Compactness property of the linearized Boltzmann collision operator for a multicomponent polyatomic gas

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Article history:
Received 29 December 2022
Available online 1 March 2024
Submitted by D. Donatelli

Keywords:
Boltzmann equation
Multicomponent mixture
Polyatomic gases
Linearized collision operator
Hilbert-Schmidt integral operator

Abstract

The linearized Boltzmann collision operator is fundamental in many studies of the Boltzmann equation and its main properties are of substantial importance. The decomposition into a sum of a positive multiplication operator, the collision frequency, and an integral operator is trivial. Compactness of the integral operator for monatomic single species is a classical result, while corresponding results for monatomic mixtures and polyatomic single species are more recently obtained. This work concerns the compactness of the operator for a multicomponent mixture of polyatomic species, where the polyatomicity is modeled by a discrete internal energy variable. With a probabilistic formulation of the collision operator as a starting point, compactness is obtained by proving that the integral operator is a sum of Hilbert-Schmidt integral operators and operators, which are uniform limits of Hilbert-Schmidt integral operators, under some assumptions on the collision kernel. The assumptions are essentially generalizations of the Grad’s assumptions for monatomic single species. Self-adjointness of the linearized collision operator follows. Moreover, bounds on - including coercivity of - the collision frequency are obtained for a hard sphere like model. Then it follows that the linearized collision operator is a Fredholm operator, and its domain is also obtained.

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1. Introduction

The Boltzmann equation is a fundamental equation of kinetic theory of gases, e.g., for computations of the flow around a space shuttle in the upper atmosphere during reentry [1]. Studies of the main properties of the linearized collision operator are of great importance in gaining increased knowledge about related problems, see, e.g., [11] and references therein, and for related half-space problems [2,5,4,8]. The linearized collision operator is obtained, by considering deviations of an equilibrium, or Maxwellian, distribution. It can in a natural way be written as a sum of a positive multiplication operator, the collision frequency, and an integral operator $-K$. Compact properties of the integral operator $K$ (for angular cut-off kernels) are
extensively studied for monatomic single species, see, e.g., \cite{10,6,12,11}, and more recently for monatomic multi-component mixtures \cite{10,6} and polyatomic single species, where the polyatomicity is modeled by either a discrete or a continuous internal energy variable \cite{6,7}. See also \cite{9} for the case of molecules undergoing resonant collisions, i.e., collisions where internal energy is transferred to internal energy, and correspondingly, translational energy to translational energy, during the collisions. The integral operator can be written as the sum of a Hilbert-Schmidt integral operator and an approximately Hilbert-Schmidt integral operator (cf. Lemma 4 in Section 4) \cite{14}, and so compactness of the integral operator $K$ can be obtained. This work extends the results of \cite{6} for monatomic multicomponent mixtures and polyatomic single species, where the polyatomicity is modeled by a discrete internal energy variable \cite{13,16}, to the case of polyatomic multi-component mixtures, where the polyatomicity is modeled by discrete internal energy variables. To consider mixtures of monatomic and polyatomic molecules are of highest relevance in, e.g., the upper atmosphere \cite{1}. The case of multicomponent mixtures of monatomic and polyatomic gases, where the polyatomicity is modeled by continuous internal energy variables was recently (after that the preprint of this paper appeared) successfully considered \cite{3}. The setting in this paper, with the possibility of different numbers of internal energy levels for different species, like in, e.g., \cite{13}, and in contrast to in, e.g., \cite{16}, allows to include also monatomic species by assuming that there is a unique energy level for such species.

Following the lines of \cite{6,7}, motivated by an approach by Kogan in \cite{19}, Sect. 2.8] for the monatomic single species case, a probabilistic formulation of the collision operator is considered as the starting point. With this approach, it is shown, based on slightly modified arguments from the ones in \cite{6}, that the integral operator $K$ can be written as a sum of compact operators in the form of Hilbert-Schmidt integral operators and approximately Hilbert-Schmidt integral operators - which are uniform limits of Hilbert-Schmidt integral operators - and so compactness of the integral operator $K$ follows. The operator $K$ is self-adjoint, as well as, the collision frequency, and thereby the linearized collision operator, as the sum of two self-adjoint operators of which one is bounded, is also self-adjoint. The probabilistic formulation of the collision operator is chosen as a starting point to be able to choose the parametrization of the velocity variables, in a way suitable for the considered term in $K$, directly based on the collision laws. Indeed, rather than starting from a standard parametrization, and then change to another one when needed, the direct choice of a suitable parametrization seems to appear more clear.

For models corresponding to hard sphere models in the monatomic case, bounds on the collision frequency are obtained. Then the collision frequency is coercive and becomes a Fredholm operator. The set of Fredholm operators is closed under addition with compact operators. Therefore also the linearized collision operator becomes a Fredholm operator by the compactness of the integral operator $K$. For hard sphere like models the linearized collision operator satisfies all the properties of the general linear operator in the abstract half-space problem considered in \cite{5}.

The rest of the paper is organized as follows. In Section 2, the model considered is presented. The probabilistic formulation of the collision operators considered and its relations to more classical formulations \cite{13,16} are accounted for in Section 2.1. Some classical results for the collision operators in Section 2.2 and the linearized collision operator in Section 2.3 are reviewed. Section 3 is devoted to the main results of this paper, while the main proofs are addressed in Sections 4-5; a proof of compactness of the integral operator $K$ is presented in Section 4, while a proof of the bounds on the collision frequency appears in Section 5. Finally, the appendix concerns a proof of a crucial - for the compactness - lemma, which is an extension of a corresponding lemma for the monatomic mixture case \cite{10,6}.

2. Model

This section concerns the model considered. A probabilistic formulation of the collision operator is considered, whose relation to a more classical formulation is accounted for. Known properties of the model and corresponding linearized collision operator are also reviewed.
Consider a multicomponent mixture of $s$ polyatomic species $a_1, …, a_s$, with masses $m_1, …, m_s$, respectively. The polyatomicity is modeled by $r_\alpha$ different internal energies $I_1^\alpha, …, I_{r_\alpha}^\alpha$ for each $\alpha \in \{1, …, s\}$. Here the internal energies $I_1^\alpha, …, I_{r_\alpha}^\alpha$ are assumed to be nonnegative real numbers; $\{I_1^\alpha, …, I_{r_\alpha}^\alpha\} \subset \mathbb{R}_+$ for $\alpha \in \{1, …, s\}$. A monatomic species $a_\alpha$ can also be considered by choosing $r_\alpha = 1$, while $s = 1$ would correspond to the case of single species.

The distribution functions are of the vector form $f = (f_1, …, f_s)$, where $f_\alpha = (f_{\alpha,1}, …, f_{\alpha,r_\alpha})$ is the distribution function for particles of species $a_\alpha$ for $\alpha \in \{1, …, s\}$. Here $f_{\alpha,i} = f_{\alpha,i}(t, x, \xi) = f_{\alpha}(t, x, \xi, I_i^\alpha)$, with temporal variable $t \in \mathbb{R}_+$, spatial variable $x = (x, y, z) \in \mathbb{R}^3$, and molecular velocity variable $\xi = (\xi_x, \xi_y, \xi_z) \in \mathbb{R}^3$, is the distribution function for particles of species $a_\alpha$ with internal energy $I_i^\alpha$ for $(\alpha, i) \in \{1, …, s\} \times \{1, …, r_\alpha\}$.

Denote by $\Omega \subset \mathbb{N}^6$,

$$
\Omega := \left\{ (\alpha, \beta, i, j, k, l) : (\alpha, \beta) \in \{1, …, s\}^2, \{(i, j), (k, l)\} \subseteq \{1, …, r_\alpha\} \times \{1, …, r_\beta\} \right\}.
$$

Moreover, denote $r = \sum_{\alpha=1}^s r_\alpha$ and consider the real Hilbert space

$$
\mathfrak{h} := (L^2(d\xi))^r,
$$

with inner product

$$
(f, g) = \sum_{\alpha=1}^s \sum_{i=1}^{r_\alpha} \int_{\mathbb{R}^3} f_{\alpha,i}g_{\alpha,i}d\xi, \ f, g \in (L^2(d\xi))^r.
$$

The evolution of the distribution functions is (in the absence of external forces) described by the (vector) Boltzmann equation

$$
\frac{\partial f}{\partial t} + (\xi \cdot \nabla_x) f = Q(f, f),
$$

where the (vector) collision operator $Q = (Q_1^1, …, Q_{r_1}^1, …, Q_1^s, …, Q_{r_s}^s)$ is a quadratic bilinear operator that accounts for the change of velocities and internal energies of particles due to binary collisions (assuming that the gas is rarefied, such that other collisions are negligible), where the component $Q_i^\alpha$ is the collision operator for particles of species $a_\alpha$ with internal energy $I_i^\alpha$ for $(\alpha, i) \in \{1, …, s\} \times \{1, …, r_\alpha\}$.

A collision can, given two particles of species $a_\alpha$ and $a_\beta$, $(\alpha, \beta) \in \{1, …, s\}^2$, respectively, be represented by two pre-collisional pairs, each pair consisting of a microscopic velocity and an internal energy, $(\xi, I_i^\alpha)$ and $(\xi_*^*, I_j^\beta)$, and two corresponding post-collisional pairs, $(\xi', I_i^\alpha)$ and $(\xi_*^*, I_j^\beta)$, for some index pairs $\{(i, j), (k, l)\} \subset \{1, …, r_\alpha\} \times \{1, …, r_\beta\}$. The notation for pre- and post-collisional pairs may be interchanged as well. Due to momentum and total energy conservation, the following relations have to be satisfied by the pairs

$$
m_\alpha \xi + m_\beta \xi_* + m_\alpha \xi' + m_\beta \xi_*^* = m_\alpha \xi + m_\beta \xi_* + m_\alpha |\xi'|^2 + 2I_i^\alpha + 2I_j^\beta = m_\alpha |\xi|^2 + m_\beta |\xi_*|^2 + 2I_k^\alpha + 2I_l^\beta.
$$

(2)

2.1. Collision operator

The (vector) collision operator $Q = (Q_1^1, …, Q_{r_1}^1, …, Q_1^s, …, Q_{r_s}^s)$ has components that can be written in the following form
\[ Q_1^\alpha (f, f) = \sum_{\beta=1}^{s} \sum_{k=1}^{r_\alpha} \sum_{j,l=1}^{r_\beta} \int_{(\mathbb{R}^3)^3} W_{\alpha\beta}(\xi, \xi_*, I_1^\alpha, I_1^\beta | \xi', \xi'_*, I_1^\alpha, I_1^\beta) \times \left( \frac{f_{\alpha,kl} f_{\beta,js}}{\varphi_1^{\alpha} \varphi_1^{\beta}} - \frac{f_{\alpha,ij} f_{\beta,js}}{\varphi_1^{\alpha} \varphi_1^{\beta}} \right) \, d\xi_1 \, d\xi'_1 \, d\xi'_1 \]  

(3)

for some positive \((\varphi_1^1, ..., \varphi_1^1, ..., \varphi_1^s, ..., \varphi_1^s) \in \mathbb{R}_+^s\), where for \((\alpha, i) \in \{1, ..., s\} \times \{1, ..., r_\alpha\}\), \(\varphi_1^\alpha\) denotes the degeneracy of the internal energy \(I_1^\alpha\) of species \(a_\alpha\) \cite{13}, i.e., the number of different states of species \(a_\alpha\) giving rise to the same internal energy \(I_1^\alpha\). To our knowledge, there is no typical choice of degeneracies for polyatomic species - for a monatomic species \(a_\alpha\) we have, remind that \(r_\alpha = 1\), \(\varphi_1^1 = 1\) - but they completely depend on the system studied. On the other hand, while considering a continuous internal energy variable \[1\], there is a typical choice in the literature - reproducing the energy law of polytropic gases - for polyatomic species \(a_\alpha\); \(\varphi_\alpha = I_1^\delta(a_\alpha)\), where \(\delta(a_\alpha)\) is the number of internal degrees of freedom \cite{3}. Here and below the abbreviations

\[ f_{\alpha,ii} = f_{\alpha,i} (t, x, \xi_*) \, f'_{\alpha,i} = f_{\alpha,i} (t, x, \xi'_*) \]  

(4)

are used. In the collision operator (3) the gain term - the term containing the product \(f'_{\alpha,kl} f_{\beta,js}\) - accounts for the gain of particles of species \(a_\alpha\) with microscopic velocity \(\xi\) and internal energy \(I_1^\alpha\) (at time \(t\) and position \(x\)) - here \((\xi, I_1^\alpha)\) and \((\xi_*, I_1^\beta)\) represent the post-collisional particles, while the loss term - the term containing the product \(f_{\alpha,i} f_{\beta,js}\) - accounts for the loss of particles of species \(a_\alpha\) with microscopic velocity \(\xi\) and internal energy \(I_1^\alpha\) - here \((\xi, I_1^\alpha)\) and \((\xi_*, I_1^\beta)\) represent the pre-collisional particles. The corresponding (signed) internal energy gap is

\[ \Delta I_{kl,ij}^{\alpha,\beta} = I_k^\alpha + I_j^\beta - I_i^\alpha - I_j^\beta. \]

The transition probabilities

\[ W_{\alpha\beta} : \left( (\mathbb{R}^3)^2 \times \{1, ..., r_\alpha\} \times \{1, ..., r_\beta\} \right)^2 \to \mathbb{R}_+ := [0, \infty), \ (\alpha, \beta) \in \{1, ..., s\}^2, \]

are of the form, cf. \cite{6},

\[ W_{\alpha\beta}(\xi, \xi_*, I_1^\alpha, I_1^\beta | \xi', \xi'_*, I_1^\alpha, I_1^\beta) \]

\[ = (m_\alpha + m_\beta)^2 m_\alpha m_\beta \sigma_{kl,ij}^{\alpha,\beta} (|g'|, \cos \theta) \frac{|g'|}{|g|} \delta_3 (m_\alpha \xi + m_\beta \xi_* - m_\alpha \xi' - m_\beta \xi') \]

\[ \times \varphi_1^{\alpha} \varphi_1^{\beta} \delta_1 \left( \frac{1}{2} \left( m_\alpha |\xi|^2 + m_\beta |\xi_*|^2 - m_\alpha |\xi'|^2 - m_\beta |\xi'|^2 \right) - \Delta I_{kl,ij}^{\alpha,\beta} \right) \]

\[ = (m_\alpha + m_\beta)^2 m_\alpha m_\beta \sigma_{kl,ij}^{\alpha,\beta} (|g'|, \cos \theta) \frac{|g'|}{|g|} \delta_3 (m_\alpha \xi + m_\beta \xi_* - m_\alpha \xi' - m_\beta \xi') \]

\[ \times \varphi_1^{\alpha} \varphi_1^{\beta} \delta_1 \left( \frac{1}{2} \left( m_\alpha |\xi|^2 + m_\beta |\xi_*|^2 - m_\alpha |\xi'|^2 - m_\beta |\xi'|^2 \right) - \Delta I_{kl,ij}^{\alpha,\beta} \right), \]

with \(\sigma_{kl,ij}^{\alpha,\beta} = \sigma_{ij,kl}^{\alpha,\beta} (|g'|, \cos \theta) > 0\) a.e., \(\cos \theta = \frac{g' \cdot g}{|g'| |g|}\), \(g = \xi - \xi_*\), \(g' = \xi' - \xi_*\), and \(\Delta I_{kl,ij}^{\alpha,\beta} = I_k^\alpha + I_j^\beta - I_i^\alpha - I_j^\beta\),

(5)
where $\delta_3$ and $\delta_1$ denote the Dirac’s delta function in $\mathbb{R}^3$ and $\mathbb{R}$, respectively; taking the conservation of momentum and total energy (2) into account. Here and below we use the (inconsistent) shorthanded expressions

$$\sigma^{\alpha\beta}_{ij,kl} = \sigma_{ij,kl}^{\alpha\beta}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j, I^\alpha_k, I^\beta_l) = \tilde{\sigma}_{ij,kl}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j)$$

and

$$\sigma^{\alpha\beta}_{kl,ij} = \sigma_{kl,ij}^{\alpha\beta}(\xi^\prime, \xi, I^\alpha_k, I^\beta_l, I^\alpha_i, I^\beta_j) = \tilde{\sigma}_{kl,ij}(\xi^\prime, \xi, I^\alpha_k, I^\beta_l),$$

for given scattering cross-section

$$\sigma_{ij}^{\alpha\beta} : ([\mathbb{R}^3]^2 \times \{1,\ldots, r_\alpha\} \times \{1,\ldots, r_\beta\})^2 \rightarrow \mathbb{R}_+,$$

or, of the form $\tilde{\sigma}_{\alpha\beta} : \mathbb{R}_+ \times [-1,1] \times \{1,\ldots, r_\alpha\} \times \{1,\ldots, r_\beta\}^2 \rightarrow \mathbb{R}_+$; assuming the pairs $(\xi, I^\alpha_i)$, $(\xi^\prime, I^\beta_j)$, $(\xi^\prime, I^\alpha_k)$, and $(\xi, I^\beta_l)$ being given - here, by the arguments of $W_{\alpha\beta}$ for $(\alpha, \beta) \in \{1,\ldots, s\}^2$.

The choice of considering this probabilistic formulation of the collision operator, rather than the standard parametrized version, as a starting point, is motivated by the possibility to choose the parametrization of the velocity variables, in a way suitable for the considered term in $K$, directly based on the collision laws. This rather than starting from a standard parametrization, and then change to another one, when needed. In this way the choice of a suitable parametrization seems to appear more clear.

The scattering cross sections $\sigma^{\alpha\beta}_{kl,ij}$, with $(\alpha, \beta, i, j, k, l) \in \Omega$, are assumed to satisfy the microreversibility conditions

$$\varphi^{\alpha\beta}_{ij} \varphi^{\beta\alpha}_{ji} \sigma_{ij,kl}^{\alpha\beta}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j, I^\alpha_k, I^\beta_l) = \varphi^{\beta\alpha}_{ki} \varphi^{\alpha\beta}_{jk} \sigma_{kl,ij}^{\alpha\beta}(\xi, \xi^\prime, I^\alpha_k, I^\beta_l, I^\alpha_i, I^\beta_j).$$

Furthermore, to obtain invariance of change of particles in a collision, it is assumed that the scattering cross sections $\sigma^{\alpha\beta}_{kl,ij}$, with $(\alpha, \beta, i, j, k, l) \in \Omega$, satisfy the symmetry relations (fixing the pairs $(\xi, I^\alpha_i)$, $(\xi^\prime, I^\beta_j)$, $(\xi^\prime, I^\alpha_k)$, and $(\xi, I^\beta_l)$)

$$\sigma^{\alpha\beta}_{ij,kl} = \sigma^{\beta\alpha}_{kl,ij}$$

while

$$\sigma^{\alpha\beta}_{ij,kl}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j) = \sigma^{\alpha\beta}_{ij,kl}(\xi^\prime, \xi, I^\alpha_i, I^\beta_j)$$

and

$$\sigma^{\alpha\beta}_{ij,kl} = \sigma^{\alpha\beta}_{ij,kl}(\xi^\prime, \xi, I^\alpha_i, I^\beta_j).$$

The invariance under change of particles in a collision, which follows directly by the definition of the transition probability (5) and the symmetry relations (7), (8) for the collision frequency, and the microreversibility of the collisions (6), implies that the transition probabilities (5) for $(\alpha, \beta) \in \{1,\ldots, s\}^2$ satisfy the relations

$$W_{\alpha\beta}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j) = W_{\beta\alpha}(\xi^\prime, \xi, I^\alpha_i, I^\beta_j)$$

$$W_{\alpha\beta}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j) = W_{\alpha\beta}(\xi^\prime, \xi, I^\alpha_i, I^\beta_j)$$

$$W_{\alpha\alpha}(\xi, \xi^\prime, I^\alpha_i, I^\beta_j) = W_{\alpha\alpha}(\xi^\prime, \xi, I^\alpha_i, I^\beta_j).$$

Applying known properties of Dirac’s delta function, the transition probabilities - aiming to obtain expressions for $G_{\alpha\beta}^\prime = \frac{m_\alpha \xi^\prime + m_\beta \xi^\prime_\alpha}{m_\alpha + m_\beta}$ and $|g^\prime|$ - in the arguments of the delta-functions - may be transformed to
\[ W_{\alpha\beta}(\xi, \xi^\prime, I_i^\alpha, I_i^\beta) \]
\[ = (m_\alpha + m_\beta)^2 m_\alpha m_\beta \varphi_k^\alpha \varphi_l^\beta \sigma_{kl,ij} \frac{|g'|}{|g|} \delta_3 \left( (m_\alpha + m_\beta) \left( G_{\alpha\beta} - G_{\alpha\beta}' \right) \right) \]
\[ \times \delta_1 \left( \frac{m_\alpha m_\beta}{2(m_\alpha + m_\beta)} \left( |g'|^2 - |g|^2 \right) - \Delta I_{\alpha\beta}^{ij} \right) \]
\[ = 2 \varphi_k^\alpha \varphi_l^\beta \sigma_{kl,ij} \frac{|g'|}{|g|} \delta_3 \left( G_{\alpha\beta} - G_{\alpha\beta}' \right) \delta_1 \left( |g'|^2 - |g|^2 - 2 \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{\alpha\beta}^{ij} \right) \]
\[ = \varphi_k^\alpha \varphi_l^\beta \sigma_{ij,kl} \frac{1}{|g|^2 |g'|^2} \delta_3 \left( G_{\alpha\beta} - G_{\alpha\beta}' \right) \delta_1 \left( \sqrt{|g|^2 - 2 \Delta I_{\alpha\beta}^{ij}} - |g'| \right) \]
\[ = \varphi_k^\alpha \varphi_l^\beta \sigma_{ij,kl} \frac{1}{|g|^2 |g'|^2} \delta_3 \left( G_{\alpha\beta} - G_{\alpha\beta}' \right) \delta_1 \left( \sqrt{|g|^2 - 2 \Delta I_{\alpha\beta}^{ij}} - |g'| \right), \]

with \( G_{\alpha\beta} = \frac{m_\alpha \xi + m_\beta \xi^\prime}{m_\alpha + m_\beta} \) and \( \Delta I_{\alpha\beta}^{ij} = \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{\alpha\beta}^{ij} \).

**Remark 1.** Note that, cf. [17],
\[
\delta_1 \left( \frac{m_\alpha m_\beta}{2(m_\alpha + m_\beta)} \left( |g'|^2 - |g|^2 \right) - \Delta I_{\alpha\beta}^{ij} \right) = \delta_1 \left( E_{ij}^{\alpha\beta} - E_{ij}^{\alpha\beta} \right),
\]
for \( E_{ij}^{\alpha\beta} = \frac{m_\alpha m_\beta}{2(m_\alpha + m_\beta)} |g|^2 + I_i^\alpha + I_j^\beta \) and \( E_{ij}^{\alpha\beta} = \frac{m_\alpha m_\beta}{2(m_\alpha + m_\beta)} |g'|^2 + I_k^\alpha + I_l^\beta \).

By a change of variables \( \left\{ g' = \xi - \xi^\prime, G_{\alpha\beta}' = \frac{m_\alpha \xi + m_\beta \xi^\prime}{m_\alpha + m_\beta} \right\} \) followed by one to spherical coordinates, noting that
\[
d\xi' \, d\xi^\prime = dG_{\alpha\beta}' \, dg' = |g'|^2 \, dG_{\alpha\beta}' \, d|g'| \, d\omega, \quad \text{with} \quad \omega = \frac{g'}{|g'|}, \quad (10)
\]
the observation that
\[
Q_1^\alpha(f, g) \]
\[ = \sum_{\beta=1}^{\alpha} \sum_{k,l=1}^{\alpha} \int_{(\mathbb{R}^3)^2 \times \mathbb{R}^+ \times S^2} W_{\alpha\beta}(\xi, \xi^\prime, I_i^\alpha, I_i^\beta) \left| \xi^\prime - \xi^\prime \right| d\xi' \, dG_{\alpha\beta}' \, d|g'| \, d\omega \]
\[ \times \left( f_{\alpha,k} f_{\beta,l}^* - f_{\alpha,l} f_{\beta,k}^* \right) \frac{\varphi_k^\alpha \varphi_l^\beta}{\varphi_i^\alpha \varphi_j^\beta} |g'|^2 \, d\xi' \, dG_{\alpha\beta}' \, d|g'| \, d\omega \]
\[ = \sum_{\beta=1}^{\alpha} \sum_{k,l=1}^{\alpha} \int_{(\mathbb{R}^3)^2 \times S^2} \sigma_{ij,kl}^{\alpha\beta} |g| \left( f_{\alpha,k} f_{\beta,l}^* - f_{\alpha,l} f_{\beta,k}^* \right) \frac{\varphi_k^\alpha \varphi_l^\beta}{\varphi_i^\alpha \varphi_j^\beta} \, d\xi' \, d\omega, \]

where
\[
\begin{cases}
\xi' = G_{\alpha\beta} + \frac{m_\beta}{m_\alpha + m_\beta} \sqrt{|g|^2 - 2 \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{\alpha\beta}^{ij}} \omega \\
\xi^\prime = G_{\alpha\beta} - \frac{m_\alpha}{m_\alpha + m_\beta} \sqrt{|g|^2 - 2 \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{\alpha\beta}^{ij}} \omega
\end{cases}
\]
can be made, resulting in a more familiar form of the Boltzmann collision operator for a mixture of polyatomic gases, where polyatomicity is modeled with a discrete energy variable, cf. e.g. [13,16].

**Remark 2.** Note that, when considering spherical coordinates, we, maybe unconventionally, often represent the direction by a vector in \( S^2 \), rather than with azimuthal and polar angels, still referring to it as spherical coordinates. By representing the direction by a unit vector, the sine of the polar angle will not appear as a factor in the Jacobian, resulting in the Jacobian to be the square of the radial length.

### 2.2. Collision invariants and Maxwellian distributions

The following lemma follows directly by the relations (9).

**Lemma 1.** For any \((\alpha, \beta, i, j, k, l) \in \Omega\), the measure

\[
dA_{ij,kl}^{\alpha\beta} = W_{\alpha\beta}(\xi, \xi^*, I_\alpha^*, I_j^\beta) |\xi', \xi^*, I_\alpha^*, I_j^\beta| d\xi d\xi'd\xi^*'
\]

is invariant under the (ordered) interchange

\[
(\xi, \xi^*, I_\alpha^*, I_j^\beta) \leftrightarrow (\xi', \xi^*, I_\alpha^*, I_j^\beta)
\]

of variables, while

\[
dA_{ij,kl}^{\alpha\beta} + dA_{ji,kl}^{\beta\alpha}
\]

is invariant under the (ordered) interchange of variables

\[
(\xi, \xi^*, I_\alpha^*, I_j^\beta) \leftrightarrow (\xi^*, \xi^*, I_j^\beta, I_l^\beta)
\]

The weak form of the collision operator \(Q(f, f)\) reads

\[
(Q(f, f), g) = \sum_{\alpha,\beta=1}^s \sum_{i,k=1}^{r_\alpha} \sum_{j,l=1}^{r_\beta} \int_{(\mathbb{R}^3)^4} \left( \frac{f'_{\alpha,k} f'_{\beta,l} - f_{\alpha,k} f_{\beta,l}}{\varphi_{\alpha}^\beta \varphi_{\beta}^l} \right) g_{\alpha,i} dA_{ij,kl}^{\alpha\beta}
\]

\[
= \sum_{\alpha,\beta=1}^s \sum_{i,k=1}^{r_\alpha} \sum_{j,l=1}^{r_\beta} \int_{(\mathbb{R}^3)^4} \left( \frac{f'_{\alpha,k} f'_{\beta,l} - f_{\alpha,k} f_{\beta,l}}{\varphi_{\alpha}^\beta \varphi_{\beta}^l} \right) g_{\beta,j} dA_{ij,kl}^{\alpha\beta}
\]

\[
- \sum_{\alpha,\beta=1}^s \sum_{i,k=1}^{r_\alpha} \sum_{j,l=1}^{r_\beta} \int_{(\mathbb{R}^3)^4} \left( \frac{f'_{\alpha,k} f'_{\beta,l} - f_{\alpha,k} f_{\beta,l}}{\varphi_{\alpha}^\beta \varphi_{\beta}^l} \right) g'_{\alpha,k} dA_{ij,kl}^{\alpha\beta}
\]

\[
- \sum_{\alpha,\beta=1}^s \sum_{i,k=1}^{r_\alpha} \sum_{j,l=1}^{r_\beta} \int_{(\mathbb{R}^3)^4} \left( \frac{f'_{\alpha,k} f'_{\beta,l} - f_{\alpha,k} f_{\beta,l}}{\varphi_{\alpha}^\beta \varphi_{\beta}^l} \right) g'_{\beta,l} dA_{ij,kl}^{\alpha\beta}
\]

for any function \(g = (g_1, ..., g_s)\), with \(g_\alpha = (g_{\alpha,1}, ..., g_{\alpha,r_\alpha})\), such that the first integrals are defined for all \((\alpha, \beta, i, j, k, l) \in \Omega\), while the following equalities are obtained by applying Lemma 1.

Denote for any function \(g = (g_1, ..., g_s)\), with \(g_\alpha = (g_{\alpha,1}, ..., g_{\alpha,r_\alpha})\),

\[
\Delta_{ij,kl}^{\alpha\beta}(g) = g_{\alpha,i} + g_{\beta,j} - g'_{\alpha,k} - g'_{\beta,l}, \quad (\alpha, \beta, i, j, k, l) \in \Omega.
\]

We have the following proposition.
Proposition 1. Let \( g = (g_1, ..., g_s) \), with \( g_\alpha = (g_{\alpha,1}, ..., g_{\alpha,r_\alpha}) \), be such that for all \( (\alpha, \beta, i, j, k, l) \in \Omega \)

\[
\int_{(\mathbb{R}^3)^4} \frac{f'_{\alpha,k} f'_{\beta,l} x}{\varphi_k^\alpha \varphi_l^\beta} - \frac{f_{\alpha,i} f_{\beta,j} x}{\varphi_i^\alpha \varphi_j^\beta} = f_{\alpha,i} f_{\beta,j} x \, dA_{ij,kl}^\alpha \beta
\]

is defined. Then

\[
(Q(f, f), g) = \frac{1}{4} \sum_{\alpha, \beta=1}^s \sum_{i=1}^{r_\alpha} \sum_{k=1}^{r_\beta} \int_{(\mathbb{R}^3)^4} \left( \frac{f'_{\alpha,k} f'_{\beta,l} x}{\varphi_k^\alpha \varphi_l^\beta} - \frac{f_{\alpha,i} f_{\beta,j} x}{\varphi_i^\alpha \varphi_j^\beta} \right) \Delta_{ij,kl}^{\alpha \beta}(g) \, dA_{ij,kl}^\alpha \beta.
\]

Definition 1. A function \( g = (g_1, ..., g_s) \), with \( g_\alpha = (g_{\alpha,1}, ..., g_{\alpha,r_\alpha}) \), is a collision invariant if

\[
\Delta_{ij,kl}^{\alpha \beta}(g) \, W_{\alpha \beta}(\xi, \xi, I_1^\alpha, I_2^\beta) = 0 \text{ a.e.}
\]

for all \( (\alpha, \beta, i, j, k, l) \in \Omega \).

Denote

\[
I = (I_1^1, ..., I_1^s, ..., I_s^s) \text{ and } e_\alpha = (0_{r_\alpha}, ..., 0_{r_{\alpha-1}}, 1_{r_\alpha}, 0_{r_{\alpha+1}}, ..., 0_{r_s}) \in \mathbb{R}^s \text{ for } \alpha \in \{1, ..., s\},
\]

where \( 0_{r_\alpha} = (0, ..., 0) \in \mathbb{R}^{r_\alpha} \) and \( 1_{r_\alpha} = (1, ..., 1) \in \mathbb{R}^{r_\alpha} \) for \( \alpha \in \{1, ..., s\} \). It is clear that \( e_1, ..., e_s, m\xi_x, m\xi_y, m\xi_z \), and \( m|\xi|^2 + 2I \), with \( m = \sum_{\alpha=1}^s m_\alpha e_\alpha \), are collision invariants - corresponding to conservation of mass(es), momentum, and total energy.

In fact, we have the following proposition, cf. [16,11].

Proposition 2. The vector space of collision invariants is generated by

\[
\{ e_1, ..., e_s, m\xi_x, m\xi_y, m\xi_z, m|\xi|^2 + 2I \}, \text{ with } m = \sum_{\alpha=1}^s m_\alpha e_\alpha.
\]

Define

\[
W[f] := (Q(f, f), \log(\varphi^{-1} f)),
\]

where \( \varphi = \text{diag}(\varphi_1^1, ..., \varphi_s^1, ..., \varphi_1^s, ..., \varphi_s^s) \). It follows by Proposition 1 that

\[
W[f] = -\frac{1}{4} \sum_{\alpha, \beta=1}^s \sum_{i=1}^{r_\alpha} \sum_{k=1}^{r_\beta} \int_{(\mathbb{R}^3)^4} f_{\alpha,i} f_{\beta,j} \varphi_k^\alpha \varphi_l^\beta \left( \frac{f'_{\alpha,k} f'_{\beta,l} x}{\varphi_k^\alpha \varphi_l^\beta} - \frac{f_{\alpha,i} f_{\beta,j} x}{\varphi_i^\alpha \varphi_j^\beta} \right) \Delta_{ij,kl}^{\alpha \beta}(g) \, dA_{ij,kl}^\alpha \beta.
\]

Since \((x - 1) \log(x) \geq 0\) for \( x > 0 \), with equality if and only if \( x = 1 \),

\[
W[f] \leq 0,
\]

with equality if and only if
\[
\left( \frac{f_{\alpha,k}f'_{\beta,l} - f_{\alpha,i}f'_{\beta,j}}{\varphi_{k}^{\alpha} \varphi_{l}^{\beta} - \varphi_{i}^{\alpha} \varphi_{j}^{\beta}} \right) W_{\alpha\beta}(\xi, \xi', \rho_{\alpha}^{i} \rho'_{\beta}^{j} \mid \xi, \xi', \rho_{\alpha}^{i} \rho'_{\beta}^{j}) = 0 \text{ a.e.} \quad (13)
\]

for all \((\alpha, \beta, i, j, k, l) \in \Omega\), or, equivalently, if and only if

\[
Q(f, f) = 0.
\]

For any equilibrium, or, Maxwellian, distribution \(M = (M_{1}, ..., M_{s})\), with \(M_{\alpha} = (M_{\alpha,1}, ..., M_{\alpha,r_{\alpha}})\), it follows by equation (13), since \(Q(M, M) = 0\), that for any \((\alpha, \beta, i, j, k, l) \in \Omega\)

\[
\left( \log \frac{M_{\alpha,i}}{\varphi_{i}^{\alpha}} + \log \frac{M'_{\beta,j}}{\varphi_{j}^{\beta}} - \log \frac{M'_{\alpha,k}}{\varphi_{k}^{\alpha}} - \log \frac{M_{\beta,l}}{\varphi_{l}^{\beta}} \right) \times W_{\alpha\beta}(\xi, \xi', \rho_{\alpha}^{i} \rho'_{\beta}^{j} \mid \xi, \xi', \rho_{\alpha}^{i} \rho'_{\beta}^{j}) = 0 \text{ a.e. .}
\]

Hence, \(\log (\varphi^{-1}M) = \left( \log \frac{M_{1,1}}{\varphi_{1}^{1}}, ..., \log \frac{M_{s,r_{s}}}{\varphi_{s}^{r_{s}}} \right)\) is a collision invariant, and the components of the Maxwellian distributions \(M = (M_{1}, ..., M_{s})\) are of the form

\[
M_{\alpha,i} = \frac{n_{\alpha} \varphi_{i}^{\alpha} \rho_{\alpha}^{i}}{(2\pi T)^{3/2} q_{\alpha}} e^{-\frac{1}{3\rho}(m_{\alpha}\xi - u^{2} + 2I_{\alpha}^{i})/(2T)},
\]

where \(n_{\alpha} = (M_{\alpha,1}, u_{\alpha}), \rho = \frac{1}{3n}(M_{\alpha}\mu_{\alpha}, T = \frac{1}{3n} \left( M_{\alpha} \mu_{\alpha} - u^{2} \right)\right),\) with \(\mu_{\alpha} = \sum_{\alpha=1}^{s} n_{\alpha}, \rho = \frac{1}{3n} \left( M_{\alpha} \mu_{\alpha} - u^{2} \right)\right),\) and

\(m = \sum_{\alpha=1}^{s} m_{\alpha}, q_{\alpha} = \sum_{i=1}^{3\rho} \varphi_{i}^{\alpha} e^{-I_{\alpha}^{i}/T},\) for \((\alpha, i) \in \{1, ..., s\} \times \{1, ..., r_{\alpha}\}\).

Note that, by equation (13), any Maxwellian distribution, or, just Maxwellian, \(M = (M_{1}, ..., M_{s})\), with \(M_{\alpha} = (M_{\alpha,1}, ..., M_{\alpha,r_{\alpha}})\), for any \((\alpha, \beta, i, j, k, l) \in \Omega\) satisfies the relation

\[
\left( \frac{M'_{\alpha,k}M'_{\beta,l} - M_{\alpha,i}M'_{\beta,j}}{\varphi_{k}^{\alpha} \varphi_{l}^{\beta} - \varphi_{i}^{\alpha} \varphi_{j}^{\beta}} \right) W_{\alpha\beta}(\xi, \xi', \rho_{\alpha}^{i} \rho'_{\beta}^{j} \mid \xi, \xi', \rho_{\alpha}^{i} \rho'_{\beta}^{j}) = 0 \text{ a.e. .} \quad (14)
\]

**Remark 3.** Introducing the \(H\)-functional

\[
H[f] = (f, \log f),
\]

an \(H\)-theorem can be obtained.

### 2.3. Linearized collision operator

Considering a deviation of a Maxwellian distribution \(M = (M_{1}, ..., M_{s})\), with \(M_{\alpha} = (M_{\alpha,1}, ..., M_{\alpha,r_{\alpha}})\),

\[
M_{\alpha,i} = \frac{n_{\alpha} \varphi_{i}^{\alpha} \rho_{\alpha}^{i}}{(2\pi)^{3/2} q_{\alpha}} e^{-m_{\alpha}\xi^{2}/2 - I_{\alpha}^{i}},
\]

of the form

\[
f = M + M^{1/2} h
\]

where \(M = \text{diag}(M_{1,1}, ..., M_{1,r_{1}}, ..., M_{s,1}, ..., M_{s,r_{s}})\), results, by insertion in the Boltzmann equation (1), in the system

\[
\frac{\partial h}{\partial t} + (\xi \cdot \nabla h) + \mathcal{L} h = \Gamma(h, h),
\]

(16)
where the components of the linearized collision operator $\mathcal{L} = (L_1, \ldots, L_s)$, with $L_\alpha = (L_{\alpha, 1}, \ldots, L_{\alpha, r_\alpha})$, are given by

$$\mathcal{L}_{\alpha,i} h = -M_{\alpha,i}^{-1/2} \left( Q^\alpha (M, \mathcal{M}^{1/2} h) + Q^\alpha (\mathcal{M}^{1/2} h, M) \right)$$

$$= \sum_{\beta=1}^s \sum_{\kappa=1}^{r_\beta} \sum_{l=1}^{r_\beta} \int_{(\mathbb{R}^3)^3} \left( M_{\beta,j*} M'_{\alpha,k} M'_{\beta,l} \alpha, \beta, i, j, k, l \right) \Delta_{ij,kl} \left( \mathcal{M}^{-1/2} h \right) d\xi_i d\xi_j d\xi_k d\xi_l$$

$$= \nu_{\alpha,i} h_{\alpha,i} - K_{\alpha,i} (h), \quad (17)$$

with

$$\nu_{\alpha,i} = \sum_{\beta=1}^s \sum_{\kappa=1}^{r_\beta} \sum_{l=1}^{r_\beta} \int_{(\mathbb{R}^3)^3} \left( M_{\beta,j*} M'_{\alpha,k} M'_{\beta,l} \alpha, \beta, i, j, k, l \right) \Delta_{ij,kl} \left( \mathcal{M}^{-1/2} h \right) d\xi_i d\xi_j d\xi_k d\xi_l,$$

$$K_{\alpha,i} = \sum_{\beta=1}^s \sum_{\kappa=1}^{r_\beta} \sum_{l=1}^{r_\beta} \int_{(\mathbb{R}^3)^3} \left( h'_{\alpha,k} \left( M'_{\alpha,k} \right)^{1/2} + h'_{\beta,l} \left( M'_{\beta,l} \right)^{1/2} - \frac{h_{\beta,j*}}{M_{\beta,j*}^{1/2}} \right) \left( M_{\beta,j*} M'_{\alpha,k} M'_{\beta,l} \alpha, \beta, i, j, k, l \right) \Delta_{ij,kl} \left( \mathcal{M}^{-1/2} h \right) d\xi_i d\xi_j d\xi_k d\xi_l\right), \quad (18)$$

for all $(\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}$, while the components of the quadratic term $\Gamma = (\Gamma_1, \ldots, \Gamma_s)$, with $\Gamma_\alpha = (\Gamma_{\alpha, 1}, \ldots, \Gamma_{\alpha, r_\alpha})$, are given by

$$\Gamma_{\alpha,i} (h, h) = M_{\alpha,i}^{-1/2} Q^\alpha (\mathcal{M}^{1/2} h, \mathcal{M}^{1/2} h) \quad (19)$$

for all $(\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}$.

The multiplication operator $\Lambda$ defined by

$$\Lambda(f) = \nu f, \text{ where } \nu = \text{diag} (\nu_1, \ldots, \nu_1, r_1, \ldots, \nu_s, 1, \ldots, \nu_s, r_s),$$

is a closed, densely defined, self-adjoint operator on $(L^2 (d\xi))^\tau$. It is Fredholm, as well, if and only if $\Lambda$ is coercive.

The following lemma follows immediately by Lemma 1.

**Lemma 2.** For any $(\alpha, \beta, i, j, k, l) \in \Omega$ the measure

$$d\tilde{A}_{\alpha,\beta} = \left( \frac{M_{\alpha,i} M_{\beta,j*} M_{\alpha,k} M_{\beta,l} \alpha, \beta, i, j, k, l}{\varphi_i^n \varphi_j^m \varphi_k^n \varphi_l^m} \right)^{1/2} dA_{\alpha,\beta}$$

is invariant under the (ordered) interchange (11) of variables, while

$$d\tilde{A}_{\alpha,\beta} + d\tilde{A}_{\beta,\alpha}$$

is invariant under the (ordered) interchange (12) of variables.
The weak form of the linearized collision operator $L$ reads

\[
(Lh, g) = \sum_{\alpha, \beta = 1}^{s} \sum_{i, k = 1}^{r_\alpha} \sum_{j, l = 1}^{r_\beta} \int_{\mathbb{R}^4} \Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) \frac{g_{\alpha, i}}{M_{\alpha, i}^{1/2}} \, d\tilde{A}_{ij, kl}^\alpha
\]

\[
= \sum_{\alpha, \beta = 1}^{s} \sum_{i, k = 1}^{r_\alpha} \sum_{j, l = 1}^{r_\beta} \int_{\mathbb{R}^4} \Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) \frac{g_{\beta, j}^*}{M_{\beta, j}^{1/2}} \, d\tilde{A}_{ij, kl}^\alpha
\]

\[
= - \sum_{\alpha, \beta = 1}^{s} \sum_{i, k = 1}^{r_\alpha} \sum_{j, l = 1}^{r_\beta} \int_{\mathbb{R}^4} \Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) \frac{g_{\alpha, k}^*}{M_{\alpha, k}^{1/2}} \, d\tilde{A}_{ij, kl}^\alpha
\]

for any function $g = (g_1, ..., g_s)$, with $g_\alpha = (g_{\alpha, 1}, ..., g_{\alpha, r_\alpha})$, such that the first integrals are defined for all $(\alpha, \beta, i, j, k, l) \in \Omega$, while the following equalities are obtained by applying Lemma 2. We have the following lemma.

Lemma 3. Let $g = (g_1, ..., g_s)$, with $g_\alpha = (g_{\alpha, 1}, ..., g_{\alpha, r_\alpha})$, be such that

\[
\int_{\mathbb{R}^4} \Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) \frac{g_{\alpha, i}}{M_{\alpha, i}^{1/2}} \, d\tilde{A}_{ij, kl}^\alpha
\]

is defined for any $(\alpha, \beta, i, j, k, l) \in \Omega$. Then

\[
(Lh, g) = \frac{1}{4} \sum_{\alpha, \beta = 1}^{s} \sum_{i, k = 1}^{r_\alpha} \sum_{j, l = 1}^{r_\beta} \int_{\mathbb{R}^4} \Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) \Delta_{ij, kl}^\beta \left( M^{-1/2} g \right) \, d\tilde{A}_{ij, kl}^\alpha.
\]

Proposition 3. The linearized collision operator is symmetric and nonnegative,

\[
(Lh, g) = (h, Lg) \quad \text{and} \quad (Lh, h) \geq 0,
\]

and the kernel of $L$, $\ker L$, is generated by

\[
\left\{ M^{1/2} e_1, ..., M^{1/2} e_s, M^{1/2} m \xi_x, M^{1/2} m \xi_y, M^{1/2} m \xi_z, M^{1/2} \left( m |\xi|^2 + 2l \right) \right\},
\]

where $m = \sum_{\alpha = 1}^{s} m_\alpha e_\alpha$ and $M = \text{diag} \left( M_{1,1}, ..., M_{1, r_1}, ..., M_{s,1}, ..., M_{s, r_s} \right)$.

Proof. By Lemma 3, it is immediate that $(Lh, g) = (h, Lg)$, and

\[
(Lh, h) = \frac{1}{4} \sum_{\alpha, \beta = 1}^{s} \sum_{i, k = 1}^{r_\alpha} \sum_{j, l = 1}^{r_\beta} \int_{\mathbb{R}^4} \left( \Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) \right)^2 \, d\tilde{A}_{ij, kl}^\alpha \geq 0.
\]

Furthermore, $h \in \ker L$ if and only if $(Lh, h) = 0$, which will be fulfilled if and only if for all $(\alpha, \beta, i, j, k, l) \in \Omega$

\[
\Delta_{ij, kl}^\alpha \left( M^{-1/2} h \right) W_{\alpha, \beta} (\xi, \xi_x, I_0^\alpha, I_0^\beta | \xi', \xi_x', I_0^\alpha, I_0^\beta) = 0 \text{ a.e.},
\]
i.e. if and only if $\mathcal{M}^{-1/2}h$ is a collision invariant. The last part of the lemma now follows by Proposition 2. □

**Remark 4.** A property of the nonlinear term, although of no relevance to the studies here, is that it is orthogonal to the kernel of $\mathcal{L}$, i.e. $\Gamma(h, h) \in (\ker \mathcal{L})^\perp$.

This follows, since any element in $\ker \mathcal{L}$ is of the form $\mathcal{M}^{1/2}g$ for some collision invariant $g$, while for any collision invariant $g$

$$
\Gamma(h, h, \mathcal{M}^{1/2}g) = \left( \mathcal{M}^{-1/2}Q(\mathcal{M}^{1/2}h, \mathcal{M}^{1/2}h), \mathcal{M}^{1/2}g \right) = \left( Q(\mathcal{M}^{1/2}h, \mathcal{M}^{1/2}h), g \right) = 0.
$$

**3. Main results**

This section is devoted to the main results, concerning a compactness property in Theorem 4 and bounds of the collision frequencies for each species in Theorem 5.

Assume that for some positive number $\gamma$, such that $0 < \gamma < 1$, there for all $(\alpha, \beta, i, j, k, l) \in \Omega$ is a bound

$$
0 \leq \sigma_{ij,kl}^{\alpha\beta}(|g|, \cos \theta) \leq \frac{C}{|g|^2} \left( \Psi_{ij,kl}^{\alpha\beta} + \left( \Psi_{ij,kl}^{\alpha\beta} \right)^{\gamma/2} \right), \quad \text{where}
$$

$$
\Psi_{ij,kl}^{\alpha\beta} = |g| \sqrt{|g|^2 - 2 \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{kl,ij}^{\alpha\beta}},
$$

(20)

for $|g|^2 > 2 (m_\alpha + m_\beta) \Delta I_{kl,ij}^{\alpha\beta} (m_\alpha m_\beta)$, on the scattering cross sections $\sigma_{ij,kl}^{\alpha\beta}$. Note that assumption (20) reduces to Grad’s assumption [15] in the case of vanishing internal energy gap, cf. [6] for the case of monatomic multicomponent mixtures.

The following result may be obtained.

**Theorem 4.** Assume that for all $(\alpha, \beta, i, j, k, l) \in \Omega$ the scattering cross sections $\sigma_{ij,kl}^{\alpha\beta}$ satisfy the bound (20) for some positive number $\gamma$, such that $0 < \gamma < 1$. Then the operator $K = (K_{1,1}, ..., K_{\alpha,1}, ..., K_{s,1}, ..., K_{s,r_z})$, with the components $K_{\alpha,i}$ given by expressions (18) is a self-adjoint compact operator on $(L^2(d\xi))^\Gamma$.

Theorem 4 will be proven in Section 4.

**Corollary 1.** The linearized collision operator $\mathcal{L}$, with scattering cross sections satisfying (20), is a closed, densely defined, self-adjoint operator on $(L^2(d\xi))^\Gamma$.

**Proof.** By Theorem 4, the linear operator $\mathcal{L} = \Lambda - K$ is closed as the sum of a closed and a bounded operator, and densely defined, since the domains of the linear operators $\mathcal{L}$ and $\Lambda$ are equal; $D(\mathcal{L}) = D(\Lambda)$. Furthermore, it is a self-adjoint operator, since the set of self-adjoint operators is closed under addition of bounded self-adjoint operators, see Theorem 4.3 of Chapter V in [18]. □

Now consider the scattering cross sections - cf. hard sphere models -

$$
\sigma_{ij,kl}^{\alpha\beta} = C_{\alpha\beta} \frac{\sqrt{|g|^2 - 2 \Delta I_{kl,ij}^{\alpha\beta}}}{|g| \varphi_i^{\alpha} \varphi_j^{\beta}} \quad \text{if} \quad |g|^2 > 2 \Delta I_{kl,ij}^{\alpha\beta}, \quad \text{with} \quad \Delta I_{kl,ij}^{\alpha\beta} = \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{kl,ij}^{\alpha\beta},
$$

(21)

for some positive constants $C_{\alpha\beta} > 0$ for all $(\alpha, \beta, i, j, k, l) \in \Omega$. Note that assumption (21) reduces to the hard sphere model for monatomic multicomponent mixtures [6] in the case of vanishing internal energy gap and unit weights $\varphi_1 = ... = \varphi_{r_1} = ... = \varphi_1^* = ... = \varphi_{r_z}^* = 1$. 
In fact, it would be sufficient with the bounds
\[
C_+ \sqrt{|g|^2 - 2\Delta l_{kl,ij}^{\alpha\beta}} \leq \sigma_{ij,kl}^{\alpha\beta} \leq C_+ \sqrt{|g|^2 - 2\Delta l_{kl,ij}^{\alpha\beta}} \quad \text{if} \quad |g|^2 > 2\Delta l_{kl,ij}^{\alpha\beta},
\]
for some positive constants \(C_+ > 0\), on the scattering cross sections.

**Theorem 5.** The linearized collision operator \(\mathcal{L}\), with scattering cross sections (21) (or (22)), can be split into a positive multiplication operator \(\Lambda\), defined by \(\Lambda f = \nu f\), minus a compact operator \(K\) on \((L^2(d\xi))^\tau\)
\[
\mathcal{L} = \Lambda - K,
\]
where \(\nu = \nu(|\xi|) = \text{diag}(\nu_{1,1}, \ldots, \nu_{1,r_1}, \ldots, \nu_{s,1}, \ldots, \nu_{s,r_s})\) and there exist positive numbers \(\nu_-\) and \(\nu_+\), with \(0 < \nu_- < \nu_+\), such that for any \((\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_s\}\)
\[
\nu_- (1 + |\xi|) \leq \nu_{\alpha,i}(|\xi|) \leq \nu_+ (1 + |\xi|) \quad \text{for all} \quad \xi \in \mathbb{R}^3.
\]

The decomposition (23) follows by the decomposition (17), (18) and Theorem 4, while the bounds (24) are proven in Section 5.

**Corollary 2.** The linearized collision operator \(\mathcal{L}\), with scattering cross sections (21) (or (22)), is a Fredholm operator, with domain
\[
D(\mathcal{L}) = (L^2((1 + |\xi|) d\xi))^\tau.
\]

**Proof.** By Theorem 5 the multiplication operator \(\Lambda\) is coercive and, hence, a Fredholm operator. The set of Fredholm operators is closed under addition of compact operators, see Theorem 5.26 of Chapter IV in [18] and its proof, so, by Theorem 5, \(\mathcal{L}\) is a Fredholm operator.

Moreover, by Theorem 5, \(D(\mathcal{L}) = D(\Lambda) = (L^2((1 + |\xi|) d\xi))^\tau\). \(\square\)

**Corollary 3.** For the linearized collision operator \(\mathcal{L}\), with scattering cross sections (21) (or (22)), there exists a positive number \(\lambda, 0 < \lambda < 1\), such that
\[
(h, \mathcal{L}h) \geq \lambda (h, \nu(|\xi|)h) \geq \lambda \nu_- (h, (1 + |\xi|)h)
\]
for any \(h \in (L^2((1 + |\xi|) d\xi))^\tau \cap \text{Im} \mathcal{L}\).

**Proof.** Let \(h \in (L^2((1 + |\xi|) d\xi))^\tau \cap (\text{ker} \mathcal{L})^\perp = (L^2((1 + |\xi|) d\xi))^\tau \cap \text{Im} \mathcal{L}\). As a Fredholm operator, \(\mathcal{L}\) is closed with a closed range, and as a compact operator, \(K\) is bounded, and so there are positive constants \(\nu_0 > 0\) and \(c_K > 0\), such that
\[
(h, \mathcal{L}h) \geq \nu_0(h, h) \quad \text{and} \quad (h, Kh) \leq c_K(h, h).
\]
Let \(\lambda = \frac{\nu_0}{\nu_0 + c_K}\). Then
\[
(h, \mathcal{L}h) = (1 - \lambda)(h, \mathcal{L}h) + \lambda(h, \nu(|\xi|) - K)h) \\
\geq (1 - \lambda)\nu_0(h, h) + \lambda(h, \nu(|\xi|)h) - \lambda c_K(h, h) \\
= (\nu_0 - \lambda(\nu_0 + c_K))(h, h) + \lambda(h, \nu(|\xi|)h) = \lambda(h, \nu(|\xi|)h). \quad \square
\]
Remark 5. By Proposition 3 and Corollary 1-3, for hard sphere like models the linearized operator $L$ fulfills the properties assumed on the linear operators in [5], and hence, the results therein can be applied for hard sphere like models.

4. Compactness

This section concerns the proof of Theorem 4.

Note that in the proof the kernels are rewritten in such a way that $\xi_*$ - and not $\xi'$ and $\xi'_*$ - always will be an argument of the distribution functions. As for single species, either $\xi_*$ is an argument in the loss term (like $\xi$) or in the gain term (unlike $\xi$) of the collision operator. However, in the latter case, unlike for single species, one has to differ between two different situations for mixtures (while considering interspecies collision operators); either $\xi_*$ is the velocity of particles of the same species as the particles with velocity $\xi$, or not. The kernels of the terms from the loss part of the collision operator will be shown to be Hilbert-Schmidt in a quite direct way. Some of the terms - for which $\xi_*$ is the velocity of particles of the same species as the particles with velocity $\xi$ - of the gain parts of the collision operators will be shown to be uniform limits of Hilbert-Schmidt integral operators, i.e., approximately Hilbert-Schmidt integral operators in the sense of Lemma 4. By applying the following lemma, Lemma 5, (for disparate masses), which is a generalization of corresponding lemma for monatomic mixtures by Boudin et al. in [10], see also [6], it will be shown that the kernels of the remaining terms - for which $\xi_*$ is the velocity of particles of a species different to the particles with velocity $\xi$ - from the gain parts of the collision operators, are Hilbert-Schmidt.

Denote, for any (non-zero) natural number $N$,

$$\mathfrak{h}_N := \left\{ (\xi, \xi_*) \in (\mathbb{R}^3)^2 : |\xi - \xi_*| \geq \frac{1}{N}; |\xi| \leq N \right\}$$

and

$$b^{(N)} = b^{(N)}(\xi, \xi_*) := b(\xi, \xi_*) \mathbf{1}_{\mathfrak{h}_N}.$$ 

Then we have the following lemma from [14], that will be of practical use for us to obtain compactness in this section.

Lemma 4. (Glassey [14, Lemma 3.5.1], Drange [12])

Assume that $b(\xi, \xi_*) \geq 0$ and let $T f (\xi) = \int_{\mathbb{R}^3} b(\xi, \xi_*) f (\xi_*) d\xi_*$. Then $T$ is compact on $L^2 (d\xi)$ if

(i) $\int_{\mathbb{R}^3} b(\xi, \xi_*) d\xi$ is bounded in $\xi_*$;
(ii) $b^{(N)} \in L^2 (d\xi d\xi_*)$ for any (non-zero) natural number $N$;
(iii) $\sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{R}^3} b(\xi, \xi_*) - b^{(N)}(\xi, \xi_*) d\xi_* \to 0$ as $N \to \infty$.

Then the operator $T$ is the uniform limit of Hilbert-Schmidt integral operators, and we say that the kernel $b(\xi, \xi_*)$ is approximately Hilbert-Schmidt, while $T$ is an approximately Hilbert-Schmidt integral operator. The reader is referred to Lemma 3.5.1 in [14] for a proof.

Lemma 5. For $(\alpha, \beta, i, j, k, l) \in \Omega$, assume that $m_\alpha \neq m_\beta$,

$$\begin{cases}
\xi' = \xi - |\xi - \xi'| \eta \\
\xi'_* = \xi_* - \frac{m_\alpha}{m_\beta} |\xi - \xi'| \eta
\end{cases}, \text{ with } \eta \in \mathbb{S}^2, \tag{25}$$
and
\[ m_\alpha = \frac{|\xi|^2}{2} + m_\beta = \frac{|\xi'|}{2} + \frac{|\xi_\star|^2}{2} + \Delta I_{k,j,il}^{\alpha \beta}, \]
where
\[ \Delta I_{k,j,il}^{\alpha \beta} = I_{k}^{\alpha} + I_{j}^{\beta} - I_{i}^{\alpha} - I_{l}^{\beta}. \]

Then there exists a positive number \( \rho, 0 < \rho < 1 \), such that
\[ m_\alpha |\xi'|^2 + m_\beta |\xi_\star|^2 \geq \rho \left( m_\alpha |\xi|^2 + m_\beta |\xi_\star|^2 \right) + (1 + \rho) \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} \Delta I_{k,j,il}^{\alpha \beta} \]
\[ \geq \rho \left( m_\alpha |\xi|^2 + m_\beta |\xi_\star|^2 \right) - 2 |\Delta I_{k,j,il}^{\alpha \beta}|. \]

A proof of Lemma 5, based on the proof of the corresponding lemma [10] for monatomic mixtures in [6], is accounted for in the appendix. The proof is constructive, in the way that an explicit value of such a number \( \rho \), namely
\[ \rho = \left( \frac{\sqrt{m_\alpha} - \sqrt{m_\beta}}{\sqrt{m_\alpha} + \sqrt{m_\beta}} \right)^2, \]
is produced in the proof.

Now we turn to the proof of Theorem 4. Note that throughout the proof \( C \) will denote a generic positive constant.

**Proof.** For \((\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}\), rewrite expression (18) as
\[
K_{\alpha,i} = (M_{\alpha,i})^{-1/2} \sum_{\beta=1}^{s} \sum_{k=1}^{r_\alpha} \sum_{j,l=1}^{r_\beta} \int_{\mathbb{R}^3} w_{\alpha \beta}(\xi, \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', I_{k}^{\alpha}, I_{l}^{\beta} \bigg\rangle \bigg\langle \xi', I_{k}^{\alpha}, I_{l}^{\beta} \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle \bigg\rangle \bigg\langle \xi', I_{k}^{\alpha}, I_{l}^{\beta} \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle
\times \left( \frac{h_{\alpha,k}'}{(M_{\alpha,k}')^{1/2}} + \frac{h_{\beta,l}'}{(M_{\beta,l}')^{1/2}} - \frac{h_{\beta,j}'}{(M_{\beta,j}')^{1/2}} \right) d\xi_\star d\xi' d\xi_\star.
\]

with
\[
w_{\alpha \beta}(\xi, \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle
= \left( \frac{M_{\alpha,i} M_{\beta,j} M_{\alpha,k} M_{\beta,l}'}{\varphi_1^\alpha \varphi_2^\alpha \varphi_3^\beta \varphi_1^\beta} \right)^{1/2} \bigg\langle W_{\alpha \beta}(\xi, \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle.
\]

Due to relations (9), the relations
\[
w_{\alpha \beta}(\xi, \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle = w_{\beta \alpha}(\xi, \xi_\star, I_{i}^{\beta}, I_{j}^{\alpha}) \bigg| \xi', \xi_\star, I_{i}^{\beta}, I_{j}^{\alpha} \bigg\rangle
\]
\[
w_{\alpha \beta}(\xi, \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle = w_{\alpha \beta}(\xi_\star, \xi, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle
\]
\[
w_{\alpha \alpha}(\xi, \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle = w_{\alpha \alpha}(\xi_\star, \xi, I_{i}^{\alpha}, I_{j}^{\beta}) \bigg| \xi', \xi_\star, I_{i}^{\alpha}, I_{j}^{\beta} \bigg\rangle
\]
are satisfied for \((\alpha, \beta, i, j, k, l) \in \Omega\).
By renaming \( \{ \xi_s, j \} \mapsto \{ \xi_s', l \} \), for \( i \in \{1, ..., r \} \) and \( \{ \alpha, \beta \} \subseteq \{1, ..., s \} \)

\[
\sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^3} w_{\alpha\beta}(\xi, \xi_s, I^\alpha, I^\beta) \left| \xi', \xi_s', I^\alpha, I^\beta \right| \frac{h_{\alpha,l}'}{M_{\beta,l}'} \, d\xi_s \, d\xi' \, d\xi_s' \n
= \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^3} w_{\alpha\beta}(\xi, \xi_s', I^\alpha, I^\beta) \left| \xi', \xi_s', I^\alpha, I^\beta \right| \frac{h_{\beta,l}'}{M_{\beta,l}'} \, d\xi_s \, d\xi' \, d\xi_s'.
\]

Moreover, by renaming \( \{ \xi_s, j \} \mapsto \{ \xi', k \} \),

\[
\sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^3} w_{\alpha\beta}(\xi, \xi, I^\alpha, I^\beta) \left| \xi', \xi, I^\alpha, I^\beta \right| \frac{h_{\alpha,k}'}{M_{\alpha,k}'} \, d\xi_s \, d\xi' \, d\xi_s'
\]

\[
= \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^3} w_{\alpha\beta}(\xi, \xi', I^\alpha, I^\beta) \left| \xi', \xi, I^\alpha, I^\beta \right| \frac{h_{\beta,l}'}{M_{\alpha,k}'} \, d\xi_s \, d\xi' \, d\xi_s',
\]

for \( i \in \{1, ..., r \} \) and \( \{ \alpha, \beta \} \subseteq \{1, ..., s \} \). It follows that

\[
K_{\alpha,i}(h) = \sum_{\beta=1}^{s} \int_{(\mathbb{R}^3)^3} k_{\alpha\beta,i}(\xi, \xi_s) \, h_s \, d\xi_s, \text{ where}
\]

\[
k_{\alpha\beta,i} h_s = \sum_{j=1}^{r_{\beta}} k_{\alpha\beta,ij}(\xi_s) \, h_{\alpha,s} + \sum_{j=1}^{r_{\beta}} k_{\alpha\beta,ij}(\xi_s) \, h_{\beta,s}
\]

\[
= \sum_{j=1}^{r_{\beta}} k_{\alpha\beta,ij}(\xi_s) \, h_{\alpha,s} + \sum_{j=1}^{r_{\beta}} \left( k_{\alpha\beta,ij}^{(2)} - k_{\alpha\beta,ij}^{(1)} \right) \, h_{\beta,s}, \text{ with}
\]

\[
k_{\alpha\beta,ij}(\xi, \xi_s) = \sum_{k,l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^2} w_{\alpha\beta}(\xi, \xi', I^\alpha, I^\beta) \left| \xi', \xi_s', I^\alpha, I^\beta \right| \frac{h_{\alpha,l}'}{M_{\beta,l}'} \, d\xi' \, d\xi_s',
\]

\[
k_{\alpha\beta,ij}^{(1)}(\xi, \xi_s) = \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^2} w_{\alpha\beta}(\xi, \xi_s, I^\alpha, I^\beta) \left| \xi', \xi_s', I^\alpha, I^\beta \right| \frac{h_{\alpha,l}'}{M_{\beta,l}'} \, d\xi' \, d\xi_s',
\]

\[
k_{\alpha\beta,ij}^{(2)}(\xi, \xi_s) = \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^2} w_{\alpha\beta}(\xi, \xi_s, I^\alpha, I^\beta) \left| \xi', \xi_s', I^\alpha, I^\beta \right| \frac{h_{\alpha,l}'}{M_{\beta,l}'} \, d\xi' \, d\xi_s',
\]

for \((\alpha, i) \in \{1, ..., s\} \times \{1, ..., r\}\).

Next we obtain some symmetry relations that will help to yield self-adjointness of the operator \( K \) below. Indeed, by applying the second relation in (27) and renaming \( \{ \xi', k \} \mapsto \{ \xi_s', l \} \),

\[
k_{\alpha\beta,ij}(\xi, \xi_s) = \sum_{k,l=1}^{r_{\beta}} \int_{(\mathbb{R}^3)^2} w_{\alpha\beta}(\xi, \xi_s, I^\alpha, I^\beta) \left| \xi', \xi_s', I^\alpha, I^\beta \right| \frac{h_{\alpha,l}'}{M_{\beta,l}'} \, d\xi' \, d\xi_s'.
\]
follows,

\[
\sum_{k,l=1}^{r^a} \int \frac{w_{\alpha \beta}(\xi_*^t, \xi_*^t, I_k^\alpha, I_k^\beta) \big| \xi_*^t, \xi_*^t, I_k^\alpha, I_k^\beta \big)}{(M_{\alpha,i}M_{\alpha,j})^{1/2}} d\xi_*^t d\xi_*^t
\]

\[
= k_{\alpha \beta, ij}^{(\gamma)}(\xi_*^t, \xi_*)
\]

for \((i, j) \in \{1, \ldots, r_\alpha\}^2\) and \((\alpha, \beta) \in \{1, \ldots, s\}^2\).

Moreover, for \((i, j) \in \{1, \ldots, r_\alpha\} \times \{1, \ldots, r_\beta\}\) and \((\alpha, \beta) \in \{1, \ldots, s\}^2\)

\[
k_{\alpha \beta, ij}^{(\gamma)}(\xi_*^t, \xi_*) = k_{\alpha \beta, ji}^{(\gamma,1)}(\xi_*^t, \xi_*) - k_{\alpha \beta, ji}^{(\gamma,2)}(\xi_*^t, \xi_*) = k_{\beta \alpha, ji}^{(\gamma)}(\xi_*^t, \xi_*)
\]

since, by applying the first relation in (27) and renaming \(\{\xi_*^t, k\} \equiv \{\xi_*^t, l\}\),

\[
k_{\alpha \beta, ij}^{(\gamma,1)}(\xi_*^t, \xi_*) = \sum_{k=1}^{r_\alpha} \sum_{l=1}^{r_\beta} \int w_{\alpha \beta}(\xi_*^t, \xi_*^t, I_j^\beta, I_l^\alpha) \big| \xi_*^t, \xi_*^t, I_j^\beta, I_l^\alpha \big) \frac{d\xi_*^t d\xi_*^t}{(M_{\alpha,i}M_{\alpha,j})^{1/2}}
\]

\[
= k_{\beta \alpha, ji}^{(\gamma)}(\xi_*^t, \xi_*)
\]

while, by applying the two first relations in (27) and renaming \(\{\xi_*^t, k\} \equiv \{\xi_*^t, l\}\),

\[
k_{\alpha \beta, ij}^{(\gamma,2)}(\xi_*^t, \xi_*) = \sum_{k=1}^{r_\alpha} \sum_{l=1}^{r_\beta} \int w_{\alpha \beta}(\xi_*^t, \xi_*^t, I_j^\beta, I_l^\alpha) \big| \xi_*^t, \xi_*^t, I_j^\beta, I_l^\alpha \big) \frac{d\xi_*^t d\xi_*^t}{(M_{\alpha,i}M_{\alpha,j})^{1/2}}
\]

\[
= k_{\beta \alpha, ji}^{(\gamma)}(\xi_*^t, \xi_*)
\]

We now continue by proving the compactness for the three different types of collision kernel separately. Note that, if \(\alpha = \beta\), by applying the last relation in (27), \(k_{\alpha \beta, ij}^{(\gamma)}(\xi_*^t, \xi_*) = k_{\alpha \beta, ij}^{(\gamma)}(\xi_*^t, \xi_*)\), and we will remain with only two cases - the first two below. Even if \(m_\alpha = m_\beta\), the kernels \(k_{\alpha \beta, ij}^{(\gamma)}(\xi_*^t, \xi_*)\) and \(k_{\beta \alpha, ij}^{(\gamma)}(\xi_*^t, \xi_*)\) are structurally equal, why we (in principle) remain with (first) two cases (the second one twice).

I. Compactness of \(k_{\alpha \beta, ij}^{(\gamma)} = \int_{\mathbb{R}^3} k_{\alpha \beta, ij}^{(\gamma,1)}(\xi_*^t, \xi_*) h_{\beta, j*} d\xi_*^t\) for the index pairs \((i, j) \in \{1, \ldots, r_\alpha\} \times \{1, \ldots, r_\beta\}\) and \((\alpha, \beta) \in \{1, \ldots, s\}^2\).

Regarding this term we can repeat the same steps like in the corresponding proof for a polyatomic single species [6]. By using a standard parametrization, it is a quite direct consequence that the kernels are Hilbert-Schmidt.

Assume the internal energy gap \(\Delta I_{kl, ij}^{\alpha \beta} = I_k^\alpha + I_l^\beta - I_i^\alpha - I_j^\beta\), as well as, the velocities \(\xi_*^t\) and \(\xi_*^t\), to be given. Then a collision will be uniquely determined by the unit vector \(\omega = g' / |g'|\), with \(g' = \xi_*^t - \xi_*^t\). This follows, since, by conservation of momentum and total energy (2), \(m_\alpha (\xi_*^t - \xi_*^t) = m_\beta (\xi_*^t - \xi_*^t)\), while also \(|g'|\) can be obtained, cf. Fig. 1.
Indeed, expression (28) of \( k_{\alpha\beta,ij}^{(\beta,1)} \) may be transformed - by a classical change of variables, cf. Fig. 1, \( \{\xi',\xi''\} \rightarrow \left\{ \left|\frac{g'}{|g'|}\right|, G_{\alpha\beta} = \frac{m_{\alpha}\xi' + m_{\beta}\xi''}{m_{\alpha} + m_{\beta}} \right\} \), noting equality (10), and using relation (14) - to

\[
k_{\alpha\beta,ij}^{(\beta,1)}(\xi,\xi) = (M_{\alpha,i}M_{\beta,j})^{1/2} |g| \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} |g|^2 \int_{\mathbb{R}^3} \left( \sigma_{ij,kl}^\alpha G_{\alpha\beta} - G_{\alpha\beta}' \right) \delta_3 \left( G_{\alpha\beta} - G_{\alpha\beta}' \right) dG_{\alpha\beta}' d|g'| d\omega
\]

with \( \cos \theta = \omega \cdot \frac{g}{|g'|} \), \( g = \xi - \xi' \), \( G_{\alpha\beta} = \frac{m_{\alpha}\xi + m_{\beta}\xi'}{m_{\alpha} + m_{\beta}} \), and \( \widetilde{I}_{kl,ij}^{\alpha\beta} = \frac{m_{\alpha} + m_{\beta}}{m_{\alpha}m_{\beta}} I_{kl,ij}^{\alpha\beta} \).

By assumption (20) and the expression

\[
m_{\alpha} \frac{|\xi|^2}{2} + m_{\beta} \frac{|\xi'|^2}{2} + I_{i}^{\alpha} + I_{j}^{\beta} = \frac{m_{\alpha} + m_{\beta}}{2} |G_{\alpha\beta}|^2 + E_{ij}^{\alpha\beta}, \text{ where}
\]

\[
E_{ij}^{\alpha\beta} = \frac{m_{\alpha}m_{\beta}}{2(m_{\alpha} + m_{\beta})} |g|^2 + I_{i}^{\alpha} + I_{j}^{\beta},
\]

for the exponent of the product \( M_{\alpha,i}M_{\beta,j'} \), the bound

\[
\left( k_{\alpha\beta,ij}^{(\beta,1)}(\xi,\xi) \right)^2 \leq \frac{C}{|g|^2} \left( \frac{m_{\alpha} + m_{\beta}}{|G_{\alpha\beta}|^2} \right)^{1/2} \left( \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \left( \psi_{ij,kl}^\alpha + \psi_{ij,kl}^\beta \right)^{\gamma/2} \right) \left( \int_{S^2} \int_{|\omega| > 2\tilde{\Delta}I_{kl,ij}^{\alpha\beta}} d\omega \right)^2
\]

\[
\leq \frac{C}{|g|^2} e^{-\left( m_{\alpha} + m_{\beta} \right) |G_{\alpha\beta}|^2} \left( \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \left( 1 + |g|^2 \right)^2 \right)
\]
may be obtained. Then, by applying the bound (32), changing variables of integration \( \{ \xi, \xi_\star \} \rightarrow \{ g, G_{\alpha\beta} \} \), with unitary Jacobian, followed by changes to spherical coordinates,

\[
\begin{align*}
\int_{(\mathbb{R}^3)^2} \left( k_{\alpha\beta,ij}^{(1)}(\xi, \xi_\star) \right)^2 d\xi d\xi_\star \\
&\leq C \int_{(\mathbb{R}^3)^2} e^{-(m_\alpha+m_\beta)|G_{\alpha\beta}|^2/2 - E_{\alpha\beta}^{ij}} \frac{1+|g|^4}{|g|^2} dg dG_{\alpha\beta} \\
&\leq C \int_0^{\infty} R^2 e^{-(m_\alpha+m_\beta)|G_{\alpha\beta}|^2/2} dR \int_0^{\infty} e^{-m_\alpha m_\beta s^2/(2(m_\alpha+m_\beta))} (1+s^4) \, ds = C.
\end{align*}
\]

Note that, here and below, we will, in general, not indicate an integration over a directional vector in \( S^2 \) of the form

\[
\int_{S^2} d\omega = 4\pi,
\]

but just integrate it in the generic constant \( C \).

Hence,

\[
K_{\alpha\beta,ij}^{(1)} = \int_{\mathbb{R}^3} k_{\alpha\beta,ij}^{(1)}(\xi, \xi_\star) h_{\beta,j_\star} d\xi_\star
\]

are Hilbert-Schmidt integral operators and as such continuous and compact on \( L^2(d\xi) \), see Theorem 7.83 in [21], for \( (i,j) \in \{1,\ldots,r_\alpha\} \times \{1,\ldots,r_\beta\} \) and \( (\alpha, \beta) \in \{1,\ldots,s\}^2 \).

II. Compactness of \( K_{\alpha\beta,ij}^{(3)} = \int_{\mathbb{R}^3} k_{\alpha\beta,ij}^{(3)}(\xi, \xi_\star) h_{\alpha,j_\star} d\xi_\star \) for \( (i,j) \in \{1,\ldots,r_\alpha\}^2 \) and \( (\alpha, \beta) \in \{1,\ldots,s\}^2 \).

For this term the proof is more or less a combination of the corresponding proofs in the polyatomic single species and monatomic multicomponent mixture cases [6], which both are generalizations of a proof
in the monatomic single species case [15,14], and as such already follow a similar structure. Here by using a different parametrization we show that the kernels are approximately Hilbert-Schmidt in the sense of Lemma 4.

Assume the internal energy gap \( \Delta I^\alpha_{ik,jl} = I^\alpha_i + I^\alpha_k - I^\alpha_j - I^\beta_l \), as well as, the velocities \( \xi \) and \( \xi_* \), to be given. Then a collision will be uniquely determined by a vector \( w \) orthogonal to \( g = \xi - \xi_* \). This follows, since, by conservation of momentum and total energy (2) (reminding the relabeling of the velocities and internal energies), the relation between \( |\xi - \xi_*| \) and \( |\xi'_* - \xi'_*| \) can be obtained, while additionally, cf. Fig. 2, \( m_\beta g' = m_\beta (\xi'_* - \xi'_\ast) = m_\alpha g \). Indeed, note that - aiming to obtain expressions for \( g' \) and \( \chi_+ \) in the arguments of the delta-functions,

\[
W_{\alpha\beta}(\xi, \xi', I_i^\alpha, I_k^\beta | \xi_*, \xi'_* , I_i^\alpha, I_k^\beta )
= (m_\alpha + m_\beta)^2 m_\alpha m_\beta \varphi_\alpha \varphi_\beta \varphi_\alpha \varphi_\beta \frac{|g|}{|g_*|} \delta_3 (m_\alpha g + m_\beta g') \\
\times \delta_1 \left( m_\alpha |g| \left( \chi_+ - \frac{m_\alpha - m_\beta}{2 m_\beta} |g| - \Delta I^\alpha_{ik,jl} \right) \right)
= \frac{(m_\alpha + m_\beta)^2}{|g_*| |g|} \varphi_\alpha \varphi_\beta \varphi_\alpha \varphi_\beta \delta_3 (m_\alpha + m_\beta) \delta_1 \left( \chi_+ - \frac{m_\alpha - m_\beta}{2 m_\beta} |g| - \Delta I^\alpha_{ik,jl} \right)
\]

where \( g = \xi - \xi_* \), \( g' = \xi' - \xi'_* \), \( \tilde{g} = \xi - \xi'_* \), \( g_* = \xi - \xi_* \), \( \Delta I^\alpha_{ik,jl} = I^\alpha_i + I^\alpha_k - I^\alpha_j - I^\beta_l \), and \( \chi_+ = (\xi'_* - \xi) \cdot n \), with \( n = \frac{g}{|g|} \). Then by performing a change of variables \( \{ \xi'_*, \xi'_\ast \} \rightarrow \{ \tilde{g}' = \xi' - \xi'_*, \tilde{g} = \xi_* - \xi \} \), where

\[
d\xi'd\xi'_* = dg'd\tilde{g} = dg'd\chi_+dw, \text{ with } w = \xi_* - \xi - \chi_+ n,
\]

the expression (28) of \( k_{\alpha\beta,ij}^{(0)} \) may be rewritten in the following way

\[
k_{\alpha\beta,ij}^{(0)}(\xi, \xi_*)
= \sum_{k,l=1}^{r_\beta} \left\{ \begin{array}{l}
\int \left( M'_{\beta,k} M'_{\beta,l} \right)^{1/2} \frac{W_{\alpha\beta}(\xi, \xi', I_i^\alpha, I_k^\beta | \xi_*, \xi'_* , I_i^\alpha, I_k^\beta )}{\varphi_\alpha \varphi_\beta \varphi_\alpha \varphi_\beta} \frac{1}{|g|} \frac{|g_\ast|}{|g|} \frac{1}{|g|} \frac{1}{|g_\ast|} \right) \frac{d\tilde{g}' \tilde{g}}{ |g| }, \\
\times \frac{(m_\alpha + m_\beta)^2}{|g_*| |g|} \varphi_\alpha \varphi_\beta \varphi_\alpha \varphi_\beta \delta_3 (m_\alpha + m_\beta) \delta_1 \left( \chi_+ - \frac{m_\alpha - m_\beta}{2 m_\beta} |g| - \Delta I^\alpha_{ik,jl} \right)
\right\}
\]

where \( \left\{ \begin{array}{l}
|g|^2 > 2\Delta I^\alpha_{ij,ik} \varphi_\alpha \varphi_\beta \varphi_\alpha \varphi_\beta, \\
\tilde{g}' \tilde{g} = \chi_+ n \end{array} \right\} \)

Here, see Fig. 2,

\[
\left\{ \begin{array}{l}
\xi' = \xi_* + w + \chi_+ n, \\
\xi'_* = \xi + w + \chi_- n, \end{array} \right\}
\]

implying that the kinetic energy part of the exponent of the product \( M'_{\beta,k} M'_{\beta,l} \ast \) equals
\[ m_\beta \frac{\xi_n^2}{2} + m_\beta \frac{\xi_n^2}{2} = m_\beta \left( \frac{\xi + \xi_s}{2} - \frac{\Delta I_{ik,jl}^{\alpha \beta}}{m_\alpha |g|} n + \mathbf{w} \right) + \frac{m_\alpha^2}{4m_\beta} |g|^2 \]

\[ = m_\beta \left( \frac{(\xi + \xi_s)_n}{2} + \mathbf{w} \right)^2 + m_\beta \left( \frac{(\xi + \xi_s)_n}{2} - \frac{\Delta I_{ik,jl}^{\alpha \beta}}{m_\alpha |g|} \right)^2 + \frac{m_\alpha^2}{4m_\beta} |g|^2 \]

\[ = m_\beta \left( \frac{(\xi + \xi_s)_n}{2} + \mathbf{w} \right)^2 + m_\beta \left( \frac{(\xi - \xi_s)}{2} \right)^2 + \frac{m_\alpha^2}{4m_\beta} |g|^2, \]

where

\[(\xi + \xi_s)_n = (\xi + \xi_s) \cdot n = \frac{|\xi|^2 - |\xi_s|^2}{|\xi - \xi_s|}, \text{ and} \]

\[(\xi + \xi_s)_\perp = \xi + \xi_s - (\xi + \xi_s)_n n. \]

Hence, by assumption (20) and the Cauchy-Schwarz inequality,

\[
\left( \frac{I_{\alpha \beta, kl}^{(n)}(\xi, \xi_s)}{|g|^2} \right)^2 \leq C \left( \sum_{k,l=1}^{r_\beta} \frac{1}{e^{\frac{|g|^2}{n_0}} + 1} \exp \left( -m_\beta \frac{\left( m_\alpha \left( |\xi_s|^2 - |\xi|^2 \right) + 2\Delta I_{ik,jl}^{\alpha \beta} \right)}{8m_\alpha^2 |g|^2} - \frac{m_\alpha^2}{8m_\beta} |g|^2 \right) \right)^2 
\]

\[
\times \int_{(\mathbb{R}^3)^n} \left( 1 + \frac{1}{|g|^2 + 2\Delta I_{ik,jl}^{\alpha \beta}} \right) \left( \frac{\exp \left( \frac{m_\beta}{2} \left( \xi + \xi_s \right)_\perp + \mathbf{w} \right)^2}{2} \right)^2 d\mathbf{w} \]

\[
\leq C \left( \sum_{k,l=1}^{r_\beta} \exp \left( -m_\beta \frac{\left( m_\alpha \left( |\xi_s|^2 - |\xi|^2 \right) + 2\Delta I_{ik,jl}^{\alpha \beta} \right)}{8m_\alpha^2 |g|^2} - \frac{m_\alpha^2}{8m_\beta} |g|^2 \right) \right)^2 
\]

\[
= C \left( \sum_{k,l=1}^{r_\beta} \exp \left( -\frac{m_\beta}{8} \left( |g| - 2 |\xi| \cos \varphi + 2\lambda_{ik,jl}^{\alpha \beta} \right)^2 - \frac{m_\alpha^2}{8m_\beta} |g|^2 \right) \right)^2 
\]

\[
\leq C \left( \sum_{k,l=1}^{r_\beta} \exp \left( -\frac{m_\beta}{8} \left( |g| - 2 |\xi| \cos \varphi + 2\lambda_{ik,jl}^{\alpha \beta} \right)^2 - \frac{m_\alpha^2}{4m_\beta} |g|^2 \right), \right) 
\]

\[
\lambda_{ik,jl}^{\alpha \beta} = \lambda_{ik,jl}^{\alpha \beta} \left( |g| \right) = \frac{\Delta I_{ik,jl}^{\alpha \beta}}{m_\alpha |g|}, \cos \varphi = \mathbf{n} \cdot \frac{\xi}{|\xi|}, \tilde{\psi}_{ik,jl}^{\alpha \beta} = |g| \cdot |g_s|, \text{ and} \]

\[
|g_s|^2 = |g|^2 - 2 \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{ik,jl}^{\alpha \beta}. \quad (33) 
\]

Here, the second inequality follows by the following bound, which can be obtained by noting that \( \min \left( |g|, |g_s| \right) \geq |\mathbf{w}| \), cf. Fig. 2, and making a change of variables \( \mathbf{w} \rightarrow \tilde{\mathbf{w}} = (\xi + \xi_s)_\perp / 2 + \mathbf{w} \) followed by one to polar coordinates,
\[
\int_{(\mathbb{R}^3)^n} \left(1 + \frac{1}{|\xi|^2} \frac{\partial^2}{\partial x_{ik} \partial x_{jl}} \right) \exp \left( - \frac{m_\beta}{2} \left( |\xi + \xi_*| \right)_{\pm n} + w \right)^2 \right) \, dw \\
\leq \int_{|w| \leq 1} 1 + |w|^{\gamma - 2} \, dw + 2 \int_{|w| \geq 1} e^{-m_\beta |\xi|^2 / 2} \, d\tilde{w} \\
= 2\pi \left( \int_{0}^{1} R + R^{\gamma - 1} \, dR + 2 \int_{0}^{\infty} R^{-m_\beta R^2 / 2} \, dR \right) = C.
\]

Then \( k^{(\alpha)}_{\alpha \beta, ij}(\xi, \xi_*) \mathbb{I}_{\mathbb{N}} \in L^2(\mathbb{R}^3 \, d\xi \, d\xi_*) \). Indeed, by changing variables \( \xi \to \mathbf{g} \), with \( \mathbf{g} = \xi - \xi_* \), and then to spherical coordinates,

\[
\int_{\mathbb{R}^3} \left( k^{(\alpha)}_{\alpha \beta, ij}(\xi, \xi_*) \right)^2 \, d\xi \, d\xi_* \leq \int_{\mathbb{R}^3} \frac{C}{|\mathbf{g}|^2} e^{-m_\alpha^2 |\mathbf{g}|^2 / (4m_\beta)} \, d\mathbf{g} \, d\xi \\
= C \int_{0}^{\infty} e^{-m_\alpha^2 R^2 / (4m_\beta)} \, dR \int_{0}^{\infty} \eta^2 \, d\eta = CN^3.
\]

Next we aim for proving that the integral of \( k^{(\alpha)}_{\alpha \beta, ij}(\xi, \xi_*) \) with respect to \( \xi \) over \( \mathbb{R}^3 \) is bounded \( \xi_* \). Indeed, directly by the bound (33) on \( \left( k^{(\alpha)}_{\alpha \beta, ij} \right)^2 \)

\[
0 \leq k^{(\alpha)}_{\alpha \beta, ij}(\xi, \xi_*) \leq \frac{C}{|\mathbf{g}|} \sum_{k,l=1}^{r_\beta} \exp \left( - \frac{m_\beta}{8} \left( |\mathbf{g}| - 2 |\xi| \cos \varphi + 2 \chi^\beta_{ik,jl} \right)^2 \right) - \frac{m_\alpha^2}{8m_\beta} |\mathbf{g}|^2.
\]

Hence, due to the symmetry \( k^{(\alpha)}_{\alpha \beta, ij}(\xi, \xi_*) = k^{(\alpha)}_{\alpha \beta, ji}(\xi_*, \xi) \) (29), by a change of variables \( \xi \to \mathbf{g} = \xi - \xi_* \), followed by one to spherical coordinates,

\[
\int_{\mathbb{R}^3} k^{(\alpha)}_{\alpha \beta, ij}(\xi, \xi_*) \, d\xi = \int_{\mathbb{R}^3} k^{(\alpha)}_{\alpha \beta, ji}(\xi_*, \xi) \, d\xi \\
\leq \int_{\mathbb{R}^3} \frac{C}{|\mathbf{g}|} \sum_{k,l=1}^{r_\beta} \exp \left( - \frac{m_\alpha^2}{8m_\beta} |\mathbf{g}|^2 \right) \, d\mathbf{g} \\
= C \int_{0}^{\infty} R^{-m_\alpha^2 R^2 / (8m_\beta)} \, dR = C.
\]

Finally, heading for proving the uniform convergence of the integral of \( k^{(\alpha)}_{\alpha \beta, ij} \) with respect to \( \xi_* \) over the truncated domain \( h_N \) to the one over all of \( \mathbb{R}^3 \), the following bound on the integral over \( \mathbb{R}^3 \) can be obtained for \( |\xi| \neq 0 \). Indeed, by bound (34), by changing variables \( \xi_* \to \mathbf{g} = \xi - \xi_* \), then to (conventional) spherical coordinates, with \( \xi \) as zenithal direction, and hence, \( \varphi \) as polar angle, followed by the change of variables \( \varphi \to \eta = R - 2 |\xi| \cos \varphi + 2 \chi^\alpha_{ik}(R) \), with \( d\eta = 2 |\xi| \sin \varphi \, d\varphi \).
\[
\begin{align*}
&\int_{\mathbb{R}^3} k^{(\alpha)}_{\alpha\beta,ij}(\xi, \xi_*) \, d\xi_* \\
&\leq \int_{\mathbb{R}^3} \frac{C}{|g|} \sum_{k,l=1}^{r_{\beta}} R \exp \left( -\frac{m_{\beta}}{8} \left( R - 2 |\xi| \cos \varphi + 2 \chi_{ik,jl}(R) \right)^2 - \frac{m_{\alpha}^2}{8m_{\beta}} R^2 \right) \, dg \\
&= C \sum_{k,l=1}^{r_{\beta}} \int_0^\pi \int_0^\infty R \exp \left( -\frac{m_{\beta}}{8} \left( R - 2 |\xi| \cos \varphi + 2 \chi_{ik,jl}(R) \right)^2 - \frac{m_{\alpha}^2}{8m_{\beta}} R^2 \right) \\
&\quad \times \sin \varphi \, d\varphi \, dR \\
&= C \frac{r_{\beta}}{|\xi|} \sum_{k,l=1}^{r_{\beta}} \int_0^\infty \int_{R+2\chi_{ik,jl}(R)-2|\xi|}^{\infty} Re^{-m_{\beta}n^2/8} e^{-m_{\alpha}^2 R^2/(8m_{\beta})} \, d\eta \, dR \\
&\leq C \frac{r_{\beta}}{|\xi|} \int_0^\infty Re^{-m_{\alpha}^2 R^2/(8m_{\beta})} \, dR \int_{-\infty}^\infty e^{-m_{\beta}n^2/8} \, d\eta = \frac{C}{|\xi|}, \tag{35}
\end{align*}
\]

Then, by the bounds (34) and (35),

\[
\begin{align*}
&\sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{R}^3} \left| k^{(\alpha)}_{\alpha\beta,ij}(\xi, \xi_*) - k^{(\alpha)}_{\alpha\beta,ij}(\xi, \xi_*) \right| 1_{B_N} \, d\xi_* \\
&\leq \sup_{\xi \in \mathbb{R}^3} \int_{|g| \leq \frac{N}{4}} \left| k^{(\alpha)}_{\alpha\beta,ij}(\xi, \xi_*) \right| \, dg + \sup_{|\xi| \geq N} \int_{\mathbb{R}^3} \left| k^{(\alpha)}_{\alpha\beta,ij}(\xi, \xi_*) \right| \, d\xi_* \\
&\leq \int_{|g| \leq \frac{N}{4}} \frac{C}{|g|} \, dg + \frac{C}{N} \leq C \left( \int_0^\infty RdR + \frac{1}{N} \right) \\
&= C \left( \frac{1}{N^2} + \frac{1}{N} \right) \to 0 \text{ as } N \to \infty.
\end{align*}
\]

Hence, by Lemma 4 the operators

\[
K^{(3)}_{\alpha\beta,ij} = \int_{\mathbb{R}^3} k^{(\alpha)}_{\alpha\beta,ij}(\xi, \xi_*) \, h_{\alpha,j*} \, d\xi
\]

are compact on \( L^2(d\xi) \) for \( (i, j) \in \{1, \ldots, r_{\alpha}\}^2 \) and \( (\alpha, \beta) \in \{1, \ldots, s\}^2 \).

### III. Compactness

Compactness of \( K^{(2)}_{\alpha\beta,ij} = \int_{\mathbb{R}^3} k^{(\alpha,2)}_{\alpha\beta,ij}(\xi, \xi_*) \, h_{\beta,j*} \, d\xi_* \) for the index pairs \( (i, j) \in \{1, \ldots, r_{\alpha}\} \times \{1, \ldots, r_{\beta}\} \) and \( (\alpha, \beta) \in \{1, \ldots, s\}^2 \).

Firstly, assume that \( m_{\alpha} \neq m_{\beta} \). This term (for disparate masses) gives rise to the main extension of the corresponding proofs in the polyatomic single species and monatomic multicomponent mixture cases [6]. Disparate masses do not exist at all for single species, while for a mixture of monatomic species it follows quite directly that the kernel is Hilbert-Schmidt, by a suitable parametrization, the monatomic version [10,6] of Lemma 5, and an obvious bound - that in Fig. 3, \( |g| = |g| \geq |g| = |g'| \), if the internal energy gap vanishes like in the monatomic case. However, if the internal energy gap is non-zero this will not be the case anymore, and, even if we can follow the proof in the monatomic case and apply Lemma 5, we don’t have that essential bound. Instead, we obtain a satisfactory bound, by considering three different cases (H1-3 below), and then correspondingly can show that the kernel is Hilbert-Schmidt.
Assume that the internal energy gap $\Delta_{il,kj}^{\alpha\beta} = I_i^{\alpha} + I_l^{\beta} - I_k^{\alpha} - I_j^{\beta}$ and the velocities $\xi$ and $\xi^*$ are given. Then a collision will be uniquely determined by a unit vector $\eta = (\xi - \xi^*) / |\xi - \xi^*|$, or, $\omega = (\xi' - \xi^*) / |\xi' - \xi^*|$. This follows, since, by conservation of momentum and total energy (2) (reminding the relabeling of the velocities and internal energies), the relation between $|\xi - \xi^*|$ and $|\xi' - \xi^*|$ (or, equivalently, between $|\xi' - \xi^*|$ and $|\xi - \xi^*|$) can be obtained, while also, cf. Fig. 3, $m_{\beta} (\xi, \xi^*) = m_{\alpha} (\xi - \xi^*)$. Indeed, note that - with the aim to obtain expressions for $|g'|$ and $g_{\alpha\beta} = m_{\alpha} \xi^*_{\alpha} - m_{\beta} \xi^*_{\beta}$ in the arguments of the delta-functions,

$$W_{\alpha\beta}(\xi, \xi^*, I_i^{\alpha}, I_l^{\beta} | \xi', \xi^*, I_k^{\alpha}, I_j^{\beta})$$

$$= (m_{\alpha} + m_{\beta})^2 m_{\alpha} m_{\beta} \varphi_i^{\alpha} \varphi_l^{\beta} \sigma_{il,kj}^{\alpha\beta} \frac{|g|}{|g|} \delta_3 \left( (m_{\alpha} - m_{\beta}) (g_{\alpha\beta} - g'_{\alpha\beta}) \right)$$

$$\times \delta_1 \left( \frac{m_{\alpha} m_{\beta}}{2 (m_{\alpha} - m_{\beta})} \left( |g'|^2 - |g|^2 \right) + \Delta_{il,kj}^{\alpha\beta} \right)$$

$$= \frac{(m_{\alpha} + m_{\beta})^2}{(m_{\alpha} - m_{\beta})^2} \varphi_i^{\alpha} \varphi_l^{\beta} \sigma_{il,kj}^{\alpha\beta} \frac{|g|}{|g|} \delta_3 \left( (m_{\alpha} - m_{\beta}) (g_{\alpha\beta} - g'_{\alpha\beta}) \right)$$

$$\times \delta_1 \left( |g| - \sqrt{|g|^2 - 2 \Delta_{il,kj}^{\alpha\beta}} \right), \text{ with } g = \xi - \xi^*, g' = \xi' - \xi^*,$$

$$g = \xi^* - \xi, \bar{g} = \xi - \xi', g_{\alpha\beta} = m_{\alpha} \xi^*_{\alpha} - m_{\beta} \xi^*_{\beta}, g'_{\alpha\beta} = m_{\alpha} \xi^*_{\alpha} - m_{\beta} \xi^*_{\beta},$$

$$\Delta_{il,kj}^{\alpha\beta} = m_{\alpha} - m_{\beta} \Delta_{il,kj}, \text{ and } \Delta_{il,kj}^{\alpha\beta} = I_j^{\alpha} + I_l^{\beta} - I_k^{\alpha} - I_j^{\beta}.$$  

Then, by a change of variables $\{\xi', \xi^*\} \rightarrow \{g' = \xi' - \xi^*, g'_{\alpha\beta} = m_{\alpha} \xi^*_{\alpha} - m_{\beta} \xi^*_{\beta}\}$, followed by one to spherical coordinates, where

$$d\xi'd\xi^* = dg'dg'_{\alpha\beta} = |g'|^2 d|g'| d\omega_{\alpha\beta}, \text{ with } \omega = \frac{g'}{|g'|}$$

the expression (28) of $k_{\alpha\beta,ij}^{(2)}$ may be transformed to
\[
\begin{align*}
k^{(\beta,2)}_{\alpha\beta,i,j}(\xi, \xi_s) &= \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{S^2} w_{\alpha\beta}(\xi, \xi', I^{\alpha}_{i}, I^{\beta}_{l}) \left| \frac{\xi' - I^{\alpha}_{i}}{2} \right| \left| \frac{\xi_s - I^{\beta}_{l}}{2} \right| \left| \frac{g'}{g} \right| \left| \frac{d|g'}{d|g_s} \right| d\omega \\
&= \left( m_{\alpha} + m_{\beta} \right)^2 \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{S^2} \left( M^{\alpha}_{k,l} M^{\beta}_{k,l} \right)^{1/2} \left| \frac{g'}{g} \right| \left( \frac{\varphi^{\alpha}_{i} \varphi^{\beta}_{j}}{\varphi^{\alpha}_{k} \varphi^{\beta}_{l}} \right)^{1/2} \\
&\times 1_{|g'|^2 > 2\Delta l^{\alpha}_{k,j,i,l}} \left( |\hat{g}|, \frac{\hat{g} \cdot \xi}{|\hat{g}|} \right) d\omega. \\
\end{align*}
\]

Here, see Fig. 3,

\[
\begin{align*}
\xi' &= \xi - \frac{|\xi - \xi'|}{\eta} \\
\xi_s &= \xi_s - \frac{m_{\alpha}}{m_{\beta}} |\xi - \xi'| \eta \\
\text{for} \quad \eta = \frac{\xi - \xi'}{|\xi - \xi'|} \in S^2.
\end{align*}
\]

Then, by Lemma 3, since relation (26) follows by energy conservation, we have the following relation between the kinetic parts of the exponents of the products \( M^{\alpha}_{k,l} M^{\beta}_{k,l} \) and \( M^{\alpha}_{i,j} M^{\beta}_{i,j} \), respectively

\[
m_{\alpha} \left| \xi' \right|^2 + m_{\beta} \left| \xi_s \right|^2 \geq \rho \left( m_{\alpha} \left| \xi \right|^2 + m_{\beta} \left| \xi_s \right|^2 \right) - 2 \left| \Delta l^{\alpha}_{k,j,i,l} \right|,
\]

for some positive number \( \rho \), where \( 0 < \rho < 1 \). However, also

\[
k^{(\beta,2)}_{\alpha\beta,i,j}(\xi, \xi_s) \\
= \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{S^2} \left| g' \right| \left( M^{\alpha}_{k,l} M^{\beta}_{k,l} \right)^{1/2} \left( \frac{\varphi^{\alpha}_{i} \varphi^{\beta}_{j}}{\varphi^{\alpha}_{k} \varphi^{\beta}_{l}} \right)^{1/2} \sigma^{\alpha\beta}_{k,j,l} (|g'|, \cos \theta) 1_{|g'|^2 > 2\Delta l^{\alpha}_{k,j,i,l}} d\omega,
\]

with \( \cos \theta = \hat{\omega} \cdot \frac{\hat{g}}{|\hat{g}|} \) and \( \tilde{\Delta} l^{\alpha}_{k,j,i,l} = \frac{m_{\alpha} + m_{\beta}}{m_{\alpha} m_{\beta}} \Delta l^{\alpha}_{k,j,i,l} \),

and

\[
k^{(\beta,2)}_{\alpha\beta,i,j}(\xi, \xi_s) \\
= \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} \int_{S^2} \left| g' \right| \left( M^{\alpha}_{k,l} M^{\beta}_{k,l} \right)^{1/2} \left( \frac{\varphi^{\alpha}_{i} \varphi^{\beta}_{j}}{\varphi^{\alpha}_{k} \varphi^{\beta}_{l}} \right)^{1/2} \sigma^{\alpha\beta}_{k,j,l} (|g'|, \cos \theta) 1_{|g'|^2 > 2\Delta l^{\alpha}_{k,j,i,l}} d\omega,
\]

with \( \cos \theta = \frac{\hat{\omega} \cdot \frac{\hat{g}}{|\hat{g}|}}{} \) and \( \tilde{\Delta} l^{\alpha}_{i,l,k,j} = \frac{m_{\alpha} + m_{\beta}}{m_{\alpha} m_{\beta}} \Delta l^{\alpha}_{i,l,k,j} \).

**H1.** If \( \min (|\hat{g}|, |\hat{g}'|) \geq |\hat{g}| \) and \( \max (|\hat{g}|, |\hat{g}'|) \geq |\hat{g}| \), then \( \Psi_{i,l,k,j} = |\hat{g}| |\hat{g}'| \geq |\hat{g}| \), and, hence, following that \( |g'| \leq C (1 + |\hat{g}|) \), by expression (36), assumption (20), and bound (37), one may obtain the following bound

\[
\left( k^{(\beta,2)}_{\alpha\beta,i,j}(\xi, \xi_s) \right)^2 \leq C e^{-\rho (m_{\alpha} |\xi|^2 + m_{\beta} |\xi_s|^2)/2} \sum_{k=1}^{r_{\alpha}} \sum_{l=1}^{r_{\beta}} (1 + |g'|)^2 \left( 1 + \frac{1}{|g|^{1-\gamma/2}} \right)^2 \left( \int_{S^2} d\omega \right)^2 \leq C e^{-\rho (m_{\alpha} |\xi|^2 + m_{\beta} |\xi_s|^2)/2} \left( |g|^2 + \frac{1}{|g'|^2} \right)
\]
**H2.** If $|\tilde{g}| \geq |g|$ and $|\tilde{g}| \leq \max(1, |g|) \leq 1 + |g|$, then

$$
\Psi_{k,j}^{i,l} \mathbf{1}_{|g|^2 > 2\Delta \Delta I_{k,j}^\beta} = |g| \sqrt{|g|^2 - 2\Delta \Delta I_{k,j}^\beta} \mathbf{1}_{|g|^2 > 2\Delta \Delta I_{k,j}^\beta} \leq C (1 + |g|)^2,
$$

and hence, by expression (39), assumption (20), and inequality (37), one may obtain the following bound

$$
\left( k_{\alpha\beta,i,j}^{(\beta,2)}(\xi, \xi_s) \right)^2 
\leq C e^{-\rho(m_a|\xi|^2 + m_\beta|\xi_s|^2)/2} \sum_{k=1}^{r_s} \sum_{l=1}^{r_s} \frac{(1 + |g|)^2 + (1 + |g|)\gamma^2}{|g|^2} \left( \int_{\mathbb{R}^2} d\omega \right)^2
\leq C e^{-\rho(m_a|\xi|^2 + m_\beta|\xi_s|^2)/2} \left( |g|^2 + \frac{1}{|g|^2} \right)
$$

**H3.** If $|\tilde{g}| \geq |g|$ and $|\tilde{g}| \leq \max(1, |g|) \leq 1 + |g|$, then

$$
\Psi_{i,j}^{k,l} \mathbf{1}_{|g|^2 > 2\Delta \Delta I_{i,j}^\beta} = |\tilde{g}| \sqrt{|g|^2 - 2\Delta \Delta I_{i,j}^\beta} \mathbf{1}_{|g|^2 > 2\Delta \Delta I_{i,j}^\beta} \leq C (1 + |g|)^2,
$$

and hence, by expression (38), assumption (20), and inequality (37), the bound

$$
\left( k_{\alpha\beta,i,j}^{(\beta,2)}(\xi, \xi_s) \right)^2 \leq C e^{-\rho(m_a|\xi|^2 + m_\beta|\xi_s|^2)/2} \left( |g|^2 + \frac{1}{|g|^2} \right)
$$

may again be obtained. However, it is clear that $|g| \leq \max(|\tilde{g}|, |g|)$, implying that all possibilities are covered by the cases H1-3 above. Therefore, we have the general bound (40) for $\left( k_{\alpha\beta,i,j}^{(\beta,2)}(\xi, \xi_s) \right)^2$.

By applying the bound (40), making a change of variables $\{\xi, \xi_s\} \rightarrow \{g, G_{\alpha\beta} = \frac{m_\alpha \xi + m_\beta \xi_s}{m_\alpha + m_\beta}\}$, with unitary Jacobian, followed by one to spherical coordinates, reminding the expression (31),

$$
\int_{(\mathbb{R}^3)^2} \left( k_{\alpha\beta,i,j}^{(\beta,2)}(\xi, \xi_s) \right)^2 d\xi d\xi_s
\leq C \int_{(\mathbb{R}^3)^2} e^{-\rho(m_a+m_\beta)|G_{\alpha\beta}|^2/(2(m_a+m_\beta))} \left( |g|^2 + \frac{1}{|g|^2} \right) d\tilde{g} dG_{\alpha\beta}
\leq C \int_0^\infty R^2 e^{-R^2} dR \int_0^\infty (1 + \eta^4) e^{-pm_a m_\beta \eta^2/(2(m_a+m_\beta))} d\eta = C.
$$

Hence,

$$
K_{\alpha\beta,i,j}^{(2)} = \int_{\mathbb{R}^3} k_{\alpha\beta,i,j}^{(\beta,2)}(\xi, \xi_s) h_{\beta,j} \, d\xi_s
$$

are Hilbert-Schmidt integral operators, and as such continuous and compact on $L^2(d\xi)$ [21, Theorem 7.83], for $(i,j) \in \{1,...,r_\alpha\} \times \{1,...,r_\beta\}$ and $(\alpha, \beta) \in \{1,...,s\}^2$. 

On the other hand, if \( m_\alpha = m_\beta \), then

\[
k_{\alpha, \beta, i, j}^{(\beta, 2)}(\xi, \xi, \eta) = \sum_{k=1}^{r_\alpha} \sum_{l=1}^{r_\beta} \int_{\mathbb{R}^2} \frac{4 \hat{g}(M'_{\alpha, k} M'_{\beta, l} \mathbf{1}^{1/2}}{||g|| |g|} \left( \frac{\varphi_{\alpha} \varphi_{\beta}}{\varphi_{\alpha} \varphi_{\beta}} \right)^{1/2} \times 1_{|g|^2 > 4 \Delta I_{\alpha, \beta, kl}^{1/2}} \mathbf{1}_{\alpha, \beta, i, j} (\hat{g}, \mathbf{g}) \mathbf{1}_{\alpha, \beta} (\hat{g}, \mathbf{g}) \right) dw, \]  

with \( \hat{g} = \xi - \xi' \) and \( g = \xi - \xi' \).

Here

\[
\begin{align*}
\xi' &= \xi + \mathbf{w} - \mathbf{\chi} \mathbf{n} \\
\xi' &= \xi' + \mathbf{w} - \mathbf{\chi} \mathbf{n}
\end{align*}
\]

with \( g = \xi - \xi' \) and \( \mathbf{n} = \frac{g}{|g|} \).

Then similar arguments to the ones for \( k_{\alpha, \beta, i, j}^{(\alpha)}(\xi, \xi, \eta) \) (with \( m_\alpha = m_\beta \)) above, can be applied.

Concluding, the operator

\[
K = (K_1, ..., K_s) = \sum_{\beta=1}^{r_\beta} \left( K^{(3)}_{i, 1}, ..., K^{(3)}_{s, \beta} \right) - \left( K^{(1)}_{i, 1}, ..., K^{(1)}_{s, \beta} \right) + \left( K^{(2)}_{i, 1}, ..., K^{(2)}_{s, \beta} \right)
\]

with \( K^{(i)}_{\alpha, \beta} = \sum_{j=1}^{r_\beta} (K^{(i)}_{\alpha, \beta, 1}, ..., K^{(i)}_{\alpha, \beta, r}) \) for \( i \in \{1, 2, 3\} \) and \((\alpha, \beta) \in \{1, ..., s\}^2\),

is a compact self-adjoint operator on \( (L^2(d\xi))^s \). Self-adjointness is due to the symmetry relations (29), (30), cf. [22, p.198].

5. Bounds on the collision frequency

This section concerns the proof of Theorem 5. The proof is mainly a combination of the corresponding proofs in the polyanalytic single species and monatomic multicomponent mixture cases [6]. Note that throughout the proof, \( C \) will denote a generic positive constant.

**Proof.** Under assumption (21) the collision frequencies \( \nu_{\alpha, i} \) for \((\alpha, i) \in \{1, ..., s\} \times \{1, ..., r_\alpha\}\), making the change of variables \( \{\xi', \xi'_i\} \rightarrow \left\{|g'|, \omega = \frac{g'}{|g'|}, G_{\alpha, \beta} = \frac{m_\alpha \xi' + m_\beta \xi'_i}{m_\alpha + m_\beta}\right\} \), equal

\[
\nu_{\alpha, i} = \sum_{\beta=1}^{r_\beta} \left( \sum_{k=1}^{r_\alpha} \sum_{l=1}^{r_\beta} W_{\alpha, \beta} m_\alpha^2 \mathbf{1}_{|g'|^2} d\xi' d|g'| dG_{\alpha, \beta} d\omega
\]

\[
= 4\pi \sum_{\beta=1}^{r_\beta} \frac{n_\beta \varphi_{\beta}^3 m_\beta^3}{2(2\pi)^{3/2}} \sum_{k=1}^{r_\alpha} \sum_{l=1}^{r_\beta} \int e^{-M_{j, k}^2 - m_\beta |\xi'_i|^2} |g'| |g| \mathbf{1}_{|g|^2 > 2 \Delta I_{\epsilon, kl}^{1/2}} d\xi'
\]

\[
= \sum_{\beta=1}^{r_\beta} \frac{2 \Delta I_{\epsilon, kl}^{1/2}}{2 \pi} \sum_{k=1}^{r_\alpha} \sum_{l=1}^{r_\beta} \int e^{-M_{j, k}^2 - m_\beta |\xi'_i|^2} \sqrt{|g|^2 - 2 \Delta I_{\epsilon, kl}^{1/2} \mathbf{1}_{|g|^2 > 2 \Delta I_{\epsilon, kl}^{1/2}} d\xi'}
\]

with \( \Delta I_{\epsilon, kl}^{1/2} = \frac{m_\alpha + m_\beta}{m_\alpha m_\beta} \Delta I_{\epsilon, kl}^{1/2} \).
Clearly, for any \((\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}\), since \(\Delta I^{\alpha}_{ii,ii} = 0\),

\[
\nu_{\alpha, i} \geq C \alpha \left( \sqrt{2} n_\alpha \frac{m_\alpha^{3/2}}{\varphi_{\alpha} q_{\alpha} \sqrt{\pi}} \int_{\mathbb{R}^3} e^{-m_\alpha |\xi|^2 / 2} |g| \, d\xi \right) \geq C \int_{\mathbb{R}^3} |g| e^{-m_\alpha |\xi|^2 / 2} \, d\xi.
\]

Now consider the two different cases \(|\xi| \leq 1\) and \(|\xi| \geq 1\) separately. Firstly, if \(|\xi| \geq 1\), then, for any \((\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}\), by a trivial estimate

\[
\nu_{\alpha, i} \geq C \int_{\mathbb{R}^3} e^{-m_\alpha |\xi|^2} ||\xi| - |\xi_*|| \, d\xi_* \geq C \int_{|\xi| \leq 1/2} e^{-m_\alpha |\xi|^2} \, d\xi_* \geq C e^{-m_\alpha/4} \frac{|\xi|}{2} \int_{|\xi| \leq 1/2} \, d\xi_* \geq C |\xi| \geq C (1 + |\xi|).
\]

Secondly, if \(|\xi| \leq 1\), then, for any \((\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}\), by trivial estimates and a change to spherical coordinates,

\[
\nu_{\alpha, i} \geq C \int_{\mathbb{R}^3} e^{-m_\alpha |\xi|^2} ||\xi| - |\xi_*|| \, d\xi_* \geq C \int_{|\xi| \geq 2} e^{-m_\alpha |\xi|^2} \, d\xi_* \geq C \int_{|\xi| \geq 2} e^{-m_\alpha |\xi|^2} \frac{|\xi|}{2} \, d\xi_* \geq C \int_{|\xi| \geq 2} \, d\xi_* = C \int_{\mathbb{R}^3} e^{-m_\alpha R^2} \, dR = C \geq C (1 + |\xi|).
\]

Hence, there is a positive constant \(\nu_- > 0\), such that \(\nu_{\alpha, i} \geq \nu_- (1 + |\xi|)\) for all \((\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}\) and \(\xi \in \mathbb{R}^3\).

On the other hand, by some estimates and a change to spherical coordinates,

\[
\nu_{\alpha, i} \leq C \sum_{\beta=1}^{s} \sum_{k=1}^{r_\beta} \sum_{l=1}^{r_\beta} \int_{\mathbb{R}^3} e^{-m_\beta |\xi|^2 / 2} \sqrt{|g|^2 - 2 \Delta \nu^{\alpha}_{kl,ij}} \, d\xi_* \leq C \sum_{\beta=1}^{s} \sum_{k=1}^{r_\beta} \sum_{l=1}^{r_\beta} \left( 1 + |\xi| + |\xi_*| \right) e^{-m_\beta |\xi|^2 / 2} \, d\xi_* \leq C |\xi| \sum_{\beta=1}^{s} \int_{0}^{\infty} R^2 e^{-m_\beta R^2} \, dR + C \sum_{\beta=1}^{s} \int_{0}^{\infty} e^{-m_\beta R^2 / 2} \left( R^2 + R^3 \right) \, dR \leq C (1 + |\xi|).
\]
Hence, there is a positive constant $\nu_+ > 0$, such that $\nu_{\alpha,i} \leq \nu_+ (1 + |\xi|)$ for all $(\alpha, i) \in \{1, \ldots, s\} \times \{1, \ldots, r_\alpha\}$ and $\xi \in \mathbb{R}^3$. □

**Declaration of competing interest**

The author has no relevant financial or non-financial interests to disclose.

**Appendix A. Proof of Lemma 5**

This appendix concerns a proof of Lemma 5.

**Proof.** Denote $q := |\xi - \xi'|$. For $\eta = (\xi - \xi') / q$ the following (unique) decomposition can be made, by the relations (25), cf. Fig. 3,

$$\begin{cases} 
\xi = w + r \eta \\
\xi' = w + (r - q) \eta 
\end{cases}$$

with $w \perp \eta$.

while

$$\begin{cases} 
\xi_* = \tilde{w} + r_* \eta \\
\xi'_* = \tilde{w} + \left( r_* - \frac{m_\alpha - m_\beta}{m_\beta} q \right) \eta
\end{cases}$$

with $\tilde{w} \perp \eta$.

where it follows by relation (26) that

$$r_* = r - \chi + \frac{m_\alpha - m_\beta}{2m_\beta} q,$$

implying that

$$\begin{cases} 
\xi_* = \tilde{w} + \left( r - \chi + \frac{m_\alpha - m_\beta}{2m_\beta} q \right) \eta \\
\xi'_* = \tilde{w} + \left( r - \chi - \frac{m_\alpha + m_\beta}{2m_\beta} q \right) \eta
\end{cases}$$

with $\tilde{w} \perp \eta$.

Then

$$m_\alpha |\xi'|^2 + m_\beta |\xi'_*|^2$$

$$= m_\alpha |\tilde{w}|^2 + m_\beta |\tilde{w}'|^2 + (m_\alpha + m_\beta) \left( r^2 + \chi q \right) - (3m_\alpha + m_\beta) qr$$

$$+ \frac{m_\alpha^2 + 6m_\alpha m_\beta + m_\beta^2}{4m_\beta} q^2 + m_\beta \left( \chi^2 - 2\chi r \right)$$

$$= m_\alpha |\tilde{w}|^2 + m_\beta |\tilde{w}'|^2 + (m_\alpha + m_\beta) \left( r - \frac{2m_\beta \chi + (3m_\alpha + m_\beta) q}{2(m_\alpha + m_\beta)} \right)^2$$

$$+ \frac{m_\alpha m_\beta}{m_\alpha + m_\beta} \left( \chi + \frac{m_\alpha - m_\beta}{2m_\beta} q \right)^2,$$

while
\[ m_\alpha |\xi|^2 + m_\beta |\xi_s|^2 = m_\alpha |w|^2 + m_\beta |\tilde{w}|^2 + (m_\alpha + m_\beta) r^2 + (m_\alpha - m_\beta) q (r - \chi) + \frac{(m_\alpha - m_\beta)^2}{4m_\beta} q^2 + m_\beta (\chi^2 - 2\chi r) = m_\alpha |w|^2 + m_\beta |\tilde{w}|^2 + (m_\alpha + m_\beta) \left( r + \frac{(m_\alpha - m_\beta) q - 2m_\beta \chi}{2(m_\alpha + m_\beta)} \right)^2 + \frac{m_\alpha m_\beta}{m_\alpha + m_\beta} \left( \chi - \frac{m_\alpha - m_\beta}{2m_\beta} q \right)^2. \]

Denote \( a := -\frac{2m_\beta \chi + (3m_\alpha + m_\beta) q}{2(m_\alpha + m_\beta)}, b := \frac{(m_\alpha - m_\beta) q - 2m_\beta \chi}{2(m_\alpha + m_\beta)}, \) and \( c^2 := \frac{(m_\alpha - m_\beta)^2}{4(m_\alpha + m_\beta)^2} m_\beta q^2. \) Then
\[
m_\alpha |\xi'|^2 + m_\beta |\xi_s'|^2 - \rho \left( m_\alpha |\xi|^2 + m_\beta |\xi_s|^2 \right) \geq (m_\alpha + m_\beta) \left( \left( r + a \right)^2 - \rho \left( r + b \right)^2 + (1 - \rho) c^2 \right) + (1 + \rho) \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} m_\alpha \chi q = (m_\alpha + m_\beta) (1 - \rho) \left( r^2 + 2 \frac{a - \rho b}{1 - \rho} r + \frac{a^2 - \rho b^2}{1 - \rho} + c^2 \right) + (1 + \rho) \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} \Delta I_{kj,il}^\alpha \]
\[
= (m_\alpha + m_\beta) \left( (1 - \rho) \left( r + a - \rho b \right)^2 + \frac{c^2}{1 - \rho} \left( \rho^2 - 2\rho \left( 1 + \frac{(a - b)^2}{2c^2} \right) + 1 \right) \right) + (1 + \rho) \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} \Delta I_{kj,il}^\alpha \]
\[
\geq c^2 \frac{m_\alpha + m_\beta}{1 - \rho} \left( \rho^2 - 2\rho \left( 1 + \frac{(a - b)^2}{2c^2} \right) + 1 \right) + (1 + \rho) \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} \Delta I_{kj,il}^\alpha \geq (1 + \rho) \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} \Delta I_{kj,il}^\alpha \geq -2 \left| \Delta I_{kj,il}^\alpha \right| \]
if \( 0 \leq \rho \leq 1 + \frac{(a - b)^2}{2c^2} - \frac{1}{2c^2} \sqrt{(a - b)^4 + 4c^2(a - b)^2} < 1. \)

Note that the second last inequality follows by the assumption on \( \rho. \) Now let
\[
\rho = 1 + \frac{(a - b)^2}{2c^2} - \frac{1}{2c^2} \sqrt{(a - b)^4 + 4c^2(a - b)^2}
= 1 + \frac{2}{1 + \frac{4c^2}{(a - b)^2}} = 1 - \frac{2}{1 + \sqrt{1 + \frac{4c^2}{(a - b)^2}}}
= 1 - \frac{2}{1 + \frac{m_\alpha - m_\beta}{4m_\alpha m_\beta}} = \left( \frac{\sqrt{m_\alpha} - \sqrt{m_\beta}}{\sqrt{m_\alpha + \sqrt{m_\beta}}} \right)^2 \geq 0 \text{ if } m_\alpha \neq m_\beta. \]
References