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ORIGINAL RESEARCH ARTICLE

Discrete Quantum Kinetic Equation

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Abstract

A semi-classical approach to the study of the evolution of bosonic or fermionic excitations is through the Nordheim—Boltzmann- or, Uehling—Uhlenbeck—equation, also known as the quantum Boltzmann equation. In some low ranges of temperatures—e.g., in the presence of a Bose condensate—also other types of collision operators may render in essential contributions. Therefore, extended— or, even other—collision operators are to be considered as well. This work concerns a discretized version—a system of partial differential equations—of such a quantum equation with an extended collision operator. Trend to equilibrium is studied for a planar stationary system, as well as the spatially homogeneous system. Some essential properties of the linearized operator are proven, implying that results for general half-space problems for the discrete Boltzmann equation can be applied. A more general collision operator is also introduced, and similar results are obtained also for this general equation.

Keywords Quantum Boltzmann equation · Discrete kinetic equation · Bosons · Fermions

Mathematics Subject Classification 81Q10 · 82C10 · 82C22 · 82C40

1 Introduction

The Nordheim—Boltzmann- or, Uehling—Uhlenbeck-equation [19, 22], the so-called quantum Boltzmann equation, is traditionally used as a semi-classical model for the evolution of distribution functions for excitations of bosons or fermions [20]. However, in some ranges of low temperatures—e.g., in the presence of a Bose condensate—also other collision operators may essentially contribute to the evolution [1, 17, 21]. This paper concerns a discrete version of the equation

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$$\frac{\partial F}{\partial t}+\left(\nabla_{\mathbf{p}}E+\mathbf{v}_c\right)\cdot\nabla_{\mathbf{x}}F=C_{12}\left(F\right)+C_{22}\left(F\right)+C_{31}\left(F\right). \tag{1}$$

Here $F=F\left(t,\mathbf{x},\mathbf{p}\right)$ denotes the distribution function of the excitations with momentum $\mathbf{p}=\left(p^1,p^2,p^3\right)\in\mathbb{R}^3$ at time $t\in\mathbb{R}_+$ and position $\mathbf{x}=\left(x,x_2,x_3\right)\in\mathbb{R}^3$, $E=E(\mathbf{p})$ denotes the Bogoliubov excitation energy, and $\mathbf{v}_c=\left(\mathrm{v}_c^1,\mathrm{v}_c^2,\mathrm{v}_c^3\right)\in\mathbb{R}^3$ the superfluid velocity. Furthermore, let $\varepsilon^2=1$, where $\varepsilon=1$ for bosons and $\varepsilon=-1$ for fermions, and assume that $0\leq F\leq 1$ for $\varepsilon=-1$. The collision integrals (cf. [17, 21] for $\varepsilon=1$) are given by

$$\begin{aligned}C_{12}\left(F\right)=\int_{\left(\mathbb{R}^3\right)^3}&\Gamma^{12}\left(\mathbf{p}_*,\mathbf{p}',\mathbf{p}_*'\right)\delta_0\delta_3\left[\left(1+\varepsilon F_*\right)F'F_*'\right.\\&\left.-F_*\left(1+\varepsilon F'\right)\left(1+\varepsilon F_*'\right)\right]d\mathbf{p}_*d\mathbf{p}'d\mathbf{p}_*',\end{aligned}$$

with

$$\begin{aligned}\delta_0&=\delta\left(\mathbf{p}_*-\mathbf{p}'-\mathbf{p}_*'\right)\delta\left(E_*-E'-E_*'\right),\\\delta_3&=\delta\left(\mathbf{p}_*-\mathbf{p}\right)-\delta\left(\mathbf{p}'-\mathbf{p}\right)-\delta\left(\mathbf{p}_*'-\mathbf{p}\right),\end{aligned}$$

and

$$\Gamma^{12}\left(\mathbf{p}_*,\mathbf{p}',\mathbf{p}_*'\right)=\Gamma^{12}\left(\mathbf{p}_*,\mathbf{p}_*',\mathbf{p}'\right);$$

while

$$\begin{aligned}C_{22}\left(F\right)=\int_{\left(\mathbb{R}^3\right)^3}&\Gamma^{22}\left(\mathbf{p},\mathbf{p}_*,\mathbf{p}',\mathbf{p}_*'\right)\delta_1\left[\left(1+\varepsilon F\right)\left(1+\varepsilon F_*\right)F'F_*'\right.\\&\left.-FF_*\left(1+\varepsilon F'\right)\left(1+\varepsilon F_*'\right)\right]d\mathbf{p}_*d\mathbf{p}'d\mathbf{p}_*',\end{aligned}$$

with

$$\delta_1=\delta\left(\mathbf{p}+\mathbf{p}_*-\mathbf{p}'-\mathbf{p}_*'\right)\delta\left(E+E_*-E'-E_*'\right),$$

and

$$\Gamma^{22}\left(\mathbf{p},\mathbf{p}_*,\mathbf{p}',\mathbf{p}_*'\right)=\Gamma^{22}\left(\mathbf{p}_*,\mathbf{p},\mathbf{p}',\mathbf{p}_*'\right)=\Gamma^{22}\left(\mathbf{p}',\mathbf{p}_*',\mathbf{p},\mathbf{p}_*\right);$$

and, finally,

$$\begin{aligned}C_{31}\left(F\right)=\int_{\left(\mathbb{R}^3\right)^4}&\Gamma^{31}\left(\mathbf{p}_*,\mathbf{p}',\mathbf{p}_*',\mathbf{p}_{**}'\right)\delta_2\delta_4\left[\left(1+\varepsilon F_*\right)F'F_*'F_{**}'\right.\\&\left.-F_*\left(1+\varepsilon F'\right)\left(1+\varepsilon F_*'\right)\left(1+\varepsilon F_{**}'\right)\right]d\mathbf{p}_*d\mathbf{p}'d\mathbf{p}_*'d\mathbf{p}_{**}',\end{aligned}$$

with—note that, here the momentum variables of the triple of excitations in a collision are all primed, with no, one, or two asterisks, $*$ —

$$\begin{aligned}\delta_2 &= \delta\left(\mathbf{p}_* - \mathbf{p}' - \mathbf{p}'_* - \mathbf{p}'_{**}\right)\delta\left(E_* - E' - E'_* - E'_{**}\right),\\ \delta_4 &= \delta\left(\mathbf{p}_* - \mathbf{p}\right) - \delta\left(\mathbf{p}' - \mathbf{p}\right) - \delta\left(\mathbf{p}'_* - \mathbf{p}\right) - \delta\left(\mathbf{p}'_{**} - \mathbf{p}\right),\end{aligned}$$

and

$$\Gamma^{31}\left(\mathbf{p}_*, \mathbf{p}', \mathbf{p}'_*, \mathbf{p}'_{**}\right) = \Gamma^{31}\left(\mathbf{p}_*, \mathbf{p}'_*, \mathbf{p}'_{**}, \mathbf{p}'\right).$$

Typical expressions for the collision kernels [1, 17, 21] are

$$\begin{aligned}\Gamma^{12}\left(\mathbf{p}_*, \mathbf{p}', \mathbf{p}'_*\right) &= k_1\left(u_* u' u'_* - v_* v' v'_* + (v_* - u_*)\left(u' v'_* + v' u'_*\right)\right)^2,\\ \Gamma^{22}\left(\mathbf{p}, \mathbf{p}_*, \mathbf{p}', \mathbf{p}'_*\right) &= k_2\left(u u_* u' u'_* + v v_* v' v'_* + (u v_* + v u_*)\left(u' v'_* + v' u'_*\right)\right)^2,\end{aligned}$$

and

$$\begin{aligned}&\Gamma^{31}\left(\mathbf{p}_*, \mathbf{p}', \mathbf{p}'_*, \mathbf{p}'_{**}\right)\\ &= k_3\left(u_*\left(u' u'_* v'_{**} + u' v'_* u'_{**} + v' u'_* u'_{**}\right) + v_*\left(v' v'_* u'_{**} + u' v'_* v'_{**} + v' u'_* v'_{**}\right)\right)^2.\end{aligned}$$

Here and below the notations $F'_* = F\left(t, \mathbf{x}, \mathbf{p}'_*\right)$, $u'_* = u\left(\mathbf{p}'_*\right)$ etc. are used. Expressions for $k_1, k_2, k_3, u, v,\ u', v'$ etc. can be found in, e.g., [1, 17]. In the Nordheim—Boltzmann [19]—-or, Uehling—Uhlenbeck [22]— collision integral $C_{22}(F)$ binary collisions between excited atoms are considered, while in the collision integral $C_{12}(F)$ binary collisions involving one condensate atom are considered [23]. For more explicit expressions of the kernels Γ^{12}, Γ^{22}, and Γ^{31} see for example [17].

If the distribution function F is close to a Planckian—i.e., a typical equilibrium distribution

$$P = \frac{1}{e^{\alpha E + \boldsymbol{\beta}\cdot\mathbf{p}} - \varepsilon},$$

with $\alpha > 0$ and $\boldsymbol{\beta} \in \mathbb{R}^3$ (such that $\alpha E + \boldsymbol{\beta}\cdot\mathbf{p} > 0$ for $\varepsilon = 1$, which might be obtained by a truncation in the momentum-space, cf. [2]), then the nonlinear equation (1) can be approximated by the linearized equation

$$\frac{\partial f}{\partial t} + \left(\nabla_{\mathbf{p}} E + \mathrm{v}_c\right)\cdot\nabla_{\mathbf{x}} f + Lf = 0 \text{ for } f = f(t, \mathbf{x}, \mathbf{p}),$$

where

$$F = P + (P(1 + \varepsilon P))^{1/2} f \text{ and } L = L_{12} + L_{22} + L_{31},$$

with

$$
\begin{aligned}
L_{12} f = {} & n(P(1+\varepsilon P))^{-1/2} \int_{(\mathbb{R}^3)^3} \Gamma^{12}\left(\mathbf{p}_*, \mathbf{p}', \mathbf{p}_*'\right) \delta_0 \delta_3 \\
& \times \left[\left(\varepsilon P_* - P' + (1-\varepsilon) P_* P'\right) (P_*'(1+\varepsilon P_*'))^{1/2} f_*' \right. \\
& + \left(\varepsilon P_* - P_*' + (1-\varepsilon) P_* P_*'\right) (P'(1+\varepsilon P'))^{1/2} f' \\
& \left. + \left(1 + \varepsilon\left(P' + P_*'\right) + (1-\varepsilon) P' P_*'\right) (P_*(1+\varepsilon P_*))^{1/2} f_* \right] d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}_*',
\end{aligned}
$$

while

$$
\begin{aligned}
L_{22} f = {} & (P(1+\varepsilon P))^{-1/2} \int_{(\mathbb{R}^3)^3} \Gamma^{22}\left(\mathbf{p}, \mathbf{p}_*, \mathbf{p}', \mathbf{p}_*'\right) \delta_1 \\
& \left[\left(\varepsilon P P_* - P'(1+\varepsilon (P + P_*))\right) (P_*'(1+\varepsilon P_*'))^{1/2} f_*' \right. \\
& + \left(\varepsilon P P_* - P_*'(1+\varepsilon (P + P_*))\right) (P'(1+\varepsilon P'))^{1/2} f' \\
& + \left(P(1+\varepsilon\left(P' + P_*'\right)) - \varepsilon P' P_*'\right) (P_*(1+\varepsilon P_*))^{1/2} f_* \\
& \left. + \left(P_*(1+\varepsilon\left(P' + P_*'\right)) - \varepsilon P' P_*'\right) (P(1+\varepsilon P))^{1/2} f \right] d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}_*',
\end{aligned}
$$

and

$$
\begin{aligned}
& L_{31} f \\
& = (P(1+\varepsilon P))^{-1/2} \int_{(\mathbb{R}^3)^4} \Gamma^{31}\left(\mathbf{p}_*, \mathbf{p}', \mathbf{p}_*', \mathbf{p}_{**}'\right) \delta_2 \delta_4 \\
& \left[\left(P_*\left(\varepsilon + P' + P_{**}'\right) - P' P_{**}'\right) (P_*'(1+\varepsilon P_*'))^{1/2} f_*' \right. \\
& + \left(P_*\left(\varepsilon + P_*' + P_{**}'\right) - P_*' P_{**}'\right) (P'(1+\varepsilon P'))^{1/2} f' \\
& + \left(P_*\left(\varepsilon + P' + P_*'\right) - P' P_*'\right) (P_{**}'(1+\varepsilon P_{**}'))^{1/2} f_{**}' \\
& \left. + \left(\left(1+\varepsilon P'\right)\left(1+\varepsilon P_*'\right) + P_{**}'\left(\varepsilon + P' + P_*'\right)\right) (P_*(1+\varepsilon P_*))^{1/2} f_* \right] \\
& \times d\mathbf{p}_* d\mathbf{p}' d\mathbf{p}_*' d\mathbf{p}_{**}'.
\end{aligned}
$$

It can be shown (cf. [2] for L_{12} and, for example, [15] for the linearized Boltzmann operator) that the linearized operators L_{12}, L_{22}, and L_{31}, and so also L, are symmetric and nonnegative operators on a suitable L^2-space.

The remaining part of the paper is organized as follows. Section 2 introduces the general system of partial differential equations of discrete Boltzmann type considered to approximate Eq. (1). The collision operators are introduced in Sect. 2.1, while collision invariants and equilibrium distributions are considered in Sect. 2.2. The trend to equilibrium is considered for a planar stationary system, as well as the spatially homogeneous system, in Sect. 3, while the linearized collision operator with some important properties is considered in Sect. 4. The trend to equilibrium in the aforementioned cases can be extended to yield also for a more general collision operator, see Sect. 3, and the generalization of the linearized collision operator to this general collision operator and its properties are considered in Sect. 5.

2 Discrete Model

Consider a general discrete model of equation (1), cf. [4, 6],

$$\frac{\partial F_i}{\partial t} + \left(\left(\nabla_{\mathbf{p}} E\right)_i + \mathbf{v}_c\right) \cdot \nabla_{\mathbf{x}} F_i = C_{12i}(F) + C_{22i}(F) + C_{31i}(F) \text{ for } 1 \leq i \leq N. \tag{2}$$

Here $P = \{\mathbf{p}_1, ..., \mathbf{p}_N\} \subset \mathbb{R}^d$ is a finite set of momentum variables, $\mathbf{v}_c \in \mathbb{R}^d$, $\left(\nabla_{\mathbf{p}} E\right)_i = \nabla_{\mathbf{p}} E\big|_{\mathbf{p}=\mathbf{p}_i}$, and $F_i = F_i(\mathbf{x}, t) = F(\mathbf{x}, \mathbf{p}_i, t)$ for $i \in \{1, ..., N\}$, where $F = F(\mathbf{x}, \mathbf{p}, t)$ is the distribution function of the excitations at time $t \in \mathbb{R}_+$ and position $\mathbf{x} \in \mathbb{R}^d$, with $0 < F_i < 1$ for $i \in \{1, ..., N\}$ if $\varepsilon = -1$. For generality, allow $\mathbf{p}$ to be of dimension d, rather than of dimension 3.

2.1 Collision Operator

The collision operators $C_{12i}(F)$ are for $i \in \{1, ..., N\}$ given by

$$C_{12i}(F) = \sum_{j,k,l=1}^{N} \left(\delta_{ij} - \delta_{ik} - \delta_{il}\right) \Gamma_{kl}^{j} \left(\left(1 + \varepsilon F_j\right) F_k F_l - F_j \left(1 + \varepsilon F_k\right)\left(1 + \varepsilon F_l\right)\right),$$

where the collision coefficients

$$\Gamma_{kl}^{j} = \Gamma_{lk}^{j} \geq 0 \tag{3}$$

for any indices $\{j, k, l\} \subseteq \{1, ..., N\}$, with equality in inequality (3) unless conservation of momentum and energy

$$\mathbf{p}_j = \mathbf{p}_k + \mathbf{p}_l \text{ and } E_j = E_k + E_l \tag{4}$$

is fulfilled, and

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

The collision operators $C_{22i}(F)$ are for $i \in \{1, ..., N\}$ given by

$$C_{22i}(F) = \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} \left(\left(1 + \varepsilon F_i\right)\left(1 + \varepsilon F_j\right) F_k F_l - F_i F_j \left(1 + \varepsilon F_k\right)\left(1 + \varepsilon F_l\right)\right),$$

where for any indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0, \tag{5}$$

with equality in inequality (5) unless conservation of momentum and energy

$$\mathbf{p}_i + \mathbf{p}_j = \mathbf{p}_k + \mathbf{p}_l \text{ and } E_i + E_j = E_k + E_l \tag{6}$$

is fulfilled. Furthermore, the collision operators $C_{31i}(F)$ are for $i \in \{1, ..., N\}$ given by

$$\begin{aligned} C_{31i}(F) = & \sum_{j,k,l,m=1}^{N} \left(\delta_{im} - \delta_{ij} - \delta_{ik} - \delta_{il}\right) \Gamma_{jkl}^{m} \\ & \times \left(F_j F_k F_l \left(1 + \varepsilon F_m\right) - \left(1 + \varepsilon F_j\right)\left(1 + \varepsilon F_k\right)\left(1 + \varepsilon F_l\right) F_m\right), \end{aligned}$$

where for any indices $\{j, k, l, m\} \subseteq \{1, ..., N\}$

$$\Gamma_{jkl}^{m} = \Gamma_{kjl}^{m} = \Gamma_{klj}^{m} \geq 0, \tag{7}$$

with equality in inequality (7) unless conservation of momentum and energy

$$\mathbf{p}_m = \mathbf{p}_j + \mathbf{p}_k + \mathbf{p}_l \text{ and } E_m = E_j + E_k + E_l \tag{8}$$

is fulfilled.

Remark 1 For a function $g = g(\mathbf{p})$ (possibly depending on more variables than $\mathbf{p}$), when considering the discrete case, identify g with its restrictions to the points $\mathbf{p} \in P$, i.e.,

$$g = (g_1, ..., g_N), \text{ with } g_i = g(\mathbf{p}_i).$$

Denote by $\mathbf{B}$ the matrix, whose rows are the transposes of $\nabla_{\mathbf{p}} E\big|_{\mathbf{p}=\mathbf{p}_1} + \mathbf{v}_c$, ..., $\nabla_{\mathbf{p}} E\big|_{\mathbf{p}=\mathbf{p}_N} + \mathbf{v}_c$, respectively. Then system (2) reads

$$\frac{\partial F}{\partial t} + (\mathbf{B}\,\nabla_{\mathbf{x}}) \cdot F = C_{12}(F) + C_{22}(F) + C_{31}(F). \tag{9}$$

The collision operator $C_{12}(F)$ in the right hand side of system (9) can be decomposed as

$$C_{12}(F) = \widetilde{L}F + \widetilde{Q}(F, F) + \widetilde{\widetilde{Q}}(F, F, F),$$

where

$$\left(\widetilde{L}F\right)_i = \sum_{j,k=1}^{N} \left(2\Gamma_{ij}^{k} F_k - \Gamma_{jk}^{i} F_i\right),$$

$$\widetilde{Q}_i(F,G) = \sum_{j,k=1}^{N} \left(2\Gamma_{ij}^{k} Q_{ij}^{k}(F,G) - \Gamma_{jk}^{i} Q_{jk}^{i}(F,G)\right), \text{ and}$$

$$\widetilde{\widetilde{Q}}_i(F,G,H) = \sum_{j,k=1}^{N} \left(2\Gamma_{ij}^{k} \widetilde{Q}_{ij}^{k}(F,G,H) - \Gamma_{jk}^{i} \widetilde{Q}_{jk}^{i}(F,G,H)\right),$$

with

$$Q_{jk}^{i}(F,G) = \frac{1}{2}\left(\varepsilon F_i \left(G_j + G_k\right) + \varepsilon G_i \left(F_j + F_k\right) - \left(F_j G_k + G_j F_k\right)\right)$$

and

$$\widetilde{Q}_{jk}^{i}(F,G,H) = \frac{1-\varepsilon}{2} F_i \left(G_j H_k + H_j G_k\right).$$

for any indices $\{i, j, k\} \subseteq \{1, ..., N\}$. Moreover, the collision operator $C_{22}(F)$ in the right hand side of system (9) can be decomposed as

$$C_{22}(F) = \overline{Q}(F,F) + \overline{\overline{Q}}(F,F,F),$$

where for $i \in \{1, ..., N\}$

$$\overline{Q}_i(F,G) = \frac{1}{2} \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} \left((F_k G_l + G_k F_l) - \left(F_i G_j + G_j F_i\right)\right),$$

and

$$\overline{\overline{Q}}_i(F,G,H) = \frac{\varepsilon}{2} \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} \left(\left(F_i + F_j\right)(G_k H_l + H_k G_l) - (F_k + F_l)\left(G_i H_j + H_i G_j\right)\right),$$

while the collision operator $C_{31}(F)$ in the right hand side of system (9) can be decomposed as

$$C_{31}(F) = \widehat{L}F + \widehat{Q}(F,F) + \widehat{\widehat{Q}}(F,F,F),$$

where

$$\left(\widehat{L}F\right)_i = \sum_{j,k,l=1}^{N} \left(3\Gamma_{ijk}^{l} F_l - \Gamma_{jkl}^{i} F_i\right),$$
$$\widehat{Q}_i(F,G) = \sum_{j,k,l=1}^{N} \left(3\Gamma_{ijk}^{l} Q_{ijk}^{l}(F,G) - \Gamma_{jkl}^{i} Q_{jkl}^{i}(F,G)\right), \text{ and}$$
$$\widehat{\widetilde{Q}}_i(F,G,H) = \sum_{j,k,l=1}^{N} \left(3\Gamma_{ijk}^{l} \widetilde{Q}_{ijk}^{l}(F,G,H) - \Gamma_{jkl}^{i} \widetilde{Q}_{jkl}^{i}(F,G,H)\right),$$

with

$$Q_{jkl}^{i}(F,G) = \frac{\varepsilon}{2}\left(F_i\left(G_j + G_k + G_l\right) + G_i\left(F_j + F_k + F_l\right)\right)$$

and

$$\widetilde{Q}_{jkl}^{i}(F,G,H)$$
$$= \frac{1}{2}\left(F_i\left(G_jH_l + H_jG_l + G_jH_k + H_jG_k + G_kH_l + H_kG_l\right) - F_j\left(G_kH_l + H_kG_l\right)\right).$$

for any indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$.

2.2 Collision Invariants and Equilibrium Distributions

A function $\phi = \phi(\mathbf{p})$ is a collision invariant if and only if

$$\phi_i = \phi_j + \phi_k \tag{10}$$

for all indices $\{i, j, k\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jk}^{i} \neq 0$,

$$\phi_i + \phi_j = \phi_k + \phi_l \tag{11}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{ij}^{kl} \neq 0$, and

$$\phi_i = \phi_j + \phi_k + \phi_l \tag{12}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jkl}^{i} \neq 0$.

The trivial— or, “physical”—collision invariants

$$\phi^1 = p^1, ..., \phi^d = p^d, \text{ and } \phi^{d+1} = E \tag{13}$$

(also including $\phi^0 = 1$ if all collision coefficients Γ_{jk}^{i} and Γ_{jkl}^{i} are zero) generate a subspace of the vector space of collision invariants. Note that by Remark 1 and in correspondence with relations (10)–(12) the collision invariants $\phi^i = \phi^i(\mathbf{p})$ given in (13) are vectors.

In the discrete case, unlike in the continuous case with suitable conditions on E, see [14], there can be spurious—or, "non-physical"—collision invariants. This is a common problem for different kinds of velocity/momentum models, cf. [13]; if there will not be enough of admissible collisions, undesired quantities $\phi = \phi(\mathbf{p})$ will be invariant under collisions—-most trivial case, with no admissible collisions at all, all $\phi = \phi(\mathbf{p})$ will be invariant. In fact, to obtain only the desired collision invariants, there must be a set of $N - p$—here p denotes the number of desired collision invariants—independent admissible collisions—i.e., collisions with non-zero collision coefficients, that can't be obtained by any chain of other collisions in the set (or their reversion). Consider below (even if this restriction is not necessary in the general context) only normal discrete models. That is, consider discrete models without spurious collision invariants, i.e., any collision invariant is of the form

$$\phi = \phi(\mathbf{p}) = \alpha E + \boldsymbol{\beta} \cdot \mathbf{p} \tag{14}$$

for some constant $\alpha > 0$ and $\boldsymbol{\beta} \in \mathbb{R}^d$. Construction of normal discrete kinetic models and, especially, discrete velocity models for the Boltzmann equation have been extensively studied, see for example [10, 12, 13] and references therein. Those models can be used in case of approximations of the Bogoliubov excitation energy $E = E(\mathbf{p})$ of the form $E = c_1 |\mathbf{p}|^2 + c_2$ for constant c_1 and c_2 (as long as the collision term $C_{22}(F)$ is included).

A Maxwellian distribution- or, Maxwellian—is of the form

$$M = e^{-\phi} = e^{-\alpha E - \boldsymbol{\beta} \cdot \mathbf{p}}$$

or, equivalently,

$$M_i = e^{-\phi_i} = e^{-\alpha E_i - \boldsymbol{\beta} \cdot \mathbf{p}_i} \text{ for } i \in \{1, ..., N\},$$

where $\phi = (\phi_1, ..., \phi_N)$ is a collision invariant. Moreover, a Planckian distribution—or, Planckian—is given by

$$P = \frac{M}{1 - \varepsilon M} = \frac{1}{M^{-1} - \varepsilon} = \frac{1}{e^{\alpha E + \boldsymbol{\beta} \cdot \mathbf{p}} - \varepsilon} \tag{15}$$

or, equivalently,

$$P_i = \frac{M_i}{1 - \varepsilon M_i} = \frac{1}{e^{\alpha E_i + \boldsymbol{\beta} \cdot \mathbf{p}_i} - \varepsilon} \text{ for } i \in \{1, ..., N\},$$

for some constant $\alpha > 0$ and $\boldsymbol{\beta} \in \mathbb{R}^d$ (such that $\alpha E + \boldsymbol{\beta} \cdot \mathbf{p} > 0$ if $\varepsilon = 1$).

Denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^n$. It is straightforward that

$$\begin{aligned}&\langle H, C_{12}(F)\rangle\\&=\sum_{i,j,k=1}^{N}\Gamma_{jk}^{i}\left(H_i-H_j-H_k\right)\left((1+\varepsilon F_i)F_jF_k-F_i\left(1+\varepsilon F_j\right)(1+\varepsilon F_k)\right),\end{aligned}\tag{16}$$

and so, assuming that $0 < F_i < 1$ for $i \in \{1, ..., N\}$ if $\varepsilon = -1$, to prevent a second type of equilibrium distribution, cf. [18],

$$\begin{aligned}&\left\langle \log\frac{F}{1+\varepsilon F}, C_{12}(F)\right\rangle\\&=\sum_{i,j,k=1}^{N}\Gamma_{jk}^{i}(1+\varepsilon F_i)\left(1+\varepsilon F_j\right)(1+\varepsilon F_k)\\&\quad\times\left(\frac{F_j}{1+\varepsilon F_j}\frac{F_k}{1+\varepsilon F_k}-\frac{F_i}{1+\varepsilon F_i}\right)\left(\log\frac{F_i}{1+\varepsilon F_i}-\log\left(\frac{F_j}{1+\varepsilon F_j}\frac{F_k}{1+\varepsilon F_k}\right)\right)\leq 0,\end{aligned}\tag{17}$$

with equality in inequality (17) if and only if

$$\frac{F_i}{1+\varepsilon F_i}=\frac{F_j}{1+\varepsilon F_j}\frac{F_k}{1+\varepsilon F_k}\tag{18}$$

for all indices $\{i, j, k\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jk}^{i} \neq 0$. Hence, there is equality in inequality (17) if and only if $\dfrac{F}{1+\varepsilon F}$ is a Maxwellian or, equivalently, if and only if F is a Planckian.

It is again straightforward that

$$\begin{aligned}\langle H, C_{22}(F)\rangle &= \frac{1}{4}\sum_{i,j,k,l=1}^{N}\Gamma_{ij}^{kl}\left(H_i+H_j-H_k-H_l\right)\\&\quad\times\left((1+\varepsilon F_i)\left(1+\varepsilon F_j\right)F_kF_l-F_iF_j(1+\varepsilon F_k)(1+\varepsilon F_l)\right),\end{aligned}\tag{19}$$

and so, again assuming that $0 < F_i < 1$ for $i \in \{1, ..., N\}$ if $\varepsilon = -1$

$$\begin{aligned}\left\langle \log\frac{F}{1+\varepsilon F}, C_{22}(F)\right\rangle &= \frac{1}{4}\sum_{i,j,k=1}^{N}\Gamma_{ij}^{kl}(1+\varepsilon F_i)\left(1+\varepsilon F_j\right)(1+\varepsilon F_k)(1+\varepsilon F_l)\\&\quad\times\left(\frac{F_k}{1+\varepsilon F_k}\frac{F_l}{1+\varepsilon F_l}-\frac{F_i}{1+\varepsilon F_i}\frac{F_j}{1+\varepsilon F_j}\right)\\&\quad\times\left(\log\left(\frac{F_i}{1+\varepsilon F_i}\frac{F_j}{1+\varepsilon F_j}\right)-\log\left(\frac{F_k}{1+\varepsilon F_k}\frac{F_l}{1+\varepsilon F_l}\right)\right)\leq 0,\end{aligned}\tag{20}$$

with equality in inequality (20) if and only if

$$\frac{F_i}{1+\varepsilon F_i}\frac{F_j}{1+\varepsilon F_j}=\frac{F_k}{1+\varepsilon F_k}\frac{F_l}{1+\varepsilon F_l} \tag{21}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{ij}^{kl} \neq 0$. Then there is equality in inequality (20) if and only if $\dfrac{F}{1+\varepsilon F}$ is a Maxwellian or, equivalently, if and only if F is a Planckian.

Furthermore, in a similar way

$$\begin{aligned}\langle H, C_{31}(F)\rangle =& \sum_{i,j,k,l=1}^{N} \Gamma_{jkl}^{i}\left(H_i - H_j - H_k - H_l\right)\\ &\times \left(\left((1+\varepsilon F_i) F_j F_k F_l - F_i \left(1+\varepsilon F_j\right)(1+\varepsilon F_k)(1+\varepsilon F_l)\right)\right),\end{aligned} \tag{22}$$

and so, once again assuming that $0 < F_i < 1$ for $i \in \{1, ..., N\}$ if $\varepsilon = -1$

$$\begin{aligned}&\left\langle \log\frac{F}{1+\varepsilon F}, C_{31}(F)\right\rangle\\ &= \sum_{i,j,k,l=1}^{N} \Gamma_{jkl}^{i}(1+\varepsilon F_i)\left(1+\varepsilon F_j\right)(1+\varepsilon F_k)(1+\varepsilon F_l)\\ &\quad\times\left(\frac{F_j}{1+\varepsilon F_j}\frac{F_k}{1+\varepsilon F_k}\frac{F_l}{1+\varepsilon F_l}-\frac{F_i}{1+\varepsilon F_i}\right)\\ &\quad\times\left(\log\frac{F_i}{1+\varepsilon F_i}-\log\left(\frac{F_j}{1+\varepsilon F_j}\frac{F_k}{1+\varepsilon F_k}\frac{F_l}{1+\varepsilon F_l}\right)\right)\leq 0,\end{aligned} \tag{23}$$

with equality if and only if

$$\frac{F_i}{1+\varepsilon F_i}=\frac{F_j}{1+\varepsilon F_j}\frac{F_k}{1+\varepsilon F_k}\frac{F_l}{1+\varepsilon F_l} \tag{24}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jkl}^{i} \neq 0$. Then, there is equality in inequality (23) if and only if $\dfrac{F}{1+\varepsilon F}$ is a Maxwellian or, equivalently, if and only if F is a Planckian.

By the relations (16), (19), and (22),

$$\langle \phi, C_{12}(F) + C_{22}(F) + C_{31}(F)\rangle$$

is zero, independently of our choice of nonnegative function F if and only if ϕ is a collision invariant, and so (for normal models) the equation

$$\langle \phi, C_{12}(F) + C_{22}(F) + C_{31}(F) \rangle = 0 \tag{25}$$

has the general solution (14).

3 Trend to Equilibrium

This section concerns the trend to equilibrium in two particular cases, a planar stationary case and the spatially homogeneous case.

Note that, by our discrete approach, we avoid some main difficulties in the continuous boson case, like mass concentration and appearance of singular measures in the models.

3.1 Planar Stationary System

Introduce the functional

$$\widetilde{\mathcal{H}}[F] = \widetilde{\mathcal{H}}[F](x) = \sum_{i=1}^{N} \left(\frac{\partial E}{\partial p^1}(\mathbf{p}_i) + \mathrm{v}_c^1 \right) \mu(F_i(x)),$$

where, cf. [19],

$$\mu(y) = y \log y - \varepsilon (1 + \varepsilon y) \log (1 + \varepsilon y) . \tag{26}$$

For the planar stationary system

$$\begin{aligned} B \frac{dF}{dx} &= C_{12}(F) + C_{22}(F) + C_{31}(F), \\ \text{where } B &= \operatorname{diag}\left(\frac{\partial E}{\partial p^1}(\mathbf{p}_1) + \mathrm{v}_c^1, ..., \frac{\partial E}{\partial p^1}(\mathbf{p}_N) + \mathrm{v}_c^1 \right), \end{aligned} \tag{27}$$

yields

$$\begin{aligned} \frac{d}{dx} \widetilde{\mathcal{H}}[F] &= \sum_{i=1}^{N} \left(\frac{\partial E}{\partial p^1}(\mathbf{p}_i) + \mathrm{v}_c^1 \right) \frac{dF_i}{dx} \log \frac{F_i}{1 + \varepsilon F_i} \\ &= \left\langle \log \frac{F}{1 + \varepsilon F}, C_{12}(F) + C_{22}(F) + C_{31}(F) \right\rangle \leq 0, \end{aligned}$$

with equality if and only if F is a Planckian. Denote the moments—if all collision coefficients Γ_{jk}^{i} and Γ_{jkl}^{i} for any $\{i, j, k, l\} \subseteq \{1, ..., N\}$, are zero, then we have to include also $\widetilde{j_0} = \langle 1, BF \rangle$— by

$$\begin{cases} \widetilde{j}_i = \left\langle Bp^i, F \right\rangle \text{ for } 1 \le i \le d \\ \widetilde{j}_{d+1} = \left\langle BE, F \right\rangle \end{cases}. \tag{28}$$

By applying equality (25) in system (27) the numbers $\widetilde{j}_1, ..., \widetilde{j}_{d+1}$ are independent with respect to x in the planar stationary case. For some fixed numbers $\widetilde{j}_1, ..., \widetilde{j}_{d+1}$—if all collision coefficients Γ^i_{jk} and Γ^i_{jkl} for any $\{i, j, k, l\} \subseteq \{1, ..., N\}$ are zero, then include also $\widetilde{j}_0$—denote by P the manifold of all Planckians $F = P$ given in (15), such that the relations (28) are fulfilled. Then the following theorem can be proven by arguments similar to the ones used for the discrete Boltzmann equation in [16] (see also [9]; cf. [8]).

Theorem 1 *Let $F = F(x)$ be a bounded positive solution to system (27), and assume that there exists a number $\eta > 0$, such that $F_i(x) \ge \eta$ for $i \in \{1, ..., N\}$, and for $\varepsilon = -1$ that, additionally, $F_i(x) \le 1 - \eta$ for $i \in \{1, ..., N\}$. Then*

$$\lim_{x\to\infty} \operatorname{dist}(F(x), \mathbb{P}) = 0,$$

where P is the Planckian manifold associated with the same invariants (28) as F. If there are only finitely many Planckians in P, then there is a Planckian P in P, such that $\lim\limits_{x\to\infty} F(x) = P$.

3.2 Spatially Homogeneous System

For the spatially homogeneous system

$$\frac{dF}{dt} = C_{12}(F) + C_{22}(F) + C_{31}(F), \tag{29}$$

similar results, presented in Theorem 2 below, can be obtained, by repeating the same arguments, considering the modified functional

$$\mathcal{H}[F] = \mathcal{H}[F](t) = \sum_{i=1}^{N} \mu(F_i(t)),$$

with μ given by equality (26), and—if all collision coefficients Γ^i_{jk} and Γ^i_{jkl} are zero for any $\{i, j, k, l\} \subseteq \{1, ..., N\}$, then we have to include also $j_0 = \langle 1, F\rangle$—the moments

$$\begin{cases} j_i = \left\langle p^i, F \right\rangle \text{ for } 1 \le i \le d \\ j_{d+1} = \left\langle E, F \right\rangle \end{cases}. \tag{30}$$

The following result is relevant in the spatially homogeneous case.

Lemma 1 *Let P and $\widetilde{P}$ be two Planckians with the same moments (30). Then $P = \widetilde{P}$.*

Proof Note that

$$-\log\left(P^{-1}+\varepsilon\right)=\log\frac{P}{1+\varepsilon P}=\sum_{i\in I}c_i\phi^i \text{ and}$$
$$-\log\left(\widetilde{P}^{-1}+\varepsilon\right)=\log\frac{\widetilde{P}}{1+\varepsilon \widetilde{P}}=\sum_{i\in I}\widetilde{c}_i\phi^i,$$

for some numbers c_i and $\widetilde{c}_i$ and that

$$\left\langle \phi^i, P\right\rangle = j_i = \left\langle \phi^i, \widetilde{P}\right\rangle \text{ for } i\in I.$$

Here $I = \begin{cases} \{0,...,d+1\} \text{ if } \Gamma^i_{jl}=\Gamma^i_{jkl}=0 \text{ for all } \{i,j,k,l\}\subseteq\{1,...,N\} \\ \{1,...,d+1\} \text{ otherwise} \end{cases}$ and ϕ^i for $i\in I$, are the collision invariants (13). Obviously,

$$\left\langle \log\left(P^{-1}+\varepsilon\right), P\right\rangle = -\sum_{i\in I}c_i j_i = \left\langle \log\left(P^{-1}+\varepsilon\right), \widetilde{P}\right\rangle \text{ and}$$
$$\left\langle \log\left(\widetilde{P}^{-1}+\varepsilon\right), P\right\rangle = -\sum_{i\in I}\widetilde{c}_i j_i = \left\langle \log\left(\widetilde{P}^{-1}+\varepsilon\right), \widetilde{P}\right\rangle,$$

and, hence,

$$\begin{aligned}
&\sum_{i=1}^{N} P_i\widetilde{P}_i\left(\widetilde{P}_i^{-1}-P_i^{-1}\right)\log\left(\frac{P_i^{-1}+\varepsilon}{\widetilde{P}_i^{-1}+\varepsilon}\right)\\
&=\sum_{i=1}^{N}\left(P_i-\widetilde{P}_i\right)\log\left(\frac{P_i^{-1}+\varepsilon}{\widetilde{P}_i^{-1}+\varepsilon}\right)\\
&=\left\langle \log\left(P^{-1}+\varepsilon\right)-\log\left(\widetilde{P}^{-1}+\varepsilon\right), P-\widetilde{P}\right\rangle = 0.
\end{aligned}\tag{31}$$

Since

$$(y-z)\log\frac{z}{y}\leq 0 \tag{32}$$

for all positive numbers $y>0$ and $z>0$, with equality in inequality (32) if and only if $y=z$, it follows that

$$\begin{aligned}
&\left(\widetilde{P}_i^{-1}-P_i^{-1}\right)\log\left(\frac{P_i^{-1}+\varepsilon}{\widetilde{P}_i^{-1}+\varepsilon}\right)=\left(\widetilde{P}_i^{-1}+\varepsilon-\left(P_i^{-1}+\varepsilon\right)\right)\\
&\log\left(\frac{P_i^{-1}+\varepsilon}{\widetilde{P}_i^{-1}+\varepsilon}\right)\leq 0 \text{ for } i\in I.
\end{aligned}\tag{33}$$

By equality (31), it follows that $P = \widetilde{P}$, since, in fact, all the inequalities in (33) must be equalities, and hence, $\widetilde{P}_i^{-1} = P_i^{-1}$ for all $i \in I$. $\square$

Theorem 2 *Let $F = F(t)$ be a bounded positive solution to equation (29), and assume that there exists a number $\eta > 0$, such that $F_i(t) \geq \eta$ for $i \in \{1, ..., N\}$, and for $\varepsilon = -1$ that, additionally, $F_i(t) \leq 1 - \eta$ for $i \in \{1, ..., N\}$. Then*

$$\lim_{t \to \infty} F(t) = P,$$

where P is the Planckian with the same moments (30) as F.

Remark 2 The above results in Theorems 1 and 2 can be generalized to a more general case. Let $I_N = \{1, ..., N\}$ and $1 \leq m \leq n \leq N - m$, and denote

$$\begin{aligned} C(F) &= \sum_{1 \leq m \leq n \leq N-m} a_{mn} C_{mni}(F), \text{ with } a_{mn} \geq 0, \\ C_{mni}(F) &= \sum_{\substack{I', I'' \subset I_N \\ |I'|=n,\, |I''|=m}} \Gamma_{I'}^{I''} \left(\sum_{k \in I'} \delta_{ik} - \sum_{k \in I''} \delta_{ik} \right) \\ &\quad \times \left(\prod_{j \in I'} F_j \prod_{j \in I''} (1 + \varepsilon F_j) - \prod_{j \in I''} F_j \prod_{j \in I'} (1 + \varepsilon F_j) \right) \\ &= \sum_{\substack{I', I'' \subset I \\ |I'|=n,\, |I''|=m}} \Gamma_{I'}^{I''} \left(\sum_{k \in I'} \delta_{ik} - \sum_{k \in I''} \delta_{ik} \right) \\ &\quad \times \prod_{j \in I' \cup I''} (1 + \varepsilon F_j) \left(\prod_{j \in I'} \frac{F_j}{1 + \varepsilon F_j} - \prod_{j \in I''} \frac{F_j}{1 + \varepsilon F_j} \right), \end{aligned} \tag{34}$$

where $\Gamma_{I'}^{I''} = 0$ if the relations

$$\sum_{k \in I'} \mathbf{p}_k = \sum_{k \in I''} \mathbf{p}_k \text{ and } \sum_{k \in I'} E_k = \sum_{k \in I''} E_k$$

are not satisfied. Then, in a similar way as above, we can obtain corresponding results for the system (2), (27), (29) with the right hand side replaced by $C(F)$. In particular, the stationary points of the systems are Planckians and Theorems 1 and 2 are still valid.

Remark 3 For generalizations to anyons the reader is referred to [8].

4 Linearized Collision Operator

Given a Planckian

$$P = \frac{1}{e^{\alpha E + \boldsymbol{\beta}\cdot\mathbf{p}} - \varepsilon},$$

with $\alpha > 0$ and $\boldsymbol{\beta} \in \mathbb{R}^d$ (such that $\alpha E + \boldsymbol{\beta}\cdot\mathbf{p} > 0$ if $\varepsilon = 1$), inserting the expression

$$F = P + R^{1/2} f, \text{ where } R = P(1 + \varepsilon P),$$

in system (9), results in the system

$$\frac{\partial f}{\partial t} + (\mathbf{B}\nabla_{\mathbf{x}}) \cdot f + Lf = S(f).$$

4.1 The Linearized and Nonlinear Operators

The linearized collision operator—$N \times N$ matrix—$L = L_{12} + L_{22} + L_{31}$ is given by

$$\begin{aligned} L_{12} f &= -R^{-1/2}\Big(\widetilde{L} R^{1/2} f \\ &\quad + 2\widetilde{Q}\big(P, R^{1/2} f\big) + \widetilde{\widetilde{Q}}\big(R^{1/2} f, P, P\big) + 2\widetilde{\widetilde{Q}}\big(P, R^{1/2} f, P\big)\Big), \qquad (35) \\ L_{22} f &= -R^{-1/2}\Big(2\overline{Q}\big(P, R^{1/2} f\big) + \overline{\overline{Q}}\big(R^{1/2} f, P, P\big) \\ &\quad + 2\overline{\overline{Q}}\big(P, R^{1/2} f, P\big)\Big), \qquad (36) \end{aligned}$$

and

$$\begin{aligned} L_{31} f &= -R^{-1/2}\Big(\widehat{L} R^{1/2} f + 2\widehat{Q}\big(P, R^{1/2} f\big) + \widehat{\widehat{Q}}\big(R^{1/2} f, P, P\big) \\ &\quad + 2\widehat{\widehat{Q}}\big(P, R^{1/2} f, P\big)\Big). \qquad (37) \end{aligned}$$

The nonlinear part $S(f) = S_{12}(f, f) + S_{22}(f, f, f) + S_{31}(f, f, f)$ is given by

$$\begin{aligned} S_{12}(f, g, h) = R^{-1/2}\Big(&\widetilde{Q}\big(R^{1/2} f, R^{1/2} g\big) + \widetilde{\widetilde{Q}}\big(P + R^{1/2} f, R^{1/2} g, R^{1/2} h\big) \\ &+ \widetilde{\widetilde{Q}}\big(R^{1/2} f, P, R^{1/2} h\big) + \widetilde{\widetilde{Q}}\big(R^{1/2} f, R^{1/2} g, P\big)\Big), \qquad (38) \end{aligned}$$

$$S_{22}(f,g,h) = R^{-1/2}\left(\overline{Q}\big(R^{1/2}f, R^{1/2}g\big) + \overline{\overline{Q}}\big(P + R^{1/2}f, R^{1/2}g, R^{1/2}h\big)\right.$$
$$\left. + \overline{\overline{Q}}\big(R^{1/2}f, P, R^{1/2}h\big) + \overline{\overline{Q}}\big(R^{1/2}f, R^{1/2}g, P\big)\right), \tag{39}$$

and

$$S_{31}(f,g,h) = R^{-1/2}\left(\widehat{Q}\big(R^{1/2}f, R^{1/2}g\big) + \widehat{\widehat{Q}}\big(P + R^{1/2}f, R^{1/2}g, R^{1/2}h\big)\right.$$
$$\left. + \widehat{\widehat{Q}}\big(R^{1/2}f, P, R^{1/2}h\big) + \widehat{\widehat{Q}}\big(R^{1/2}f, R^{1/2}g, P\big)\right). \tag{40}$$

In more explicit forms, the operators (35) and (38) read

$$(L_{12}f)_i = \sum_{j,k=1}^{N} \frac{\Gamma_{jk}^{i} L_{jk}^{i} f - 2\Gamma_{ij}^{k} L_{ij}^{k} f}{R_i^{1/2}} \text{ for } i \in \{1, ..., N\}, \tag{41}$$

where

$$L_{jk}^{i} f = \big(1 + \varepsilon\big(P_j + P_k\big) + (1-\varepsilon) P_j P_k\big) R_i^{1/2} f_i$$
$$+ (\varepsilon P_i - P_k + (1-\varepsilon) P_i P_k) R_j^{1/2} f_j + \big(\varepsilon P_i - P_j + (1-\varepsilon) P_i P_j\big) R_k^{1/2} f_k$$

and

$$S_{12i}(f,f) = \sum_{j,k=1}^{N} \frac{\Gamma_{jk}^{i} S_{jk}^{i}(f,f) - 2\Gamma_{ij}^{k} S_{ij}^{k}(f,f)}{R_i^{1/2}} \text{ for } i \in \{1, ..., N\},$$

with for any indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$

$$S_{jk}^{i}(f,f)$$
$$= (1 + (\varepsilon - 1) P_i) R_j^{1/2} R_k^{1/2} f_j f_k + R_i^{1/2} f_i$$
$$\times \left(((\varepsilon - 1) P_k - \varepsilon) R_j^{1/2} f_j - ((\varepsilon - 1) P_l - \varepsilon) R_k^{1/2} f_k + (\varepsilon - 1) R_j^{1/2} R_k^{1/2} f_j f_k\right).$$

Moreover, the operators (36) and (39) read, in more explicit forms,

$$(L_{22}f)_i = \sum_{j,k,l=1}^{N} \frac{\Gamma_{ij}^{kl}}{R_i^{1/2}} \left(L_{ij}^{kl} f_i + L_{ji}^{kl} f_j - L_{kl}^{ij} f_k - L_{lk}^{ij} f_l\right) \text{ for } i \in \{1, ..., N\}, \tag{42}$$

where

$$L_{ij}^{kl} = \big(P_j\,(1 + \varepsilon P_k + \varepsilon P_l) - \varepsilon P_k P_l\big) R_i^{1/2}$$

and

$$S_{22i}(f,f,f) = \sum_{j,k,l=1}^{N} \frac{\Gamma_{ij}^{kl}}{R_i^{1/2}} \left(S_{ij}^{kl}(f,f,f) - S_{kl}^{ij}(f,f,f) \right) \text{ for } i \in \{1,...,N\},$$

with for any indices $\{i,j,k,l\} \subseteq \{1,...,N\}$

$$\begin{aligned} S_{ij}^{kl}(f,f,f) = &\left(1+\varepsilon P_i + \varepsilon P_j\right) R_k^{1/2} R_l^{1/2} f_k f_l \left(R_i^{1/2} f_i + R_j^{1/2} f_j \right) \\ &\times \left(P_k R_l^{1/2} f_l + P_l R_k^{1/2} f_k + R_k^{1/2} R_l^{1/2} f_k f_l \right). \end{aligned}$$

Furthermore, the operators (37) and (40) read, in more explicit forms,

$$(L_{31} f)_i = \sum_{j,k,l=1}^{N} \frac{\Gamma_{jkl}^{i} L_{jkl}^{i} f - 3\Gamma_{ijk}^{l} L_{ijk}^{l} f}{R_i^{1/2}} \text{ for } i \in \{1,...,N\}, \tag{43}$$

where

$$\begin{aligned} L_{jkl}^{i} = &\left(\left(1+\varepsilon\left(P_j+P_k\right)\right)\left(1+\varepsilon P_l\right) + P_j P_k\right) R_i^{1/2} f_i \\ &+ \left(P_i\left(\varepsilon + P_k + P_l\right) - P_k P_l\right) R_j^{1/2} f_j \\ &+ \left(P_i\left(\varepsilon + P_j + P_l\right) - P_j P_l\right) R_k^{1/2} f_k + \left(P_i\left(\varepsilon + P_j + P_k\right) - P_j P_k\right) R_l^{1/2} f_l \end{aligned}$$

for any indices $\{i,j,k,l\} \subseteq \{1,...,N\}$, and

$$S_{31i}(f,f,f) = \sum_{j,k,l=1}^{N} \frac{\Gamma_{jkl}^{i} S_{jkl}^{i} - 3\Gamma_{ijk}^{l} S_{ijk}^{l}}{R_i^{1/2}} \text{ for } i \in \{1,...,N\},$$

with for any indices $\{i,j,k,l\} \subseteq \{1,...,N\}$

$$\begin{aligned} S_{jkl}^{i}(f,f,f) = &\left(R_j^{1/2} R_k^{1/2} f_j f_k + R_j^{1/2} R_l^{1/2} f_j f_l + R_k^{1/2} R_l^{1/2} f_k f_l \right)\left(P_i + R_i^{1/2} f_i \right) \\ &+ \left(\left(\varepsilon + P_k + P_l \right) R_j^{1/2} f_j \right. \\ &\left. + \left(\varepsilon + P_j + P_l \right) R_k^{1/2} f_k + \left(\varepsilon + P_j + P_k \right) R_l^{1/2} f_l \right) R_i^{1/2} f_i \\ &- R_j^{1/2} f_j \left(P_l R_k^{1/2} f_k + P_k R_l^{1/2} f_l + R_k^{1/2} R_l^{1/2} f_k f_l \right) \\ &- P_j R_k^{1/2} R_l^{1/2} f_k f_l. \end{aligned}$$

.

4.2 Some Properties of the Linearized Collision Operator

By equalities (4), (41), and the relations

$$P_j(1+\varepsilon P_j)(P_k-\varepsilon P_i+(\varepsilon-1)\,P_iP_k) = P_i(1+\varepsilon P_j)(1+\varepsilon P_k)=P_jP_k\,(1+\varepsilon P_i)\,,$$
$$\text{and } P_jP_k\,(1+(\varepsilon-1)\,P_i) = P_i\Big(1+\varepsilon\left(P_j+P_k\right)\Big)$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jk}^{i} \neq 0$, follows, for any functions $g = g(t, \mathbf{x}, \mathbf{p})$ and $f = f(t, \mathbf{x}, \mathbf{p})$, the equality

$$\begin{aligned}\langle g, L_{12}f\rangle = &\sum_{i,j,k=1}^{N} \Gamma_{jk}^{i} P_i\left(1+\varepsilon P_j\right)(1+\varepsilon P_k)\\ &\times\left(\frac{f_i}{R_i^{1/2}}-\frac{f_j}{R_j^{1/2}}-\frac{f_k}{R_k^{1/2}}\right)\left(\frac{g_i}{R_i^{1/2}}-\frac{g_j}{R_j^{1/2}}-\frac{g_k}{R_k^{1/2}}\right).\end{aligned}$$

Similarly, by equalities (6), (42), and the relations

$$P_iP_j(1+\varepsilon P_k)(1+\varepsilon P_l) = P_kP_l(1+\varepsilon P_i)(1+\varepsilon P_j)$$
$$\text{and } L_{ij}^{kl} = P_kP_l(1+\varepsilon P_j)\frac{\sqrt{1+\varepsilon P_i}}{\sqrt{P_i}}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{ij}^{kl} \neq 0$, follows, for any functions $g = g(t, \mathbf{x}, \mathbf{p})$ and $f = f(t, \mathbf{x}, \mathbf{p})$, the equality

$$\begin{aligned}\langle g, L_{22}f\rangle = &\frac{1}{4}\sum_{i,j,k,l=1}^{N} \Gamma_{ij}^{kl} P_iP_j(1+\varepsilon P_k)(1+\varepsilon P_l)\\ &\times\left(\frac{f_i}{R_i^{1/2}}+\frac{f_j}{R_j^{1/2}}-\frac{f_k}{R_k^{1/2}}-\frac{f_l}{R_l^{1/2}}\right)\left(\frac{g_i}{R_i^{1/2}}+\frac{g_j}{R_j^{1/2}}-\frac{g_k}{R_k^{1/2}}-\frac{g_l}{R_l^{1/2}}\right).\end{aligned}$$

Finally, by equalities (8), (43), and the relations

$$P_jP_kP_l(1+\varepsilon P_i) = P_i(1+\varepsilon P_j)(1+\varepsilon P_k)\,(1+\varepsilon P_l) \text{ and}$$
$$\left(P_j-P_i\right)P_kP_l = P_i(1+\varepsilon P_j)(1+\varepsilon\,(P_k+P_l))$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jkl}^{i} \neq 0$, follows, for any functions $g = g(t, \mathbf{x}, \mathbf{p})$ and $f = f(t, \mathbf{x}, \mathbf{p})$, the equality

$$\langle g, L_{31} f\rangle = \sum_{i,j,k,l=1}^{N} \Gamma_{jkl}^{i} P_i \left(1+\varepsilon P_j\right)(1+\varepsilon P_k)(1+\varepsilon P_l)$$
$$\times \left(\frac{f_i}{R_i^{1/2}} - \frac{f_j}{R_j^{1/2}} - \frac{f_k}{R_k^{1/2}} - \frac{f_l}{R_l^{1/2}}\right)\left(\frac{g_i}{R_i^{1/2}} - \frac{g_j}{R_j^{1/2}} - \frac{g_k}{R_k^{1/2}} - \frac{g_l}{R_l^{1/2}}\right).$$

The following proposition follows.

Proposition 3 *The matrix L is symmetric and nonnegative, i.e.,*

$$\langle g, Lf\rangle = \langle Lg, f\rangle \text{ and } \langle f, Lf\rangle \geq 0$$

for all functions $g = g(t, \mathbf{x}, \mathbf{p})$ and $f = f(t, \mathbf{x}, \mathbf{p})$.

Furthermore, $\langle f, Lf\rangle = 0$ if and only if

$$\frac{f_i}{R_i^{1/2}} = \frac{f_j}{R_j^{1/2}} + \frac{f_k}{R_k^{1/2}} \tag{44}$$

for all indices $\{i, j, k\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jk}^{i} \neq 0$,

$$\frac{f_i}{R_i^{1/2}} + \frac{f_j}{R_j^{1/2}} = \frac{f_k}{R_k^{1/2}} + \frac{f_l}{R_l^{1/2}} \tag{45}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{ij}^{kl} \neq 0$, and

$$\frac{f_i}{R_i^{1/2}} = \frac{f_j}{R_j^{1/2}} + \frac{f_k}{R_k^{1/2}} + \frac{f_l}{R_l^{1/2}} \tag{46}$$

for all indices $\{i, j, k, l\} \subseteq \{1, ..., N\}$ such that $\Gamma_{jkl}^{i} \neq 0$. Denote $f = R^{1/2}\phi$ in equalities (44)–(46), obtaining the relations (10)–(12), respectively. Hence, since L is nonnegative,

$$Lf = 0 \text{ if and only if } f = R^{1/2}\phi,$$

where ϕ is a collision invariant (14), and the following proposition follows.

Proposition 4 *The kernel of the linearized operator L is given by*

$$\ker L = \left\{ f \mid f = \sqrt{P\,(1+\varepsilon P)}\phi, \text{ where } \phi \text{ is a collision invariant} \right\}.$$

Note that $S(f) \in \mathrm{Im}L = (\ker L)^{\perp}$, since

$$\left\langle S(f), R^{1/2}\phi \right\rangle = \langle C_{12}(F) + C_{22}(F) + C_{31}(F), \phi\rangle + \left\langle f, LR^{1/2}\phi \right\rangle = 0$$

for all collision invariants ϕ.

Remark 4 The general results obtained for planar stationary half-space problems for the discrete equations obtained in [3, 6, 11], yield also for the extended discrete quantum Boltzmann equation presented here. Indeed, consider the planar stationary system (27)—for the linearized collision operator, possibly also with an inhomogeneous term, see [3, 6, 11], or in a weakly non-linear setting, see [6]—for $x > 0$. Assume the components $F_i(0)$ of the distribution function at $x = 0$ for which $\dfrac{\partial E}{\partial p^1}(\mathbf{p}_i) + \mathrm{v}_c^1$ is positive to be given - possibly linearly depending on the components of $F(0)$ for which $\dfrac{\partial E}{\partial p^1}(\mathbf{p}_i) + \mathrm{v}_c^1$ is negative. Then results concerning the number of conditions needed for existence and/or uniqueness of solutions-based on the signature of the restriction of the quadratic form $\langle \cdot, B\cdot \rangle$ to the kernel of L in [3, 6, 11] can be applied. We stress that the results presented in [3, 6, 11] can be applied also for the Cauchy problem in the spatially homogenous case.

Remark 5 The results can be extended to mixtures, including the case of mixtures containing both bosons and fermions—i.e. not necessarily with the same ε for different species, and particles with a discrete number of different energy levels, with the approaches presented in [5–7, 10], see Remark 7 below.

5 Linearized Collision Operator for a General Collision Operator

Now consider the general collision operator (34). Denoting

$$F = P + R^{1/2} f, \text{ where } R = P(1 + \varepsilon P),$$

in system (2), with the right hand side replaced by the general collision operator (34), and linearizing in f, one obtain

$$\frac{\partial f_i}{\partial t} + \left((\nabla_{\mathbf{p}} E)_i + \mathrm{v}_c \right) \cdot \nabla_{\mathbf{x}} f_i + (Lf)_i = 0$$

where L is the linearized collision operator ($N \times N$ matrix) given by

$$(Lf)_i = \sum_{1 \leq m \leq n \leq N-m} a_{mn} (L_{mn} f)_i \;\text{ for } i \in \{1, ..., N\}, \tag{47}$$

with $a_{mn} \geq 0$ and

$$(L_{mn}f)_i = \sum_{\substack{I',I'' \subset I_N \\ |I'|=n,\, |I''|=m}} \frac{\Gamma_{I'}^{I''}}{R_i^{1/2}} \left(\sum_{k \in I'} \delta_{ik} - \sum_{k \in I''} \delta_{ik} \right) \left(\sum_{j \in I'} \left(P_{I'}^{I''} \right)_i f_j - \sum_{j \in I''} \left(P_{I'}^{I''} \right)_i f_j \right). \tag{48}$$

Here, denoting

$$\Pi_{I'}^{I''}(g) = \prod_{j \in I'} g_j \prod_{j \in I''} \left(1 + \varepsilon g_j\right) - \prod_{j \in I''} g_j \prod_{j \in I'} \left(1 + \varepsilon g_j\right),$$

follows that for $\Gamma_{I'}^{I''} \neq 0$,

$$\begin{aligned}
\left(P_{I'}^{I''} \right)_i &= \left. \frac{\partial \Pi_{I'}^{I''}\left(P + R^{1/2} f\right)}{\partial f_i} \right|_{f=0} \\
&= R_i^{1/2} \left(\frac{1}{P_i} \prod_{j \in I'} P_j \prod_{j \in I''} \left(1 + \varepsilon P_j\right) - \frac{\varepsilon}{1 + \varepsilon P_i} \prod_{j \in I''} P_j \prod_{j \in I'} \left(1 + \varepsilon P_j\right) \right) \\
&= \frac{1}{R_i^{1/2}} \prod_{j \in I''} P_j \prod_{j \in I'} \left(1 + \varepsilon P_j\right) \frac{1}{P_i \left(1 + \varepsilon P_i\right)} R_i \\
&= \frac{1}{R_i^{1/2}} \prod_{j \in I''} P_j \prod_{j \in I'} \left(1 + \varepsilon P_j\right),
\end{aligned} \tag{49}$$

since (cf. relations (18), (21), and (24))

$$\prod_{j \in I'} \frac{P_j}{1 + \varepsilon P_j} = \prod_{j \in I''} \frac{P_j}{1 + \varepsilon P_j} \tag{50}$$

for $\Gamma_{I'}^{I''} \neq 0$. Hence, by the equalities (47)–(50), follows the equality

$$\langle g, Lf \rangle = \sum_{1 \leq m \leq n \leq N-m} a_{mn} \Gamma_{I'}^{I''} \prod_{j \in I''} P_j \prod_{j \in I'} \left(1 + \varepsilon P_j\right) \times \left(\sum_{j \in I'} \frac{f_j}{R_j^{1/2}} - \sum_{j \in I''} \frac{f_j}{R_j^{1/2}} \right) \left(\sum_{j \in I'} \frac{g_j}{R_j^{1/2}} - \sum_{j \in I''} \frac{g_j}{R_j^{1/2}} \right). \tag{51}$$

From equality (51), it is easy to see that the matrix L is symmetric and nonnegative. Furthermore, by the equality (51), $\langle f, Lf \rangle = 0$ if and only if

$$\prod_{j \in I'} \frac{f_j}{R_j^{1/2}} = \prod_{j \in I''} \frac{f_j}{R_j^{1/2}} \tag{52}$$

for all sets I' and I'' such that $\Gamma_{I'}^{I''} \neq 0$. Denote $f = R^{1/2}\phi$ in the relation (52), obtaining

$$\prod_{j \in I'} \phi_j = \prod_{j \in I''} \phi_j$$

for all sets I' and I'' such that $\Gamma_{I'}^{I''} \neq 0$. Hence, since L is nonnegative,

$$Lf = 0 \text{ if and only if } f = R^{1/2}\phi,$$

where ϕ is a collision invariant, and the following proposition follows.

Proposition 5 *The linearized operator L is symmetric and nonnegative, and its kernel is given by*

$$\ker L = \left\{ f \mid f = \sqrt{P\,(1 + \varepsilon P)}\phi, \textit{ where } \phi \textit{ is a collision invariant} \right\}.$$

Remark 6 We note that the general results obtained for linearized half-space problems for the discrete Boltzmann equation obtained in [3, 11], and for a discrete quantum Boltzmann equation in [4], yield also for the discrete Boltzmann equation for the general linearized collision operator considered here, cf. Remark 4. Again we stress that the results presented in [3, 4, 11] can be applied also for the Cauchy problem in the spatially homogenous case.

Remark 7 The results can again be extended to mixtures, including the case of mixtures containing both bosons and fermions, and particles with a discrete number of different energy levels, with the approaches presented in [5–7, 10]. Indeed, the key feature is that to each component F_i of the distribution function F there will be assigned not only a momentum $\mathbf{p}_i$, but also a species α_i with species-dependent ε_{α_i}, where $\varepsilon_{\alpha_i}^2 = 1$, and possibly also an internal energy I_i. The sets of admissible momentums—and possible sets of internal energies—may vary for different species. At a formal level, this extension seems merely to be a matter of notation. Regarding implementations, it is another story: e.g., finding models without spurious collision invariants will be a more delicate task, even if known models for discrete velocity models of the Boltzmann equation, see [5, 7, 10] and references therein, again can be used in case of approximations of the Bogoliubov excitation energy $E = E(\mathbf{p})$ of the form $E = c_1 |\mathbf{p}|^2 + c_2$ for constant c_1 and c_2 (as long as the collision term $C_{22}(F)$ is included). Also, even if not restricted to the discrete case, there will be several different collision operators of each kind, e.g., for s different species there will be s^2—one for each ordered pair of species—collision operators of type C_{22}, $2\,s^2 - s$—for each ordered pair of different species there will be two different—collision operators of type C_{12}, etc..

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