Gibbs Phenomenon for Fourier-Legendre Series

Gibbs fenomen för Fourier-Legendre serier

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Abstract

In this thesis, the main objective is to study the presence of Gibbs phenomenon and the Gibbs constant in Fourier-Legendre series. The occurrence of The Gibbs phenomenon is a well known consequence when approximating functions with Fourier series that have points of discontinuity. Consequently, the initial focus was to examine Fourier series and the occurrence of Gibbs phenomenon in this context. Next, we delve into Legendre polynomials, showing their applicability to be expressed as a Fourier series due to their orthogonality in $[-1, 1]$. We then continue to explore Gibbs phenomenon for Fourier-Legendre series. The findings proceeds to confirm the existence of the Gibbs phenomenon for Fourier-Legendre series, but most notebly, the values of the error seem to converge to the same number as for Fourier series which is the Gibbs constant.

Sammanfattning

I denna uppsats är målet att studera förekomsten av Gibbs fenomen och Gibbs konstanten i Fourier-Legendre-serier. Gibbs fenomenet är en välkänd konsekvens när man approximerar funktioner med Fourier-serier som har punkter av diskontinuitet. Det ursprungliga fokuset var därför att undersöka Fourier-serier och förekomsten av Gibbs fenomen i detta sammanhang. Därefter går vi in på Legendre-polynom och visar deras tillämplighet att uttryckas som en Fourier-serie på grund av deras ortogonalitet i intervallet $[-1, 1]$. Vi fortsätter sedan att utforska Gibbs fenomen för Fourier-Legendre-serier. Resultaten bekräftar förekomsten av Gibbs fenomenet för Fourier-Legendre-serier och anmärkningsvärt verkar värdet för felet konvergera till samma värde som för Fourier-serier, vilket är Gibbs konstanten.

Acknowledgements

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Contents

Chapter 1

Introduction

In the early 1800's, Joseph Fourier caused major controversy in the scientific world by claiming that an arbitrary function could be written as an infinite series of trigonometric functions. His claim was that a given function defined on an interval can be expanded in terms of an infinite sum of sines and cosines [1].

The aim was to solve partial differential equations, more precisely, to find solutions to the heat equation. At the time, Fourier's approach was considered nonsense but has later been developed into one of the oldest and most major part of mathematical analysis. The study of Fourier series, also known as *harmonic analysis* has large applications in areas such as signal processing, quantum mechanics, neuroscience or any other event with a recurrent nature. However, there is an undesirable consequence of these approximations when truncating the series, namely Gibbs Phenomenon.

The Gibbs phenomenon could roughly be described as oscillations, or *overshoots* and *undershoots* that occur near the discontinuities of a function when approximated by a Fourier series. This phenomenon was first discovered by Henry Wilbraham in 1848 [2], but it didn't get a lot of attention and was largely forgotten until the late 1800s when Josiah W. Gibbs discussed its existence [3]. Initially, Gibbs phenomenon was discussed predominately in its relation to the trigonometric system. However, research demonstrated that this phenomenon extends beyond this system.

In 2010, researchers investigated the Gibbs phenomenon for Fourier-Bessel series expansions. The findings of this research indicated that the Gibbs phenomenon, does indeed emerge in Fourier-Bessel series expansions and exhibits similar behaviour to the Gibbs phenomenon observed in the trigonometric system. More interestingly, the research highlighted that the amplitude of the over- and undershoots near jump discontinuities in the truncated Fourier-Bessel series tends toward a distinct limit, mirroring the behavior of the Gibbs phenomenon in Fourier series.

For Fourier series, this limit is often referred to as "Gibbs Constant", and can be computed by

$$\frac{2c}{\pi}\int_0^\pi \frac{\sin(x)}{x}dx - c$$

where c is one-half the magnitude of the jump discontinuity. It is indeed intriguing that this constant would also appear in an entirely different series expansion, characterized by its own unique set of functions and properties.

In the conclusion of this research, the question was raised of whether the Gibbs constant is also present in Fourier-Legendre series as it is for Fourier-Bessel series[4]. This thesis delves into this matter to explore the existence of the Gibbs phenomenon and the Gibbs constant in Fourier-Legendre series.

This thesis is divided into 6 chapters:

In Chapter 2 we present a set of definitions and properties of Fourier Series and lay the groundwork for the thesis, that will be relevant in later chapters. Additionally,we explore Gibbs phenomenon graphically and analytically for the general Fourier Series.

In Chapter 3 we discuss the Legendre polynomials, exploring their orthogonality while also introducing and defining the Fourier-Legendre expansion.

In Chapter 4 we finish this paper with a numerical experiment using python, exploring the Gibbs phenomenon for Fourier-Legendre series.

Chapter 5 compiles some computations done throughout the thesis, while Chapter 6 presents the python code that was used for the numerical experiments.

Chapter 2

Gibbs Phenomenon for Fourier Series

2.1 Fourier series: Definition and Properties

A real valued function can be expressed as a Fourier Series given it is periodic and piecewise continuous. We give a formal definition.

Definition 2.1.1. *Let $f(x)$ be an arbitrary integrable periodic function defined on $(-l, l), l > 0$. Then the Fourier series associated to f is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

where a_n and b_n are the Fourier coefficients given by

$$\begin{aligned} a_n &= \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx, \qquad n = 0, 1, 2, ... \\ b_n &= \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx, \qquad n = 1, 2, 3, ... \qquad . \end{aligned}$$

Some basic important properties of the Fourier series worth noting include:

- Periodicity: The Fourier Series of a periodic function f is also periodic.
- Linearity: If f and g are two periodic functions and F_f and G_g is their respective Fourier series, then $\alpha f + \beta g \sim \alpha F_f + \beta G_g$.
- Even and odd symmetry: The Fourier series of a periodic odd(even) function includes only sine(cosine) terms.

The issue of convergence in Fourier series is frequently of interest. That is, it is not always immediately apparent what form of convergence, if any, is occurring. Throughout this thesis, we assume every function to be *regulated*. A regulated function is a function such that

$$f(x) = \frac{f(x^+) + f(x^-)}{2},$$

where

$$f(x^+) = \lim_{x \to x_0^+} f(x) \neq \pm\infty$$
$$f(x^-) = \lim_{x \to x_0^-} f(x) \neq \pm\infty.$$

When addressing the convergence we have to examine the partial sum of the expansion. Suppose that f is periodic, then the N:th partial sum S_N of the Fourier series for f is

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right).$$

One of the most significant tools for representing the partial sum was introduced by Peter Gustav Lejeune Dirichlet, a German mathematician and a student of Joseph Fourier. This is known as *Dirichlet's integral form*, and is derived in "Fourier Analysis" by James S. Walker(pp 45-48) from the $N:th$ partial sum of the Fourier series for f at $x = x_0$ [5]. It is shown to be

$$S_N(f; x_o) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + t) \frac{\sin\left(N + \frac{1}{2}\right)t}{\sin\frac{t}{2}} dt,$$

where the function

$$D_N(t) = 1 + 2\sum_{n=1}^{N} \cos nt = \frac{\sin\left(N + \frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

is called *Dirichlet's kernel*.

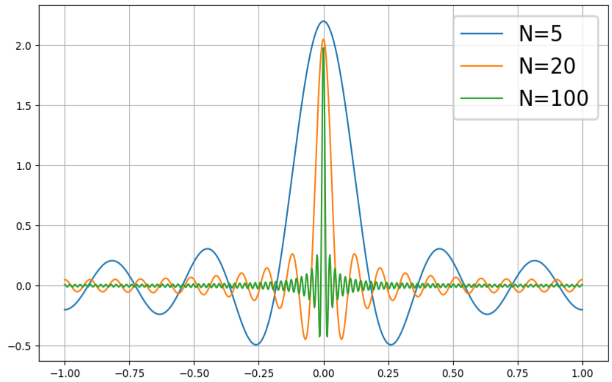


Figure 2.1.1: The Dirichlet Kernel.

2.1.1 Orthogonal systems

The inner product of two vectors can be extended to functions as follows

Definition 2.1.2. *A system (or set) of functions $\{g_n\}_{n=0}^{\infty}$ over the closed interval $[-l, l]$ is called orthogonal if*

$$\int_{-l}^{l} g_m(x)g_n(x)dx = 0 \qquad \textit{if } m \neq n$$
$$\int_{-l}^{l} g_n^2(x)dx > 0 \qquad \textit{for each } n \in \mathbb{Z}_0^+$$

If $\int_{-l}^{l} g_n^2(x)dx = 1$ for each n, then the set of functions $\{g_n\}_{n=0}^{\infty}$ is called an orthonormal system.

Furthermore, an orthogonal system of functions is said to be *complete* if for all g which is orthogonal to every function g_n, we have that $g = 0$ for all n. In other words, for a set of functions to form a complete orthogonal system, any function g that is orthogonal to all functions g_n in the set must be the zero function. This property is of great importance since any set of functions $\{g_n\}_n$ that form complete a orthogonal system have a corresponding generalized Fourier series.

Definition 2.1.3. *Let $\{g_n\}_n$ be an orthonormal set of functions over $[-l, l]$, then the generalized Fourier coefficient of a integrable function f is given by*

$$c_n = \int_{-l}^{l} f(x)g_n(x)dx.$$

The generalized Fourier series for f is then

$$\sum_{n=0}^{\infty} c_n g_n(x).$$

2.2 Gibbs Phenomenon

It is well-established that when f is a piece-wise smooth function with a period $2l$ in a closed interval devoid of discontinuity points, the Fourier series of f converges uniformly for all values of x. Put simply, if x_0 denotes a point where f is continuous, then the Fourier series converges to $f(x_0)$ at x_0. However, if x_0 denotes a point of discontinuity for f, then the Fourier series converges to the average of the right-hand and left-hand limits of f at x_0 [5]. That is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) = \frac{f(x^+) + f(x^-)}{2}.$$

Nonetheless, this framework falls short in addressing the excessive oscillations observed near a point of discontinuity while approximating a function using a finite number of Fourier coefficients. Consequently, it becomes apparent that these oscillations are indicative of approximation errors. Essentially, the occurrence of the Gibbs phenomenon arises when points of discontinuity exist within the interval, and the series is truncated. An interesting inquiry then arise, can we pick any point x_0 within an interval and ensure that a number α will consistently be confined in a given set. Additionally, is it possible to ensure that the approximation error for a given point will not lead to α surpassing the set, even when that point is a point of discontinuity?

Moving forward to address these inquiries, we examine the sawtooth function $\phi(x)$ defined by

$$\phi(x) = \frac{\pi - x}{2}, \qquad x \in (0, 2\pi)$$

and extend to $\mathbb{R}$ by periodicity 2π. Hence, $\phi(x)$ will have discontinuity points at $x = 0, \pm 2\pi, \pm 4\pi ...$

Lemma 2.2.1. *The Fourier series of $\phi(x)$ is given by*

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \qquad x \neq 0, \pm 2\pi, \pm 4\pi ...$$

Proof. The function $\phi(x)$ is odd, so in essence we only need to find the Fourier coefficient b_n. However for the sake of being instructive, we compute all coefficients. For a_0, we have that

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) dx = \frac{1}{\pi} \left[\frac{\pi x}{2} - \frac{x^2}{4} \right]_0^{2\pi} = 0.$$

For a_n,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) \cos(nx) dx.$$

We can solve this integral by applying Bernoulli's formula, which states

$$\int u\ v\ dx = uv_1 - u'v_2 + u''v_3 - ...$$

Note that u is a polynomial function of x, thus the successive derivative for u will eventually be zero. We have for u and v

$$\begin{cases} u = \pi - x \\ u' = -1 \\ u'' = 0. \end{cases} \quad \begin{cases} v = \cos(nx) \\ v_1 = \frac{\sin(nx)}{n} \\ v_2 = \frac{-\cos(nx)}{n^2}. \end{cases}$$

This gives us

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\cos(nx)dx &= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin(nx)}{n} - (-1)\frac{-\cos(nx)}{n^2}\right]_0^{2\pi} \\ &= 0. \end{aligned}$$

Lastly, for b_n

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin(nx)dx.$$

Once again we apply Bernoulli's formula and

$$\begin{cases} u = \pi - x \\ u' = -1 \\ u'' = 0. \end{cases} \quad \begin{cases} v = \sin(nx) \\ v_1 = \frac{-\cos(nx)}{n} \\ v_2 = \frac{-\sin(nx)}{n^2}. \end{cases}$$

We get

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin(nx)dx &= \left[-(\pi - x)\frac{\cos(nx)}{n} - (-1)\frac{-\sin(nx)}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right] \\ &= \frac{1}{n}. \end{aligned}$$

Hence, the Fourier series of $\phi(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n}.$$

□

A commonly encountered integral while studying the Gibbs phenomenon is the sine integral [6], $Si(x)$. It is given by the following definition

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt \qquad \forall x \in \mathbb{R}.$$

Furthermore, this sine integral has its maximum and minimum at $x = \pi$, and $x = -\pi$ respectively. We can convince ourselves for this to be true, by simply applying the first derivative test. We compute the derivative with respect to x and set it equal to zero

$$Si'(x) = \frac{\sin(x)}{x} = 0.$$

Note that $Si'(x)$ is equal to zero for integer multiples of π. Also, note that

$$\begin{aligned} Si'(x) > 0 \qquad & \text{for } x \in (0, \pi) \\ Si'(x) < 0 \qquad & \text{for } x \in (\pi.2\pi). \end{aligned}$$

Essentially, this means that we have a maximum at $x = \pi$. Since $Si(x)$ is odd, we have that $Si(-x) = -Si(x)$ and

$$Si(-\pi) \leq Si(x) \leq Si(\pi) \qquad \forall x \in \mathbb{R}$$

Now, let $\mathcal{G} := Si(\pi)$. We now proceed to define the concept of the *Gibbs set.*

Definition 2.2.1. *The Gibbs set of ϕ at x is the set consisting of all possible limits of the sequence $\{S_n(\phi; x + \delta_n)\}_{n=1}^{\infty}$ where δ_n is a sequence such that $\delta_n \overset{n\to\infty}{\longrightarrow} 0$*

Theorem 2.2.2. *For any $\alpha \in [-\mathcal{G}, \mathcal{G}]$ there exists $\{\delta_n\}_{n=1}^{\infty}$ with $\delta_n \overset{n\to\infty}{\longrightarrow} 0$ such that*

$$\lim_{n\to\infty} S_n(\phi; \delta_n) = \alpha.$$

Consequently, this is false if $|\alpha| > \mathcal{G}$.

Proof. Let $\phi(x)$ be the sawtooth function previously defined. We have that

$$S'_N(\phi; x) = \sum_{n=1}^{N} \cos(nx).$$

We recognize this as *Dirichlet's kernel* and

$$S'_N(\phi; x) = D_N(x) - \frac{1}{2} = \frac{\sin\left(\left(N + \frac{1}{2}\right)\right) x}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2}.$$

Using the fundamental theorem of calculus we have

$$S_N(\phi; x) = \int_0^x S'_N(\phi; t) dt = \int_0^x \frac{\sin\left(\left(N + \frac{1}{2}\right)\right) t}{2\sin\left(\frac{t}{2}\right)} dt - \frac{x}{2}.$$

By substitution and Taylor development, we get that

$$S_N(\phi; x) = \int_0^{(N+\frac{1}{2})x} \frac{\sin(t)}{t} dt + \xi_N(x)$$

where

$$\xi_N(x) = \mathcal{O}(|x|) \tag{1}$$

for all N. For more details, please refer to the Appendix 5.1 and 5.2.

Since $Si(x)$ is a continuous function, for any $\alpha \in [-\mathcal{G}, \mathcal{G}]$ we can select $\beta \in [-\pi, \pi]$ such that $Si(\beta) = \alpha$. A visual argument is given in figure 2.2.1.

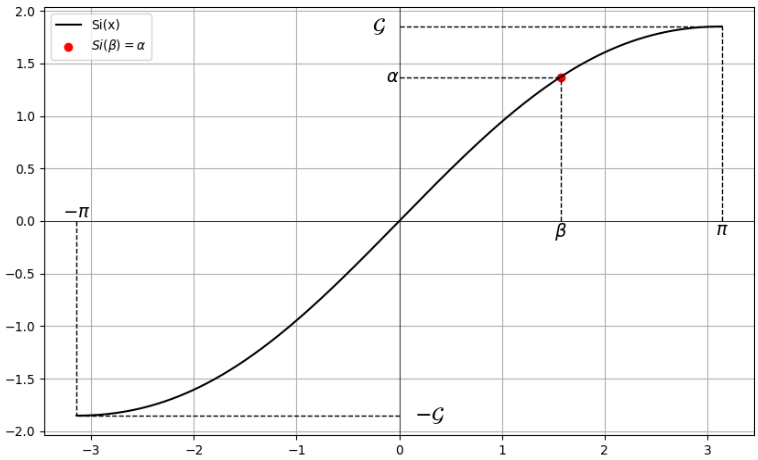


Figure 2.2.1: $Si(\beta) = \alpha$.

Let $\delta_N = \frac{\beta}{N+\frac{1}{2}}$. Then

$$\begin{aligned} S_N(\phi; \delta_N) &= \int_0^{(N+\frac{1}{2})\delta_N} \frac{\sin(t)}{t} dt + \xi_N(\delta_N) \\ &= \int_0^{\beta} \frac{\sin(t)}{t} dt + \xi_N\left(\frac{\beta}{N+\frac{1}{2}}\right) \\ &= Si(\beta) + \xi_N\left(\frac{\beta}{N+\frac{1}{2}}\right) \\ &= \alpha + \xi_N\left(\frac{\beta}{N+\frac{1}{2}}\right). \end{aligned}$$

Because of (1) we have

$$\xi_N\left(\frac{\beta}{N+\frac{1}{2}}\right) \leq C \cdot \frac{|\beta|}{N+\frac{1}{2}}.$$

Taking the limit we get

$$\lim_{n\to\infty} S_n(\phi; \delta_n) = \lim_{n\to\infty} \left(Si(\beta) + \xi_n\left(\frac{\beta}{n+\frac{1}{2}}\right)\right) = \alpha + 0 = \alpha. \tag{2}$$

Now, assume that $\alpha > \mathcal{G}$ and that (2) holds. We then have

$$\alpha = \lim_{n\to\infty} S_n(\phi;\delta_n) = Si(\beta) + \lim_{n\to\infty} \xi_n(\delta_n) \leq Si(\pi) + \lim_{n\to\infty} \xi_n(\delta_n) \leq \mathcal{G} + \lim_{n\to\infty} C \cdot |\delta_n| = \mathcal{G},$$

which is a contradiction. A similar argument can be made for the case $\alpha < -\mathcal{G}$. $\square$

Now that we have established that for the function ϕ, the approximation error will not cause $S_N(\phi;x)$ to surpass the Gibbs set, we want to explore what this means for an arbitrary piecewise smooth function g. We first want to know the behavior of the partial sums when g does not possess any points of discontinuity. In other words, we are interested in finding the outcome of

$$\lim_{n\to\infty} S_n(g;x+\delta_n)$$

for all δ_n with $\delta_n \to 0$. We can explore this matter from a broader perspective, entirely unrelated to Fourier analysis.

Lemma 2.2.3. *Let $\{g_n\}$ be a sequence of real-valued functions on a set $\mathcal{A}$ that converges uniformly to $g : \mathcal{A} \to \mathbb{R}$. Let $\{\delta_n\}$ be an arbitrary sequence with $\delta_n \to 0$ as $n \to \infty$. For any fixed $x \in \mathcal{A}$ there holds*

$$g_n(x+\delta_n) \to g(x).$$

Proof. We know that $g_n \to g$ uniformly, so given $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$, we have

$$|g_n(x) - g(x)| < \frac{\epsilon}{2} \qquad \forall x \in \mathcal{A}. \tag{3}$$

We also know that g is continuous, so we can choose a number $M > 0$ such that

$$|g(x+\delta_n) - g(x)| < \frac{\epsilon}{2}. \tag{4}$$

whenever $|\delta_n| < M$.

We then have that

$$\begin{aligned} |g_n(x+\delta_n) - g(x)| &= |g_n(x+\delta_n) - g(x+\delta_n) + g(x+\delta_n) - g(x)| \\ &\leq \underbrace{|g_n(x+\delta_n) - g(x+\delta_n)|}_{(*)} + \underbrace{|g(x+\delta_n) - g(x)|}_{(**)}. \end{aligned}$$

$(*) |g_n(x+\delta_n) - g(x+\delta_n)| < \frac{\epsilon}{2}$ because of (3),

$(**)$ since $g(x+\delta_n)$ is continuous, we can choose M so that (4) holds.

Hence,

$$|g_n(x+\delta_n) - g(x)| \leq |g_n(x+\delta_n) - g(x+\delta_n)| + |g(x+\delta_n) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

Now, suppose that an arbitrary piecewise smooth function $g(x)$ is discontinuous at $x_0 \in (0, 2\pi)$. We define two new functions $f(x)$ and $\lambda(x)$ where we remove a discontinuity point at x_0 from $g(x)$ using $\lambda(x)$. Let

$$\lambda(x) = \frac{d}{\pi}\phi(x - x_0)$$
$$f(x) = g(x) - \lambda(x)$$

where $\phi(x)$ is the same function as previously defined and $d = f(x_0^+) - f(x_0^-)$.

From this setup, the following observations can be made:

- f is differentiable everywhere, most notably at the point x_0;
- Since f is smooth in a closed interval, we have that $S_n(f; x + \delta_n) \to f$;
- Linearity property gives us that $S_n(g; x) = S_n(f + \lambda; x) = S_n(f; x) + S_n(\lambda; x)$.

Consequently, if the function λ shows the Gibbs phenomenon at x_0, then so does any arbitrary function $g(x)$.

Theorem 2.2.4. *Let g be piecewise smooth, 2π periodic and discontinuous at $x_0 \in (0, 2\pi)$. Let $d = g(x_0^+) - g(x_o^-)$, then the Gibbs set for g at a point x_0 is*

$$\left[g(x_0) - \frac{|d|\mathcal{G}}{\pi}, g(x_0) + \frac{|d|\mathcal{G}}{\pi} \right].$$

Proof. Let δ_n be a sequence such that $\delta_n \to 0$, we then have

$$S_n(g; x_0 + \delta_n) = S_n(f; x_0 + \delta_n) + S_n(\lambda; x_0 + \delta_n).$$

We already know that $S_n(f; x_0 + \delta_n) \to f(x_0)$. However it is not immediately apparent what the convergence of $S_n(\lambda; x_0 + \delta_n)$ is.

Hence, we look at the N :th partial sum for $S_n(\lambda; x)$ using Dirichlet's integral form

$$\begin{aligned} S_N(\lambda; x) &= \frac{1}{2\pi} \int_0^{2\pi} \lambda(x + t) D_N(t) dt \\ &= \frac{d}{2\pi^2} \int_0^{2\pi} \phi(x - x_0 + t) D_N(t) dt. \end{aligned}$$

But

$$S_N(\phi; x - x_0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(x - x_0 + t) D_N(t) dt,$$

thus

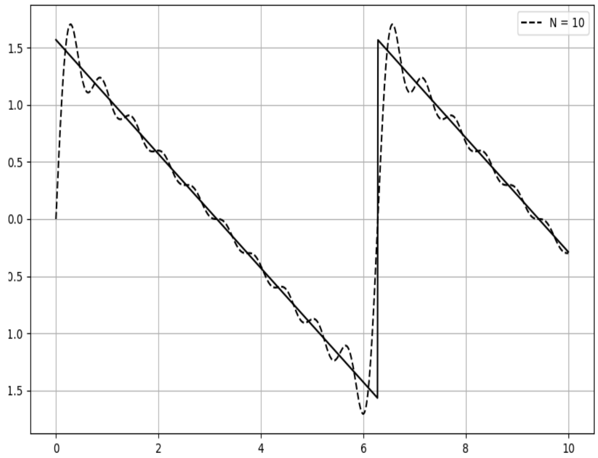
$$S_N(\lambda; x) = \frac{d}{\pi} S_N(\phi; x - x_0).$$

Then by theorem 2.2.2, we can see that

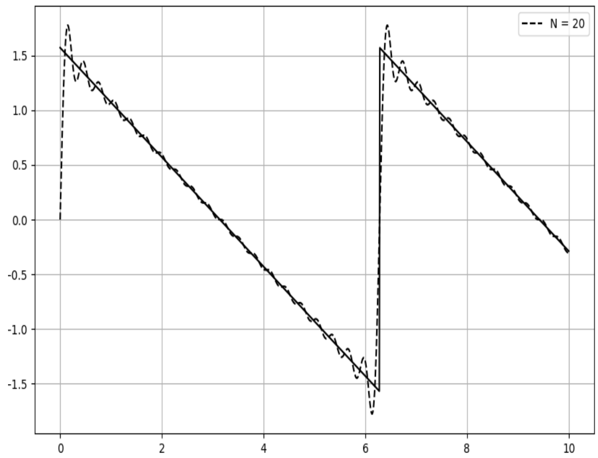
$$S_n(\lambda; x_0 + \delta_n) = \frac{d}{\pi} S_n(\phi; x + \delta_n) \overset{n\to\infty}{\longrightarrow} \frac{d\alpha}{\pi}$$

where $\alpha \in [-\mathcal{G}, \mathcal{G}]$. □

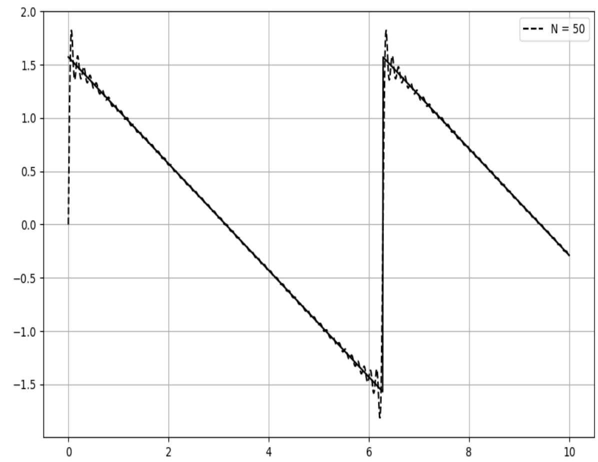
If we graph the function $\phi(x)$ alongside its corresponding Fourier series, we will see oscillations around each discontinuity point developed by the approximation errors of the partials sums.



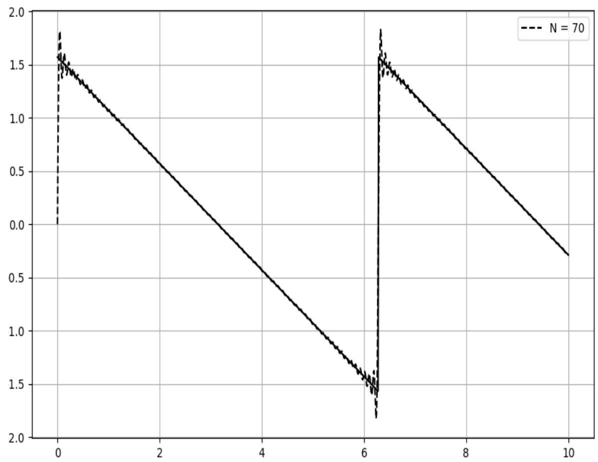
$N = 10$.



$N = 20$.



$N = 50$.



$N = 70$.

Figure 2.2.2: Comparison of $\phi(x)$ and $S_N(\phi; x)$ for different values of N.

Observing these graphs clearly reveals the oscillations occurring due to the approximation errors. Importantly, it becomes evident that the oscillations, both above and below the discontinuity point of the true function, persist even as the number of terms in the Fourier series increases.

N	Overshoot	Relative Error	Execution time(Seconds)
10	1.7069	0.0433	0.1609
20	1.7756	0.0652	0.1657
50	1.8200	0.0793	0.1898
70	1.8288	0.0821	0.2076
N	Undershoot	Relative Error	Execution time(Seonds)
10	-1.7069	0.0433	0.1609
20	-1.7761	0.0654	0.1657
50	-1.8117	0.0767	0.1898
70	-1.8241	0.0806	0.2076

Figure 2.2.3: Comparison of some numerical results for different values of N - Fourier series - $\phi(x)$.

Figure 2.2.3 shows the over/undershoot in the vicinity of the discontinuity point $x_0 = 2\pi$ and the relative error. The relative error is computed by

$$\frac{|\text{Over/undershoot value - Maximum/minimum value for } \phi|}{\text{Magnitude of the jump discontinuity}}$$

where $\phi_{max} = \frac{\pi}{2}$ and $\phi_{min} = -\frac{\pi}{2}$. Furthermore, an additional column has been included in order to display the program's execution time in seconds for each value of N.

Evidently, the errors in approximation tend to approach a value of around 9% relative to the size of the jump discontinuity. More interestingly to note is that the over- and undershoots does seem to converge towards Gibbs constant. If we denote the Gibbs constant as κ, then κ for this function is equal to

$$\kappa = \int_0^\pi \frac{\sin(x)}{x} dx - \frac{\pi}{2} \approx 0.2811.$$

Hence, if these oscillations do converge to the Gibbs constant, then we would add/subtract κ to the maximum/minimum value. We can see that

$$\phi_{max} + \kappa \approx 1.8519$$
$$\phi_{min} - \kappa \approx -1.8519.$$

As can be seen from figure 2.2.3, this indeed appears to hold true. These convergence results provide a compelling explanation for the upper and lower bounds of the Gibbs set. In essence, it is the existence of this Gibbs constant that ensures that the approximation errors of the truncated Fourier series will always be contained within the Gibbs set.

Chapter 3

Legendre polynomials and Fourier-Legendre Series

Harmonic functions of three variables, *spherical harmonics*, have proven invaluable in solving a wide range of problems within natural sciences and engineering. Application areas such as geosciences, astronomy, heat-transfer theory and quantum mechanics to name a few, all have befitted from the study of spherical harmonics [7]. One of the most important system of functions in spherical harmonics, is the Legendre polynomials. Adrian-Marie Legendre, a french mathematician who lived from the mid-18th century to the early 19th century discovered these polynomials when studying solutions to the Laplace equation in spherical coordinates. These polynomials can be obtained through the recursive formula:

$$\begin{aligned} &P_0(x) = 1, \\ &P_1(x) = x, \\ &P_n(x) = (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + xP_{n-1}(x) = 0, \qquad n \geq 1. \end{aligned} \tag{5}$$

Hence, the first five Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

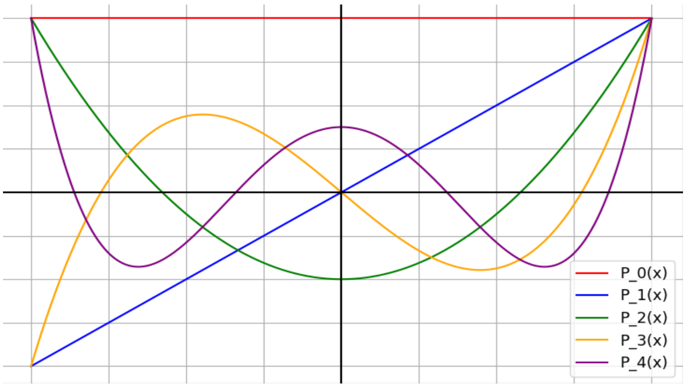


Figure 3.0.1: The first five Legendre polynomials.

Notebly, each $P_n(x)$ will always have its highest degree polynomial to be equal to n. We can also see that if n is odd(even), then $P_n(x)$ is an odd(even) function sharing the odd/even symmetry of the trigonometric system.

To define the Legendre-Fourier series, we must first ensure the orthogonality of the Legendre polynomials. Then by definition 2.1.4, we can form the Fourier series with them. It is well known that Legendre polynomials are orthogonal over the interval $[-1, 1]$. We prove their orthogonality using *Legendre's equation* to our advantage, which the Legendre polynomials provide a solution for.

Definition 3.0.1. *The second-order ordinary differential equation*

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \qquad n \in \mathbb{Z}_0^+ \tag{6}$$

is called Legendre's equation.

Theorem 3.0.1. *The Legendre polynomials are orthogonal over the interval* $[-1, 1]$ *with weight* $f(x) = 1$*, and satisfy*

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n. \end{cases}$$

Proof. We begin by multiplying (6) with $P_m(x)$

$$(1-x^2)P_n''(x)P_m(x) - 2xP_n'(x)P_m(x) + n(n+1)P_n(x)P_m(x) = 0. \tag{7}$$

Rewrite (6) for m and multiply with $P_n(x)$, then subtract (6) from (7) and obtain the expression

$$\frac{d}{dx}\left[(1-x^2)P_m'(x)P_n(x) - P_n'(x)P_m(x)\right] + (m-n)(m+n+1)P_m(x)P_n(x) = 0. \tag{8}$$

A more detailed calculation can be found in the Appendix 5.3.

If we Integrate (8) over [-1,1] we get

$$(m-n)(m+n+1)\int_{-1}^{1} P_m(x)P_n(x) = 0. \tag{9}$$

Hence, if $m \neq n$ the Legendre polynomials are orthogonal.

We must now look at the case where $m = n$. We begin by replacing n by $n-1$ in the recursive formula (5), and then multiply the result with $(2n+1)P_n$ and obtain

$$n(2n+1)P_n^2(x) - (2n-1)(2n+1)xP_{n-1(x)}P_n(x) + (n-1)(2n+1)P_{n-2}(x)P_n(x) = 0.$$

Multiply (5) once again with $(2n-1)P_{n-1}(x)$ and subtract the resulting equation from the equation above and get

$$\begin{aligned} n(2n+1)P_n^2(x) + (n-1)(2n+1)P_{n-2}(x)P_n(x) \\ -(n+1)(2n-1)P_{n+1}(x)P_{n-1}(x) - n(2n-1)P_{n-1}^2 = 0 \qquad n \geq 2. \end{aligned}$$

Once again, we integrate this equation over $[-1,1]$ and we have

$$\begin{aligned} & n(2n+1)\int_{-1}^{1} P_n^2(x)dx \\ & + (n-1)(2n+1)\int_{-1}^{1} P_{n-2}(x)P_n(x)dx \\ & - (n+1)(2n-1)\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx \\ & - n(2n-1)\int_{-1}^{1} P_{n-1}^2 dx = 0. \end{aligned}$$

The second and third terms are equal to 0 due to (9), and hence we get

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2 dx.$$

By iteration and induction we get for $n \geq 2$

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}.$$

□

A small proof of the last identity is given in Appendix 5.4. Furthermore, it can be shown that the Legendre polynomials also form a complete orthogonal system. For a comprehensive proof of this property, interested readers can refer to "Fourier Analysis" by James S. Walker, chapter 8, section 8 [5].

Now that we have established the orthogonality of Legendre polynomials over $[-1,1]$, as stated in definition 2.1.4, we can form Fourier series with them.

Definition 3.0.2. *Given a periodic function $f(x)$ defined on $[-1,1]$, the Fourier-Legendre series for $f(x)$ is then*

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x)$$

where the coefficients c_n are given by

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx.$$

Chapter 4

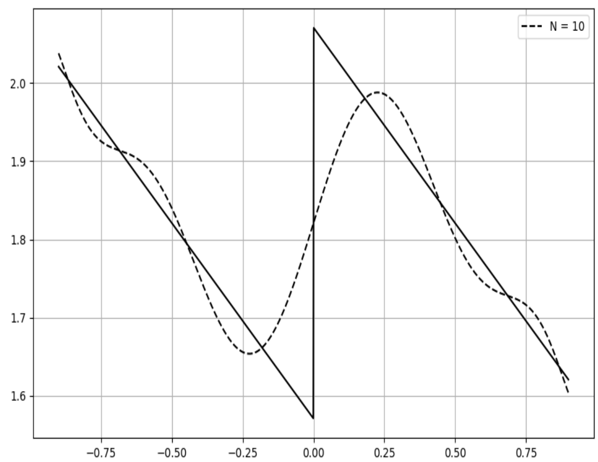
Gibbs Phenomenon for Fourier-Legendre Series

We will now look at the truncated Fourier-Legendre series numerically, and try to spot any similarities to the Gibbs phenomenon in Fourier series.

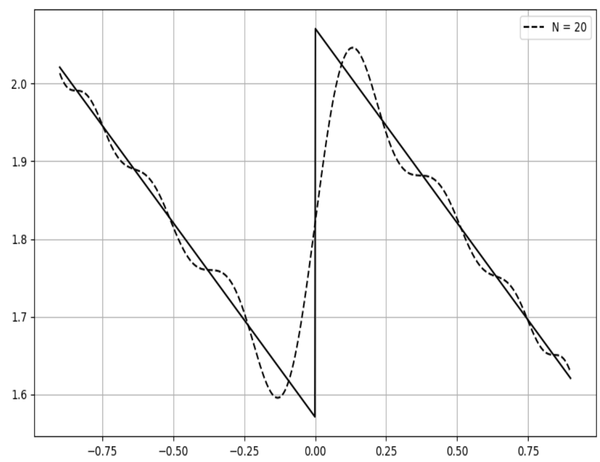
We go back to the the sawtooth function $\phi(x)$, but define it as

$$\phi(x) = \frac{\pi - x}{2}, \qquad x \in [-1, 0)$$

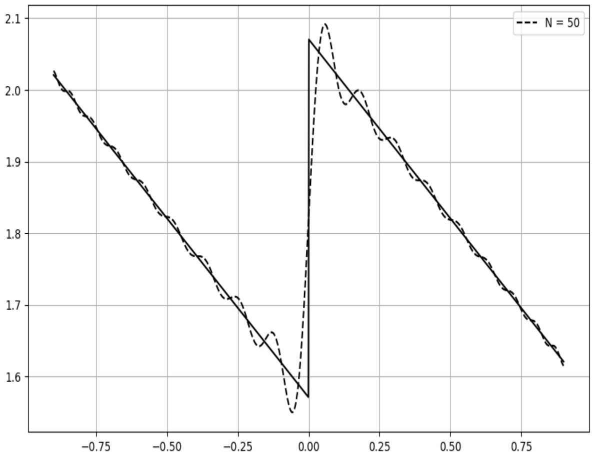
and extent to $[0, 1)$ by periodicity 1. This function has a discontinuity point at $x_0 = 0$, and should display the Gibbs phenomenon at this point. If we look at this function and its corresponding Fourier-Legendre series strictly from a visual stand-point in figure 4.0.2, we can see that the approximation errors does indeed cause oscillations around the discontinuity point.



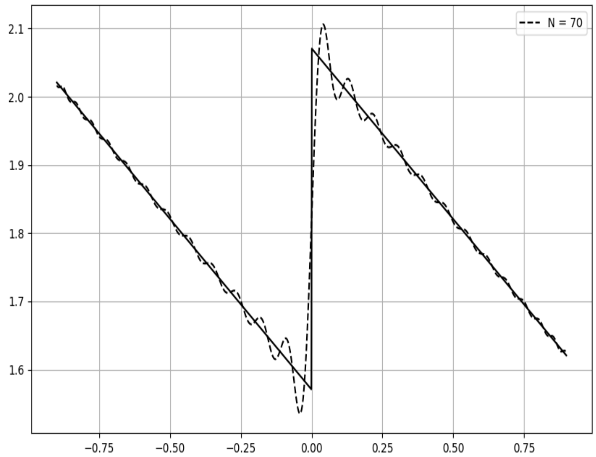
$\boldsymbol{N = 10}$.



$\boldsymbol{N = 20}$.



$\boldsymbol{N = 50}$.



$\boldsymbol{N = 70}$.

Figure 4.0.2: Comparison of $\phi(x)$ and $\Lambda_N(\phi; x)$ for different values of N.

The rate of convergence appears to be slower in comparison to the Fourier series, as evident from both figure 4.0.2 and figure 4.0.3 where the execution time was recorded. Notably, while the execution times for the Fourier series remained consistently under one second for all values of N, the Fourier-Legendre series increased quickly in execution time as more terms were added in the series. A possible explanation for this could be the fact that ϕ is neither even or odd, thus lacking the even or odd symmetry in the Legendre polynomials in the interval $[-1, 1]$. Consequently, more Legendre polynomials needs to be added to the approximation, leading to extended execution times. Thus, an argument could be made that ϕ seem to be a well-behaved function for Fourier series, but not so well-behaved for Fourier-Legendre series.

N	Overshoot	Relative Error	Execution time(seconds)
10	1.9879	-	2.5272
20	2.0460	-	5.5980
50	2.0916	0.0426	20.0320
70	2.1059	0.0712	36.2811
N	Undershoot	Error	Execution time(Seconds)
10	1.6537	-	2.5272
20	1.5956	-	5.5980
50	1.5500	0.0424	20.0320
70	1.5248	0.071	36.2811

Figure 4.0.3: Comparison of some numerical results for different values of N - Fourier-Legendre series - $\phi(x)$.

Even if that would be the case, the approximation errors do still seem to converge to a value of approximately 9% of the jump discontinuity, albeit at a slower rate. Once more, we verify whether the oscillations converge towards the Gibbs constant or not.

We have that $\phi_{max} = 2.0703, \phi_{min} = 1.5712$, so the distance of the jump is approximately $\frac{1}{2}$. If the over-/ and undershoots do converge towards the Gibbs constant, where

$$\kappa = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin(x)}{x} dx - \frac{1}{4} \approx 0.0447,$$

then they should converge towards a value of

$$\phi_{max} + \kappa \approx 2.1150$$

for the overshoot and

$$\phi_{min} - \kappa \approx 1.5265$$

for the undershoot.

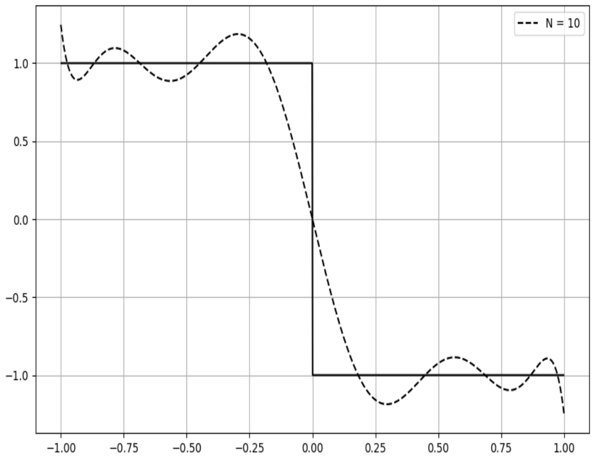
We can see from figure 4.0.3 that this does indeed seem to be the case, which is rather intriguing. When dealing with Fourier series, the emergence of this constant might not be too surprising. Fourier series are approximations of periodic functions using the trigonometric system. Consequently, when the Gibbs phenomenon gained recognition it might have been expected that these oscillations would converge to a trigonometric function as well. But by that reasoning, the truncated Fourier-Legendre series would be expected to converge to a function incorporating Legendre polynomials, but this doesn't align with these results. Instead the oscillations converge to the same constant.

However, one might argue that this constant arises solely due to the behaviour of the function ϕ, since we used ϕ for both Fourier-/ and Fourier-Legendre series. Hence, we examine a simpler function.

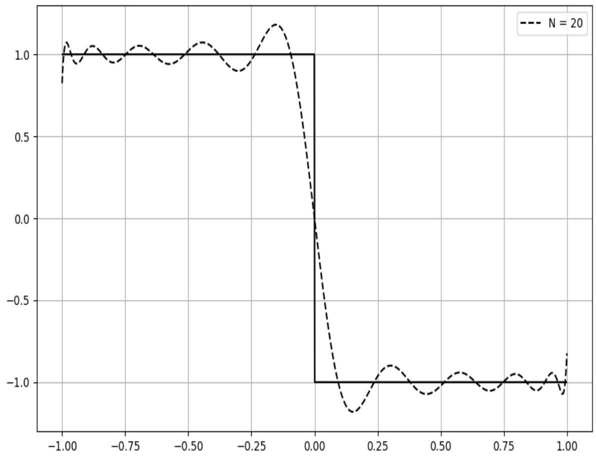
Consider the step function f, defined as

$$f(x) = \begin{cases} 1 & -1 < x < 0 \\ -1 & 0 < x < 1. \end{cases}$$

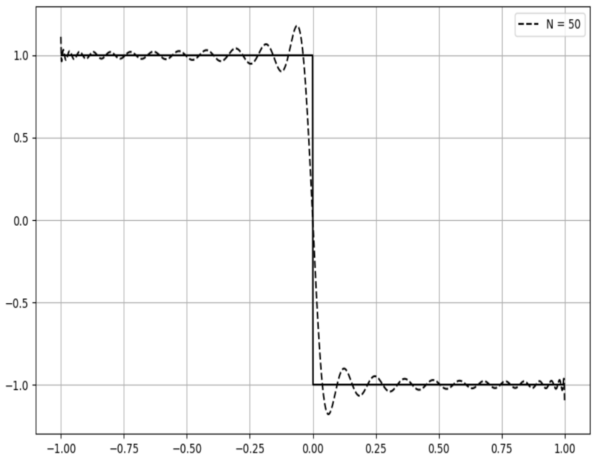
Hence, we will find a point of discontinuity at $x_0 = 0$. Figure 4.0.4 and 4.0.5 show the graphical results and numerical results respectively.



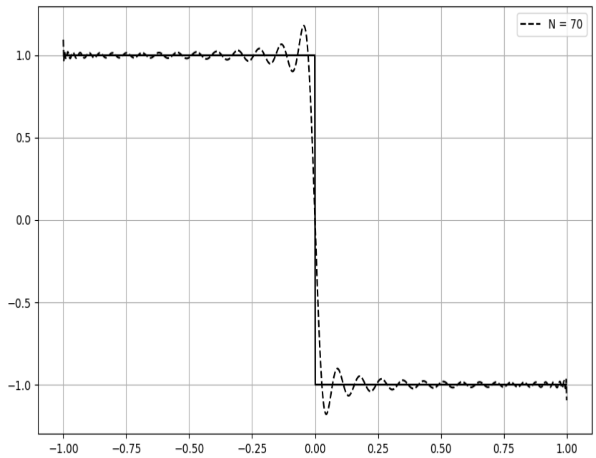
$\boldsymbol{N = 10}$.



$\boldsymbol{N = 20}$.



$\boldsymbol{N = 50}$.



$\boldsymbol{N = 70.}$

Figure 4.0.4: Comparison of step function and $\Lambda_N(\phi; x)$ for different values of N.

As expected, we can see from figure 4.0.4 the Gibbs phenomenon developing around $x_0 = 0$. Now, if these oscillations do converge towards the same Gibbs constant, which for f is equal to

$$\kappa = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x} dx - 1 \approx 0.1789,$$

then the overshoot should tend to a value of

$$f_{max} + \kappa \approx 1.1789$$

while the undershoot should tend to a value of

$$f_{min} - \kappa \approx -1.1789.$$

N	Overshoot	Relative Error	Execution time(Seconds)
10	1.1868	0.0934	2.4684
20	1.1810	0.0905	5.6120
50	1.1795	0.0897	20.1862
70	1.1796	0.0898	35.9075
N	Undershoot	Error	Execution time(Seconds)
10	-1.1868	0.0934	2.4684
20	-1.1810	0.0905	5.6120
50	-1.1795	0.0897	20.1862
70	-1.1796	0.0898	35.9075

Figure 4.0.5: Comparison of some numerical results for different values of N - Fourier-Legendre series - Step function.

Observing the results from figure 4.0.5, it becomes apparent that the oscillations do converge towards the Gibbs constant here as well. It is noteworthy that the step function seem to converge to this value at a faster rate than the saw tooth function ϕ, arguing it is a more well-behaved function for Fourier-Legendre series. It is remarkable to see that despite the distinct characteristics of the two systems of functions - the trigonometric system and Legendre polynomials - there is still a sense of unity. Although these deductions were primarily drawn from numerical findings, it shows the need for additional research where a more analytical approach is taken. A suggestion could be that one would examine if there is a kernel similar to the Dirichlet kernel could be formulated, and if so, does it exhibit the same type of behaviour? Additionally, it is possible that Gibbs phenomenon simply relates to the nature of approximation itself. If this is the case, it would be intriguing to explore whether the Gibbs constant emerges in other methods of approximations.

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Chapter 5

Appendix , Computations

5.1 $S_n(\phi; x) = \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin(t)}{t} dt + \xi_n(x)$

We have

$$S_n(\phi; x) = \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} dt - \frac{x}{2}.$$

If we only look at the left term

$$\begin{aligned}
\int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} dt &= \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} dt + \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{t} dt - \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{t} dt \\
&= \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{t} dt + \int_0^x \left(\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)t}{t} \right) dt.
\end{aligned}$$

We then get by substitution

$$\begin{aligned}
S_n(\phi; x) &= \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{t} dt + \underbrace{\int_0^x \left(\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)t}{t} \right) dt - \frac{x}{2}}_{\xi_n(x)} \\
&= \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin(t)}{t} dt + \xi_n(x).
\end{aligned}$$

5.2 $\xi_n(x) = \mathcal{O}(|x|)$

We have

$$\begin{aligned}
\xi_n(x) &= \int_0^x \left(\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)t}{t} \right) dt - \frac{x}{2} \\
&= \int_0^x \left(\sin\left(n+\frac{1}{2}\right) t \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t} \right) dt \right) - \frac{x}{2}. \\
&= \int_0^x \sin\left(n+\frac{1}{2}\right) t\, h(t) dt - \frac{x}{2}.
\end{aligned}$$

Since $\sin\left(n+\frac{1}{2}\right)t \leq 1$ we only need to concern ourselves with $h(t)$. We have that

$$\begin{aligned}
|\xi_n(x)| &= \left| \int_0^x \sin\left(n+\frac{1}{2}\right) t\, h(t) dt - \frac{x}{2} \right| \\
&\leq \int_0^x |h(t)|\, dt + \frac{|x|}{2}.
\end{aligned}$$

Consider the Maclaurin expansion of $h(t)$ when t is close to zero, we get

$$\begin{aligned}
h(t) = \frac{1}{2\sin\frac{t}{2}} - \frac{1}{t} &= \frac{t - 2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} \\
&= \frac{t - 2\left(\frac{t}{2} - \frac{\left(\frac{t}{2}\right)^3}{3!} + \frac{\left(\frac{t}{2}\right)^5}{5!} - \ldots\right)}{2t\left(\frac{t}{2} - \frac{\left(\frac{t}{2}\right)^3}{3!} + \frac{\left(\frac{t}{2}\right)^5}{5!} - \ldots\right)} \\
&= \frac{\frac{t^3}{24} + \frac{t^5}{1920} - \ldots}{t^2 - \frac{t^4}{24} + \frac{t^6}{1920} - \ldots} \\
&= \frac{\frac{t}{24} + \frac{t^3}{1920} - \ldots}{1 - \frac{t^2}{24} + \frac{t^4}{1920} - \ldots}
\end{aligned}$$

We can see that $h(t) \xrightarrow{t\to 0} 0$. Hence, $h(t)$ is gonna be less than or equal to some constant k and we have that

$$\begin{aligned}
|\xi_n(x)| &\leq \int_0^x |h(t)| dt + \frac{|x|}{2} \\
&\leq \int_0^x k\, dt + \frac{|x|}{2} \\
&\leq k|x| + \frac{|x|}{2} \\
&= \underbrace{(k + \frac{1}{2})}_{=C}|x| \\
&= C|x|.
\end{aligned}$$

5.3 Orthogonality of Legendre polynomials calculations

We have the following expressions

$$
\begin{aligned}
&(*) \quad (1-x^2)P_n''(x)P_m(x) - 2xP_n'(x)P_m(x) + n(n+1)P_n(x)P_m(x) = 0 \\
&(**) \quad (1-x^2)P_m''(x)P_n(x) - 2xP_m'(x)P_n(x) + m(m+1)P_m(x)P_n(x) = 0.
\end{aligned}
$$

Subtract (**) with (*) yields

$$
\begin{aligned}
&(1-x^2)P_m''(x)P_n(x) - (1-x^2)P_n''(x)P_m(x) - 2xP_m'(x) + 2xP_n'(x)P_m(x) \\
&+ m(m+1)P_m(x)P_n(x) - n(n+1)P_n(x)P_m(x) = 0
\end{aligned}
$$

Factor out the expression

$$
\begin{aligned}
&(1-x^2)\left[P_m''(x)P_n(x) - P_n''(x)P_m(x)\right] - 2x[P_m'(x)P_n(x) - P_n'(x)P_m(x)] \\
&+ (m-n)(m+n+1)P_m(x)P_n(x) = 0.
\end{aligned}
$$

Note that due to the chain rule, we have that

$$
\begin{aligned}
&\frac{d}{dx}\left[(1-x^2)(P_m'(x)P_n(x) - P_n'(x)P_m(x)\right] \\
&= (1-x^2)\left[\cancel{P_m'(x)P_n'(x)} + P_m''(x)P_n(x) - \cancel{P_n'(x)'P_m(x)} - P_n''(x)P_m(x)\right] \\
&- 2x\left[P_m'(x)P_n(x) - P_n'(x)P_m(x)\right] \\
&= (1-x^2)\left[P_m''(x)P_n(x) - P_n''(x)P_m(x)\right] - 2x[P_m'(x)P_n(x) - P_n'(x)P_m(x)].
\end{aligned}
$$

Thus, the expression becomes

$$
\frac{d}{dx}\left[(1-x^2)(P_m'(x)P_n(x) - P_n'(x)P_m(x)\right] + (m-n)(m+n+1)P_m(x)P_n(x) = 0.
$$

5.4 $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$

We prove this equality by induction. We want to show that $\forall n \geq 2$, we have that

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}.$$

We use the fact that $\int_{-1}^{1} P_n^2(x)dx = \frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)dx$.

We first check that the equality holds for $n = 2$. We get

$$\frac{2n-1}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)dx = \frac{3}{5}\int_{-1}^{1} P_1^2 dx = \frac{2}{5}$$
$$\frac{2}{2n+1} = \frac{2}{5}.$$

Hence, the equality holds for $n = 2$. Assume that the equality also holds for $n = k$, giving us

$$\frac{2k-1}{2k+1}\int_{-1}^{1} P_{k-1}^2(x)dx = \frac{2}{2k+1}.$$

For $k+1$ we obtain

$$\frac{2(k+1)-1}{2(k+1)+1}\int_{-1}^{1} P_k^2(x)dx = \frac{2}{2(k+1)+1}$$
$$\frac{2k+1}{\cancel{2k+3}}\int_{-1}^{1} P_k^2(x)dx = \frac{2}{\cancel{2k+3}}$$
$$\Leftrightarrow \int_{-1}^{1} P_k^2(x)dx = \frac{2}{2k+1}.$$

Chapter 6

Appendix , Python Code

6.1 Gibbs Phenomenon $\phi(x)$ - Fourier Series

```
import numpy as np
import matplotlib.pyplot as plt
import time

## Start recording time
start_time = time.time()

def phi(x, n):
    return np.sum([np.sin(n*x)/n for n in range(1, n+1)])

def Fourier_phi(x):
    return (np.pi - (x % (2 * np.pi))) / 2

x_vals = np.linspace(0, 10, 1000)
n_values = [70]

plt.figure(figsize=(10, 6))
for n in n_values:
    y_vals = [phi(x, n) for x in x_vals]
    plt.plot(x_vals, y_vals, label='N = 70', linestyle='dashed', color='
   black')

y_F_vals = [Fourier_phi(x) for x in x_vals]

## Find and mark the maximum point for phi(x)
x_range_max_f = x_vals[(x_vals >= 6.28) & (x_vals <= 6.36)]
y_range_max_f = [phi(x, n_values[0]) for x in x_range_max_f]
max_index_f = np.argmax(y_range_max_f)
max_x_f = x_range_max_f[max_index_f]
max_y_f = y_range_max_f[max_index_f]

## Find and mark the minimum point for phi(x)
x_range_min_f = x_vals[(x_vals >= 5.5) & (x_vals <= 7)]
y_range_min_f = [phi(x, n_values[0]) for x in x_range_min_f]
```

```
min_index_f = np.argmin(y_range_min_f)
min_x_f = x_range_min_f[min_index_f]
min_y_f = y_range_min_f[min_index_f]

## Find and mark the maximum point for Fourier series
x_range_max_F = x_vals[(x_vals >= 6.17) & (x_vals <= 6.30)]
y_range_max_F = [Fourier_phi(x) for x in x_range_max_F]
max_index_F = np.argmax(y_range_max_F)
max_x_F = x_range_max_F[max_index_F]
max_y_F = y_range_max_F[max_index_F]

## Find and mark the minimum point for Fourier series
x_range_min_F = x_vals[(x_vals >= 6.17) & (x_vals <= 6.30)]
y_range_min_F = [Fourier_phi(x) for x in x_range_min_F]
min_index_F = np.argmin(y_range_min_F)
min_x_F = x_range_min_F[min_index_F]
min_y_F = y_range_min_F[min_index_F]


## Record the end time
end_time = time.time()

runtime = end_time - start_time
print(f"Runtime: {runtime:.4f} seconds")

## Plotting

plt.plot(x_vals, y_F_vals, color='black')
plt.title('Partial Sums of  (sin(nx)/n) and phi(x)')
plt.scatter(max_x_f, max_y_f, color='green', label=f'({max_x_f:.4f}, {
    max_y_f:.4f})')
plt.scatter(min_x_f, min_y_f, color='blue', label=f'({min_x_f:.4f}, {
    min_y_f:.4f})')
plt.scatter(max_x_F, max_y_F, color='purple', label=f'({max_x_F:.4f}, {
    max_y_F:.4f})')
plt.scatter(min_x_F, min_y_F, color='orange', label=f'({min_x_F:.4f}, {
    min_y_F:.4f})')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()
```

6.2 Gibbs Phenomenon $\phi(x)$ - Fourier-Legendre Series

```
import numpy as np
import scipy.special as sp
from scipy.integrate import fixed_quad
import matplotlib.pyplot as plt
import time

## Start recording time
start_time = time.time()

## Extract Legendre Polynomials
def legendre_polynomial(x, n):
    return sp.legendre(n)(x)

def phi(x):
     return (np.pi - (x%-1)) / 2

## Compute the Fourier-Legendre coefficients and series
def fourier_legendre_coefficients(phi, N):
    coefficients = []
    for n in range(N + 1):
        def integrand(x):
            return phi(x) * legendre_polynomial(x, n)
        coefficient, _ = fixed_quad(integrand, -1, 1, n=100)
        coefficient *= (2 * n + 1) / 2
        coefficients.append(coefficient)
    return coefficients

N = 20
def fourier_legendre_expansion(x, coefficients):
    expansion = 0
    for n, coefficient in enumerate(coefficients):
        expansion += coefficient * legendre_polynomial(x, n)
    return expansion


coefficients = fourier_legendre_coefficients(phi, N)

x_interval = np.linspace(-0.9, 0.9, 1000)

y_interval_expansion = [fourier_legendre_expansion(x, coefficients) for x
    in x_interval]

## x-values for interval of interest (close to 0)
x_interval_interest = np.linspace(-0.5, 0.5, 1000)

y_interval_interest_expansion = [fourier_legendre_expansion(x, coefficients
    ) for x in x_interval_interest]


```

```
## Find and mark the maximum and minimum point for phi(x)
y_interval_original = [phi(x) for x in x_interval]
max_index_interval_original = np.argmax(y_interval_original)
min_index_interval_original = np.argmin(y_interval_original)
max_x_interval_original = x_interval[max_index_interval_original]
min_x_interval_original = x_interval[min_index_interval_original]
max_y_interval_original = y_interval_original[max_index_interval_original]
min_y_interval_original = y_interval_original[min_index_interval_original]

## Find and mark the maximum and minimum point for Fourier-Legendre series
max_index_interval_interest_expansion = np.argmax(
    y_interval_interest_expansion)
min_index_interval_interest_expansion = np.argmin(
    y_interval_interest_expansion)
max_x_interval_interest_expansion = x_interval_interest[
    max_index_interval_interest_expansion]
min_x_interval_interest_expansion = x_interval_interest[
    min_index_interval_interest_expansion]
max_y_interval_interest_expansion = y_interval_interest_expansion[
    max_index_interval_interest_expansion]
min_y_interval_interest_expansion = y_interval_interest_expansion[
    min_index_interval_interest_expansion]


## Record the end time
end_time = time.time()

runtime = end_time - start_time
print(f"Runtime: {runtime:.4f} seconds")

## Plotting

plt.figure(figsize=(10, 6))


plt.plot(x_interval, y_interval_original, color='black')
plt.plot(x_interval, y_interval_expansion, label='N = 20', linestyle='
    dashed', color='black')
plt.scatter(max_x_interval_original, max_y_interval_original, color='red',
    label=f'{max_x_interval_original:.4f}, {max_y_interval_original:.4f})')
plt.scatter(min_x_interval_original, min_y_interval_original, color='blue',
     label=f'({min_x_interval_original:.4f}, {min_y_interval_original:.4f})
    ')
plt.scatter(max_x_interval_interest_expansion,
    max_y_interval_interest_expansion, color='green', label=f'({
    max_x_interval_interest_expansion:.4f}, {
    max_y_interval_interest_expansion:.4f})')
plt.scatter(min_x_interval_interest_expansion,
    min_y_interval_interest_expansion, color='purple', label=f'({
    min_x_interval_interest_expansion:.4f}, {
    min_y_interval_interest_expansion:.4f})')
plt.xlabel('x')
plt.ylabel('y')
```

```
plt.title('Original Function and Fourier-Legendre Expansion')
plt.legend()
plt.grid(True)
plt.show()

print(f"Maximum Point of Original Function: x = {max_x_interval_original:.4f}, y = {max_y_interval_original:.4f}")
print(f"Minimum Point of Original Function: x = {min_x_interval_original:.4f}, y = {min_y_interval_original:.4f}")
print(f"Maximum Point of Fourier-Legendre Expansion: x = {max_x_interval_interest_expansion:.4f}, y = {max_y_interval_interest_expansion:.4f}")
print(f"Minimum Point of Fourier-Legendre Expansion: x = {min_x_interval_interest_expansion:.4f}, y = {min_y_interval_interest_expansion:.4f}")
```

6.3 Gibbs Phenomenon Step Function - Fourier-Legendre Series

```
import numpy as np
import scipy.special as sp
from scipy.integrate import fixed_quad
import matplotlib.pyplot as plt
import time

## Start recording time
start_time = time.time()

## Extract Legendre Polynomials
def legendre_polynomial(x, n):
    return sp.legendre(n)(x)


def f(x):
    return np.where((0 < x) & (x < 1), -1, np.where((-1 < x) & (x < 0), 1, np.nan))

## Compute the Fourier-Legendre coefficients and series
def fourier_legendre_coefficients(f, N):
    coefficients = []
    for n in range(N + 1):
        def integrand(x):
            return f(x) * legendre_polynomial(x, n)
        coefficient, _ = fixed_quad(integrand, -1, 1, n=1000)
        coefficient *= (2 * n + 1) / 2
        coefficients.append(coefficient)
    return coefficients
N = 10
def fourier_legendre_expansion(x, coefficients):
    expansion = 0
    for n, coefficient in enumerate(coefficients):
        expansion += coefficient * legendre_polynomial(x, n)
    return expansion

coefficients = fourier_legendre_coefficients(f, N)

x_interval = np.linspace(-1, 1, 1000)

y_interval_expansion = [fourier_legendre_expansion(x, coefficients) for x in x_interval]

## x-values for interval of interest (close to 0)
x_interval_interest = np.linspace(-0.5, 0.5, 1000)

y_interval_interest_expansion = [fourier_legendre_expansion(x, coefficients) for x in x_interval_interest]

y_interval_original = [f(x) for x in x_interval]
```

```python

## Find and mark the maximum and minimum point for Fourier-Legendre series
max_index_interval_interest_expansion = np.argmax(
    y_interval_interest_expansion)
min_index_interval_interest_expansion = np.argmin(
    y_interval_interest_expansion)
max_x_interval_interest_expansion = x_interval_interest[
    max_index_interval_interest_expansion]
min_x_interval_interest_expansion = x_interval_interest[
    min_index_interval_interest_expansion]
max_y_interval_interest_expansion = y_interval_interest_expansion[
    max_index_interval_interest_expansion]
min_y_interval_interest_expansion = y_interval_interest_expansion[
    min_index_interval_interest_expansion]

## Record the end time
end_time = time.time()

runtime = end_time - start_time
print(f"Runtime: {runtime:.4f} seconds")

## Plotting
plt.figure(figsize=(10, 6))

plt.plot(x_interval, y_interval_original, color='black')
plt.plot(x_interval, y_interval_expansion, label='N = 10', linestyle='
    dashed', color='black')
plt.scatter(max_x_interval_interest_expansion,
    max_y_interval_interest_expansion, color='green', label=f' ({
    max_x_interval_interest_expansion:.4f}, {
    max_y_interval_interest_expansion:.4f})')
plt.scatter(min_x_interval_interest_expansion,
    min_y_interval_interest_expansion, color='purple', label=f'({
    min_x_interval_interest_expansion:.4f}, {
    min_y_interval_interest_expansion:.4f})')
plt.xlabel('x')
plt.ylabel('y')
plt.title('Original Function and Fourier-Legendre Expansion')
plt.legend()
plt.grid(True)
plt.show()


print(f"Maximum Point of Fourier-Legendre Expansion: x = {
    max_x_interval_interest_expansion:.4f}, y = {
    max_y_interval_interest_expansion:.4f}")
print(f"Minimum Point of Fourier-Legendre Expansion: x = {
    min_x_interval_interest_expansion:.4f}, y = {
    min_y_interval_interest_expansion:.4f}")
```

List of Figures