Design of truss structures with multiple eigenfrequency constraints via rank minimization
Anton Tkachuk, Mykola M. Tkachuk

Keywords: Truss structures, Eigenvalue constraints, Dynamic stiffness matrix, Rank minimization heuristics, Semi-definite programming

Abstract
Rank deficiency of the dynamic stiffness matrix is an indicator for resonance of a structure at a given frequency. This indicator can be exploited as a heuristic optimization objective to achieve resonance at several frequencies. Log-det heuristic provides a tractable surrogate function for matrix rank in the case of affine dependency of stiffness and mass matrices on design parameters, which applies to truss structures. Reducing the rank of the dynamic stiffness matrix for higher frequencies implies that the matrix is not semi-positive definite. For this case, the log-det heuristic is valid with a combination of interior-point methods and Fazel’s semi-definite embedding via linear matrix inequalities. Further constraints on the fundamental frequency and compliance can be easily added within the framework as linear matrix inequalities. Several successful numerical examples illustrate the performance of the approach.

1. Introduction
Design of truss structures with multiple eigenfrequency constraints is a well-known problem [18,11]. Usually, the total mass of the structure is minimized under equality or inequality constraints on the few lowest eigenfrequencies. Design variables are the cross-sectional areas of individual members and, sometimes, locations of the nodes. One type of relevant application is ensuring a sufficient distance between eigenfrequencies of a structure and the frequencies of the external periodic forcing, which is often associated with resonance avoidance. Design of a support structure for a rotating machine with an imbalance would be a typical engineering example. Another type of relevant application is ensuring resonance for amplification of amplitudes. The majority of ultrasonic fatigue testing machines [3] and ultrasound knives [5] use one of the longitudinal resonant modes of their structures. Operating these machines at a few distinctive frequencies requires resonance at these frequencies. Furthermore, repeated resonances can be beneficial to design a complex motion for the tip of the knife [37]. This contribution is dedicated to the latter type of application.

Classical approaches for truss structure design with multiple eigenfrequency constraints use gradient-based methods with accurate sensitivities of eigenfrequencies [18,11]. These approaches require eigenfrequency computation, sensitivities for eigenfrequencies, mode tracking and mode ordering. Moreover, obtained truss designs are often local minima as the problem is highly nonlinear and non-convex [17]. Therefore, numerous heuristic approaches are proposed with genetic algorithms [20], particle swarm algorithms [10,17] or teaching-learning based optimization [30,2] being few non-exhaustive examples. These heuristic approaches are often expensive and require a large number of forward analyses for convergence. Furthermore, they lack a solid mathematical background compared to semi-definite programming (SDP) approaches known for optimizations with constraints on the fundamental frequency [27,1]. Therefore, it is reasonable to formulate constraints on higher eigenfrequencies in the context of SDP and rank minimization.

Methods for numerical rank computation rely on LU or QR factorizations with and without column pivoting [9,6,15,13]. Several applications for numerical rank computations are listed in [13]. Independently, rank minimization formulations gained substantial interest in the last 30 years for various problems in engineering and mathematics. Initially, applications in optimal control and finding lowest-dimension embedding of points in Euclidean space from noisy distance data were demonstrated in [24,8]. Further cases include low-rank matrix completion [25,19], image denoising [23,14] and customization of finite element discretizations for lowest dispersion error [31]. Recently, design problems in structural mechanics were solved via rank

* Corresponding author.
E-mail address: anton.tkachuk@kau.se (A. Tkachuk).

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minimization, e.g., for discrete periodic structures [33] and tensegrity structures [36,32].

Rank minimization approaches are well established for an affine dependency of matrix on free parameters. Common rank minimization approaches use tractable relaxations of the matrix rank. Various relaxations are valid depending on the matrix type. For a symmetric semi-definite matrix, trace of the matrix is the simplest valid relaxation of rank [24,36]. The logarithm of the determinant of the matrix (log-det) is valid for symmetric semi-definite square and general rectangular matrices [8]. This more general relaxation requires a trick in the case of rectangular matrices $\mathbb{R}^{m \times n}$ or square matrices with negative eigenvalues $\mathbb{R}^{n \times n}$. They are embedded in symmetric semi-definite square matrices with dimension $\mathbb{R}^{(n+m) \times (n+m)}$ and $\mathbb{R}^{2n \times 2n}$, respectively. Log-det approach proved to be efficient for low- and high-rank matrices and large dimensions, and it is used below. Further relaxation functions for rectangular matrices include nuclear norm [7,28] and Schatten $p$-norm [21], which prove to be very efficient for low-rank matrices. The latter two relaxation functions are related to the iterative re-weighted least squares approach [25,19] or Bregman iterative method [23].

In this contribution, we consider a problem of cross-section sizing for a truss structure with multiple equality constraints on higher eigen-frequencies and inequality constraints on the fundamental eigen-frequency, compliance, and total volume. This problem is solved via the log-det heuristic initially proposed for affine matrix rank minimization in [8]. This approach reduces the original design problem to a series of convex semi-definite optimization problems solved by the interior-point method. On the one hand, the proposed formulation extends semi-definite approaches, e.g. given in [27] to multiple frequency matching objective. On the other hand, the proposed formulation provides an alternative to standard gradient-based approaches that require eigenfrequency ordering and eigenmode tracking. The following structure is adopted in the contribution. The design procedure is described in Section 2. The implementation details for the design procedure are given in Section 3. Three illustrative numerical examples are presented in Section 4. Conclusions are given in Section 5.

2. Design of procedure

2.1. Problem statement for design of truss structures with multiple eigenfrequency constraints

Given is a truss structure with fixed topology and with the mass and stiffness matrices $(\mathbf{M}, \mathbf{K}) \in \mathbb{R}^{n \times n}$ affinely depending on cross-sectional areas $\mathbf{s} \in \mathbb{R}^p$

$$\mathbf{K}(\mathbf{s}) = \mathbf{K}_0 + \sum_{i=1}^{p} \mathbf{K}_i s_i, \quad \mathbf{M}(\mathbf{s}) = \mathbf{M}_0 + \sum_{i=1}^{p} \mathbf{M}_i s_i. \quad (1)$$

Here, $n$ and $p$ are the number of degrees of freedom and truss members in the structure, respectively. The mass matrix $\mathbf{M}_0$ represents non-structural point masses connected to some nodes and the matrix $\mathbf{K}_0$ represents the part of the stiffness independent of the parameters. Assume that the admissible designs $\mathbf{s}$ belong to a convex set $\mathcal{G} \subset \mathbb{R}^p$; the total volume of the structure is limited from above by $\tilde{V}$ and each cross-sectional area has a strictly positive lower limit $\underline{s}_j$ and an upper limit $\overline{s}_j$, i.e.

$$\mathcal{G} = \{ \mathbf{s} : \sum_{i=1}^{p} L_i s_i \leq \tilde{V}, \underline{s}_j \leq s_j \leq \overline{s}_j, j = 1, p \}. \quad (2)$$

where $L_i$ is the length of $i^{th}$ truss. Angular frequencies and corresponding eigenvectors of the truss structure are denoted with $\omega_j$ and $\phi_j$, respectively

$$\mathbf{K}(\mathbf{s}) - \omega_j^2 \mathbf{M}(\mathbf{s}) \phi_j = 0. \quad (3)$$

The ordered list of eigenfrequencies forms the spectrum of the structure $\{\omega_j\}_{j=1}^n$. A single static load case is given by external load vector $\mathbf{p}$. We seek a realization $s^* \Gamma$ from a convex admissible set $\mathcal{G}$ that satisfies three conditions:

C1 each of $q$ target frequencies $\omega_i^* \Gamma$ belongs to the spectrum of the structure $\{\omega_j\}_{j=1}^n$ (the order to which a target frequency match and multiplicity of any frequency is not important)

C2 the fundamental frequency of the structure is greater or equal to the lower bound $\omega$

C3 the compliance for the given load vector is less or equal to the upper bound $\mathcal{F}$.

The problem statement above imposes one inequality (C2) and multiple equality constraints (C1) on the spectrum. This problem statement does not put constraints on the stress level in the members due to plastic yield, local buckling or global buckling. Furthermore, flexural vibrations of the pinned trusses as beams or damping are disregarded.

Squares of angular eigenfrequencies are used below for conciseness of the expressions

$$\omega_i^2 = \Omega_i, \quad i = \overline{1,n}. \quad (4)$$

$$\omega_i^* \Gamma = \Omega_i^*, \quad i = \overline{1,q}. \quad (5)$$

$$\omega^2 = \Omega. \quad (6)$$

The ordinary frequencies are denoted with letter $\omega$.

2.2. Formalisation via rank minimization heuristic

Due to the non-linear and non-convex nature of the constraints on higher eigen-frequencies in C1, the required solution should not necessarily provide the minimum volume but we rather seek at least one feasible solution. Therefore, we are free to transform part of the constraints into a minimization objective. Rank minimization provides a possibility for formalization of the frequency matching constraint. If a frequency $\omega_i^* \Gamma$ belongs to the spectrum $\{\omega_j\}_{j=1}^n$ then the rank of the dynamic stiffness matrix has deficit

$$\text{rank} (\mathbf{K}(\mathbf{s}) - \Omega_i^* \mathbf{M}(\mathbf{s})) < \text{size} (\mathbf{K}(\mathbf{s})) = n. \quad (7)$$

Minimizing sum of ranks of dynamic stiffness evaluated for each target eigenfrequency does not guarantee that each target frequency is matched but it can improve the design sufficiently. Thus, a heuristic objective can be defined

$$J(\mathbf{s}) = \sum_{i=1}^{q} \text{rank} (\mathbf{K}(\mathbf{s}) - \Omega_i^* \mathbf{M}(\mathbf{s})). \quad (8)$$

Two other design constraints in the problem statement (C2, C3) can be formulated as linear matrix inequalities (LMI) as shown for truss structures in [27,16,4]. The constraint on the fundamental frequency can be formulated

$$| (\mathbf{K}(\mathbf{s}) - \Omega \mathbf{M}(\mathbf{s})) | \geq O \quad (9)$$

and the constraint on compliance

$$\left[ \begin{array}{c} \mathbf{r} \\ \mathbf{p} \end{array} \right] \mathbf{K}(\mathbf{s}) \geq O, \quad r \leq \tilde{r}. \quad (10)$$

The symbol $\geq O$ denotes that on the left side is semi-definite matrix and $r$ is auxiliary compliance variable. Two LMI’s given in eq. (9) and (10) define a convex set $\mathcal{F'}$ that satisfies constraints (C2, C3) with

$$\mathcal{F'} = \{ s, r \ : | (\mathbf{K}(\mathbf{s}) - \Omega \mathbf{M}(\mathbf{s})) | \geq O, \left[ \begin{array}{c} \mathbf{r} \\ \mathbf{p} \end{array} \right] \mathbf{K}(\mathbf{s}) \geq O, r \leq \tilde{r} \}. \quad (11)$$

Combining the objective (8) and conditions on $(s, r)$ yields a minimization problem
The problem $P_1$ is still difficult to solve numerically. Therefore, a tractable relaxation of the rank objective is used below to reduce it to series of SDPs.

### 3.4. Log-det relaxation of the rank minimization problem

There are many rank minimization relaxations for semi-definite symmetric square matrices. Minimizing rank of a square matrix $X(s) = K(s) - Q^s M(s)$ with some positive and some negative eigenvalues is more challenging.

A standard way is to use an embedding of the original rank minimization problem in a semi-definite problem and log-det surrogate of the rank function [8]. The embedding requires two additional square matrices $(Y_i, Z_i)$ for each target frequency as variables of the optimization. Then the surrogate function for matrix rank reads

$$\mathcal{F}(Y_i, Z_i) = \sum_{i=1}^{q} \log \det \left( \text{diag}(Y_i, Z_i) + \delta I \right)$$

with $\delta$ being a small positive regularization parameter and $\text{diag}$ being constructor of a block-diagonal matrix from the list of arguments. The square matrices $(Y_i, Z_i)$ must satisfy $q$ LMI’s. For conciseness, a convex admissible set $\mathcal{W}$ is introduced to collect the LMI’s with

$$\mathcal{W} = \{ (s, Y_i, Z_i) : \begin{bmatrix} Y_i & K(s) - Q^s M(s) \ \ K^T(s) - Q^T M^T(s) & Z_i \end{bmatrix} \succeq O, i = 1, q \}$$

Combining the surrogate function for matrix rank in eq. (13) with constraints defines an optimization problem

$$P_2 : \begin{array}{ll}
\text{min} & \mathcal{F}(Y_i, Z_i) \\
\text{s.t.} & (s, Y_i, Z_i) \in \mathcal{W}, (s, r) \in \mathcal{W}, s \in \mathcal{G}.
\end{array}$$

The objective function is concave. Fortunatelly, a successive minimization of its linearization usually leads to satisfactory solutions [8]. Taylor series of the objective in (13) reads

$$\mathcal{F}(Y_i, Z_i) \approx \mathcal{F}(Y_i^{(k)}, Z_i^{(k)}) + \sum_{i=1}^{q} \left( \text{tr}(W_{Y_i}^{(k)} \Delta Y_i) + \text{tr}(W_{Z_i}^{(k)} \Delta Z_i) \right)$$

where a superscript in brackets $^{(k)}$ denotes $k^{th}$ iteration number, $\Delta Y_i = Y_i - Y_i^{(k)}$ and $\Delta Z_i = Z_i - Z_i^{(k)}$ denote increments of matrices and the weighting matrices are updated each iteration

$$W_{Y_i}^{(k)} = (Y_i^{(k)} + \delta I)^{-1}, \quad W_{Z_i}^{(k)} = (Z_i^{(k)} + \delta I)^{-1}$$

At first iteration the weight can be initialized with identity matrices.

Extending the Fasel’s approach to a sum of rank in problem (12) leads to inner minimization problem

$$P_{2_l} : \begin{array}{ll}
\text{min} & \mathcal{F}(Y_i^{(k)}, Z_i^{(k)}) \\
\text{s.t.} & (s, r, Y_i, Z_i) \in \mathcal{K}, (s, s) \in \mathcal{G}, (s, r) \in \mathcal{G}.
\end{array}$$

where the admissible convex set $\mathcal{K}$ combines all constraints with

$$\mathcal{K} = \{ s, r, Y_i, Z_i : (s, r, Y_i, Z_i) \in \mathcal{W}, (s, r) \in \mathcal{W}, s \in \mathcal{G} \}$$

The objective in the problem $P_{2_l}$ is equivalent to eq. (16) up to a constant $\mathcal{F}(Y_i^{(k)}, Z_i^{(k)}) - \sum_{i=1}^{q} \left( \text{tr}(W_{Y_i}^{(k)} Y_i^{(k)}) + \text{tr}(W_{Z_i}^{(k)} Z_i^{(k)}) \right)$, which is omitted for simplicity. The problem $P_{2_l}$ has a linear objective and a convex admissible set, which qualifies the problem for solution with interior-point method.

A disadvantage of the minimization problem $P_{2_l}$ is in the number of independent variables. The symmetric matrices $Y_i$ and $Z_i$ in the semi-definite embedding are dense and have dimension $n$, i.e. $qn(n+1)$ additional variables are added for $q$ matching frequencies on top to $p$ variables for design $s$ and one variable for the compliance. Thus, the dimension of the minimization problem $P_{2_L}$ reads

$$N_p = qn(n+1) + p + 1.$$  

The dimension of the dual problem includes variables for constrains defining the admissible sets $\mathcal{X}$. The set is intersection of three convex sets $\mathcal{W}$, $\mathcal{W}$ and $\mathcal{G}$. The dual variables for semi-definiteness constraints in set definitions $\mathcal{W}$ and $\mathcal{W}$ are also dense symmetric matrices with dimension $2n$ and $n+1$, respectively. Thus, the dual problem has dimension

$$N_d = q(n+1)(n+2) + n(n+1) + 2p + 2.$$  

The primary and dual problems are prohibitively expensive for system dimensions $n$ beyond 200 or 300 even if the initial topology of the truss structure is sparse.

### 3. Implementation details

The algorithm for the design of truss structures with multiple eigenfrequency constraints is implemented in Matlab relying on the open-source package for disciplined convex optimization CVX. The main advantage is easy access to powerful SDP solvers such as SDPT3 and SeDuMi. Only lumped mass matrix approximation is implemented and used in the numerical experiments below.

The design algorithm is presented in Box 1. Input parameters include frequency values for each frequency constraint, the external force vector $p$, the maximum admissible compliance and volume. The interior-point method performs better if the constraints and variables are scaled in proper way [12, Sec. 10.4] and [26, Sec. 7.3]. Herein, semi-definiteness constraints stemming from frequency constraints are divided with a scaling factor $K_{\text{max}} = 0.1 \max_{1 \leq i \leq s} |K_{ij}(s)|$

$$\frac{1}{K_{\text{max}}} \begin{bmatrix} Y_i & K(s) - Q^s M(s) \ \ K^T(s) - Q^T M^T(s) & Z_i \end{bmatrix} \succeq O,$$

where $s$ is some reference design given in the beginning of the optimization process. The design variables for areas are scaled uniformly such that the maximal upper bound of cross-sectional areas is one. The external force vector $p$ is scaled such that admissible compliance value $\bar{v}$ is in range 0.05 to 10. Herein, the external force vector $p$ is given with physical dimension of Newton and the corresponding value of the compliance is measured in $N/m^2$. Furthermore, the input requires values for the regularization for the log-det heuristic $\delta$, the tolerance for numerical rank computation tol and the maximum number of iterations. The regularization parameter in log-det heuristic is also scaled with the stiffness matrix $\delta = a_k K_{\text{max}}$, where $a_k$ value of 5.0 · 10$^{-4}$ is found to be reasonable for majority of the cases. Weighting matrices $W_{Y_i}$ and $W_{Z_i}$ are always initialized with identity matrices. Noteworthy, the algorithm does not require an initial design values $s^{(0)}$. The interior-point method finds a feasible point within the admissible set $\mathcal{K}$ and uses it as a starting point.

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1 Minimizing rank of $X^* X$ changes the problem to symmetric semi-definite case, but the latter problem suffers from bad conditioning in many practical application.

2 Herein, a combination MATLAB version: 9.11 (R2021b) and CVX version 2.2 on Win10 with SDPT3 version 4.0 [34] or SeDuMi version 1.3.4 [29] is used. High precision is used and the maximum number of the interior-point iterations is increased to 120. Other settings are the default for the CVX package. Double precision is used for all computations.
Algorithm 1 Design of truss via rank minimization based on log-det heuristic.

1: procedure SPECTRDESIGNTRUSSVIAMINRANK($Ω, Ω^1, p, \tilde{r}, δ, n_{var}, tol$)
2: Initialize weighting: $W^{(1)}_Y \leftarrow 1$, $W^{(1)}_Z \leftarrow 1$, $i \leftarrow 1, q$
3: $k \leftarrow 1$
4: repeat
5: $\{k, Y_{ij}, Z_{ij}, x^{(k)}\} = \arg \min_{k, Y_{ij}, Z_{ij}, x^{(k)}} \sum_{i,j} \left( \text{tr}(W^{(k)}_Y Y_{ij}) + \text{tr}(W^{(k)}_Z Z_{ij}) \right)$
6: Update weighting: $W^{(k+1)}_Y = \left(Y^{(k)} + δI\right)^{-1}, W^{(k+1)}_Z = \left(Z^{(k)} + δI\right)^{-1}, i = 1, q$
7: Compute rank deficit: $r^{(k)}_I = \text{rank} (K(s^{(k)}) - \Omega^1 \text{Mi}(s^{(k)}), tol), i = 1, q$
8: Surrogate obj.: $\mathcal{F}^{(k)} = \sum_{i=1}^Q \sum_{j=1}^N \left( \log \lambda_j (Y^{(k)} + δI) + \log \lambda_j (Z^{(k)} + δI) \right)$
9: $k \leftarrow k + 1$
10: until ($k > n_{iter}$)
11: return $s^{(k-1)}, x^{(k-1)}, \mathcal{F}, r$
12: end procedure

Fig. 1. Setup of example 1 (left) and the obtained design (right). Numbers in circles designate member number (corresponding cross-section is $s_j$).

Table 1
Parameters for the design problem in example 1.

<table>
<thead>
<tr>
<th>$f_i$, Hz</th>
<th>$q$</th>
<th>$f_{s_i}$, Hz</th>
<th>$f_{s_j}$, Hz</th>
<th>$V_i$, m$^3$</th>
<th>$E$, GPa</th>
<th>$\rho$, kg/m$^3$</th>
<th>$\tilde{r}$, N/m$^2$</th>
<th>$a_j$</th>
<th>$n_{var}$</th>
<th>$s_i$, cm$^2$</th>
<th>$s_j$, cm$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>3</td>
<td>200</td>
<td>400</td>
<td>0.3</td>
<td>210</td>
<td>7800</td>
<td>0.15</td>
<td>5.0 $\cdot$ 10$^{-4}$</td>
<td>12</td>
<td>500</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 2
Properties of obtained structure in example 1. The obtained structure is symmetric, i.e. $s_j = s_{i(s,j)}$ and values up to $j = 8$ are given.

<table>
<thead>
<tr>
<th>$f_i$, Hz</th>
<th>$f_{s_i}$, Hz</th>
<th>$f_{s_j}$, Hz</th>
<th>$V_i$, m$^3$</th>
<th>$r$, N/m$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>124.903</td>
<td>176.291</td>
<td>200.000</td>
<td>277.999</td>
<td>300.000</td>
</tr>
<tr>
<td>$s_i$, cm$^2$</td>
<td>$s_j$, cm$^2$</td>
<td>$s_i$, cm$^2$</td>
<td>$s_j$, cm$^2$</td>
<td>$s_i$, cm$^2$</td>
</tr>
<tr>
<td>56.1104</td>
<td>129.158</td>
<td>51.5320</td>
<td>50.0000</td>
<td>54.9012</td>
</tr>
</tbody>
</table>

The output includes the final design and its compliance value. Normal output of the algorithm does not require the surrogate of the objective function $\mathcal{F}(Y^{(k)}, Z^{(k)})$ and the rank deficit of the dynamic stiffness matrix at $i^{th}$ target frequency $r_i$. For research purposes such an output is added in form

$$\mathcal{F}(Y^{(k)}, Z^{(k)}) = \sum_{i=1}^Q \sum_{j=1}^N \left( \log \lambda_j (Y^{(k)} + δI) + \log \lambda_j (Z^{(k)} + δI) \right),$$

$$r^{(k)}_I = \text{rank} (K(s^{(k)}) - \Omega^1 \text{Mi}(s^{(k)}), tol), i = 1, q,$$

where $\lambda_j(\square)$ denotes $j^{th}$ eigenfrequency of an argument matrix $\square$. The tolerance $tol$ for the rank computations is taken in examples as $5 \cdot 10^{-6} K_{max}$ to indicate vicinity to a simple or multiple resonance with sufficient accuracy. This information is not used to stop the iterations in the considered algorithm.

The proposed algorithm has a nested loop: the outer loop is the log-det heuristic iterations (lines 4-10) and the inner loop is inside the interior-point method call at line 5. This results in a cumulative number of iterations for the interior-point method at level of several hundreds.

4. Numerical examples

4.1. Example 1. 15-member structure with three target frequencies

Consider a 2D steel truss structure with 15 members and without non-structural masses ($M_0 = 0$) shown in Fig. 1. The structure spans over 5 m, and it is simply supported on both ends. Each truss member can have any cross-sectional area between 50 and 500 cm$^2$. The structure is loaded at two nodes around the middle of the span with force $F = 5$ kN. Values for the fundamental frequency constraint, three target frequencies, maximum admissible volume and maximum admissible compliance are specified in Table 1. The defined design problem is solved via the log-det heuristic.

The number of variables are $N_p = 646$ and $N_I = 1475$ for primary and dual problem, respectively. The optimization procedure requires 213 interior point iterations and ca. 24 $s$ CPU time. A result for such a design with a truss with three target frequencies is shown in Fig. 1 and values of the design parameters are given in Table 2. The obtained structure is symmetric with respect to the vertical axis. Cross-sectional areas of members 2 and 14 are the largest as they provide the most ef-
path between applied forces and supports. At small value of $\tilde{\tau} = 0.06$ algorithm fails because no feasible solution is possible with the given volume ($V \leq \tilde{V}$) and frequency ($f_1 \geq f$) constraints. Error messages from SDPT3 and SeDuMi suggest the dual problem of being infeasible. The maximum number of required log-det iterations is observed for the case with the admissible compliance $\tilde{\tau} = 0.075$, which is close to the infeasible parameters.

### 4.2. Example 2. Robustness check for 15-member structure

Another variant of the design problem is to put three equally spaced target frequencies closer to each other and keep the rest parameters as in example 1; see Table 1. The first target frequency is taken for all cases the same 200 Hz while the distance between target frequencies $\Delta f$ is varied between 5 and 100 Hz, e.g., three target frequencies are 200 Hz, 205 Hz and 210 Hz for $\Delta f = 5$ Hz. The goal of this example is to check the robustness of the proposed approach w.r.t. combinations of target frequencies.

The results for the design for the different distances between target frequencies $\Delta f$ are given in Table 5. Feasible designs are not obtained for the distance between target frequencies below 80 Hz. Exemplary parameters of such a structure for $\Delta f = 5$, which does not match all target frequencies, are given in Table 6. In this case, only frequency 200 Hz is matched. This illustrates that the formulation does not always converge to a feasible design. Further study of these design parameters reveals the similarity with example 1: cross-sectional areas of members 2 and 14 are the largest to provide the most efficient increase of the fundamental frequency and compliance, whereas other design parameters are close to lower limit $\gamma$. As a consequence, the usage of material in the structure is small. Noteworthy, a design with a repeating frequency 360 Hz is observed $\Delta f = 80$ Hz.

### 4.3. Example 3. 33-member truss with a point mass

Consider a 2D grid structure with 33-members and one nonstructural mass attached to the node shown in Fig. 5. The structure is fixed at three utmost left nodes. Each truss member can have any cross-sectional area between 1 and 10 cm$^2$. The structure is loaded at the center node of the right row with a horizontal force $F = 5$ kN. Values for one fundamental frequency constraint, two target frequencies, maximum admissible volume, and maximum admissible compliance are specified in Table 7.

The number of variables are $N_p = 718$ and $N_p = 1761$ for primary and dual problem, respectively. The optimization procedure requires 2116 interior point iterations and ca. 360 s CPU time. A result for such a design for a truss with two target frequencies is shown in Fig. 5 and values of the design parameters are given in Table 8. The obtained structure is symmetric relative to a horizontal axis. Material in the design is invested to pursuit two goals: 1) in horizontal trusses on the shortest path between the external load and supports to reduce the compliance 2) bracing close to the supports and horizontal trusses on the outer belts to increase flexural eigenfrequencies. The fundamental eigenmode corresponds to the global flexural mode, and it is given in Fig. 7. The eigenmodes corresponding to two target frequencies are given in Fig. 8. Target frequencies 300 Hz and 450 Hz are matched by a pair of eigenfrequencies (2,3) and (5,6), respectively. One of the eigenmodes in the pair is flexural and one is longitudinal. Evolution of the objective and its log-det surrogate in Fig. 6. The objective reduces monotonically. A steep reduction of the objective is observed close to iteration, where rank deficit increases. Volume and compliance constraints are met every iteration according to the properties of the interior-point methods. This example illustrates that designs with a repeating target frequency can be as well obtained by the proposed approach without any numerical issues or adjustments.
Table 4
Sweep over one parameter (admissible compliance) in example 1 listing the number of iterations till convergence, the obtained volume and compliance of the structure.

<table>
<thead>
<tr>
<th>τ, N/m²</th>
<th>0.06</th>
<th>0.075</th>
<th>0.09</th>
<th>0.105</th>
<th>0.12</th>
<th>0.135</th>
<th>0.15</th>
<th>0.165</th>
<th>0.18</th>
<th>0.195</th>
</tr>
</thead>
<tbody>
<tr>
<td>#iter</td>
<td>fall</td>
<td>37</td>
<td>15</td>
<td>23</td>
<td>25</td>
<td>17</td>
<td>5</td>
<td>7</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>V, m³</td>
<td>-</td>
<td>0.2771</td>
<td>0.2366</td>
<td>0.2095</td>
<td>0.1890</td>
<td>0.1729</td>
<td>0.1596</td>
<td>0.1532</td>
<td>0.1532</td>
<td>0.1532</td>
</tr>
<tr>
<td>τ</td>
<td>-</td>
<td>0.075</td>
<td>0.09</td>
<td>0.105</td>
<td>0.12</td>
<td>0.135</td>
<td>0.15</td>
<td>0.167</td>
<td>0.164</td>
<td>0.1613</td>
</tr>
</tbody>
</table>

Fig. 4. The obtained design for different admissible compliance values.

Fig. 5. Setup of example 3 (left) and the obtained design (right).

Fig. 6. The evolution of rank objective and its log-det relaxation in example 3. The rank of (K(s) - Ω ∗ i M(s)) is computed with tolerance tol = 5·10⁻⁶ K max. The rank J(s) for an arbitrary design is 36.

Table 5
Convergence for the different distances between target frequencies Δf in example 2 for triples of target frequencies (200, 200 + Δf, 200 + 2Δf) Hz.

<table>
<thead>
<tr>
<th>Δf, Hz</th>
<th>n = rank (K(s) - Ω ∗ i M(s), tol)</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>j = 1</td>
<td>j = 2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>80</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>90</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

4.4. Example 4. 51-member structure

Consider a 2D steel truss structure with 51 members and without non-structural masses (M₀ = 0) shown in Fig. 9. Each truss member can have any cross-sectional area between 100 and 500 cm². The structure is loaded at two nodes around the middle of the span with force F = 15 kN. Values for the fundamental frequency constraint, three tar-
Table 6
Properties of the obtained structure in example 2 for a triple of target frequencies 200 Hz, 205 Hz and 210 Hz. Obtained structure is symmetric, i.e. \( s_j = s_{8-n} \) and values up to \( j = 8 \) are given.

<table>
<thead>
<tr>
<th>( f_1 ), Hz</th>
<th>( f_2 ), Hz</th>
<th>( f_3 ), Hz</th>
<th>( f_4 ), Hz</th>
<th>( f_5 ), Hz</th>
<th>( f_6 ), Hz</th>
<th>( V_1 ), m³</th>
<th>( r ), N/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>125.390</td>
<td>176.696</td>
<td>200.000</td>
<td>277.782</td>
<td>300.146</td>
<td>401.194</td>
<td>0.1598</td>
<td>0.150</td>
</tr>
<tr>
<td>( s_1 ), cm²</td>
<td>( s_2 ), cm²</td>
<td>( s_3 ), cm²</td>
<td>( s_4 ), cm²</td>
<td>( s_5 ), cm²</td>
<td>( s_6 ), cm²</td>
<td>( s_7 ), cm²</td>
<td></td>
</tr>
<tr>
<td>56.432</td>
<td>133.390</td>
<td>50.000</td>
<td>50.000</td>
<td>54.332</td>
<td>50.000</td>
<td>50.000</td>
<td>50.000</td>
</tr>
</tbody>
</table>

Table 7
Parameters for the design problem in example 3.

<table>
<thead>
<tr>
<th>( f_1 ), Hz</th>
<th>( q )</th>
<th>( f_2 ), Hz</th>
<th>( f_3 ), Hz</th>
<th>( f_4 ), Hz</th>
<th>( f_5 ), Hz</th>
<th>( \bar{V} ), m³</th>
<th>( E ), GPa</th>
<th>( \rho ), kg/m³</th>
<th>( \bar{r} ), N/m²</th>
<th>( a_1 )</th>
<th>( n_{max} )</th>
<th>( n_{min} ), kg</th>
<th>( s ), cm²</th>
<th>( \bar{s} ), cm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>2</td>
<td>300</td>
<td>450</td>
<td>50.012</td>
<td>210</td>
<td>7800</td>
<td>0.4</td>
<td>5.0 \cdot 10^{-4}</td>
<td>15</td>
<td>5.0</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8
Properties of the obtained structure in example 3.

<table>
<thead>
<tr>
<th>( f_1 ), Hz</th>
<th>( f_2 ), Hz</th>
<th>( f_3 ), Hz</th>
<th>( f_4 ), Hz</th>
<th>( f_5 ), Hz</th>
<th>( f_6 ), Hz</th>
<th>( V_1 ), m³</th>
<th>( r ), N/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>120.000</td>
<td>300.000</td>
<td>431.709</td>
<td>450.000</td>
<td>0.012</td>
<td>0.4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.5. Example 5. 110-member grid structure

Consider a 2D grid structure with 110 members shown in Fig. 11. Each truss member can have any cross-sectional area between 100 and 500 cm². The structure is loaded at a node at a top layer with a force \( F = 15 \text{ kN} \). Values for the fundamental frequency constraint, three target frequencies, maximum admissible volume and maximum admissible compliance are specified in Table 11. This example doubles the number of members compared to example 4.

The number of variables are \( N_p = 12981 \) and \( N_j = 30123 \) for primary and dual problem, respectively. The optimization procedure requires 325 interior point iterations and ca. 4370 s CPU time. A feasible design is shown in Fig. 11 and the relevant properties of the structure for the design constrains are given in Table 12. All target frequencies are matched once and the corresponding eigenmodes are shown in Fig. 12. The compliance of the structure is far below the upper limit. Therefore, material is not invested into structure on the shortest path between the load and the supports. Several trusses close to supports have larger cross-section to increase few lowest eigenfrequencies. This example illustrates a size of a problem that is still possible to obtain with the proposed method on a laptop within one hour. This problem size is small compared to the design problems with a constraint on the fundamental frequency [27] or with a compliance constraint [16]. The main reason for this is that the multiple equality constraints (C1) are more difficult to impose within the framework. Here, a version with \( n(n+1) \) additional variables for each target frequency is used, as explained in eq. (20). The second reason is a structure of the algorithm with a nested loop as explained in the last paragraph of Section 3.

5. Conclusions

A rank minimization formulation is presented for the design of truss structures with multiple eigenfrequency constraints. The eigenfrequency constraints include an inequality on the fundamental frequency and equality for the few higher eigenfrequencies. The formulation relies on the rank deficit of the dynamic stiffness matrix of the structure at given target frequencies. The stiffness and mass matrices for truss structures have an affine dependency on the design parameters (cross-sectional areas). Thus, the formulation defines an affine rank minimization problem as the core of the design process. This rank minimization problem is solved using a tractable relaxation of the rank function. A valid log-det surrogate for the rank function is selected, which reduces the design search to a series of semi-definite programming problems. Obtained semi-definite programming problems are solved using the interior-point method. A proper regularization parameter for log-det surrogate and scaling of inequalities and variables are proposed for illustrative examples. As the method does not compute explicitly eigenvalues and eigenmodes in the objective, it does not provide any control over eigenmode shapes for the final design. Additional constraints on the total volume and compliance are considered within the problem.
Table 9
Parameters for the design problem in example 4.

<table>
<thead>
<tr>
<th>$f_1$, Hz</th>
<th>q</th>
<th>$f_2^*$, Hz</th>
<th>$f_3^*$, Hz</th>
<th>$f_4^*$, Hz</th>
<th>$\bar{V}$, m$^3$</th>
<th>$E$, GPa</th>
<th>$\rho$, kg/m$^3$</th>
<th>$\bar{\tau}$, N/m$^2$</th>
<th>$s_i$</th>
<th>$n_{ext}$</th>
<th>$s$, cm$^2$</th>
<th>$\bar{s}$, cm$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>3</td>
<td>120</td>
<td>240</td>
<td>400</td>
<td>1.8</td>
<td>210</td>
<td>7800</td>
<td>0.24</td>
<td>5.0 $\cdot$ 10$^{-4}$</td>
<td>12</td>
<td>100</td>
<td>500</td>
</tr>
</tbody>
</table>

Fig. 9. Setup of example 4 (left) and the obtained design (right).

Fig. 10. The fundamental eigenmode, the first and the second target eigenmodes for the final design in example 4.

Fig. 11. Setup of example 5 (left) and the obtained design (right).
Fig. 12. The fundamental eigenmode, the three target eigenmodes for the final design in example 5.

Table 10
Properties of the obtained structure in example 4.

<table>
<thead>
<tr>
<th>$f_1$, Hz</th>
<th>$f_2$, Hz</th>
<th>$f_{3,4}$, Hz</th>
<th>$f_5$, Hz</th>
<th>$V$, m$^3$</th>
<th>$\tau$, N/m$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>85.9268</td>
<td>120.000</td>
<td>240.000</td>
<td>400.000</td>
<td>0.8122</td>
<td>0.24</td>
</tr>
</tbody>
</table>

The latter constraint is formulated via a semi-definiteness condition and treated similarly by the interior-point method.

The validity of the proposed method is demonstrated through five examples. Feasible designs are obtained for most of the combinations of parameters in a reasonable number of iterations. Sweep over parameters shows sufficient robustness of the method. An issue of the formulation with targeting several close, equally spaced frequencies is identified by an example. In the third example, repeated frequencies were obtained without any special measures or numerical issues. Noteworthy, setups with a symmetric layout of truss structure and symmetric loads in compliance constraint lead to a symmetric distribution of cross-sectional areas. In the last two examples, the growth of computational cost with increasing number of members and free degrees of freedom is shown. The current algorithm is not recommended for problems with more than 200 members. The obtained designs can be interpreted, e.g., material is invested to change the base and target eigenfrequencies or to create loadpath from the external forces to the supports.

The current approach relies on affine dependency on the design parameter. In future work, the formulation may be extended for a polynomial dependency of the stiffness and mass matrices on design parameters, like in frames and plate structures [35]. The latter non-linear semi-definiteness programming problem can be treated with a more advanced solver, e.g., YALMIP [22].

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Anton Tkachuk reports financial support was provided by Karlstad University.

Data availability

No data was used for the research described in the article.

Table 11
Parameters for the design problem in example 5.

<table>
<thead>
<tr>
<th>$f_1$, Hz</th>
<th>$q$</th>
<th>$f_{1)}$, Hz</th>
<th>$f_{2)}$, Hz</th>
<th>$f_{3)}$, Hz</th>
<th>$\bar{V}$, m$^3$</th>
<th>$E$, GPa</th>
<th>$\rho$, kg/m$^3$</th>
<th>$\tau$, N/m$^2$</th>
<th>$a_x$</th>
<th>$n_{area}$</th>
<th>$b$, cm$^2$</th>
<th>$\bar{b}$, cm$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>3</td>
<td>140</td>
<td>180</td>
<td>240</td>
<td>8.0</td>
<td>210</td>
<td>7800</td>
<td>0.8</td>
<td>5.0 x 10$^{-4}$</td>
<td>12</td>
<td>100</td>
<td>500</td>
</tr>
</tbody>
</table>
Table 12
Properties of the obtained structure in example 5.

<table>
<thead>
<tr>
<th>$f_1$, Hz</th>
<th>$f_2$, Hz</th>
<th>$f_3$, Hz</th>
<th>$f_4$, Hz</th>
<th>$V$, m$^3$</th>
<th>$r$, N/m$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80.000</td>
<td>140.000</td>
<td>180.000</td>
<td>240.000</td>
<td>1.8679</td>
<td>0.4160</td>
</tr>
</tbody>
</table>

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References
