a F_σ-set.

Proof. We first prove *(i)*. Let $A^{(i)}$ be a countable collection of G_δ sets. Let

$$A^{(i)} = \bigcap_{k=0}^{\infty} G_k^{(i)},$$

where $G_k^{(i)}$ are open sets. Now form the intersection

$$A = \bigcap_{i=0}^{\infty} A^{(i)} = \bigcap_{i=0}^{\infty} (\bigcap_{k=0}^{\infty} G_k^{(i)}) = \bigcap_{i,k=0}^{\infty} G_k^{(i)}.$$

So, A is a countable intersection of open sets, thus we have that A is a G_δ-set. Secondly, we prove *(ii)*. Let $B^{(i)}$ be a countable collection of F_σ sets. Let

$$B^{(i)} = \bigcup_{k=0}^{\infty} F_k^{(i)},$$

where $F_k^{(i)}$ are closed sets. Now form the union

$$B = \bigcup_{i=0}^{\infty} B^{(i)} = \bigcup_{i=0}^{\infty} \bigcup_{k=0}^{\infty} F_k^{(i)} = \bigcup_{i,k=0}^{\infty} F_k^{(i)}.$$

So, B is a countable union of closed sets, thus we have that B is a F_σ-set. ☐

Theorem 2.28. Let I be a closed interval in $\mathbb{R}$, and let A be some subset of I. Then A is a G_δ-set if and only if $I \setminus A$ is a F_σ-set.

Proof. Assume that A is a G_δ-set. Prove that $I \setminus A$ is a F_σ-set. By assumption we can express A as the intersection of open sets

$$A = \bigcap_{i=0}^{\infty} G_i.$$

Then consider the complement of A relative to I,

$$I \setminus A = I \setminus \bigcap_{i=0}^{\infty} G_i = \bigcup_{i=0}^{\infty} I \setminus G_i = \bigcup_{i=0}^{\infty} I \cap G_i^c.$$

Since G_i are open sets, then $I \cap G_i^c$ are closed sets. Hence $I \setminus A$ is the union of closed sets, thus a F_σ-set.

Conversely, assume that $I \setminus A$ is a F_σ-set. Let

$$I \setminus A = \bigcup_{i=0}^{\infty} F_i,$$

where F_i are closed sets. Then A can be expressed in terms of F_i,

$$A = I \setminus \bigcup_{i=0}^{\infty} F_i = I \cap \Big(\bigcap_{i=0}^{\infty} F_i{}^c \Big). \tag{2}$$

We observe that any closed interval $[a,b]$ is a G_δ-set, since it can be expressed as the countable intersection of open sets

$$[a,b] = \bigcap_{n=0}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}).$$

By definition, the intersection of the open sets, $F_i{}^c$, is a G_δ-set. Thus, by theorem 2.27, it follows that A is a G_δ-set. □

Theorem 2.29. Let $A \subset [0,1]$ be a countable set, dense in $[0,1]$. Then A is not of the type G_δ.

Proof. Assume that A is countable. Prove that A is not a G_δ-set. Suppose that A is a G_δ-set. Then there exists open sets, G_i, such that

$$A = \bigcap_{i=0}^{\infty} G_i.$$

Since A is dense in $[0,1]$, every G_i is dense in $[0,1]$. The complement of a dense open set is nowhere dense, so

$$[0,1] \setminus G_i$$

are nowhere dense sets. The complement of A

$$[0,1] \setminus A = [0,1] \setminus \bigcap_{i=0}^{\infty} G_i = \bigcup_{i=0}^{\infty} [0,1] \setminus G_i$$

is therefore of the first category since it is the union of nowhere dense sets. But that would lead us to the conclusion that A is residual, and thus by 2.19, can not be countable. So A can not be a G_δ-set. □

Theorem 2.30. The set of all rational points in $[0,1]$ is of type F_σ but not G_δ.

Proof. Since $\mathbb{Q}$ is countable, so is $S = [0,1] \cap \mathbb{Q}$. By previous theorem S is not a G_δ-set. However, S can be described as a countable union of singleton sets, and therefore S is a F_σ-set. □

Theorem 2.31. The set of all irrational points in $[0,1]$ is of type G_δ but not F_σ.

Proof. Let $S = [0,1] \setminus \mathbb{Q}$. Since $\mathbb{Q}$ is a countable set we can define $\mathbb{Q}$ as a countable sequence $\mathbb{Q} = \{q_1, q_2, \dots\}$. So

$$S = [0,1] \setminus \bigcup_{n=1}^{\infty} \{q_n\} = \bigcap_{n=1}^{\infty} [0,1] \setminus \{q_n\} = \bigcap_{n=1}^{\infty} (0,1) \setminus \{q_n\}$$

since $(0,1) \setminus \{q_n\}$ is open for every n, we have expressed S as a countable intersection of open sets. Thus S is a G_δ-set.

Suppose that S is a F_σ-set,

$$S = \bigcup_{n=1}^{\infty} F_n.$$

Since S is residual in $[0,1]$. By Baire category theorem (2.16) there exists a F_i and an open interval $I \subset [0,1]$ such that F_i is dense in I. But since F_i is closed we have that $I \subset F_i$, and therefore contains both rational and irrational numbers, in contradiction to S containing only irrationals. Thus, S is not a F_σ-set. ☐

2.5 Uniform boundedness

The Uniform boundedness theorem, a classical theorem in analysis, can be proved by using the Baire category theorem. The proof illustrates a usage pattern for the Baire category theorem which will appear in the proof of other statements throughout this thesis, as in the examples 2.36, 3.19 and in the proof of theorems 3.22, 4.21, 4.24.

The key idea is that the Baire category theorem is used to show that some pointwise property holds on a larger set.

Definition 2.32 (Pointwise bounded)**.** A collection $\mathcal{F}$, of functions defined on $E \subset \mathbb{R}$, is said to be *pointwise bounded* if, for each $x \in E$, the set

$$\{f(x) : f \in \mathcal{F}\}$$

is bounded. This means that for every fixed $x \in E$ there exists a number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$.

Definition 2.33 (Uniformly bounded)**.** A collection $\mathcal{F}$, of functions defined on $E \subset \mathbb{R}$, is said to be *uniformly bounded* if there exists a number $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in E$ and for each $f \in \mathcal{F}$.

Lemma 2.34. Let f be a continuous function on a closed interval $I \subset \mathbb{R}$. Then the set

$$F = \{x : |f(x)| \leq \alpha\}$$

is closed for any positive real number α.

Proof. Assume that f is continuous on I. Prove that F is closed. Let x be a limit point of F. Then there exists a sequence $\{x_n\}$ in F such that $x_n \to x$. By assumption

$$|f(x_n)| \leq \alpha.$$

Since f is continuous at x we have that $f(x_n) \to f(x)$ and therefore we have

$$|f(x)| \leq \alpha.$$

So $x \in F$, and since x was an arbitrary accumulation point of F, we have that F is closed. $\square$

Theorem 2.35 (Uniform boundedness)**.** Let $\mathcal{F}$ be a pointwise bounded collection of continuous functions on a closed interval $E \subset \mathbb{R}$ (possibly the whole $\mathbb{R}$). Then there exists an open interval $I \subset E$ and a constant M such that

$$|f(x)| \leq M \text{ for every } f \in \mathcal{F} \text{ and every } x \in I.$$

Proof. The idea of the proof is to construct a sequence of subsets of E in a special way such that we can use the Baire category theorem to show that interval I exists. Construct F_n as

$$F_n = \{x : |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is pointwise bounded (for every $x \in E$) the sequence will eventually fill E as $n \to \infty$. So

$$E = \bigcup_{n=1}^{\infty} F_n.$$

Since each $f \in \mathcal{F}$ is continuous, each F_n is closed (according to lemma 2.34). Now, E is the countable intersection of closed sets, so E is closed. Since E is of the second category, we know from the Baire Category Theorem that not all F_n can be nowhere dense. At least one set, say F_{n_0} must therefore be dense in some open interval $I \subset E$. Hence,

$$\overline{F_{n_0}} \supset I.$$

But since F_{n_0} is closed we have that $I \subset F_{n_0}$. Finally, this leads us to the conclusion that

$$|f(x)| \leq n_0 \text{ for every } f \in \mathcal{F} \text{ and every } x \in I$$ $\square$

The following example uses the Baire Category theorem in a similar way as in the proof of the Uniform boundedness theorem.

Example 2.36. Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$ and suppose that

$$\lim_{n \to \infty} f_n(x) = 0$$

for all $x \in [0,1]$. Show that there exists an interval $[c,d] \subset [0,1]$ so that, for all sufficiently large n, $|f_n(x)| < 1$ for all $x \in [c,d]$

To prove this statement, we will use a similar reasoning as in the proof of the Uniform boundedness theorem. Define a sequence of closed sets

$$F_N = \{x : |f_n(x)| \leq 1/2 \text{ for every } n \geq N\}.$$

We must ensure that the union of F_N fills the unit interval. Suppose that there exists a point $p \in [0,1]$ such that $p \notin F_N$ for every N. But we also know that the sequence $\{f_n(p)\}$ converges to 0. From the definition of limit we can for any ε, say $1/2$, find a number N_0 such that

$$n > N_0 \Rightarrow |f_n(p) - 0| < 1/2.$$

But that would mean that $p \in F_{N_0}$. So, the union of all F_N must fill the interval.

$$[0,1] = \bigcup_{N=0}^{\infty} F_N$$

Applying, the Baire category theorem, we can obtain a set F_M, such that F_M is dense in some subinterval $[c,d] \subset [0,1]$. Since F_M is closed we have $[c,d] \subset F_M$. So, for every $x \in [c,d]$ we have that $x \in F_M$ and therefore $|f(x)| < 1$.

Chapter 3

Functions of the first class

One of the main problems that René Baire addressed in his doctoral thesis, was to characterize discontinuous real functions that can be represented by a series of continuous real functions. It was in this context that he introduced the notion of category. In this chapter we will present some explicit examples of these functions, and present a theorem due to Baire.

3.1 Basic definitions

Definition 3.1 (Pointwise convergence)**.** Let $\{f_n\}$ be a sequence of functions defined on a common domain D. If the limit

$$\lim_{n\to\infty} f_n(x)$$

exists for all $x \in D$, we say that $\{f_n\}$ *converge pointwise* on D. This limit defines a function on D

$$f(x) = \lim_{n\to\infty} f_n(x),$$

we write $f_n \to f$, to denote pointwise convergence.

It is well known that the pointwise limit of a sequence of continuous functions need not be continuous. But, the continuity of the limit function, can be guaranteed under additional assumptions on the character of convergence.

Definition 3.2 (Uniform convergence)**.** Let $\{f_n\}$ be a sequence of functions defined on a common domain D. We say that $\{f_n\}$ *converge uniformly* to f on D, if for every $\varepsilon > 0$ there exists N such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in D.$$

A classical theorem states that the limit of a uniformly convergent sequence of continuous functions is continuous. However, pointwise limits also have some traces of continuity, as we will see in this chapter.

Definition 3.3 (Function of the first class)**.** A function f, is said to be a *function of the first class* or a *Baire 1 function* if it is the pointwise limit of some sequence of continuous functions.

The importance of this definition follows, in particular, from the fact that any derivative is a function of the first class.

Proposition 3.4. Let F be a differentiable function on $\mathbb{R}$. The derivative function

$$F'(x) = \lim_{n\to\infty} \frac{F(x+1/n) - F(x)}{1/n}$$

can be expressed as a pointwise convergent sequence, where

$$f_n(x) = \frac{F(x+1/n) - F(x)}{1/n}$$

and

$$f_n \to F'.$$

Each f_n is continuous on $\mathbb{R}$ and $f_n \to F'$. So, F' is a function of the first class.

Example 3.5. This is an example of the fact that a derivative of a continuous function need not be continuous. Let f be a function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

It is well known that $x\sin(1/x) \to 0$, as $x \to 0$. The derivative of f at $x = 0$ is

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h\to 0} h \sin(1/h) = 0.$$

For any $x \neq 0$ we can calculate the derivative algebraically

$$f'(x) = 2x\sin(1/x) - \cos(1/x).$$

The first term converges to zero as $x \to 0$. For the second term, $\cos(1/x)$, consider two sequences $\{\frac{1}{2n\pi}\}$ and $\{\frac{1}{(2n+1)\pi}\}$. These sequences converge to zero as $n \to \infty$. But $\cos(2n\pi) = 1$ and $\cos((2n+1)\pi) = -1$ for all n, thus $\cos(1/x)$ does not have a limit as $x \to 0$. This means that the limit of $f'(x)$ as $x \to 0$ does not exists, so the derivative of f is discontinuous at $x = 0$.

3.2 Examples of first class functions

To determine that some function f is of the first class we can try to construct a pointwise convergent sequence of continuous functions, f_n, such that $f_n \to f$. In the following examples, we will consider some real valued functions of one variable, and we will use a piecewise linear curve in $\mathbb{R}^2$ to approximate the function. By refining the points in the curve we can achieve pointwise convergence.

Definition 3.6. Given a finite set of points A, where $\{a_0, \ldots, a_n\}$ is an increasingly ordered rearrangement of A, and a real valued function, f, defined on the interval $[a_0, a_n]$. We define $\Phi(f, A)$ to be the piecewise linear plane curve connecting the points $(a_i, f(a_i))$ for every $i = 0, \ldots, n$.

Since the curve is piecewise linear, and the set $\{a_0, \ldots, a_n\}$ is ordered increasingly, the curve describes a continuous function from $[a_0, a_n]$ into $\mathbb{R}$.

Example 3.7. A step function, f, defined on a closed interval $[0,1]$ have the property that the interval can be subdivided into a finite number of intervals

$$0 = a_0 < a_1 < \cdots < a_k = 1,$$

where $f(x) = c_i$ on the open interval (a_i, a_{i+1}) for every $i = 0, 1, \ldots, k-1$, and $f(a_i) = d_i$ for every $i = 0, 1, \ldots, k$. It follows that the set of points of discontinuity of f is a subset of $\{a_0, \ldots, a_k\}$. Figure 3.1 illustrates a step function on $[0,1]$ having four points of discontinuity.

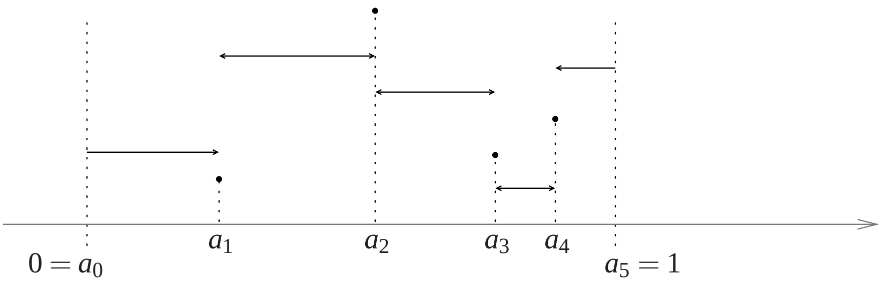

Figure 3.1: Step Function

To show that f is a function of the first class, we will create a sequence of continuous functions, $\{f_n\}$, such that f is the pointwise limit of f_n on $[0,1]$. Since our goal is to use a piecewise linear curves to approximate f, we shall define a sequence of sets A_n such that $\Phi(f, A_n)$ converge pointwise to f for increasing n.

Let A_n consists of the points

$$a_0, \ldots, a_n \ ; \ a_0 + \frac{\delta}{n}, \ldots, a_{n-1} + \frac{\delta}{n} \ ; \ a_1 - \frac{\delta}{n}, \ldots, a_n - \frac{\delta}{n},$$

where δ is defined as

$$0 < \delta = \min_{i=0,\ldots,k-1} \frac{|a_i - a_{i+1}|}{4}.$$

Observe, that we can order the points in every set A_n, such that

$$a_i \ < \ a_i + \frac{\delta}{n} \ < \ a_{i+1} - \frac{\delta}{n} \ < \ a_{i+1}$$

holds for every $i = 0, \ldots, k-1$. Let f_n be the function described by the piecewise linear curve $\Phi(f, A_n)$. Figure 3.2 illustrates the behavior of the curve at some point a_i.

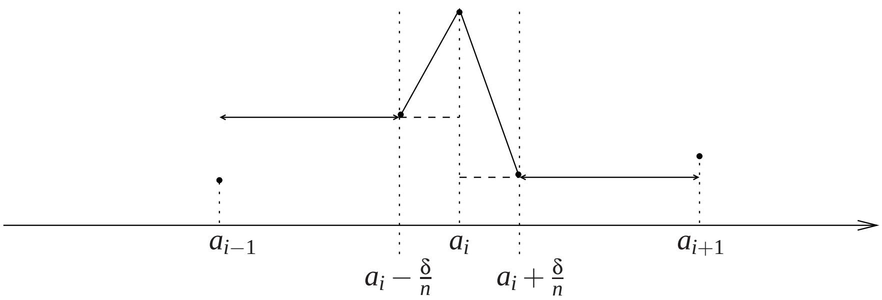

Figure 3.2: Piecewise linear curve at a_i

Now we show that $f_n \to f$ on $[0,1]$. Let x be any point in $[0,1]$. If $x \in \{a_0,\dots,a_n\}$ then $x = a_i$ for some i. Thus

$$f_n(x) = f_n(a_i) = f(a_i) = f(x).$$

If $a_i < x < a_{i+1}$ for some i, then we must ensure that

$$a_i + \frac{\delta}{n} < x < a_{i+1} - \frac{\delta}{n} \tag{1}$$

for sufficiently large n. Let

$$\gamma = \min\{|x-a_i|, |x-a_{i+1}|\}.$$

Then, for any $n > \delta/\gamma$ we have that (1) holds. Thus $f_n(x) = c_i = f(x)$. So f_n converge pointwise to f on $[0,1]$, and thus f is of the first class.

Example 3.8. In example 3.20 we showed that the Riemann function, defined on $[0,1]$, is continuous at every point except at every rational point in the interval $[0,1]$. We now show that this function is a function of the first class.

Let $D = \{d_0, d_1, \dots\}$ be the set of all rational numbers in the interval $[0,1]$. We can assume that d_0 and d_1 are the endpoints, 0 and 1. For every n we define $D_n = \{d_0, d_1, \dots, d_n\}$ to be a finite subset of D. Let

$$\delta_n < \min_{0\le i<j\le n} \left\{ \frac{|d_i - d_j|}{4} \right\}.$$

Since the set D is dense on the interval $[0,1]$ we have that δ_n converge to zero as n grows infinitely large.

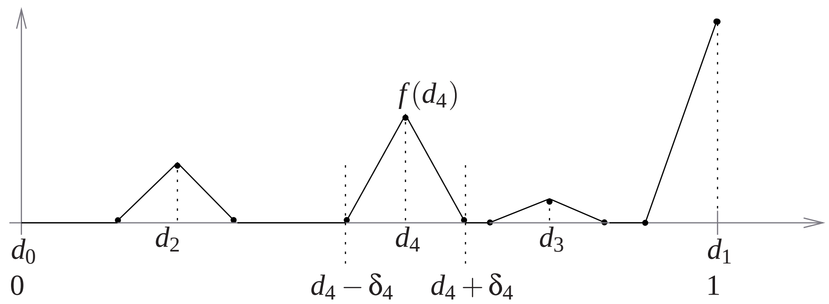

Figure 3.3: Fourth step of Riemann function approximation

For any number $n > 0$, let A_n consists of the points

$$d_0,\dots,d_n \ ; \ d_2+\delta_n,\dots,d_{n-1}+\delta_n \ ; \ d_1-\delta_n,\dots,d_n-\delta_n.$$

Though we have not assumed any order of the points in each A_n, we still have that

$$0 < d_i < d_j < 1 \quad \Rightarrow \quad d_i < d_i+\delta_n < d_j-\delta_n < d_j$$

for every $i, j = 0,\dots,n$. From the piecewise linear curve $\Phi(f, A_n)$, we define a continuous function, f_n, describe by the curve. Figure 3.3, illustrates an example of the continuous function $f_4(x)$.

We will now show that $f_n \to f$. Let x be any point in $[0,1]$. If x is rational, then $x = d_N$ for some index N, such that

$$n > N \Rightarrow f_n(x) = f_n(d_N) = f(d_N) = f(x).$$

If x is irrational, we know that f is continuous at x and $f(x) = 0$. Thus for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| = |f(y)| < \varepsilon.$$

Since f_n are piecewise linear, we have that for every n there exists two consecutive points $a_n, b_n \in A_n$ such that $I_n = [a_n, b_n]$ is an interval of linearity containing x. Also the maximal length of intervals of linearity tends to zero, for n sufficiently big. Thus there exists some N, such that for $n \geq N$ we have $I_n \subset (x-\delta, x+\delta)$. Now, this means that

$$n \geq N \Rightarrow f_n(x) \leq \max\{(f(a_n), f(b_n))\} < \varepsilon.$$

Since, ε can be arbitrary small, we have that $f_n(x) \to 0$ at any irrational point. Thus, we have showed that for every point in $[0,1]$, f is the pointwise limit of a sequence of continuous functions. So, by definition f is a function of the first class.

Example 3.9. Now we will consider the Dirichlet function defined on $\mathbb{R}$, and show that this function have no points of continuity and that this function is not a function of the first class. Let

$$f(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

First, to show that this function is discontinuous at every point, we must show that for every $x_0 \in \mathbb{R}$, there exists an $\varepsilon > 0$, say $\varepsilon = 1/2$ such that we can not find any δ such that

$$|x_0 - x| < \delta \Rightarrow |f(x_0) - f(x)| < \varepsilon = 1/2$$

but since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, any interval $|x_0 - x| < \delta$ contains both rational points and irrational points, this means that we can always find a point x such that $|x_0 - x| < \delta$ and $|f(x_0) - f(x)| = 1 > \varepsilon$ no matter how small δ we choose.

Secondly, we prove that the Dirichlet function is not a function of the first class. Assume that there exists a sequence of continuous functions f_n having f as pointwise limit.

Let $x_1 \in \mathbb{Q}$, by assumption $f_n(x_1) \to 1$, so there exists an index n_1, such that

$$f_{n_1}(x_1) > 3/4.$$

Since f_{n_1} is continuous, there exists a closed neighborhood $I_1 = [x_1 - \delta_1, x_1 + \delta_1]$ of x_1, such that

$$x \in I_1 \Rightarrow f_{n_1}(x) > 3/4.$$

Now, continue by letting $x_2 \in \mathring{I}_1 \setminus \mathbb{Q}$, and since $f_n(x_2) \to 0$ there exists index $n_2 > n_1$, such that

$$f_{n_2}(x_2) < 1/4.$$

Since f_{n_2} is continuous, there exists a closed neighborhood $I_2 = [x_2 - \delta_2, x_2 + \delta_2]$ of x_2, such that

$$\begin{aligned} & I_2 \subset I_1 \\ & |I_2| < \frac{1}{2}|I_1| \\ & x \in I_2 \Rightarrow f_{n_2}(x) < 1/4. \end{aligned}$$

Continue this process choosing rational points for odd indexes and irrational points for even indexes. Let $x \in \mathbb{R}$ be the intersection of all closed intervals $\{I_n\}$. Thus, we have two subsequences of $\{f_n(x)\}$

$$f_{n_{2i+1}}(x) \to a \geq 3/4 \text{ as } i \to \infty$$

and

$$f_{n_{2i}}(x) \to b \leq 1/4 \text{ as } i \to \infty.$$

The sequence $\{f_n\}$ does not converge pointwise, since we have found two subsequences of $\{f_n(x)\}$ converging to different values. Thus f is not a function of the first class.

Theorem 3.10. Let f be defined on $[a,b]$ and continuous at every point except a countable set of points of discontinuity. Then f is a function of the first class.

Proof. To prove that f is a function of the first class we will create a sequence of continuous functions having f as pointwise limit.

Let D be the countable set of points of discontinuity of f. Let C be some countable dense subset of the points of continuity of f. Let

$$S = C \cup D = \{s_1, s_2, s_3, \dots\}.$$

We can enforce s_1 and s_2 to be the endpoints of the interval $[a,b]$. Now we define a nested sequence of finite sets

$$S_1 = \{s_1, s_2, \dots s_{k_1}\}$$
$$S_2 = \{s_1, s_2, \dots s_{k_2}\}$$
$$S_n = \{s_1, s_2, \dots s_{k_n}\}$$

under the condition, that for any index n and for every $x \in S_n$, there exists $y \in S_n$ such that $|x-y| < \frac{1}{2^n}$. This will ensure that given any interval $I \subset [a,b]$ we can always find S_m such that I contains points from S_m, formally

$$|I| \geq \frac{1}{2^m} \Rightarrow I \cap S_m \neq \emptyset. \tag{2}$$

Let f_n be the continuous function described by the piecewise linear curve $\Phi(f, S_n)$. Now we need to prove that the sequence f_n converge pointwise to f.

Let $x_0 \in [a,b]$. If $x_0 \in S$ then there exists some index j such that $x_0 = s_{k_j}$ and thus for any $n > j$ we have that

$$f_n(x) = f_n(s_{k_j}) = f(s_{k_j}) = f(x).$$

If $x_0 \notin S$ then we know that f is continuous at x. Thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x_0 - x| < \delta \Rightarrow |f(x_0) - f(x)| < \varepsilon. \tag{3}$$

Let m be such that $\delta > 1/2^m$. By (2) we know that

$$(x_0 - \delta, x_0) \cap S_m \neq \emptyset$$
$$(x_0, x_0 + \delta) \cap S_m \neq \emptyset.$$

Thus there exist two consecutive points $s', s'' \in S_m$, such that

$$x_0 - \delta < s' < x_0 < s'' < x_0 + \delta,$$

and that no other points in S_m are closer to x_0. By (3) we have that

$$|f(x_0) - f_m(s')| = |f(x_0) - f(s')| < \varepsilon. \tag{4}$$

Since f_m is linear on the interval $[s', s'']$, we can estimate $f_m(x_0)$ by

$$|f_m(x_0) - f_m(s')| \leq |f_m(s'') - f_m(s')| = |f(s'') - f(s')| < 2\varepsilon. \tag{5}$$

Now, combine (4) and (5) to deduce

$$|f_m(x_0) - f(x_0)| \leq |f_m(x_0) - f_m(s')| + |f_m(s') - f(x_0)| < 3\varepsilon.$$

Since the point x_0 and ε were chosen arbitrary, we have proved that f_n converges pointwise to f on $[a,b]$. Thus, f is a function of the first class. ☐

Corollary 3.11. Let f be a monotone function on an interval $[a,b]$, then f is a function of the first class.

Proof. In Rudin [4] page 96, we find a proof of the well known fact that every monotonic function, defined on an interval, has at most a countable set of points of discontinuity. It now follows from theorem 3.10 that f is a function of the first class. ☐

3.3 Oscillation and continuity

In this section we will examine the points of continuity and points of discontinuity of real functions. The oscillation of a function, provides means by which we can measure the discontinuity of a function.

Definition 3.12. Let f be defined on a non-degenerate interval I. We define the *oscillation of f on I* as the quantity

$$\omega(f;I) = \sup_{x,y \in I} \Big| f(x) - f(y) \Big|.$$

From this definition we see that for any sets A and B, and for some a function f defined on A and B, we have

$$A \subset B \Rightarrow f(A) \subset f(B) \Rightarrow \omega(f;A) \leq \omega(f;B). \tag{6}$$

This is true since ω is defined to be the supremum of the image of A and B of f. Also the ω function is bounded below by zero.

Definition 3.13. Let f be defined on a neighborhood of x_0. We define the *oscillation of f at x_0* as the quantity

$$\omega_f(x_0) = \lim_{\delta \to 0} \omega(f;(x_0 - \delta, x_0 + \delta)).$$

Theorem 3.14. Let f be defined on a closed interval I (may be all of $\mathbb{R}$). Then, for any $\gamma > 0$, then the set

$$F = \{x : \omega_f(x) \geq \gamma\},$$

is closed.

Proof. Let x be a limit point of F. Then there exists a sequence $\{x_n\}$ in F such that $x_n \to x$. By assumption we have that

$$\omega_f(x_n) \geq \gamma$$

for every n. Suppose now that $x \notin F$, then $\omega_f(x) = \alpha < \gamma$. By definition there exists a $\delta > 0$, such that

$$\omega(f; N_\delta(x) \cap I) < \gamma. \tag{7}$$

Since x is a limit point of x_n, we can find some $x_i \in N_\delta(x)$. Also, x_i is an interior point of $N_\delta(x)$, thus we can find a neighborhood of x_i, such that

$$N_r(x_i) \subset N_\delta(x). \tag{8}$$

Since x_i is a point in F, we have

$$\omega(f; N_r(x_i)) \geq \omega_f(x_i) \geq \gamma.$$

By (6) and (8), it follows that

$$\omega(f; N_\delta(x)) \geq \omega(f; N_r(x_i)) \geq \gamma.$$

But this contradicts (7). So $x \in F$, and thus F is closed. □

Theorem 3.15 (Baire)**.** Let f be a bounded function on $[a,b]$. Assume that $\omega_f(x_0) \leq \alpha$ for any $x \in [a,b]$ $(\alpha \geq 0)$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$, such that for every interval $I \subset [a,b]$ with $|I| < \delta$, we have that

$$\omega(f; I) < \alpha + \varepsilon.$$

Proof. By assumption $\omega_f(x_0) \leq \alpha$ for every $x \in [a,b]$. Thus, for any point $x \in [a,b]$ and given any $\varepsilon > 0$ there exists a neighborhood $N_r(x)$ such that

$$\omega(f, N_r(x)) < \alpha + \varepsilon.$$

From this we define a real valued function $\delta(x)$ for every $x \in [a,b]$, such that

$$\omega(f, N_{\delta(x)}(x)) < \alpha + \varepsilon.$$

Now, the union of all $N_{\delta(x)}(x)$ is an open cover of $[a,b]$. Moreover, the union of all $N_{\frac{\delta(x)}{2}}(x)$ is still an open cover. Since $[a,b]$ is closed and bounded we know that there exists a finite subcover, thus

$$[a,b] \subset N_{\frac{\delta(x_1)}{2}}(x_1) \cup \cdots \cup N_{\frac{\delta(x_k)}{2}}(x_k).$$

Now let

$$\delta = \min\left\{\frac{\delta(x_1)}{2}, \ldots, \frac{\delta(x_k)}{2}\right\}.$$

Given any interval $I \subset [a,b]$ such that $|I| < \delta$. Let $N_{\frac{\delta(x_i)}{2}}(x_i)$ be one set covering the midpoint $(\frac{a+b}{2})$ of the interval. Since $N_{\frac{\delta(x_i)}{2}}(x_i)$ must cover more than half of the interval I, we have that

$$I \subset N_{\delta(x_i)}(x_i).$$

Thus we have proved that $\omega(f, I) < \alpha + \varepsilon$. □

Theorem 3.16. Let f be defined on an interval I and let $x_0 \in I$. Then f is continuous at x_0 if and only if

$$\omega_f(x_0) = 0.$$

Proof. Assume that f is continuous at x_0. Prove that $\omega_f(x_0) = 0$. Since f is continuous at x_0, then for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon/2.$$

We can express this in terms of a supremum

$$\sup_{x \in N_\delta(x_0)} |f(x) - f(x_0)| < \varepsilon/2. \tag{9}$$

Now, we need to ensure that the oscillation of the function f in the neighborhood of x_0 is bounded by ε

$$\begin{aligned}\omega(f; N_\delta(x_0)) = &\sup_{x,y \in N_\delta(x_0)} |f(x) - f(y)| \leq \\ &\leq \sup_{x \in N_\delta(x_0)} |f(x_0) - f(x)| + \sup_{y \in N_\delta(x_0)} |f(x_0) - f(y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Since ε was arbitrary, and

$$\omega_f(x_0) \leq \omega(f; N_\delta(x_0)) < \varepsilon,$$

it follows that

$$\omega_f(x_0) = 0.$$

Conversely, assume that $\omega_f(x_0) = 0$. Prove that f is continuous at x_0. By theorem 3.15 we know that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\omega(f; N_\delta(x_0)) < \varepsilon.$$

We observe that

$$\sup_{x \in N_\delta(x_0)} |f(x_0) - f(x)| \leq \sup_{x,y \in N_\delta(x_0)} |f(x) - f(y)| < \varepsilon$$

so, we can conclude that

$$|x - x_0| < \delta \Rightarrow |f(x_0) - f(x)| < \varepsilon.$$

Thus, f is continuous at x_0. □

Theorem 3.17. Let f be defined on a closed interval I (which may be all of $\mathbb{R}$). Then the set C of points of continuity of f, is a G_δ-set, and the set D of points of discontinuity of f is a F_σ-set.

Proof. Let

$$D_n = \{x : \omega_f(x) \geq \frac{1}{n}\}.$$

We know (by theorem 3.14) that D_n are closed sets. We need to show that the countable union of all D_n fills D. Suppose that $x \in D$, then

$$\omega_f(x) = \alpha > 0.$$

Let n be a number such that $\alpha > 1/n$. This means that $x \in D_n$. So we have that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Hence D is a F_σ-set. Since $C = I \setminus D$, we have that C is a G_δ-set (by theorem 2.28). ☐

Theorem 3.18. Show that a function is continuous except at the points of a first category set if and only if it is continuous at a dense set of points.

Proof. Let f be continuous at every point in C, and discontinuous at every point in D. Assume that C is dense in $\mathbb{R}$. Show that D is a set of the first category. We know that C is a G_δ-set, so

$$C = \bigcap_{n=0}^{\infty} G_n.$$

where G_n are open sets. Also, C is dense, so every G_n must be dense. Thus C is a countable intersection of open dense sets. We can define D as

$$D = \mathbb{R} \setminus C = \bigcup_{n=0}^{\infty} \mathbb{R} \setminus G_n.$$

that is, D is the countable union of nowhere dense sets, so D is of the first category.

Now, assume that D is of the first category. Show that C is dense. We know that D is a F_σ-set of the first category, so D is a countable union of closed nowhere dense sets, thus the complement, C is a residual set in $\mathbb{R}$. By the Baire category theorem, C is dense in $\mathbb{R}$. ☐

Example 3.19. Assume that a function f is discontinuous at every point of an interval $[a,b]$. Then there exists a number $\gamma > 0$ and an interval $I \subset [a,b]$ such that

$$\omega_f(x) \geq \gamma \text{ for any } x \in I.$$

Define a sequence of closed sets

$$B_n = \left\{ x : w_f(x) \geq \frac{1}{n} \right\},$$

and let

$$B = \bigcup_{n=0}^{\infty} B_n.$$

We know that B_n is closed, and also that $B = [a,b]$. From the Baire category theorem, we know that at least one, B_M, is dense in some interval $I \subset [a,b]$. Moreover, since B_M is closed we have that

$$I \subset B_M.$$

Hence, $\omega_f(x) \geq \frac{1}{M}$ for every $x \in I$ and by letting $\gamma = \frac{1}{M}$, we have proved the statement.

Example 3.20. This example will show that there exists a function, discontinuous on a countable and dense set of points. The Riemann function defined on the open interval $[0,1]$ as

$$f(x) = \begin{cases} 1/q & : x = p/q \text{ where } p/q \text{ is an irreducible rational number} \\ 0 & : x \text{ is irrational or } x = 0 \end{cases}$$

We will now show that this function is continuous at every irrational point, and discontinuous at every rational point.

To show that $f(x_0)$ is continuous at any irrational point x_0. We must ensure that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x_0 - x| < \delta \Rightarrow |f(x_0) - f(x)| < \varepsilon.$$

Let m be a number such that $\varepsilon > 1/m$ and consider the set of points having $|f(x)| > 1/m$. This set is finite, since there are only finitely many irreducible rationals $p/q \in [0,1]$ with $p \leq q < m$.

We have $\varepsilon > 1/m$, so we can choose some δ such that the closed interval $|x_0 - x| < \delta$ contains no point from the finite set having $f(x) > 1/m$. Thus we know that

$$|x_0 - x| < \delta \Rightarrow |f(x_0) - f(x)| = |0 - f(x)| \leq 1/m < \varepsilon.$$

Hence f is continuous at every irrational number. Consider any rational point p/q in $[0,1]$. Then, since every neighborhood of p/q contains irrational numbers, we have that

$$|f(x) - f(p/q)| = 1/q$$

for every irrational number x. So, f is discontinuous at every rational point.

3.4 Baire theorem

We started this chapter, by stating that a function of the first class has traces of continuity. The previous examples have indicated that the set of discontinuity points may be dense on the domain of the function. The next lemma, and the proceeding theorem, due to Baire, shows that a first class function can not be everywhere discontinuous.

Lemma 3.21. Let f be a function of the first class on $[a,b]$. Then, for any $\gamma > 0$, the set

$$E = \{x : \omega_f(x) \geq \gamma \quad \text{for every } x \in [a,b]\},$$

is closed and nowhere dense.

Proof. By theorem 3.14 it follows that the set E is closed. Assume that E is dense in some interval I, then since E is closed we have that $I \subset E$. Thus the oscillation of f at every point in I is greater or equal to γ.

Let f_n be a sequence of continuous functions converging pointwise to f on $[a,b]$. Given $0 < \varepsilon < \gamma/8$ and some arbitrary point $x_0 \in \mathring{I}$, by pointwise convergence of f_n, we have that there exists a $n_0 > 0$, such that

$$|f_{n_0}(x_0) - f(x_0)| < \varepsilon. \tag{10}$$

Since every f_n is continuous we have that there exists a closed neighborhood $I_0 \subset I$ of x_0, such that

$$x \in I_0 \Rightarrow |f_{n_0}(x_0) - f_{n_0}(x)| < \varepsilon. \tag{11}$$

By (10) and (11) we have that

$$x \in I_0 \Rightarrow |f(x_0) - f_{n_0}(x)| < 2\varepsilon. \tag{12}$$

Since $x_0 \in I_0 \subset E$, there exists some other point $x_1 \in \mathring{I}_0$, such that

$$|f(x_0) - f(x_1)| \geq \gamma. \tag{13}$$

We repeat the same reasoning with the point x_1. Since f_n converge pointwise we can find $n_1 > n_0$, such that

$$|f_{n_1}(x_1) - f(x_1)| < \varepsilon. \tag{14}$$

Since every f_n is continuous we have that there exists a closed neighborhood I_1 of x_1, such that $I_1 \subset I_0$, $|I_1| < |I_0|/2$ and

$$x \in I_1 \Rightarrow |f_{n_1}(x_1) - f_{n_1}(x)| < \varepsilon. \tag{15}$$

By (14) and (15)

$$x \in I_1 \Rightarrow |f(x_1) - f_{n_1}(x)| < 2\varepsilon. \tag{16}$$

Now, we combine (12), (13) and (16)

$$x \in I_1 \Rightarrow |f_{n_0}(x) - f_{n_1}(x)| > \gamma - 4\varepsilon > \gamma/2. \tag{17}$$

Now, we proceed by induction. Assume that for some index number k, we have obtained

$$I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k \tag{18}$$

$$x_0 \in \mathring{I}_0 \ , \ x_1 \in \mathring{I}_1 \ , \ \ldots \ , x_{k-1} \in \mathring{I}_{k-1} \ , \ x_k \in \mathring{I}_k \tag{19}$$

$$f_{n_0}, f_{n_1}, \ldots, f_{n_{k-1}}, f_{n_k}, \tag{20}$$

such that

$$|f(x_k) - f(x_{k-1})| \geq \gamma \tag{21}$$

$$|f_{n_k}(x_k) - f(x_k)| < \varepsilon \tag{22}$$

$$x \in I_k \Rightarrow |f_{n_k}(x) - f(x_k)| < 2\varepsilon. \tag{23}$$

Since I_k is a subset of E there exists a point $x_{k+1} \in \mathring{I}_k$, such that

$$|f(x_{k+1}) - f(x_k)| \geq \gamma. \tag{24}$$

This satisfies the induction hypothesis (21). Since f_n converges pointwise to f, we can find $n_{k+1} > n_k$, such that

$$|f_{n_{k+1}}(x_{k+1}) - f(x_{k+1})| < \varepsilon. \tag{25}$$

This satisfies the induction hypothesis (22). By continuity of $f_{n_{k+1}}$, there exists $I_{k+1} \subset I_k$, such that

$$|I_{k+1}| < |I_k|/2 \tag{26}$$

$$x_{k+1} \in \mathring{I}_{k+1} \tag{27}$$

$$x \in I_{k+1} \Rightarrow |f_{n_{k+1}}(x_{k+1}) - f_{n_{k+1}}(x)| < \varepsilon. \tag{28}$$

By (25) and (28) we have that

$$x \in I_{k+1} \Rightarrow |f_{n_{k+1}}(x) - f(x_{k+1})| < 2\varepsilon. \tag{29}$$

By assertion (29), we have satisfied the induction hypothesis (23) for $k+1$, and the construction can proceed. Also, for each induction step we use assertion (25) and (28), together with the induction assumption (23) to deduce

$$x \in I_{k+1} \Rightarrow |f_{n_{k+1}}(x) - f_{n_k}(x)| > \gamma - 2\varepsilon - 2\varepsilon > \gamma/2.$$

Let the point x be the intersection of all I_n. Then for every k, we have

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| > \gamma/2.$$

But this means that $\{f_{n_k}(x)\}$ diverge. So, under the assumption made, f_n can not converge pointwise on $[a,b]$. Thus, E can not be dense in I, it follows that E is nowhere dense. ☐

Theorem 3.22 (Baire theorem)**.** Let f be a function of the first class on $[a,b]$, then f is continuous except at a set of points of the first category.

Proof. Let D be the set of points of discontinuity of f. Define a sequence of sets

$$E_n = \{x : \omega_f(x) \geq 1/n \quad \text{for every } x \in [a,b]\}.$$

By previous theorem 3.21, we have that every E_n is nowhere dense. Since, the countable union of all E_n fills D (as showed in proof of theorem 3.17), we have that D is of the first category. ☐

Example 3.23. This example illustrates that two functions need not share the first class property, although the two functions are defined on the same domain and have the same set of points of continuity. Thus the converse of theorem 3.22 is false. Let K be the Cantor set, and define

$$f(x) = \begin{cases} 1 & : x \in K \\ 0 & : \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} 1 & : x \in K^{2nd} \\ 0 & : \text{otherwise} \end{cases}$$

Prove that f and g are continuous on every point of $[0,1] \setminus K$. Suppose that $x_0 \in [0,1] \setminus K$. Then x_0 is an interior point, since $[0,1] \setminus K$ is open. Thus there exists a neighborhood $N_\delta(x_0)$ contained in $[0,1] \setminus K$. Then, for every point $x \in N_\delta(x_0)$ we have that $f(x) = 0$ and $g(x) = 0$. So f and g are continuous at every point in $[0,1] \setminus K$.

Prove that f and g are discontinuous at every point in K. For any $x \in K$, we have that any neighborhood of x contains points from K, K^{1st} and K^{2nd}. Thus there will always exist a point

y in any neighborhood of x, such that $|f(x)-f(y)|=1$ and likewise for g. So f and g are discontinuous on K.

Prove that f is a function of the first class. We will create a sequence of continuous functions $\{f_n\}$ having f as the pointwise limit on $[0,1]$. The construction of each f_n will follow the steps in the construction of the Cantor set (section 2.3).

Let $K_n^{1st}=\{a_0,a_1,\dots,a_k\}$ be increasingly ordered (finite) set of points of the first kind in K_n. Let A_n consist of all points in K_n^{1st} together with the following points

$$(a_i+\frac{1}{3^{n+1}}) \text{ for every } i=1,3,\dots k$$
$$(a_i-\frac{1}{3^{n+1}}) \text{ for every } i=2,4,\dots k-1.$$

We can order the points in A_n as

$$a_{2i-2}<a_{2i-1}<a_{2i-1}+\frac{1}{3^{n+1}}<a_{2i}-\frac{1}{3^{n+1}}<a_{2i}<a_{2i+1}$$

for every $i=1,\dots,(k-1)/2$. We also note that

$$a_{2i-2}\le x\le a_{2i-1} \qquad \Rightarrow f_n(x)=1$$
$$a_{2i-1}+\frac{1}{3^{n+1}}\le x\le a_{2i}-\frac{1}{3^{n+1}} \qquad \Rightarrow f_n(x)=0.$$

Let f_n be the continuous function described by the piecewise linear curve $\Phi(f,A_n)$.

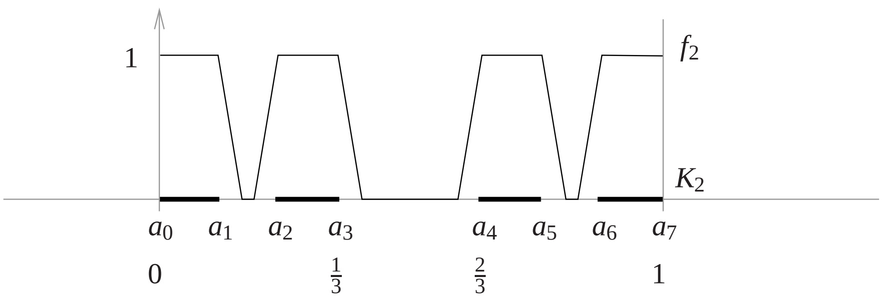

Figure 3.4: f_2 of the function sequence

Now we show that f is the pointwise limit of f_n. If $x\in K$ then $f_n(x)=1$ for every n. If on the other hand $x\notin K$ we have that for some index i, and for sufficiently large n we have

$$a_{2i-1}+\frac{1}{3^{n+1}}<x<a_{2i}-\frac{1}{3^{n+1}}.$$

Thus, $f_n(x)=0$. So f is the pointwise limit of $\{f_n\}$, and therefore f is of the first class.

Prove that g is not a function of the first class. We apply the same reasoning as in example 3.9 (Dirichlet function). Assume there exists a sequence of continuous functions $\{g_n\}$ such that $g_n\to g$. Let $x_1\in K^{1st}$, then there exists an index n_1 such that

$$g_{n_1}(x_1)<1/4,$$

and since g_{n_1} is continuous, there exists a closed neighborhood I_1 of x_1, such that

$$p \in I_1 \Rightarrow g_{n_1}(p) < 1/4.$$

Now, let $x_2 \in K^{2nd} \cap \mathring{I}_1$, then there exists an index n_2 such that

$$g_{n_2}(x_2) > 3/4,$$

and since g_{n_2} is continuous there exists a closed neighborhood I_2 of x_2, such that

$$p \in I_2 \Rightarrow g_{n_2}(p) > 3/4$$

having $I_2 \subset I_1$ and $|I_2| < \frac{1}{2}|I_1|$. Now we continue this construction to get the sequence $\{g_{n_i}\}$. Let $x = \bigcap_{i=1}^{\infty} I_i$. But since

$$g_{n_{2i}}(x) \to a \geq 3/4$$

and

$$g_{n_{2i+1}}(x) \to b \leq 1/4.$$

We have defined two subsequences of $\{g_n(x)\}$, converging to different values. Thus $\{g_n(x)\}$ have no unique limit point. So, the assumption that g is a function of the first class is false.

Chapter 4

Metric spaces

In section 2.1, we defined some basic topological properties of points and sets on the real line. All these definitions relied on the fact that the real line can be equipped with a distance function: the absolute value function.

For example, consider the neighborhood of a point, it is defined as all points within a certain distance of that point. An open set is defined in terms of neighborhoods. A nowhere dense set is defined in terms of open sets.

We will in this chapter see that all these definitions are valid in more general sets, sets that are equipped with some distance function. We also present Baire category theorem for arbitrary complete metric spaces and some applications of the theorem.

4.1 Basic definitions

Definition 4.1. A *metric space* is a tuple (X,d) where X is a set and d is a real valued *distance function* or a *metric*, defined on $X \times X$, satisfying the following conditions:

(*i*) $d(x,y) \geq 0$

(*ii*) $d(x,y) = d(y,x)$

(*iii*) $d(x,y) = 0 \Leftrightarrow x = y$

(*iv*) $d(x,z) \leq d(x,y) + d(y,z)$

On the real line, the metric induced by the absolute value function, is commonly referred to as the *usual metric*. Thus, $(\mathbb{R}, |\bullet|)$ is a metrics space, where condition (iv) is the well known triangle inequality.

Definition 4.2 (Neighborhood)**.** Let (X,d) be a metric space. For any point $x_0 \in X$ and any real number $\varepsilon > 0$, the set of points

$$\{x : d(x_0, x) < \varepsilon\}$$

is called a *neighborhood* of x_0 or *open ball* centered at x_0 of radius ε. This set is often denoted as $B(x_0, r)$.

The basic topological definitions given in 2.2, all rely on the notion of neighborhoods. So, without any changes we can reuse these definition in relation to some general metric space (X,d), considering sets and points in X instead of the real line.

Definition 4.3 (Dense)**.** Given a set $A \subset X$. We say that A is *dense* if every open set G intersects A.

Definition 4.4 (Nowhere dense)**.** Given a set $A \subset X$. We say that A is *nowhere dense* provided that every open set G contains an open subset $J \subset G$, such that $J \cap A = \emptyset$.

Definition 4.5 (Category)**.** The definitions we have given of the first category and the second category, ultimately rely on the notion of neighborhoods. Thus, we can state these definitions for any metric space (X,d).

(i) $S \subset X$ is said to be of the *first category* if it can be represented as a countable union of nowhere dense sets.

(ii) A set is said to be of the *second category* if it is not of the first category.

(iii) A set is said to be *residual* in X if its complement is of the first category.

Definition 4.6 (Cauchy sequence)**.** Let (X,d) be a metric space and let $\{x_k\}$ be a sequence in X. If, for any $\varepsilon > 0$, there exists a $N > 0$, such that

$$n,m > N \Rightarrow d(x_m,x_n) < \varepsilon.$$

We say that $\{x_k\}$ is a *Cauchy sequence* or that $\{x_k\}$ is *Cauchy*.

Definition 4.7 (Complete)**.** A metric space (X,d) is *complete* if every Cauchy sequence converge.

Particular important metric spaces are obtained if we consider *linear spaces* (or *vector spaces*) having metric functions defined by their *norms*.

Definition 4.8. A *normed space* X is a linear space with a norm defined on it. A *Banach space* is a complete normed space. A *norm* on a linear space is a real-valued function on X, denoted by

$$||x||$$

satisfying the following conditions for any $x,y \in X$ and any scalar α

(i) $||x|| \geq 0$

(ii) $||\alpha x|| = |\alpha|||x||$

(iii) $||x|| = 0 \Leftrightarrow x = 0$

(iv) $||x+y|| \leq ||x|| + ||y||$

A norm defines a metric on $X \times X$, by

$$d(x,y) = ||x-y||.$$

Definition 4.9. Let (X,d) and (Y,e) be metric spaces. A bijective mapping f is called a *homeomorphism* if both f and f^{-1} are continuous. If such mapping exists (X,d) and (Y,e) are *homeomorphic*.

Theorem 4.10. Let (X,d) and (Y,e) be homeomorphic metric spaces, and f a homeomorphism. Then A is nowhere dense in X if and only if $f(A)$ is nowhere dense in Y.

Proof. We first we make a general observation. Since f is a homeomorphism we know that a is a limit point of A if and only if $f(a)$ is a limit point of $f(A)$. Thus

$$\overline{f(A)} = f(\overline{A}).$$

Now, assume that A is nowhere dense and suppose that $f(A)$ is not nowhere dense, then there exists an open ball B, such that

$$B \subset \overline{f(A)} = f(\overline{A}).$$

By applying f^{-1}, we get

$$f^{-1}(B) \subset \overline{A},$$

Since f is continuous, $f^{-1}(B)$ is open. So, A is dense in the open ball $f^{-1}(B)$, in contradiction to A being nowhere dense. Thus, $f(A)$ is nowhere dense.

Conversely, assume that $f(A)$ is nowhere dense. Suppose that A is not nowhere dense, then there exists an open ball B, such that

$$B \subset \overline{A}$$

By applying f, we get

$$f(B) \subset f(\overline{A}) = \overline{f(A)},$$

Since f^{-1} is continuous, $f(B)$ is open. So, $f(A)$ is dense in the open ball $f(B)$, in contradiction to $f(A)$ being nowhere dense. Thus, A is nowhere dense. ☐

Corollary 4.11. Let (X,d) and (Y,e) be homeomorphic metric spaces, and f a homeomorphism. Then A is of the first category if and only if $f(A)$ is of the first category.

Proof. Let

$$A = \bigcup A_n,$$

thus we have

$$f(A) = f(\bigcup A_n) = \bigcup f(A_n).$$

By previous theorem it follows that A_n is nowhere dense if and only if $f(A_n)$ is nowhere dense. Thus, A is of the first category if and only if $f(A)$ is of the first category, as required. ☐

Remark 4.12. This corollary shows that the category of a set is preserved under a homeomorphism. Such properties are called a *topological properties*.

4.2 Examples of metric spaces

Example 4.13. Consider $C[0,1]$, the set of all continuous functions on $[0,1]$. Show that

$$d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|$$

is a metric. This metric is called the *uniform metric*.

We have to show that d satisfies all four conditions in 4.1. We see that $d(f,g) \geq 0$, since it is the supremum of an absolute value. Also, we have that

$$d(f,g) = \sup|f(t) - g(t)| = \sup|g(t) - f(t)| = d(g, f).$$

Suppose that $f = g$. Then, for every t we have that $f(t) = g(t)$. So $d(f,g) = 0$. Conversely, suppose that $d(f,g) = 0$, then we know that $|f(t) - g(t)| = 0$ for every t, thus $f = g$ on the interval. Finally, we consider the triangle inequality.

$$\begin{aligned} d(f,h) =& \sup|f(t) - h(t)| \leq \sup|f(t) - g(t)| + |g(t) - h(t)| \\ \leq& \sup|f(t) - g(t)| + \sup|g(t) - h(t)| = d(f,g) + d(g,h). \end{aligned}$$

Example 4.14. Show that $C[0,1]$, with the uniform metric, is a complete metric space.

Let $\{f_n\}$ be a Cauchy sequence. We need to prove that this sequence converges to a function in $C[0,1]$. Given some $\varepsilon > 0$ there exists a number N, such that

$$|f_i(x) - f_j(x)| < \varepsilon \text{ for every } x \in [0,1] \text{ and for every } i, j \geq N.$$

So, for any $x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, and therefore has a limit in $\mathbb{R}$. Let

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Thus, f is the pointwise limit of f_n. But we need to ensure uniform convergence. Since f_n is a Cauchy sequence, there exists a number N, such that for every $n > N$, we have

$$|f_N(x) - f_n(x)| < \varepsilon \text{ for every } x \in [0,1]. \tag{1}$$

Then for every $x \in [0,1]$,

$$|f_N(x) - f(x)| = \lim_{n\to\infty} |f_N(x) - f_n(x)| \leq \varepsilon. \tag{2}$$

From (1) and (2) it follows that

$$|f_n(x) - f(x)| = |f_n(x) - f_N(x)| + |f_N(x) - f(x)| < 2\varepsilon,$$

for every $x \in [0,1]$ and $n > N$. Thus, f_n converges uniformly to f on $[0,1]$. Since, a uniformly convergent sequence of continuous functions converges to a continuous function on a closed interval (see [5] page 384). If follows that f is continuous, and therefore, $C[0,1]$ is a complete metric space.

Example 4.15. Show that the set of all piecewise linear functions is a dense subset of $C[0,1]$.

For any f, we need to show that every open ball $B(f,r)$ contains a piecewise linear function. Since, we have the uniform metric, it is sufficient to find a sequence of piecewise linear functions converging uniformly to f.

Since, f is continuous on the closed interval $[0,1]$, it is uniformly continuous. Thus for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \tag{3}$$

Let A_n be a set defined as

$$A_n = \{\frac{i}{2^n} : i = 0, \ldots, 2^n\}.$$

Let n be a number, such that $\delta > 1/2^n$. Let f_n to be the piecewise linear function defined by the curve $\Phi(f, A_n)$. Now, we shall estimate the distance between the function f and the sequence of piecewise linear functions. Let x_0 be any point from $[0,1]$. Then,

$$(x_0 - \delta, x_0) \cap A_n \neq \emptyset$$
$$(x_0, x_0 + \delta) \cap A_n \neq \emptyset.$$

This means that there exist two consecutive points, $\frac{m}{2^n}$ and $\frac{m+1}{2^n}$ in A_n, such that

$$x_0 - \delta < \frac{m}{2^n} < x_0 < \frac{m+1}{2^n} < x_0 + \delta.$$

The function f_n is linear on the interval $[\frac{m}{2^n}, \frac{m+1}{2^n}]$, so

$$\left| f_n(x_0) - f_n(\frac{m}{2^n}) \right| \leq \left| f_n(\frac{m+1}{2^n}) - f_n(\frac{m}{2^n}) \right| = \left| f(\frac{m+1}{2^n}) - f(\frac{m}{2^n}) \right| < \varepsilon. \tag{4}$$

By (3) and (4), we have

$$\left| f(x_0) - f_n(x_0) \leq \left| f(x_0) - f(\frac{m}{2^n}) \right| + \left| f_n(\frac{m}{2^n}) - f_n(x_0) \right| < 2\varepsilon.$$

Since x_0 was arbitrary, and ε and δ was chosen independent of x_0, we can state that for any n

$$\delta > 1/2^n \Rightarrow \sup \left| f(t) - f_n(t) \right| < 2\varepsilon.$$

Thus, f_n converges uniformly to f. So, the set of all piecewise linear curves is a dense subset of $C[0,1]$.

Example 4.16. Show that the set of piecewise linear functions is a set of the first category.

Let P denote the set of all piecewise linear functions. Let $P_n \subset P$ be the set of all piecewise linear functions having n intervals of linearity. We can express

$$P = \bigcup_{n=1}^{\infty} P_n.$$

So, we need to prove that P_n are a nowhere dense sets. Fix arbitrary n. Let $f \in P_n$ and consider $B(f,r)$ of any radius. Let

$$g(t) = \frac{r}{2} \sin(2\pi\, 4n\, t) + f(t).$$

Then $d(f,g) = \frac{r}{2}$, thus g is in $B(f,r)$. Now, define the neighborhood $B(g, r/2)$. We aim to prove that this neighborhood contains no piecewise linear function from P_n. Let

$$h \in B(f,r) \cap P_n.$$

and suppose that $d(g,h) < r/2$, thus

$$d(g,h) = \sup |g(t) - h(t)| = \sup \left| \frac{r}{2} \sin(2\pi\, 4n\, t) + f(t) - h(t) \right| < r/2. \tag{5}$$

Observing that the term $\frac{r}{2}\sin(2\pi\, 4n\, t)$ oscillate between the points $(\frac{r}{2})$ and $(-\frac{r}{2})$, $4n$ times on the unit interval. Thus, the term $f(t)-h(t)$ must also oscillate between strictly positive and strictly negative values, $4n$ times, for (5) to hold. But, since the term $f(t)-h(t)$ is a piecewise linear function with at most $2n-1$ intervals of linearity, this is impossible.

So, the open ball $B(g,r/2)$ contains no functions from P_n. Since n was arbitrary, we have that P_n are nowhere dense sets and P is a set of the first category.

4.3 Baire category theorem

Theorem 4.17. Let (X,d) be a complete metric space. If A is of the first category, then $X\setminus A$ is dense in X.

Proof. We shall prove that an arbitrary open ball $B_0(x_0,r_0)$ in X, intersects $X\setminus A$. We may assume that $r_0 < 1/2$. Since A is of the first category, we have

$$A = \bigcup_{n=1}^{\infty} A_n,$$

where every A_n is nowhere dense. Since A_1 is nowhere dense, there exists some open ball $B_1(x_1,r_1)$, such that

$$\begin{aligned}&\overline{B_1} \subset B_0\\ &\overline{B_1}\cap A_1 = \emptyset\\ &r_1 < r_0/2\end{aligned}$$

Continue by induction. Assume that $B_k(x_k,r_k)$ is a non-empty open ball, such that

$$\begin{aligned}&\overline{B_k} \subset B_{k-1}\\ &\overline{B_k}\cap A_k = \emptyset\\ &r_k < \frac{1}{2}r_{k-1}.\end{aligned}$$

Let $B_{k+1}(x_{k+1},r_{k+1})$ be a non-empty open ball, such that

$$\begin{aligned}&\overline{B_{k+1}} \subset B_k\\ &\overline{B_{k+1}}\cap A_{k+1} = \emptyset\\ &r_{k+1} < \frac{1}{2}r_k.\end{aligned}$$

Since A_{k+1} is nowhere dense, we now that such B_{k+1} exists. By this, we have that the induction hypothesis is satisfied, so the construction can proceed. The center of each open ball B_n defines a sequence $\{x_n\}$, this sequence is Cauchy, since for any $\varepsilon > 0$ and N, such that $2^{-N} < \varepsilon$, we have

$$d(x_i,x_j) \le d(x_i,X_N) + d(x_j,x_N) \le \frac{1}{2^N} < \varepsilon \quad \text{for } i,j \ge N.$$

By assumption X is complete, so $x_n \to x$. Since $x_n \in B_n$ for every n, we have that $x = \bigcap \overline{B_n}$. But this means that $x \in B_0$ and $x \notin A$. So, $B_0 \cap (X\setminus A)$ is non-empty. Thus $X\setminus A$ is dense. $\square$

A frequently used, and equivalent version, of the Baire category theorem is formulated as follows:

Theorem 4.18. Let (X,d) be a non-empty complete metric space, then it is of the second category in itself.

Proof. **(4.17) $\Rightarrow$ (4.18)** We proceed by an indirect proof. Assume that X is of the first category. Then, by (4.17), $X^c = \emptyset$ is dense in X. Thus we have arrived at a contradiction. So, X must be of the second category. □

Proof. **(4.18) $\Rightarrow$ (4.17)** Let X be a complete metric space, then by (4.18) we have that X is of the second category. We shall prove that for any $A \subset X$ of the first category, $X \setminus A$ is dense. Let $B_0(x,r)$ be any open ball in X, and let $B[x,r/2] \subset B_0$ be a non-degenerate closed ball. Thus, B is a complete metric space, and by (4.18) in itself of the second category. Since, A is of the first category in X we can express A as

$$A = \bigcup A_k,$$

where A_k are nowhere dense sets. Moreover, $A \cap B$ is of the first category in B, since otherwise some A_k would be dense in some open ball $B' \subset B$, and this would mean that A_k are not nowhere dense in X. Since, B is of the second category, it follows that B must contain some point from A^c. Thus $B_0 \cap A \neq \emptyset$, so $X \setminus A$ is dense. □

Corollary 4.19. Suppose that A is a subset of the second category in a complete metric space (X,d), and

$$A = \bigcup_{k=1}^{\infty} A_k$$

where A_k are closed sets. Then at least one of A_k contains an open ball.

Proof. By Baire category theorem, at least one A_k is not nowhere dense. This means that the closure of A_k can not have an empty interior, thus containing some open ball. It follows, since A_k are closed, that this open ball is a subset of A_k. □

4.4 Decomposition of intervals

Theorem 4.20. The closed interval $[0,1]$, can not be expressed as a countable union of non-empty disjoint closed sets.

Proof. We will proceed by an indirect proof. $X = [0,1]$, endowed with the usual metric, is a complete metric space. Assume that X can be expressed as

$$X = [0,1] = \bigcup_{n=1}^{\infty} F_n,$$

where F_n are closed disjoint sets. Let $D_n = F_n \setminus \mathring{F}_n$ be the set of boundary points of F_n. Each D_n is a non-empty set. Since, by hypothesis, F_n is a closed, non-empty proper subset of X, F_n and F_n^c both intersect the open interval $(0,1)$ and by theorem 2.11 we know that the interval contains boundary points from F_n. Thus D_n are non-empty sets.

Let K be defined as the countable union of these boundary sets

$$K = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} F_n \setminus \mathring{F}_n = X \setminus \bigcup_{n=1}^{\infty} \mathring{F}_n.$$

K is closed, since it is the complement of a countable union of open sets. Thus, K is a complete metric space. Applying Baire category theorem on

$$K = \bigcup_{n=1}^{\infty} D_n,$$

we know that at least one D_m contains some open ball B_K. We can define $B_K(x,r) = B(x,r) \cap K$, where B is a corresponding open ball in X. Thus,

$$B_K \subset D_m, \tag{6}$$

where we know that D_m does not contain the whole B, since by theorem 2.9, D_m contains no interior. Thus, we can split B into a union of subsets of K and K^c, as follows

$$\begin{aligned} B &= (B \cap K) \cup (B \cap K^c) \\ &= (B \cap D_m) \cup (B \cap \bigcup \mathring{F}_n). \end{aligned}$$

Suppose, that $B \cap \mathring{F}_i$ is non-empty, for some $i \neq m$. We know that B is not a subset of $\mathring{F}_i$, since $B \cap D_m \neq \emptyset$. So, B must contain at least one boundary point from $\mathring{F}_i$, by theorem (2.10). But, this is impossible since B_K only contains boundary points from F_m. So, we have that

$$B = (B \cap D_m) \cup (B \cap \mathring{F}_m).$$

We have found an open ball B contained in F_m. But this means that B is a subset of the interior of F_m, and therefore can not intersect K. This contradicts B_K being non-empty. Thus, $[0,1]$ can not be expressed as a countable union of disjoint closed sets. ☐

4.5 Algebraic polynomials

Consider a real valued continuous function f defined on $[0,1]$. We know that if f has an nth derivative that is identically zero on $[0,1]$, then f is a polynomial of degree at most $n-1$.

Theorem 4.21. Let f be a real valued function having derivatives of all orders on $[0,1]$. Suppose that for every x there is an integer $n(x)$, such that

$$f^{(n(x))}(x) = 0.$$

Then f coincides with some polynomial on $[0,1]$.

Proof. Let $X = [0,1]$, then X is a complete metric space. Let

$$E_n = \{x : f^{(n)}(x) = 0\}.$$

Since the inverse image of $\{0\}$ is closed for continuous functions, we have that E_n are closed sets. By hypothesis, for every x there exists an integer n, such that $f^{(n)}(x) = 0$, thus

$$X = \bigcup_{n=0}^{\infty} E_n.$$

By Baire category theorem we know that at least one of E_n has a non-empty interior. Let

$$\sigma = \{n : \mathring{E}_n \neq \emptyset\}.$$

Now we consider the union of these interiors, let

$$G = \bigcup_{n\in\sigma} \mathring{E}_n. \tag{7}$$

Since $\mathring{E}_n$ are open sets in X, we know (see [5] page 155-156) that there exists a unique sequence $\{I_k^{(n)}\}$ of open disjoint intervals whose union is $\mathring{E}_n$

$$\mathring{E}_n = \bigcup_k I_k^{(n)}. \tag{8}$$

We know that $f^{(n)} = 0$ for every $x \in I_k^{(n)}$, thus f coincides with a polynomial of degree less than n on $I_k^{(n)}$. We also have

$$G = \bigcup_{n,k} I_k^{(n)}. \tag{9}$$

We prove that G is dense. Let $I \subset X$ be any open interval. We can express

$$\bar{I} = \bigcup_n \bar{I} \cap E_n,$$

and it follows from the Baire category theorem that at least one of the $\bar{I} \cap E_n$ contains some open interval I'. Thus, $I' \subset I$ and $I' \subset \mathring{E}_n$, and therefore $I' \subset G$. This means that $G \cap I$ is non-empty, and since I was arbitrary interval, G is dense.

Let $n_0 = \min(\sigma)$, then we state that $\mathring{E}_{n_0} = [0,1]$. We continue by an indirect proof in three steps, assuming

$$\mathring{E}_{n_0} \neq [0,1]. \tag{10}$$

First, we prove $G \neq [0,1]$, by showing that at least one boundary point of $\mathring{E}_{n_0}$, is in none of the open sets $\mathring{E}_n$. Given some interval component $I_k^{(n_0)}$, we know by (10) that $I_k^{(n_0)} \neq [0,1]$. Let α be one of the endpoints of the interval $I_k^{(n_0)}$, such that $0 < \alpha < 1$. Suppose that $\alpha \in G$, then α must be an interior point of some open interval $I_l^{(n_1)}$, for $n_1 > n_0$. So, α is both an interior point of $I_l^{(n_1)}$ and a boundary point of $I_k^{(n_0)}$. Let J' be some neighborhood of α, contained in $I_l^{(n_1)}$. Then, since α is a boundary point of $I_k^{(n_0)}$, we know that J' has a non-empty intersection with $I_k^{(n_0)}$. Thus $I_k^{(n_0)}$ and $I_l^{(n_1)}$ have a non-empty intersection. Let J be an open interval, such that

$$\emptyset \neq J \subset I_k^{(n_0)} \cap I_l^{(n_1)}.$$

We know that $f^{(n_1)}(x) = 0$ on $I_l^{(n_1)}$, but also that $f^{(n_0)}(x) = 0$ on J. It follows that f coincides with some polynomial of degree less than n_0 on $I_l^{(n_1)}$, thus

$$I_l^{(n_1)} \subset \mathring{E}_{n_0}.$$

Any two intervals in $\mathring{E}_{n_0}$ are either equal or disjoint, and since we arrived at J being non-empty, we have

$$I_l^{(n_1)} = I_k^{(n_0)},$$

This lead us to the conclusion that α is both a boundary point and an interior point of $I_k^{(n_0)}$, which is impossible. Thus, $\alpha \notin G$ and it follows that $G \neq [0,1]$.

Secondly, we prove that $H = X \setminus G$ is prefect. We have seen that G is an open and dense proper subset of $[0,1]$, thus H is non-empty, closed and nowhere dense in $[0,1]$. Suppose that H is not perfect. Then there exists a point $y \in H$, such y is an isolated point of H. This means that y is the common endpoint of two disjoint intervals, $I_i^{(n)}$ and $I_j^{(m)}$. Suppose that $n > m$, then by continuity of $f^{(n)}$ we have that $f^{(n)}(y) = 0$, and since f coincides with a polynomial of degree less that n on $I_j^{(m)}$, we have $f^{(n)} = 0$ on the open interval $I_i^{(n)} \cup \{y\} \cup I_j^{(m)}$, thus $y \in \mathring{E}_n$, so $y \notin H$. Hence, H is closed and contains no isolated points, by definition H is prefect.

Third and last, we shall arrive at a contradiction, thus falsifying assumption (10). Since H is a closed subset of X, H is a complete metric space. We can express H, as

$$H = \bigcup_n E_n \cap H.$$

By the Baire category theorem, we have that for some fixed n, $E_n \cap H$ contains an open ball $B_H = B \cap H$, where B is the corresponding open ball in X. Since B_H is a subset of E_n, we have that $f^{(n)} = 0$ on B_H. By the definition of the derivative

$$f^{(n+1)}(x) = \lim_{\substack{y \to x \\ y \in B_H}} \frac{f^{(n)}(x) - f^{(n)}(y)}{x - y} = \lim_{\substack{y \to x \\ y \in B_H}} \frac{0 - 0}{x - y} = 0,$$

we have that $f^{(m)}(x) = 0$ for every $x \in B_H$ and every $m \geq n$. We observe that this limit exists for every $x \in B_H$, since B_H is perfect.

Since H is nowhere dense, B has a non-empty intersection with some of the open intervals defining G. Let K be such open interval, then since K is a subset of E_m for some m. We have that $f^{(m)}(x) = 0$ for every $x \in K$.

If $m \leq n$. Then by differentiating $f^{(m)}(x)$, $n - m$ times, we get that $f^{(n)}(x) = 0$ on K.

If $m > n$. Then any boundary point α of $K \cap B$ is in B_H. Thus we have that

$$f^{(n)}(\alpha) = f^{(n+1)}(\alpha) = \cdots = f^{(m-1)}(\alpha) = f^{(m)}(\alpha) = 0.$$

Since $f^{(m)}(x) = 0$ on the interval $K \cap B$, we can calculate the integral from α to some arbitrary point $x \in K \cap B$,

$$0 = \int_\alpha^x f^{(m)}(t)dt = f^{(m-1)}(x) - f^{(m-1)}(\alpha) = f^{(m-1)}(x),$$

thus $f^{(m-1)}(x) = 0$ on $K \cap B$. We can repeat this integration, to derive that $f^{(n)}(x) = 0$ on $K \cap B$. Since this holds for any interval K, intersecting B, we have that $f^{(n)}(x) = 0$ on B. It follows that $B \cap H$ is empty, but we have arrived at B_H being non-empty under the assumption (10). Thus we can conclude that $E_{n_0} = [0,1]$.

So, the set of intervals $\{I_k^{n_0}\}$ contains only one interval $I_0^{n_0} = [0,1]$, and therefore $f^{n_0}(x) = 0$ throughout the interval $[0,1]$. It follows that f coincides with a single polynomial on $[0,1]$ of degree $n_0 - 1$. □

4.6 Nowhere differentiable functions

We know that a continuous function need not be differentiable at every point of its domain. For example $f(x) = |x|$, is continuous but not differentiable at 0. We start this section by defining the *sawtooth function*, a periodic continuous function having countably many points of nondifferentiability.

During the later half of the nineteeth century, several examples of nowhere differentiable continuous functions were presented. One, which we will present in this thesis, is due to Van der Waerden.

Definition 4.22. The *sawtooth function*, is a real valued continuous function defined on $\mathbb{R}$, as follows

$$\Phi(x) = \min\{x - [x], [x] + 1 - x\}.$$

It is a periodic function of period 1. For any integer i we have that $\Phi(i) = 0$ and $\Phi(i + \frac{1}{2}) = \frac{1}{2}$, where $\frac{1}{2}$ is the maximum value. We also note that, for any $m > 0$, the function $\Phi(mx)$ have a one side derivative numerically equal to $|m|$, at every point. However $\Phi(x)$ is not differentiable at points from $\mathbb{Z}$ since the right and left derivative does not coincide.

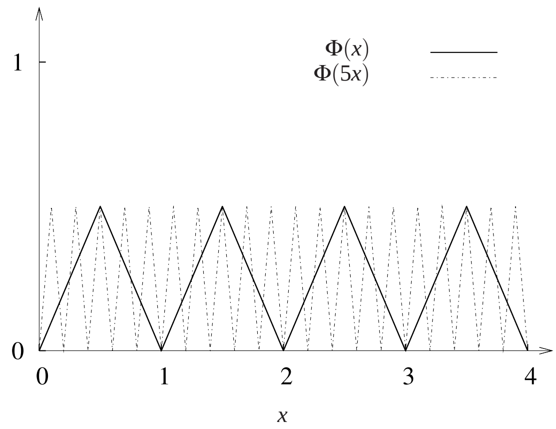

Figure 4.1: Sawtooth function $\Phi(x)$

Example 4.23. This example of a nowhere differentiable continuous function was presented by Van der Waerden. It is based on a series of sawtooth functions of decreasing period length.

Let

$$f(x) = \sum_{n=1}^{\infty} 10^{-n}\, \Phi(10^n x) \tag{11}$$

By Weierstrass M-test (see [5] page 377), we have that the infinite series (11) converge uniformly to f on $\mathbb{R}$, thus f is continuous on $\mathbb{R}$.

We shall now prove that f is nowhere differentiable. Fix any x, and let m be arbitrary positive number. Define

$$\delta_m = \pm\frac{1}{10}10^{-m},$$

where the sign is chosen such that Φ is linear between $10^m(x + \delta_m)$ and $10^m x$. This is possible since $|10^m\delta_m| = 1/10$. We also observe, that if $10^n\delta_m$ is an natural number, we can use the fact

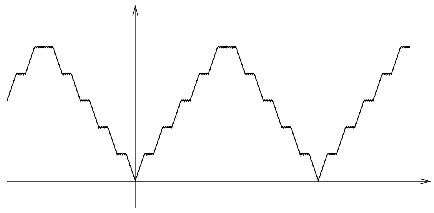

Figure 4.2: Van der Waerden nowhere differentiable function

that Φ is periodic, to obtain

$$\Phi(10^n(x+\delta_m)) - \Phi(10^n x) = 0, \tag{12}$$

whenever $n > m$. We conclude that

$$\begin{aligned}\Big|\frac{f(x+\delta_m)-f(x)}{\delta_m}\Big| &= \Big|\sum_{n=1}^{\infty}\frac{10^{-n}}{\delta_m}\Big(\Phi(10^n(x+\delta_m)) - \Phi(10^n x)\Big)\Big| \\ &= \Big|\sum_{n=1}^{m}\frac{10^{-n}}{\delta_m}\Big(\Phi(10^n(x+\delta_m)) - \Phi(10^n x)\Big)\Big| \\ &= \Big|\sum_{n=1}^{m}\frac{10^{-n}}{\delta_m}\Big(10^n\delta_m\Big)\Big| = \sum_{n=1}^{m} 1.\end{aligned}$$

Note that we used (12) to reduce the infinite series to a finite sum of m terms. By letting $m \to \infty$, we have that $\delta_m \to 0$. Thus f is not differentiable at x. Since x was arbitrary, it follows that f is nowhere differentiable.

The following theorem is a transcript from [3] pages 45-46.

Theorem 4.24. The set of all nowhere differentiable functions in $C[0,1]$ is dense.

Proof. To prove this we will consider the complement. That is, we will prove that the set of functions that have a finite derivative at some point, is a set of the first category.

Let E_n be the set of functions f, such that for some $0 < x < 1-1/n$, we have that

$$|f(x+h) - f(x)| \leq nh,$$

for every $0 < h < 1-x$. The union of all E_n will contain all functions having a finite right hand derivative at some point in $[0,1)$.

First, we need to show that E_n is closed. Let f be any function in the closure of E_n, and let $\{f_k\}$ be a sequence in E_n that converges to f. Since f_k is in E_n there exists a sequence of $\{x_k\}$, such that for each k,

$$\begin{gathered} 0 \leq x_k \leq 1-1/n \\ |f_k(x_k+h) - f_k(x_k)| \leq hn \end{gathered}$$

for all $0 < h < 1-x_k$. Since $\{x_k\}$ is a bounded sequence of real numbers, there exists a subsequence that converge to some $0 \leq x \leq 1-1/n$. For increased readability, we may assume that the index variable k carry this subsequence, thus

$$x_k \to x.$$

Observe, that if $0 < h < 1-x$, we have that $0 < h < 1-x_k$ holds for sufficiently large k. By repeated use of the triangle inequality, we derive

$$\begin{aligned}|f(x+h)-f(x)| \leq &|f(x+h)-f(x_k+h)|+|f(x_k+h)-f_k(x_k+h)|+\\ &+|f_k(x_k+h)-f_k(x_k)|+|f_k(x_k)-f(x_k)|+|f(x_k)-f(x)| \leq\\ \leq &|f(x+h)-f(x_k+h)|+d(f,f_k)+nh+d(f_k,f)+|f(x_k)-f(x)|.\end{aligned}$$

By letting $k \to \infty$, and using that fact that f is continuous at x and $x+h$, it follows that

$$|f(x+h)-f(x)| \leq nh.$$

Hence, by definition, f is in E_n, so E_n is closed.

Secondly, we need to show that E_n is nowhere dense. Recall that the set of all piecewise linear functions is dense in $C[0,1]$. By using the fact that E_n is closed, it is sufficient to show that, given a piecewise linear function, f, an arbitrary open ball $B(f,\varepsilon)$ will contain elements from $X \setminus E_n$.

We will use the sawtooth function Φ (see definition 4.22) to create a function in $B(f,\varepsilon)$, that can not be in E_n. Let M be the maximum slope of the piecewise linear function f, and let m be some number, such that

$$m\varepsilon > M+n$$

and let

$$g(x) = f(x)+\varepsilon\Phi(mx).$$

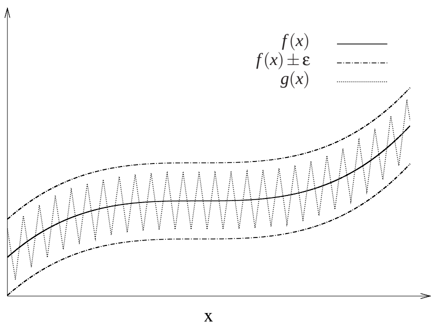

Figure 4.3: Functions $f(x)$ and $g(x)$

We see that $d(f,g) = \frac{1}{2}\varepsilon < \varepsilon$, so g is in the open ball $B(f,\varepsilon)$. But, since the term $f(x)$ at most have slope numerically equal to M, and $\varepsilon\phi(mx)$ has slope numerically equal to εm, which is by definition greater then $M+n$ for every x, it follows that, g will have right derivative greater than n for every x in $[0,1-1/n]$. Therefore g is not in E_n. Hence E_n is nowhere dense.

Now, we can conclude that the set of functions, with a right derivatives at some point, is a set of the first category. Analogous, we can repeat these steps for left derivatives. Thus, the set of nowhere differentiable functions is a residual set, and by Baire category theorem a dense set. □

Remark 4.25. It should be noted that each E_n in the previous proof, may contain nowhere differentiable functions. Thus, we can not say that complement of the union of all E_n contains all nowhere differentiable functions. What we do know, is that the set of all nowhere differentiable functions, in $C[0,1]$, contains a G_δ dense residual subset.

4.7 Lipschitz-Hölder continuous functions

Definition 4.26. A function, defined on $[0,1]$, is said to be *Lipschitz-Hölder continuous* or α-*Hölder* if for some $0 < \alpha \leq 1$, and for some constant A,

$$|f(x) - f(y)| \leq A|x-y|^{\alpha} \tag{13}$$

holds for every $x, y \in [0,1]$. If $\alpha = 1$, the function is said to be *Lipschitz continuous* and the constant A is called *Lipschitz constant*.

Definition 4.27. Let *Lip* α denote the the set of all Lipschitz-Hölder continuous functions for a fixed $0 < \alpha \leq 1$.

It follows directly from (13) that any Lip α function is continuous on $[0,1]$, thus Lip $\alpha \subset C[0,1]$. Before we proceed our investigation of the Lip α functions, we will need the following inequality.

Lemma 4.28. Let x, y be any non-negative real number and $0 < \alpha < 1$, then

$$(x+y)^{\alpha} \leq x^{\alpha} + y^{\alpha}. \tag{14}$$

Or equivalent, when $y > 0$, if we substitute $t = x/y$

$$(t+1)^{\alpha} \leq t^{\alpha} + 1. \tag{15}$$

Proof. Since (14) is reduced to an equality, if $y = 0$, we need only consider the case when $y > 0$. Equivalence between the two versions, follows from,

$$\begin{aligned} &(t+1)^{\alpha} \leq t^{\alpha} + 1 \\ &\Leftrightarrow (\frac{x}{y} + 1)^{\alpha} \leq (\frac{x}{y})^{\alpha} + 1 \\ &\Leftrightarrow (x+y)^{\alpha} \leq x^{\alpha} + y^{\alpha}. \end{aligned}$$

To prove the inequality (15), we define g as the right hand side minus the left hand side,

$$g(t) = t^{\alpha} + 1 - (t+1)^{\alpha}$$

Now, g is greater or equal to zero for all $t \geq 0$, since $g(0) = 0$, and g has a non-negative derivative for every $t > 0$. Indeed,

$$\begin{aligned} g'(t) &= \alpha t^{\alpha-1} - \alpha(t+1)^{\alpha-1} \\ &= \alpha t^{\alpha-1}\Big(1 - (1 + \frac{1}{t})^{\alpha-1}\Big) \\ &= \alpha t^{\alpha-1}\Big(1 - (\frac{1}{1+1/t})^{1-\alpha}\Big) > 0. \end{aligned}$$

Thus for $t > 0$, it follows that g is a non-negative function. So, the inequality (15) holds. □

Example 4.29. For a fix $0 < \alpha < 1$, the function $f(x) = x^{\alpha}$ is defined on $[0,1]$. It is not *Lipschitz continuous*, since it has an unbounded derivative as x tends to zero. However, it is Lip α, as we now prove. We note that f is monotonically increasing since $f'(x) > 0$ on $(0,1)$.

Let $0 \leq x < y \leq 1$, and let $h = y - x$, then

$$0 \leq |f(x) - f(y)| = (x+h)^{\alpha} - x^{\alpha} \leq h^{\alpha} = |x-y|^{\alpha},$$

by using the inequality from lemma (4.28). Thus, f is in Lip α.

Theorem 4.30. Lip α is a set of the first category in $C[0,1]$.

Proof. Let A_n be the set of continuous functions on the unit interval, such that

$$|f(x+h) - f(x)| \leq nh^{\alpha} \tag{16}$$

for any $x \in [0, 1-1/n]$ and for every $0 < h < 1/n$. Since any function in Lip α will be in some A_n for n sufficiently large, we have that the countable union of all A_n contains Lip α. Next, we will prove that A_n is closed.

Let f be in the closure of A_n, and let $\{f_n\}$ be a sequence of functions in A_n, converging to f. Thus for given $\varepsilon > 0$ there exists N, such that

$$n > N \Rightarrow d(f, f_n) < \varepsilon.$$

For any $x \in [0, 1-1/n]$ we have that

$$\begin{aligned} &|f(x+h) - f(x)| \leq \\ &\qquad \leq |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| + |f_n(x) - f(x)| \leq \\ &\qquad \leq 2\varepsilon + nh^{\alpha}, \end{aligned}$$

for every $0 < h < 1/n$. Since ε was arbitrary we have that f is in A_n. So, A_n is closed.

Finally, we will prove that A_n is nowhere dense. Since A_n is closed, it is sufficient to show that any open ball centered at f contains functions from the complement of A_n. Let $B(f, \varepsilon)$ be an open ball. Let

$$g(x) = f(x) + \varepsilon\Phi(mx)^{\alpha},$$

where we use the sawtooth function to the power of α, and let $m > 0$ be a real number, such that

$$\varepsilon m^{\alpha} = 2n. \tag{17}$$

Since $d(f, g) = \varepsilon(\frac{1}{2})^{\alpha} < \varepsilon$, we have that $g \in B(f, \varepsilon)$. Now, we show that g can not be in A_n, by proving that (16) does not hold for at least one point. Pick $x = 0$, and proceed by

$$|g(0+h) - g(0)| = \Big|(f(h) + \varepsilon\Phi(mh)^{\alpha}) - (f(0) + \varepsilon\Phi(0)^{\alpha})\Big| \geq \tag{18}$$

$$\geq \Big||f(h) - f(0)| - |\varepsilon\Phi(mh)^{\alpha}|\Big| = \tag{19}$$

$$= \Big||f(h) - f(0)| - \varepsilon|\Phi(mh)^{\alpha}|\Big|. \tag{20}$$

We now consider the two terms in the expression (20). If h is sufficiently small we have that Φ is linear, and using (17), we have

$$\varepsilon|\Phi(mh)^{\alpha}| = \varepsilon m^{\alpha}\, h^{\alpha} = 2n\, h^{\alpha}.$$

Also, since

$$|f(h) - f(0)| \leq nh^{\alpha},$$

we can deduce, that for h sufficiently small, we have

$$|g(0+h) - g(0)| > |nh^{\alpha} - 2nh^{\alpha}| = nh^{\alpha}.$$

Thus, g is not in A_n. So, A_n is nowhere dense. This proves that Lip α is a subset of a set of the first category in $C[0,1]$, and therefore a set of the first category. ☐

4.8 Hilbert sequence space l^p

Definition 4.31. Let $p \geq 1$ be a fixed real number. Each element of the space l^p is a sequence $x = \{x_n\}$ of numbers, such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty. \tag{21}$$

The metric is induced by the norm

$$\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p}.$$

It can be proved that l^p is a complete metric space (see [2] pages 11-15 for details). Since the l^p metric space consists of convergent series we will frequently use inequalities of series and convergence tests when dealing with objects in a sequence space. We continue this chapter by stating some inequalities that we need to further study the sequence space. The following inequality reveals that $l^p \subset l^q$.

Theorem 4.32 (Jensen's inequality)**.** Let $1 \leq p < q < \infty$, then for any real valued sequence x_n, we have

$$(\sum_{n=1}^{\infty} |x_n|^q)^{1/q} \leq (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}. \tag{22}$$

or by substituting and $r = p/q$ and $y_n = x_n^q$, we get

$$(\sum_{n=1}^{\infty} |y_n|)^r \leq \sum_{n=1}^{\infty} |y_n|^r, \tag{23}$$

Proof. We use induction to prove the inequality (23). Let $k = 1$, then

$$|y_1| = (\sum_{n=1}^{1} |y_n|)^r \leq \sum_{n=1}^{1} |y_n|^r = |y_1|.$$

Now assume that the inequality holds for some k. By repeated use of the inequality from lemma (4.28), it follows

$$(\sum_{n=1}^{k+1} |y_n|)^r = (\sum_{n=1}^{k} |y_n| + |y_{k+1}|)^r \leq \sum_{n=1}^{k} |y_n|^r + |y_{k+1}|^r = \sum_{n=1}^{k+1} |y_n|^r.$$

This satisfies our induction hypothesis, thus the inequality holds for any k. □

Finally, we will also need the *Minkowski inequality* for sums.

$$\Big(\sum_{n=1}^{\infty} |x_n + y_n|^p\Big)^{1/p} \leq \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p} + \Big(\sum_{n=1}^{\infty} |y_n|^p\Big)^{1/p},$$

or if we express this inequality in term of the norm, we get the triangle inequality

$$\|x - y\|_p \leq \|x\|_p + \|y\|_p.$$

The proof of this inequality can be found in ([2] pages 14-15).

Example 4.33. Let l^p and l^q be sequence spaces, where $p < q$. Show that l^p is a proper subset of l^q.

We already know that l^p is a subset of l^q. To see that l^p is a proper subset, we need to find a convergent sequence in l^q that diverge in l^p. Let

$$x = \{\frac{1}{n^{1/p}}\}.$$

It follows that

$$\sum_{n=1}^{\infty}(\frac{1}{n^{1/p}})^p = \sum_{n=1}^{\infty}\frac{1}{n},$$

which is the form of the divergent harmonic series. On the other hand

$$\sum_{n=1}^{\infty}(\frac{1}{n^{1/p}})^q = \sum_{n=1}^{\infty}\frac{1}{n^{q/p}} < \infty,$$

where $q/p > 1$, which is a convergent p-harmonic series. Thus x is in l^q but not l^p.

Example 4.34. Given $1 \leq p < q < \infty$ and the corresponding sequence spaces l^p and l^q. Show that l^p is a set of the first category in l^q.

To prove this, we will show that l^p can be expressed as a countable union of nowhere dense sets. Let A_n be a subset of l^p, defined by

$$A_n = \{x \in l^p : \|x\|_p \leq n\}.$$

Since every point in l^p will be in some A_n, l^p is equal to the countable union of all A_n. First, we show that A_n is a closed set in l^q. Let a be in the closure of A_n. Let $\{a_k\}$, from A_n, be a sequence, such that

$$\|a_k - a\|_q \to 0.$$

By definition of the l^q norm, we have that for every $i = 1, 2\dots$

$$|a_k^{(i)} - a^{(i)}| \leq \|a_k - a\|_q,$$

thus

$$\lim_{k\to\infty} a_k^{(i)} = a^{(i)}. \tag{24}$$

Since a_k is in A_n, we have that for every number N

$$\sum_{i=0}^{N} |a_k^{(i)}|^p \leq n^p.$$

By letting $k \to \infty$, we get by (24)

$$\sum_{i=0}^{N} |a^{(i)}|^p \leq n^p.$$

And since this holds for any number N, we can pass N to infinity,

$$\sum_{i=0}^{\infty} |a^{(i)}|^p \leq n^p.$$

Thus $a \in l^p$ and $a \in A_n$. So A_n is closed.

Secondly, we show that A_n is nowhere dense in l^q. Since A_n is closed it is sufficient to show that any open ball, $B(a,r)$ in l^q contains points from A_n^c. If $a \in A_n^c$ this follows immediately, thus we only have to consider the case when $a \in A_n$.

Let $b = \xi - a$, where $\xi \in l^q \setminus l^p$, such that

$$\|\xi\|_q = r/2.$$

Then, $\|b-a\|_q = \|\xi\|_q = r/2 < r$. Thus b is in $B(a,r)$. However, b is not in A_n, since that would imply that ξ must be in l^p, by

$$\|\xi\|_p \leq \|\xi - a\|_p + \|a\|_p = \|b\|_p + \|a\|_p.$$

So, A_n is a nowhere dense subset of l^q for every n, thus l^p is a set of the first category in l^q.

Chapter 5

Some basic principles of functional analysis

Baire category theorem has various applications in functional analysis. In this chapter we will examine three theorems of great importance, which are derived from the Baire category theorem.

5.1 Open mapping theorem, Banach theorem

Definition 5.1 (Open mapping)**.** Suppose that Λ maps X into Y, where X and Y are metric spaces. If $\Lambda(A)$ is open for every open set $A \subset X$, then we say that Λ is an *open mapping*.

We know that, for continuous mappings, the inverse images of open sets are open, but that the image of open sets need not be open for continuous mapping. The next theorem, the Open mapping theorem, states conditions under which bounded linear operators are open mappings. We first need to present two lemmas.

Lemma 5.2. Let X be a Banach space and Y be a normed space. If $\Lambda : X \to Y$ is a bounded linear operator, and if $\Lambda(x)$ is of second category in Y, then there exists an open ball $B_Y(0,\varepsilon)$ in Y, such that

$$B_Y(0,\varepsilon) \subset \overline{\Lambda B_X(0,1)}$$

Proof. We know that X can be expressed as the union of open balls of increasing radius, by

$$X = \bigcup_{k=1}^{\infty} k\, B(0,\frac{1}{2}).$$

Thus, by linearity of Λ, we have

$$\Lambda(X) = \bigcup_{k=1}^{\infty} k\, \Lambda(B(0,\frac{1}{2})).$$

As $\Lambda(X)$, is of the second category, at least one $k\, \Lambda(B(0,\frac{1}{2}))$ is dense in some open ball. Since $y = kx$ is a homeomorphism from Y onto itself, we have that $\Lambda B(0,\frac{1}{2})$ is dense in some open ball. Thus, there exists $B_Y(y,\varepsilon)$, such that

$$B_Y(y,\varepsilon) \subset \overline{\Lambda(B(0,1/2))}.$$

By the invariance of the metric, we can move this open ball to be centered about 0.

$$B_Y(0,\varepsilon) = B_Y(y,\varepsilon) - y \subset \overline{\Lambda(B(0,1/2))} - y.$$

Let $y_0 \in B_Y(0,\varepsilon)$, then it follows that $y_0 + y \in \overline{\Lambda(B(0,1/2))}$ and $y \in \overline{\Lambda(B(0,1/2))}$. Thus there exists two sequences a_n, b_n in $B(0,1/2)$, such that

$$\Lambda a_n \to y_0 + y$$
$$\Lambda b_n \to y.$$

Since $\|a_n - b_n\|_X < 1/2 + 1/2 = 1$, we have that $a_n - b_n \in B(0,1)$, and by linearity of Λ we have

$$\Lambda(a_n - b_n) = \Lambda a_n - \Lambda b_n \to y_0 + y - y = y_0$$

As, y_0 was arbitrary, and since we have found a sequence in $B(0,1)$ whose image converge to y_0, it follows that

$$y_0 \in \overline{\Lambda B_X(0,1)}.$$

Therefore $\overline{\Lambda B_X(0,1)}$ contains the open ball $B_Y(0,\varepsilon)$. □

Lemma 5.3. Let X be a Banach space and Y be a normed space, and let $\Lambda : X \to Y$ be a bounded linear operator. Suppose there exists an open ball $B_Y(0,\varepsilon)$ in Y, such that

$$B_Y(0,\varepsilon) \subset \overline{\Lambda B_X(0,1)}$$

then

$$B_Y(0,\varepsilon) \subset \Lambda B_X(0,2).$$

Proof. We need to prove that for any $y \in B_Y(0,\varepsilon)$, y is contained in $\Lambda B_X(0,2)$. Let $U = \Lambda B_X(0,1)$, and let $y_0 \in U$, such that

$$\|y - y_0\| < \varepsilon\frac{1}{2}, \tag{1}$$

Such y_0 exists in U, since every point in $B_Y(0,\varepsilon)$ can be arbitrary close, approximated by points from U. Also, there exists a $x_0 \in B_X(0,1)$ such that $\Lambda x_0 = y_0$. By (1), we have

$$y - y_0 \in \frac{1}{2}\overline{U},$$

and by linearity of Λ, we have that

$$\frac{1}{2}U = \frac{1}{2}\Lambda B(0,1) = \Lambda B(0,\frac{1}{2}).$$

Choose $y_1 \in \frac{1}{2}U$ and $x_0 \in B_X(0,\frac{1}{2})$ such that $\Lambda x_1 = y_1$, and

$$\|y - y_0 - y_1\| < \varepsilon\frac{1}{4},$$

this means that $y - y_0 - y_1 \in \frac{1}{4}\overline{U}$. We can continue this process to arrive at two sequences $\{x_n\}$ and $\{y_n\}$, such that

$$y_n \in \frac{1}{2^n}U \quad ; \quad x_n \in B_X(0,\frac{1}{2^n})$$
$$y_n = \Lambda x_n$$
$$\|y - y_0 - \cdots - y_n\| < \varepsilon\frac{1}{2^n}.$$

Let $z_n = x_0 + \cdots + x_n$. Since $\|x_n\| < 1/2^n$, we have that z_n is a Cauchy sequence in the complete metric space X, thus have a limit in X,

$$z_n \longrightarrow z.$$

Moreover, $z \in B_X(0,2)$, since

$$\|z\| = \| \sum_{n=0}^{\infty} x_n \| \leq \sum_{n=0}^{\infty} \|x_n\| < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Finally, we need to show that $y = \Lambda z$. Consider

$$\begin{aligned} \|y - \Lambda z_m\| = \|y - \Lambda \sum_{n=0}^{m} x_n\| = \|y - \sum_{n=0}^{m} \Lambda x_n\| = \\ = \|y - \sum_{n=0}^{m} y_n\| \leq \varepsilon \frac{1}{2^m}, \end{aligned}$$

by passing $m \to \infty$, we have $\|y - \Lambda z\| = 0$. Thus $y = \Lambda z$, and since y was arbitrary point from $B_Y(0,\varepsilon)$, we have that

$$B_Y(0,\varepsilon) \subset \Lambda B_X(0,2).$$

□

Theorem 5.4 (Open mapping theorem)**.** Let X be a Banach space and Y a normed space, and let $\Lambda : X \to Y$ be a linear bounded operator. If $\Lambda(X)$ is of the second category in Y, then Λ is an open mapping.

Proof. Let A be any open set in X. We prove the theorem by showing that for any $y = \Lambda x \in \Lambda(A)$ the set $\Lambda(A)$ contains an open ball about $y = \Lambda x$. Let $B(x,r) \subset A$. By, invariance of the norm, we have

$$B(0,1) = \frac{1}{r}(B(x,r) - x).$$

By lemma 5.2 and 5.3, there exists an open ball $B_Y(0,\varepsilon)$, such that

$$B_Y(0,\varepsilon) \subset \Lambda(B(0,1)),$$

thus

$$\begin{aligned} & B_Y(0,\varepsilon) \subset \frac{1}{r}\Lambda(B(x,r) - x) \\ \Leftrightarrow\ & B_Y(0,r\varepsilon) \subset \Lambda B(x,r) - \Lambda x \\ \Leftrightarrow\ & B_Y(y,r\varepsilon) \subset \Lambda B(x,r). \end{aligned}$$

It follows that y is an interior point of $\Lambda(A)$. And since y was arbitrary, it follows that $\Lambda(A)$ is open in Y. □

Corollary 5.5. If we, in addition to the Open mapping theorem, require Λ to be one-to-one. Then Λ^{-1} is a continuous linear operator.

Proof. By assumption Λ is both onto and one-to-one, thus the inverse exists and is a linear operator. Since Λ is an open map it follows that the inverse is continuous. □

Theorem 5.6 (Banach theorem)**.** Let X be a Banach space and Y a normed space, and let $\Lambda : X \to Y$ be a linear bounded operator. If $\Lambda(X)$ is of the second category in Y, then $\Lambda(X) = Y$

Proof. As in the proof of lemma 5.2, we know we can express $\Lambda(X)$ as

$$\Lambda(X) = \bigcup_{k=1}^{\infty} k\, \Lambda(B(0,1)),$$

and since $\Lambda(X)$ is of the second category, we have that $\overline{\Lambda(B(0,1)}$ contains an open ball $B_Y(0,\varepsilon)$. Lemma 5.2 ensures that this ball can be centered about 0, and lemma 5.3 implies that

$$B_Y(0,\varepsilon) \subset \Lambda(B(0,2).$$

This means that we can express Y as

$$Y = \bigcup_{k=1}^{\infty} kB_Y(0,\varepsilon) \subset \Lambda\Big(\bigcup_{k=1}^{\infty} k\,(B(0,2)\Big) = \Lambda(X).$$

But, by assumption, $\Lambda(X)$ is a subset of Y. It follows that $\Lambda(X) = Y$. □

Remark 5.7. Another approach (see Kreyszig [2]), when presenting the Banach theorem is to assume that $\Lambda(X)$ is onto Y, where Y is a Banach space. This assumption allow us to apply the Baire category theorem on the complete metric space $\Lambda(X)$ to find the open ball $B_Y(0,\varepsilon)$.

5.2 Closed graph theorem

Definition 5.8 (Closed graph)**.** Let X and Y be normed spaces, and let Λ be a linear operator, mapping X into Y. Then Λ is a *closed linear operator* if the *graph* of Λ

$$G(\Lambda) = \{(x,\Lambda x) : x \in X\},$$

is a closed set in the normed space $X \times Y$.

Theorem 5.9 (Closed graph theorem)**.** Let X and Y be Banach spaces and Λ be a closed linear operator from X into Y, then Λ is a bounded linear operator.

Proof. Since X and Y are complete normed spaces, so is $X \times Y$, and since G is a closed set in $X \times Y$, we have that G is complete. We define two projections onto X and Y respectively

$$\begin{aligned} \pi_X : \quad G(\Lambda) &\longrightarrow X \\ (x,\Lambda x) &\longmapsto (x) \end{aligned}$$

$$\begin{aligned} \pi_Y : \quad X \times Y &\longrightarrow Y \\ (x,y) &\longmapsto (y) \end{aligned}$$

Both π_X and π_Y are bounded, since

$$\|\pi_X\| \leq \frac{\|\pi_X((x,\Lambda x))\|}{\|(x,\Lambda x)\|} = \frac{\|x\|}{\|(x,\Lambda x)\|} \leq \frac{\|x\| + \|\Lambda x\|}{\|(x,\Lambda x)\|} = \frac{\|(x,\Lambda x)\|}{\|(x,\Lambda x)\|} = 1,$$

holds for any $x \in X$. Similar for π_Y, by

$$\|\pi_Y\| \leq \frac{\|\pi_Y(x,y)\|}{\|(x,y)\|} = \frac{\|x\|}{\|(x,y)\|} \leq \frac{\|x\| + \|y\|}{\|(x,y)\|} = \frac{\|(x,y)\|}{\|(x,y)\|} = 1.$$

Since, π_X is a bijective mapping the inverse, π_X^{-1}, exists and is a linear operator,

$$\begin{aligned} \pi_X^{-1}: \quad X &\longrightarrow G(\Lambda) \\ (x) &\longmapsto (x, \Lambda x). \end{aligned}$$

Moreover, G is complete, so π_X is an open mapping (by the Open mapping theorem), and thus the inverse is a bounded linear operator. Finally, we see that

$$\Lambda = \pi_Y \circ \pi_X^{-1},$$

where the product of two bounded linear operators is bounded, it follows that Λ is bounded. ☐

Remark 5.10. If the domain of Λ is not the whole of X, we can still use the Closed graph theorem, but we need the additional assumption that the domain of Λ is closed in X.

Example 5.11. An example of an unbounded linear operator whose graph is closed is the differential operator on $C[0,1]$. The proof of this can be found in ([2] pages 294). A consequence of the closed graph theorem is that the domain of the differential operator can not be closed in $C[0,1]$.

5.3 Uniform boundedness theorem

Theorem 5.12 (Uniform boundedness theorem)**.** Let $\{\Lambda_n\}$ be a sequence of bounded linear operators from a Banach space X into a Banach space Y. Suppose that the sequences $\{\|\Lambda_n x\|\}$ are bounded for every x, in some set X_0 of the second category in X, by some real valued constant c_x

$$\|\Lambda_n x\| \leq c_x \text{ for every } n = 1, 2, \ldots$$

Then the sequence $\{\|\Lambda_n\|\}$ is bounded.

Proof. Let

$$A_k = \{\, x : \|\Lambda_n x\| \leq k \quad \text{for every } n = 1, 2, \ldots \}.$$

First we show that A_k is closed. Let $a \in \overline{A_k}$ and $a_i \in A_k$, such that a_i converge to a. Fix n. We have that Λ_n is continuous, thus

$$\|\Lambda_n a_i\| \to \|\Lambda_n a\|,$$

and since $\|\Lambda_n a_i\| \leq k$, we have that $\|\Lambda_n a\| \leq k$, it follows that $a \in A_k$. Secondly, since every x in X_0 will be in some A_k, we have

$$X_0 = \bigcup_{k=1}^{\infty} A_k.$$

Since, X_0 is of the second category in X, at least one A_k contain an open ball $B(x_0, r)$. Let x be arbitrary point in X, and define

$$z = \frac{x}{\|x\|} \frac{r}{2} + x_0.$$

From this we obtain

$$\frac{x}{\|x\|} = \frac{2}{r}(z - x_0).$$

We see that $z \in B(x_0, r)$. Now, boundedness of Λ_n follows from

$$\|\Lambda_n\| \leq \|\Lambda_n \frac{x}{\|x\|}\| = \|\Lambda_n(\frac{2}{r}(z - x_0))\| = \frac{2}{r}\|\Lambda_n z - \Lambda_n x_0\| \leq \frac{4}{r}k,$$

since this yields for any n, and x was arbitrary point in X. □

5.4 Examples

Given a set X and some subset Y we can determine if Y is a set of the first category by trying to explicitly express Y as a countable union of nowhere dense sets. We have throughout this thesis seen examples of this technique; in (4.34) we showed that $l^p \subset l^q$ is a subset of the first category; in (4.30) we showed that Lip α is a subset of the first category in $C[0,1]$.

We now add another example to this list. The Lebesgue spaces, L^p on the unit interval, where $1 \leq p < \infty$ can be studied with respect to category. The norm is defined as

$$\|f\|_p = \Big(\int_0^1 |f(t)|^p dt\Big)^{1/p}.$$

Example 5.13. Let L^1 and L^2 be Lebesgue spaces on the unit interval. Show that L^2 is a subset of the first category in L^1.

Let

$$A_n = \{f : \|f\|_2 \leq n\}.$$

Every function in L^2 will be in some A_n, thus the union of all A_n is L^2. We first prove that A_n is closed. Let $f \in \overline{A_n}$ and $f_k \in A_n$, such that

$$\|f - f_k\|_1 \to 0.$$

It follows that $\{f_k\}$ converges in measure to f. By Riesz Theorem, there exists a subsequence $\{f_{k_j}\}$ converging to f almost everywhere. We have

$$\int_0^1 f_{k_j}(t)^2 dt \leq n^2,$$

for every $j = 1, 2, \ldots$. Applying Fatou's Lemma, we obtain

$$\int_0^1 f(t)^2 dt \leq n^2.$$

Thus $f \in A_n$. Hence, A_n is closed. Next, we show that A_n is nowhere dense. Let $B(f, \varepsilon)$ be any open ball in L^1. As A_n is closed it is sufficient to find one point in $B(f, \varepsilon)$ not in A_n to prove that A_n is nowhere dense. Let

$$h(t) = \frac{1}{\sqrt{t}}\frac{\varepsilon}{4}.$$

We know that $h \in L^1$ and $\|h\|_1 = \varepsilon/2$, and that $h \notin L^2$. Let $g = f + h$, then

$$\|f - g\|_1 = \|h\|_1 = \varepsilon/2,$$

Thus, $g \in B(f, \varepsilon)$, but $g \notin A_n$. So, A_n is nowhere dense. It follows that L^2 is of the first category.

Another approach to determine if a subset X is of the first or the second category in some set Y, is to use the Uniform boundedness theorem.

Example 5.14. Show that L^2 is a subset of the first category in L^1.

Set

$$g_n(x) = n\,\chi_{[0,n^{-2}]}(x).$$

Then $\|g_n\|_2 = 1$. We now define a sequence of linear functionals on L^1, as follows

$$\Lambda_n f = \int_0^1 g_n(t) f(t) dt.$$

For any $f \in L^1$, we have that

$$|\Lambda_n f| = \left| \int_0^{n^{-2}} n f(x) dx \right| \le n \int_0^{n^{-2}} |f(x)| dx \le n \|f\|_1.$$

Thus, Λ_n is a bounded, linear functional on L^1. For any $f \in L^2$, it follows from Cauchy's inequality, that

$$|\Lambda_n f| \le \|f\|_2 \|g_n\|_2 = \|f\|_2.$$

for every $n = 1,2,\ldots$. Thus $\{|\Lambda_n f|\}$ is bounded at any point $f \in L^2$. If we assume that L^2 is of the second category in L^1, then by the Uniform boundedness theorem, $\|\Lambda_n\|$ is bounded. This means that there exists a number $C > 0$, such that

$$|\Lambda_n f| \le C \|f\|_1, \tag{2}$$

for every $n = 1,2,\ldots$ and for every $f \in L^1$. Now we define

$$f^{(n)}(x) = n^2\,\chi_{[0,n^{-2}]}(x),$$

it follows that $\|f^{(n)}\|_1 = 1$ for every $n = 1,2,\ldots$. However, applying

$$\Lambda_n f^{(n)} = n^3 \frac{1}{n^2} = n,$$

we have that $\Lambda_n f^{(n)} \to \infty$, in contradiction with (2). So, L^2 can not be of the second category, hence L^2 is a subset of the first category in L^1.

Example 5.15. Show that l^1 is a subset of the first category in l^2.

Define a sequence of linear functionals on l^2, as follows

$$\Lambda_n x = \sum_{i=1}^{n} x_i.$$

By the inequality of algebraic mean and quadratic mean, it follows that

$$|\Lambda_n x| \le \sqrt{n} \sqrt{\sum_{i=1}^{n} x_i^2} \le \sqrt{n} \|x\|_2.$$

Thus, each Λ_n is a bounded linear function on l^2. For any $x \in l^1$ we have

$$|\Lambda_n x| = \sum_{i=1}^{n} x_i \leq \|x\|_1.$$

Thus $\{|\Lambda_n x|\}$ is bounded at any point $x \in l^1$. If we assume that l^1 is of the second category in l^2, then by the Uniform boundedness theorem, $\|\Lambda_n\|$ is bounded. This means that there exists a number $C > 0$, such that

$$|\Lambda_n x| \leq C\|x\|_2, \tag{3}$$

for every $n = 1, 2, \dots$ and for every $x \in l^2$. Now we define

$$x^{(n)} = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}, 0, \dots),$$

such that $\|x^{(n)}\|_2 = 1$ for every $n = 1, 2, \dots$. But

$$\Lambda_n x^{(n)} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

It follows that $\Lambda_n x^{(n)} \to \infty$, in contradiction with (3). So, l^1 can not be of the second category, hence l^1 is a subset of the first category in l^2.

Also, the Open mapping theorem can be used in a similar way. Given a normed space X and Banach space Y where $X \subset Y$, and if there exists a bounded linear open mapping I from X into Y, such that the mapping is not onto. Then by the Open mapping theorem, $I(X)$ must be of the first category in Y.

Example 5.16. Show that L^q is a subset of the first category in L^p on the unit interval, when $1 \leq p < q < \infty$.

Since L^q is continuously embedded in L^p, there exists a real valued constant C, such that

$$\|f\|_p \leq C\|f\|_q,$$

for every $f \in L^q$. Let I be the inclusion map,

$$\begin{aligned} I: \quad & L^q \longrightarrow L^p \\ & (f) \longmapsto (f). \end{aligned}$$

Since, L^q is a proper subset of L^p, we have that I is not onto. Also, I is bounded, since

$$\|I\| = \sup_{f \in L^q} \frac{\|If\|_p}{\|f\|_q} = \sup_{f \in L^q} \frac{\|f\|_p}{\|f\|_q} \leq \sup_{f \in L^q} \frac{C\|f\|_q}{\|f\|_q} = C.$$

So, by the Open mapping theorem, $L^q = I(L^q)$ can not be of the second category, thus L^q is of the first category in L^p.

Example 5.17. Show that l^p is a subset of the first category in l^q where $1 \leq p < q < \infty$.

We have seen that l^p is a proper subset of l^q. Also, we recall that the norms are such that

$$\|x\|_p \geq \|x\|_q,$$

for all $x \in l^p$. Let I be the inclusion map,

$$\begin{aligned} I: \quad l^p &\longrightarrow l^q \\ (x) &\longmapsto (x). \end{aligned}$$

Since, l^p is a proper subset of l^q, we have that I is not onto. Also, I is bounded, since

$$\|I\| = \sup_{x \in l^p} \frac{\|Ix\|_q}{\|x\|_p} = \sup_{x \in l^p} \frac{\|x\|_q}{\|x\|_p} \le \sup_{x \in l^p} \frac{\|x\|_p}{\|x\|_p} = 1.$$

So, by the Open mapping theorem, $l^p = I(l^p)$ can not be of the second category, thus l^p is of the first category in l^q.

The last two examples can be generalized into the following theorem, by realizing that the norm have certain properties.

Theorem 5.18. Let X, Y be Banach spaces, where $X \subset Y$. Assume that the norms are such that

$$\|x\|_Y \le \|x\|_X,$$

for every $x \in X$, and for every real number $A > 0$, there exists $x \in X$, such that

$$A\|x\|_Y < \|x\|_X. \tag{4}$$

Then X is a subset of the first category.

Proof. Suppose that X is of the second category. Let I be the inclusion map

$$I : X \to Y.$$

I is one-to-one. Also, I is bounded, since

$$\|I\| = \sup_{x \in X} \frac{\|Ix\|_Y}{\|x\|_X} = \sup_{x \in X} \frac{\|x\|_Y}{\|x\|_X} \le \sup_{x \in X} \frac{\|x\|_X}{\|x\|_X} = 1.$$

By the Banach theorem, we have that $I(X) = Y$. Hence I is onto. Therefore the inverse, I^{-1}, exists and is a linear operator. By the Open mapping theorem, I^{-1} is bounded. Then for any $x \in X$, we have

$$\begin{aligned} \|x\|_X = \|I^{-1}x\|_X \le \|I^{-1}\| \; \|x\|_X \le \\ \le \|I^{-1}\| \; \|x\|_Y. \end{aligned}$$

Let $A = \|I^{-1}\| + 1$. Then according to (4) there exists some $x \in X$, such that

$$(\|I^{-1}\| + 1)\|x\|_Y \; < \; \|x\|_X \; \le \; \|I^{-1}\|\|x\|_Y.$$

But this means that,

$$\|x\|_Y < 0,$$

which is impossible. The assumption that X is of the second category is false, thus X is of the first category. ☐

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