Research Article

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Narrative Didactics in Mathematics Education: Results from a University Geometry Course

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Abstract: In a Swedish second semester course on Euclidean and non-Euclidean geometry in mathematics education at Karlstad University, a bridging teaching strategy comprising elements from Mathematics, Didactics, History, Literature, and Technology was applied by using the concept of Narrative Didactics and Digital Interactive Mathematical Maps developed by the Professorship for Didactics of Mathematics at the University of Passau, Germany. The complete assessment of the teaching strategy comprised both an evaluation of the technological and the narrative didactical scaffolding and was partly analysed previously concerning especially the technology acceptance aspect, while here we broaden and deepen the evaluation analysis of the application of narrative didactic elements. Regarding the latter, at the end of the course, students were asked to formulate a short historical-oriented narrative motivation for a school topic of own choice with the help of information provided by the timeline of the Mathematical Map. A representative example of these art-combining products is presented and evaluated according to elements of narrative didactics. Results indicate a fruitful, promising, and synergetic connection between different fields of Science, Technology, Engineering, the Arts, and Mathematics that can lead to a richer and more sustainable learning process in mathematics lessons both at university and school level.

Keywords: narrative didactics, geometry, teacher education, mathematical maps, technology

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1 Introduction

Teaching mathematics is not only teaching mathematics. Teaching mathematics always represents a complex and multi-faceted process of didactical reduction and preparation of an abstract mathematical content resulting in a hopefully successful way of students constructing new elements of knowledge and further development of corresponding competencies. Already Bruner (1986) contrasted the logico-scientific to the narrative mode of thinking, addressing not only rational thought but also emotional aspects of learning. The principles of narrative didactics as shown in Klassen (2006, 2009) or Norris, Guilbert, Smith, Shahram, and Phillips (2005) try to address these aspects and were brought to the context of an *Artful STEM* teaching among others by Brandl (2009, 2010, 2016, 2017) and Kubli (2002, 2005a,b). We combine this didactical approach with a technological tool for the visualisation of historical and biographical aspects of mathematics and mathematicians, respectively, called Digital Interactive Mathematical Map (Brandl, 2009; Brandl, Kaiser, Przybilla, & Hackstein, in press; Datzmann, Przybilla, Brandl, & Kaiser, 2020; Przybilla, Brandl, Vinerean, & Liljekvist, 2021; Przybilla, Brandl, Vinerean, & Liljekvist, 2022; Vinerean, Brandl, & Liljekvist, 2023; Vinerean, Liljekvist, Brandl, & Przybilla, in press) to support the student with a meaningful connection regarding the development of mathematical knowledge by “the trials, the triumphs, the tribulations” (I.I. Rabi in Holton, Rutherford, & Watson, 1970, cited in Kubli, 2005b) of the people behind this science. After presenting these blocks of theory and description, we show results mainly of a single case study from a Swedish Geometry course where this teaching strategy was taught and analyse the student’s use of specific narrative elements in a text produced in a corresponding assignment. In a summarizing conclusion, we collect first findings and provide a brief outlook of future research desiderata.

2 Making Connections to Context and Using Scaffoldings in a Bridging Teaching Strategy

Making connections is important for successful learning processes, at least in the sense of Bruner's idea of scaffoldings (Wood, Bruner, & Ross, 1976). In the following, we shortly recapitulate our understanding of the effect of making connections as described and illustrated in Brandl and Nordheimer (2017) in order to make visible the bridges built in the intended teaching strategy and to motivate the structural origin of the didactic tool Interactive Mathematical Map.

A proceeding without an additional supporting method in teaching is illustrated in Figure 1: an isolated topic that has to be understood and learnt by the students.

The isolation of the topic can be counteracted by making connections to related areas and establishing mental anchor points within those (Figure 2).

These connections now serve as a scaffolding process as described by Wood et al. (1976, p. 90) to support individual successful learning processes by anchoring in context (Brinkmann, Maaß, Ossimitz, & Siller, 2017). We illustrate this by a shift of understanding onto a higher level in Figure 3.

This may lead to a kind of "enlightenment" (Wagenschein, 1968) of the topic, making it and its connected context more dominant and obvious in the learning process (Figure 4).

By letting time go by while moving on to other topics in the learning course, the enlightening energy of the elevated topic will probably fade, leading to a natural lowering of the helping anchor areas in the scaffolding (Figure 5).

However, certain psychological holding structures may remain (Figure 6). The central topic and corresponding

Figure 1: Illustration of an isolated topic and its corresponding level of understanding.

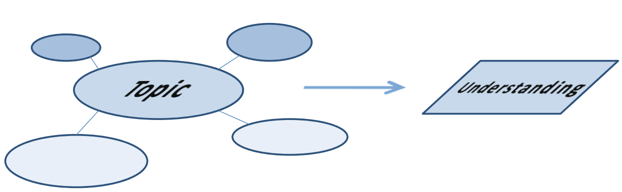

Figure 2: Connections to related areas establish mental anchor points to serve as a scaffolding.

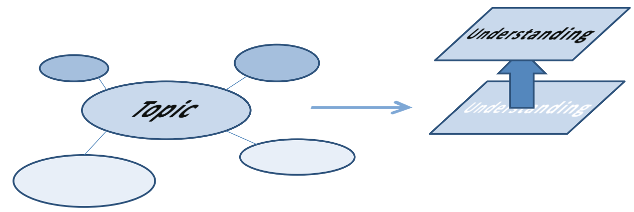

Figure 3: Shift to a higher level of understanding.

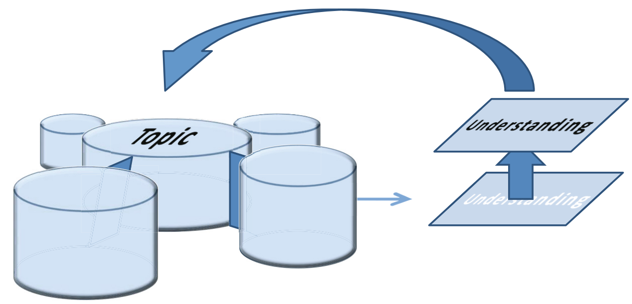

Figure 4: "Enlightenment" (Wagenschein) of the topic and its contexts.

competencies then may not be lost so easily. On the contrary, the holding structure serves for a "becoming at home" ("heimisch werden," dt.) (Wagenschein, 1968, p. 42), resulting from a metaphorical weaving into memory. Furthermore, this kind of knowledge later may be easier to address and activate again as originally connected to several neighbouring aspects. Referring to Simone Weil's "Enracinement," Martin Wagenschein calls this "routed-in knowledge" ("eingewurzeltes Wissen," dt.) (Wagenschein, 1968, p. 65) in contrast to "uneducational, unrealistic, unrouted knowledge" ("bildungswidriges, wirklichkeitsfremdes, entwurzeltes Wissen: Scheinwissen," dt., transl. by authors) (Wagenschein, 1968, p. 66). Sousa (2017) called this an "integration into the existing neuron network, which increases meaning and retention" (Przybilla et al., 2021, p. 210).

The scaffolding context not necessarily needs to only comprise contents in the form of facts or corresponding

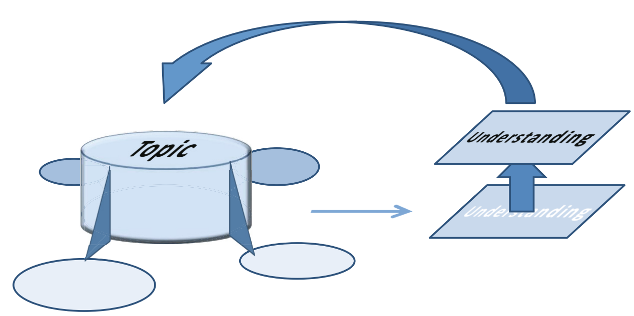

Figure 5: Fading of the scaffolding context areas.

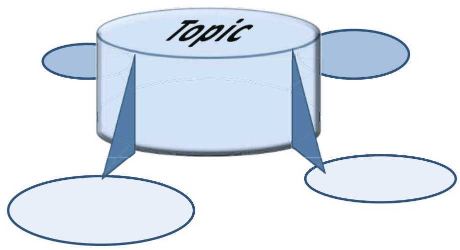

Figure 6: "Routed-in knowledge" (Wagenschein)/ "Enracinement" (Weil).

competencies. Also the emotional aspect of the learning process can be addressed in this way. Klassen (2006) refers to the important role of affect in the context of the learning processes and points out that "emotions act as an arbitrator in rational decision-making; without access to one's emotions, it is impossible to plan and make rational decisions (Damasio, 1994)," and "unless pupils are willing to take the risk of some emotional commitment they are unlikely to learn" (Barnes, 1992, p. 87); furthermore, "[e]xperience arouses emotion, which fixes attention and leads to understanding and insight, which results in memory" (Howard, 2000, p. 549).

Both aspects of connection to a context, the fact-based and the affect-based, are addressed by our teaching strategy by using elements from the history of mathematics in narrative didactic methodology and parallelly visualizing the way of connection to related aspects of the topic by the help of technology in the form of Digital Interactive Mathematical Maps. The mathematical contents serve as the basis and the goal of the teaching strategy, didactically and digitally supported by scaffolding methods (Figure 7).

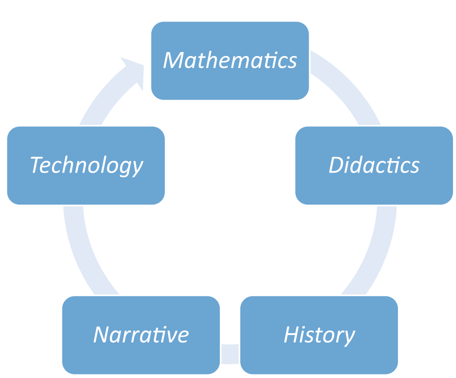

Figure 7: Illustration of the bridging teaching strategy (transformation of mathematical contents by didactical methods using context elements from history of mathematics and the scaffoldings of narrative didactics and a technological tool).

2.1 Scaffolding 1: Narrative Didactics

In the following, we provide an introductory summary of narrative didactics as described comprehensively in Brandl (2017).

Narrative didactics in mathematics, or natural sciences at all, is a rather young method in Science, Technology, Engineering, and Mathematics (STEM) didactics trying to improve teaching quality by considering elements from the science of literature and storytelling (e.g., Brandl, 2010, 2016, 2017; Kubli, 1999, 2002, 2005a,b; Klassen, 2006, 2009; Norris et al., 2005). However, it is not only telling stories or using narrative structures for analysing lessons. The method uses narrative elements to support the mathematical content.

In mathematics and natural sciences, causal arguments loaded with heavy symbolism are used. They transport the proofs and logical reasoning by always guaranteeing the necessary exactness and strength of the exact sciences (Norris et al., 2005). Narrative perspectives focus on the unique and are the opposite of the more generally orientated logical-discursive theories (Kubli, 2002) and are suited to explain unique and unrepeatable events (Moffett, 1983).

Klassen (2006) criticizes common school books by asserting that the main intention behind the conception of school books was "to strip all unnecessary material and leave only the bare decontextualized scientific facts, theories, and laws along with the exemplar problems that demonstrate them" (Klassen, 2006, p. 33), still being connected to Locke's idea of the mind as an *a priori* tabula rasa, justifying a pure instructional teaching and learning method like the "Nürnberger Trichter". The corresponding dominance of isolated task design comes from behavioural psychology and is at odds with a modern idea of learning based on modern learning theory of integrated learning, i.e. making connections. A behavioural psychological learning theory is characterized by the two central assumptions of "decomposability" and "decontextualization" (Resnick & Resnick, 1992), allowing for and resulting in atomistic single-step teaching and learning procedures in linear time and isolated topics with single solution tasks. Both assumptions are in fundamental contradiction to the idea of connected learning processes making use of various contexts.

Cognitive learning theory (Vygotsky) leads to constructivism and a paradigmatic change. "Knowledge is the result of an individual subject's constructive activity, not a commodity that somehow resides outside the knower and can be conveyed or instilled by diligent perception or linguistic communication" (von Glasersfeld, 1990, p. 37).

Modern teaching methods based on this take into account the demand for a "forum for scientific discourse" (Ebenezer & Fraser, 2001, p. 513) and reflect implicitly and explicitly the networking character both in relation to the content and to the teaching process. The consideration of contexts proves to be essential here.

Klassen (2006) among others emphasizes the historical and affective teaching and learning context. Repeating his criticism of traditional school books, he asserts that students may see lessons as boring and irrelevant. The liveliness of science manifesting in the processes of research, discovery, and creativity is erased from the content and nowhere to be seen. Here, a recourse to authentic elements from the history of mathematics could provide remedy. "The humanizing and clarifying influence of history of science brings the science to life and enables the student to construct relationships that would have been impossible in the traditional decontextualized manner in which science has been taught" (Klassen, 2006, p. 48, referring to Cohen, 1993; Jung, 1994; Kipnis, 1996; Koul & Dana, 1997). One of the most important things is the "humanistic element in the learning process: Portraying scientists as human beings and giving students the opportunity to become effectively involved in the story of science are worthy goals in themselves" (Klassen, 2006, p. 51). Kubli (1999, 2002) also pointed out that students react much more positive to historical material if it is presented in a narrative form.

The importance of the emotional context has already been addressed previously and is mentioned by Klassen (2006) as well as by Egan (1989a,b). "Egan argues that the story form of presentation of curriculum materials stimulates the imagination and evokes emotional response, thereby producing learning that students more easily assimilate with long-term memory than learning produced by drilling and memorizing" (Klassen, 2006, p. 53). Referring to Mott, Callaway, Zettlemoyer, Lee, and Lester (1999) Klassen says: "The listener to, or reader of, a story engages with the story because he or she is encouraged to participate vicariously in the experiences of the protagonist. The kind of motivation produced by story is intrinsic, as opposed to the extrinsic motivation produced by a prescriptive teaching and learning episode." Here Klassen is supported by teachers, citing Noddings and Witherell (1991, pp. 279/280): We learn from stories. More important, we come to understand – ourselves, others, and even the subjects we teach and learn. Stories engage us ... Stories can help us to understand by making the abstract concrete and accessible. What is only dimly perceived at the level of principle may become vivid and powerful in the concrete. Further, stories motivate us. Even that which we understand at the abstract level may not move us to action, whereas a story often does.

Table 1: Narrative elements and their meaning (based on Norris et al., 2005, Table 1)

Narrative element	Meaning
Event-tokens	– Particular occurrences involving particular actors at a particular place and time – Are chronologically related – Involve a unified subject and are interconnected – Later events seen as significant in light of earlier events – Lead to changes of state
Narrator	– The agent relating a narrative (foregrounded or backgrounded) – Determines the point and purpose of the story to be told – Selects events and the sequence in which they are told – Fashions a sequence of events into a significant whole
Narrative appetite	– Desire created in readers and listeners to know what will happen – Based on a range of possibilities that create anticipation and suspense
Past time	– Narratives concern the past – Narratives can manipulate time in relating narratives
Structure	– Narratives typically start with imbalances, introduce complications, and end in success or failure – Narratives are structured around two independent time sequences – the sequence of plot events and the sequence in which the events are related – Narratives are tied together by satisfying expectations that are established previously
Agency	– Actors cause and experience events in narratives – Actors are responsible for their actions – Narratives involve human beings or other moral agents
Purpose	– To help us better understand the natural world and humans' place in it – To help us imagine and feel the experience of others
Reader	– The reader must interpret the text as a narrative in order to approach it with appropriate expectations and anticipations

One of the broadest definitions of narratives is given by Phelan (1996, p. 18), attesting “the purpose of communicating knowledge, feelings, values and beliefs.” Norris et al. (2005) see eight typical components of narratives listed in Table 1 together with their meanings.

For the didactical implementation in a lesson’s setting design, Klassen (2006) suggests a so-called story-driven contextual approach (SDCA). The goal of this “heuristic teaching device” on the one hand is a humanization of science by the use of historical (and biographical) elements, and on the other hand an affective engagement of the students within the learning process: “it is the affective context, created by a story, which can produce the incentive desired at the beginning of a lesson. The story focuses attention and motivates, which is why it is a good starting point for any learning episode” (Klassen, 2006, p. 54).

Klassen’s definition of context is “the entities that connect to or surround a focal entity and contribute to the meaningfulness of the whole” (Klassen, 2006, p. 54). From a system-theoretical view as in Brandl (2017), this corresponds to a structural coupling between the system focal entity and the so-called environment 2, i.e. a world for which the system has a purpose (Krieger, 1996, p. 81). The remaining environmental factors that despite having constructive effect on the system, too, but don’t provide meaning for it, are collected in environment 1. “The contexts relevant to learning could be viewed either from the perspective of the curriculum and the teacher or from the perspective of the student – originating either with the knowledge being taught or with the way students learn that knowledge” (Klassen, 2006, p. 35).

Klassen’s own schematic depiction of the SDCA looks similar to Figure 8.

As it seems obvious that the teacher’s ability to feel joy when delivering narratives or telling biographical elements of a mathematician’s story of life is necessary for the success of teaching, narrative didactics assumes the human being to be a homo narrans. As Breithaupt (2022) – examining “The Narrative Brain” – mentions “because we think, experience and live in narratives: Are we narrative creatures?” (Breithaupt, 2022, p. 11, transl. by authors), this seems at least not to be too far-fetched. It opposes the traditional accusation against mathematics and natural sciences to be cold, unattractive, and boring while asking for the human factor in the science of mathematics. Bruner (1986) already contrasted the “logico-scientific mode” to the “narrative mode.” While the logical-discursive mode “looks for universal truth conditions … the narrative mode looks for particular connections between events” (Richardson, 1990, p. 118). Table 2 differentiates the two modes as done in Norris et al. (2005) and summarized in Brandl (2017).

According to Kubli, teaching with narrative didactics means to make a step towards poetics. Addressing the context of a scientific discourse leads to an often-neglected aspect of science and scientists, concerning the emotional side of research such as expectations, illusions, hopes and dreams, success and failure; in the words of Physics Nobel Prize winner I.I. Rabi: “I propose that science should be taught at whatever level, from the lowest to the highest, in the humanistic way. It should be taught with a certain philosophical understanding and a human understanding in the sense of biography, the nature of the people who made this construction, the triumphs, the trials, the tribulations” (in Holton et al., 1970, cited in Kubli, 2005b).

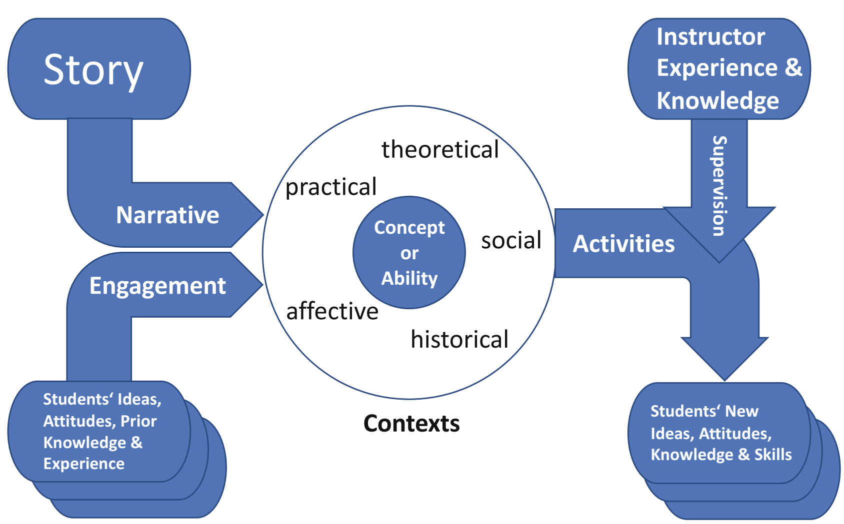

Figure 8: Scheme of the SDCA (based on Klassen, 2006, Figure 1).

Table 2: Comparison of the two modes of teaching

	Logico-scientific mode	Narrative mode
Events	Simple recitation of a series of events (linear, “hence,” “then,” “thus,” ...)	Looking for particular connections between events
	Prediction	Retrodiction
Time	Irrelevant	Concerns the past
Structure	No discourse time	Double time structuring including discourse time
	Logical chronological ordering	(flashbacks, flash forwards, ...)
	Strictly connected	Possibility of blanks
Agency	No agency/actors/characters necessary	Characters/actors as human beings necessary
Explanations	Intrinsic (facts, knowledge)	Extrinsic (“about,” “how”)

Another possibility to use and provide narrative structures in teaching can also be done by widening the approach to pictures as it is done, for example, in Brandl (2016, 2017) or Baptist (2008) and to a lesser degree in Enzensberger (1997), for instance, casually touching aspects and trends regarding the “iconic turn”.

2.2 Scaffolding 2: Digital Interactive Mathematical Maps

Based on first ideas and supplemented by an exemplary historical development part of mathematical knowledge in Brandl (2009), a digital didactical tool named “Interactive Mathematical Map” has been developed at the professorship for Didactics of Mathematics at the University of Passau. It is freely accessible under the address https://math-map.fim.uni-passau.de and offers a visual representation of the historical development of mathematical knowledge in time as well as the interrelation of different knowledge items according to each other (Figure 9). In order to realize this, the map is made up as a three-dimensional net, starting from one initial problem and expanding like a tree from node to node. The relationship between the nodes representing pieces of mathematical knowledge (often a famous discovery resulting in a central theorem) is computed by an algorithm from graph theory using certain weights given when putting a new content into the database (for details e.g. see Przybilla et al., 2021, 2022). Clicking on a node opens the corresponding content on a linked timeline (Figure 10), where milestones of the considered mathematical area are clearly and consecutively displayed.

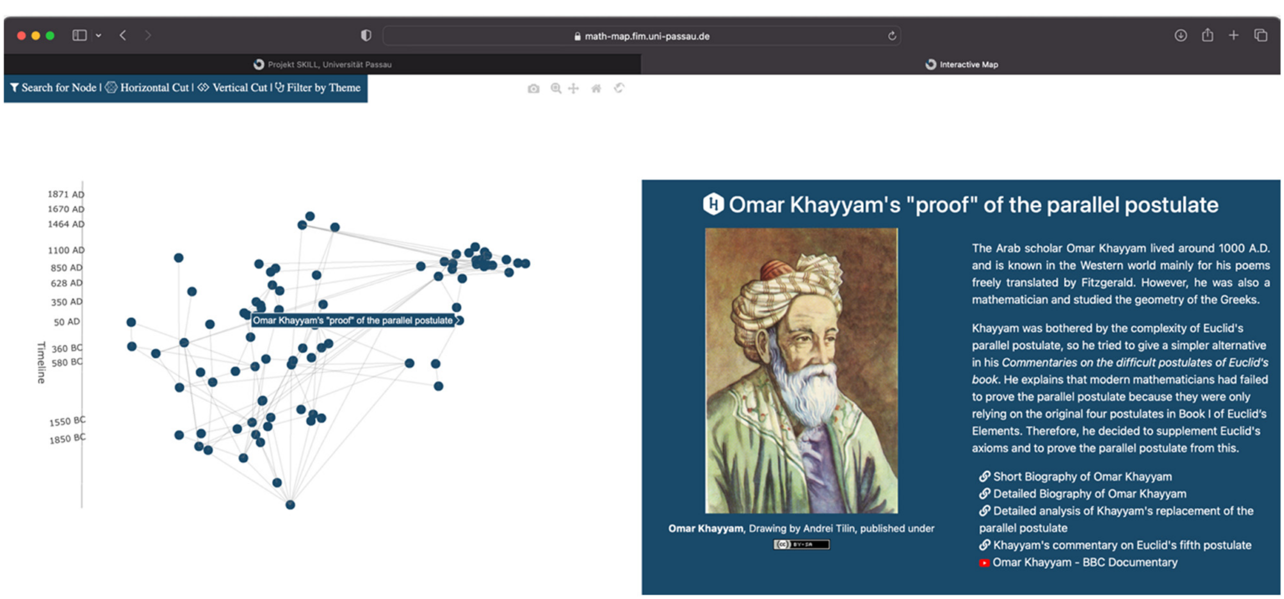

Figure 9: The Digital Interactive Mathematical Map (Status: 23.10.2022).

Each node or element of the map or timeline, respectively, comprises information about the piece of mathematics and the person responsible for the discovery, flanked by providing biographical and factual information via texts, links to further publicly available sites, and free videos. These elements can be collected by a shopping cart system and exported for further use in lessons or tasks.

A projection into the plane of all nodes selected in a certain period of time and/or filtered by a specific topic is also possible and called the "horizontal cut" functionality. It allows for the visualization of inner-mathematical relationships between the different content nodes (Figure 11). Contrarily flanking this functionality is the "vertical cut" mechanism resulting in a reduction of all nodes to those representing the historical development of the mathematical idea or resulting in the form of a two-dimensional directed graph (Figure 12). Clicking on the nodes then allows for an efficient and meaningful connected way through the genesis of the mathematical aspect and its explorers. Furthermore, a summary of all nodes being part of the evolutionary line can be created and downloaded, in order to be distributed to other learners and worked with in offline mode.

Regarding aspects of narratives and their corresponding didactical theory, the offered content elements of the map nodes serve as providers of the basic bricks for the construction of a narrative for school lessons.

Especially, the vertical cut function of the map can help to create narratives covering several points in time. The "main goal of the vertical cut is to promote an accurate picture of mathematics as an emerging science. It is a consequence of its very nature that mistakes and misunderstandings can and probably will occur when doing or creating mathematics" (Przybilla et al., 2022, p. 4796).

In the geometry course at Karlstad university, for instance, the centuries-long discussion about Euclid's parallel postulate was picked as a suitable example (Figure 12), which also connects school mathematics (Euclidean geometry) with university mathematics (non-Euclidean geometry).

3 Methodological Approach: Qualitative Evaluation of a Geometry Course

Concerning the methodological approach used, the field of application and research was a second semester geometry course of the teacher training program at Karlstad

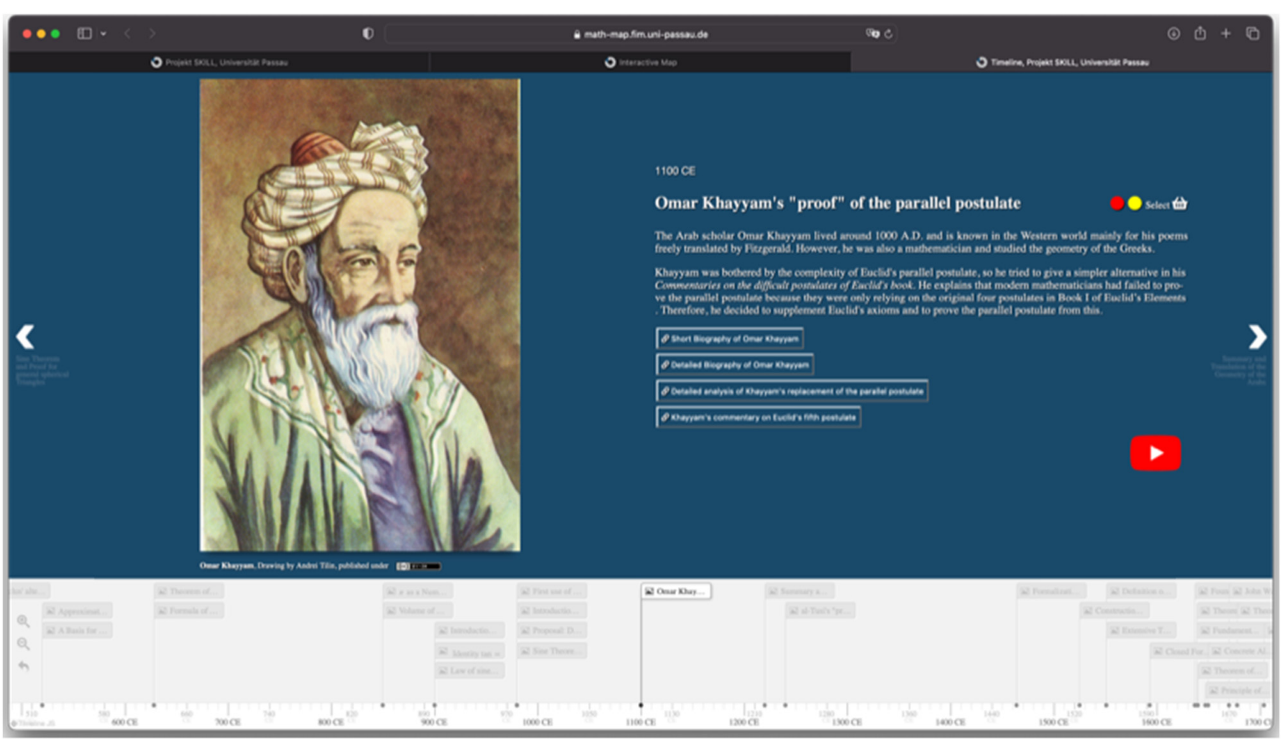

Figure 10: The timeline (Status: 23.10.2022).

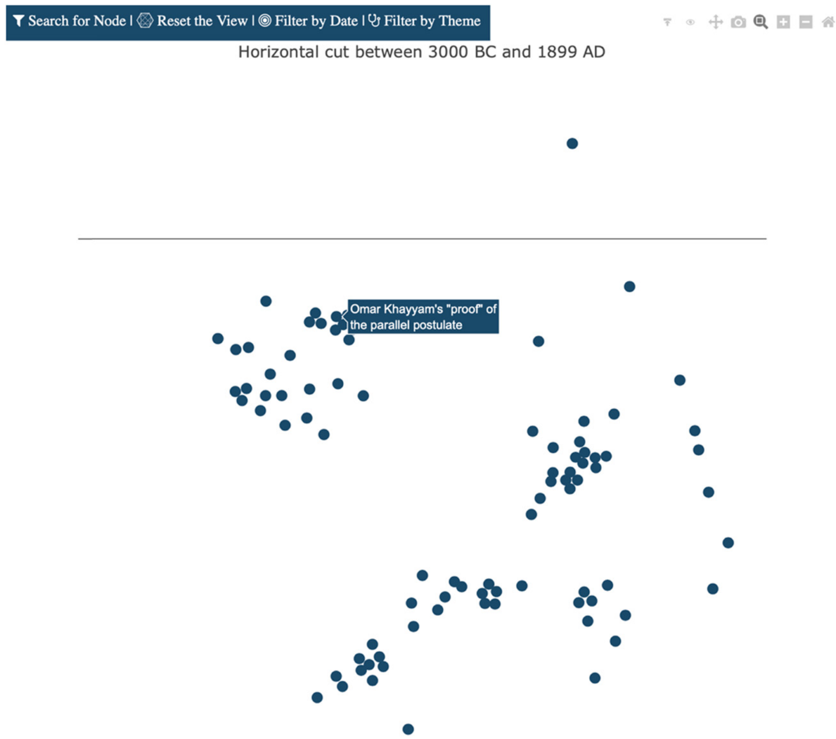

Figure 11: The horizontal cut (Status: 23.10.2022).

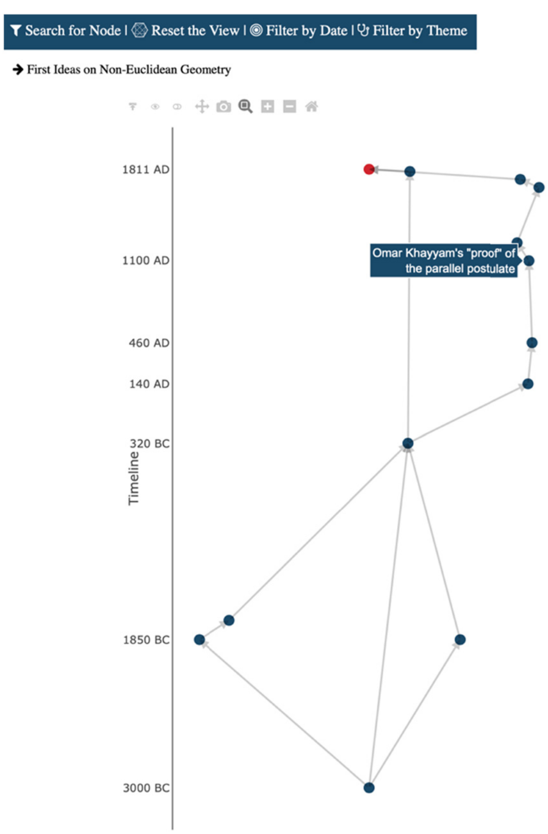

Figure 12: The vertical cut (Status: 23.10.2022).

University in Sweden, which was designed beforehand cooperatively by Swedish and German docents and researchers of mathematics and mathematics education. Central contents from Euclidean geometry represent the main part of the course by additionally introducing key concepts from non-Euclidean geometry, flanked by the necessity for fostering competencies with regard to the evaluation of digital learning tools. In sum, 44 students (27 teachers and 17 prospective teachers) actively took part in the course.

Weekly lectures were followed up by weekly evaluative interventions in the form of work assignments[1] concerning the timeline, narrative didactics as an area of application of the contents in the Mathematical Map, the three-dimensional Mathematical Map, the vertical cut, the horizontal cut, the discussion around the parallel postulate, and a general evaluation of the didactical tool.

The assessment of the teaching strategy was based on several pillars, just as the teaching strategy combines different aspects like the use of technology and narratives. We evaluated the two scaffoldings provided (see earlier discussion) according to their main characteristics

1 The weekly assignments are freely accessible at https://docs.google.com/document/d/e/2PACX-1vTq_w6A2MqresPr5PQLGujDt3NIS6i7aiL7-ShsmaR841TtI-hrsyTM2zLY6UEhbULBumIx0zEtTOz6V/pub

(Digital Interactive Mathematical Maps: Technology; Narrative Didactics: Text/Stories/Narratives) and analysed for a more general assessment of the teaching strategy the effect on the students' beliefs concerning the nature of mathematics as these "are a crucial part of the professional competence of mathematics teachers" (Felbrich, Müller, & Blömeke, 2008, p. 763).

For the analysis of the evaluation results, we proceeded in two steps, starting in the first part with the presentation of conclusions derived for the technology scaffolding, the influence on students' beliefs, and first global indications concerning the use of narrative didactics in Vinerean et al. (in press) and partly already sketched in Przybilla et al. (2022) and Vinerean et al. (2023). Hence, for a detailed description of the proceeding and results especially of the technology evaluation and the effect on the support of favourable beliefs we refer to the mentioned references, as within the discussion here we want to focus in detail on the second step of the evaluation, i.e. broaden and deepen the analysis of application of elements from narrative didactics illustrated by a single case study as presented at the BBC'22 conference (cf. Brandl & Vinerean, 2022).

For the first part of our survey, as it is described and evaluated in Vinerean et al. (in press) and sketched partly in Przybilla et al. (2022), Vinerean et al. (2023) and Brandl et al. (in press), it was important to see whether the application of our holistic teaching strategy would be able to impact on a transition of beliefs towards favourable beliefs, which are represented by a dynamic view on mathematics consisting of an understanding of mathematics as an emerging science allowing for errors and failures in the creative process. In contrast, a static perception does not promote the accurate interpretation of doing mathematics, consequently preventing the possibility of an open error culture in class and standing in contradiction to the actual nature of mathematics (cf. Brandl et al., in press; Przybilla et al., 2022; Vinerean et al., 2023, in press).

Regarding the procedure of the survey, the students got to know each of the features, namely three-dimensional map, two-dimensional timeline, vertical cut functionality, and horizontal cut functionality, by short explanatory introductory episodes from the docent. In a second step, the students always should try and work with the tool by themselves and test the components according to specific work assignments given. Then, each time the evaluation of the different aspects was carried out in the form of a quiz containing single choice questions (answerable on a 5-point Likert scale) and free-text questions. Furthermore, there was always the possibility to give feedback on potential opportunities for improvement. For further details, we refer to Vinerean et al. (in press).

Regarding the evaluation aspect of narrative didactics as an area of application of the contents in the mathematical map, an extra unit about the specifics of narrative didactics, especially in STEM education, was carried out. Within the corresponding work assignment, students then had to produce a short narrative designed to serve as a motivational lesson start for a suitable topic free of choice from the school curriculum by using information provided by the nodes of the three-dimensional map or the elements in the timeline of the Digital Interactive Mathematical Map.

4 Results and Findings

In summary, the results concerning the first part of the evaluation analysis, derived and described in Przybilla et al. (2022) and Vinerean et al. (in press), showed that the specific technical functionalities of the Digital Interactive Mathematical Map were generally evaluated positively. Some few negative remarks mainly concerned the three-dimensional interactive map regarding the complexity of usability.

The desired support of favourable beliefs with respect to the nature of mathematics could also be observed by analysis of the free-text answers (Vinerean et al., 2023, in press). "About 70% of the student responses reveal signs of a process-related and application-related orientation" (Przybilla et al., 2022, p. 4799) and "[s]igns of how the map may open up for dynamic understanding of mathematics can be seen in, for instance, students' reasoning about the struggle when developing mathematical concepts" (Przybilla et al., 2022, p. 4799).

Apart from this and as the focal point of the second part of the evaluation analysis here, we will focus next on the narrative didactic aspects and principles, i.e. if and in which way they were addressed in the texts produced by the students. The use of narrative didactics in combination with the Digital Interactive Mathematical Maps was implemented in the course in two ways with regard to methodology of evaluation:

(A) Formulation of a short historical-orientated narrative motivation for a free-of-choice topic from the school curriculum by using information provided in the nodes or the timeline of the Digital Interactive Mathematical Map.

(B) Description of the historical evolvement of non-Euclidean geometry from Euclidean geometry by using information provided in the nodes or the timeline of the Digital Interactive Mathematical Map.

While (B) also made a weak use of elements from storytelling, the main focus for the application of elements from narrative didactics lay on (A) where a short narrative should provide the effect of a scaffolding for teaching mathematics to learners at school.

The short texts could be about subjects free-of-choice, requiring only to be both element of the school curriculum and the Digital Interactive Mathematical Map.

In order to illustrate the successful use of elements from narrative didactics as taught in the context of the teaching strategy, in the following, we perform a single case study via qualitative content analysis of the text taking into account the narrative elements from Tables 1 and 2 as categories to look for in the text. Thus, in the following, we present the exemplary excerpt from a student's assignment and answer to the feedback possibilities on the narrative didactics aspect[2] in order to illustrate successful and not successful use of the desired elements.

"Student 5" gives a short introduction concerning the motivation and intention to use narrative elements and pieces of biographies collected with the help of the map in an imaginary mathematics lesson:

> To introduce 4–5 graders to coordinates and the number plane I think that the story about Descartes seeing the flies on the ceiling can really help to visualize and understand it. By now, the students should already be familiar with the number line. To expand on that knowledge I will tell a very abridged version of the story of Descartes and tie it to previous discoveries by other people.

Analysing these lines, we detect the conviction of the student that using an anecdote ("story") from the biographical description of the life of the famous mathematician René Descartes can "help" the "4–5 graders" to "understand" a specific new topic from the mathematics curriculum, i.e. "coordinates and the number plane." So, he/she identified a curriculum topic and intents to produce a historical scaffolding. Regarding the psychologically important learning aspect of making connections to other elements in the learner's mind, the student attests the necessity of previous knowledge ("the number line") which is intended to be connected to the new and "expand [ed]" idea of "coordinates and the number plane." As the vehicle selected to generate this connection, the student wants to "tell a very abridged version of the story of Descartes" and "tie it to the previous discoveries by other people," which is essentially what a historical scaffolding tries to offer didactically. Significantly, the student's words point much more into the direction of the narrative mode (instead of the traditional logico-scientific mode) of Table 2 (right column) by *looking for particular connections of events* ("Descartes seeing the flies on the ceiling") in a *retrodiction* in a time mode *concerning the past* and using *human being as necessary characters* ("Descartes," "other people") as an agency. Also, the explanations are seen as extrinsic ("story *about*") additional to the intrinsic facts ("number line," "coordinates," "number plane"). Referring to Klassen's SDCA, the student successfully entered the cycle of Figure 8 by paying respect to the learners' prior knowledge and the selection of a suitable story. Furthermore, the historical and theoretical contexts are fixed. According to Table 1 and the narrative elements, several aspects already have been addressed, too (Table 3).

The student then approaches the task of generating a short narrative by stepping into the story:

> Descartes was a French philosopher, lawyer, and mathematician. He is mostly known nowadays for his idea of radical doubt, but for now, we will only look at his mathematics.

So, the protagonist is introduced as a real past-time human character ("was," "philosopher, lawyer, and mathematician"), place and time are concretized ("French"), and the narrator selects the events to be told ("we will only look at his mathematics"). To a certain extent, the narrative appetite may be aroused, too, by indicating a specific character description as an important scientific celebrity ("known nowadays," "his idea of radical doubt").

Now the action within the story begins:

> One evening as Descartes was lying in his bed looking up at the ceiling, as he has done so many nights before, he noticed a fly. Rather than getting annoyed, he had an idea. If he were to pick a point where two lines cross each other he could with a unique pair of numbers describe how far away the fly was from that point.

The anecdote unfolds around an everyday situation in the life of René Descartes ("lying in his bed looking up the ceiling," "as he has done so many nights before") and puts the place and time to a very specific moment creating listeners' anticipation based on a range of possibilities and their desire of what will happen ("he noticed a fly"). With regard to the structure of the narrative, it typically starts with an imbalance compared to the usual everyday incidents and an unusual reaction of the protagonist towards it ("noticed a fly," "Rather than getting annoyed," "he had an idea"). Now further narrative appetite is created in the

2 Translation from Swedish to English by the authors.

Table 3: Identification of first narrative elements collected by Norris et al. (2005) in the assignment text of Student 5

Narrative element	Meaning	Student 5's choice
Event-tokens	Particular occurrences involving particular actors at a particular place and time	René Descartes, "Descartes seeing the flies on the ceiling," Lifetime of René Descartes
Event-tokens	Are chronologically related; involve a unified subject and are interconnected	"tie it to previous discoveries of other people"; coordinates and the number plane
Narrator	The agent …; determines the point and purpose of the story to be told; selects events and the sequence…	"I will tell …"; "abridged version of the story of Descartes"
Past time	Concerns the past	Lifetime of Descartes
Agency	Actors experience events; human beings	"Descartes seeing the flies on the ceiling"; Descartes and "other people"
Purpose	To help us better understand the natural world…	"help to […] understand" the "coordinates and the number plane"

listeners raising the desire to know what will happen, i.e. which annoyance-avoiding idea Descartes could have had caused by an ordinary fly on the ceiling. By now, the learners' interest should be sufficiently caught and their (narrative) interest aroused. We notice that up to now no mathematics has been introduced, only the historical scaffolding has been created by the narrator. No whatsoever loather of mathematics should be driven away by now; instead, it rather might be possible to catch their intention by a seemingly contrary approach not tricking but helping them to an emotionally more positive encounter with logico-scientific elements of mathematics, which Student 5 then starts to implement into the narrative ("pick a point where two lines cross each other," "a unique pair of numbers," "describe how far away"). So, a story-driven motivated very vivid idea of a Cartesian coordinate system with Euclidean distance suddenly appears, which is immediately picked up by the narrator in the role of the teacher and brought into classroom reality directly from the story context:

> This I would demonstrate by drawing a grid, without any numbers or an origin, on the board. I would add a fly to this sketch, on one of the nodes. When the sketch is done, I will pick another arbitrary point as the origin. By picking a point as the origin, the beginning, we can now describe the point the fly is on by counting the amount of steps we will take sideways and how many we will take upwards. We will now think of a way we can write that down. If I say (steps sideways, steps up) I can perfectly tell someone else where that fly is located.

In Klassen's SDCA, the teaching student applies the instructor's experience and knowledge to create new ideas, knowledge, and skills on the side of the learners (right side of Figure 8). Student 5 describes the methodical-didactical approach of the lesson introducing the mathematical learning content ("drawing a grid, without any numbers or an origin, on the board"). Still, the connection to the scaffolding narrative context is maintained ("add a fly to the sketch"); so psychologically seen, the learning content is attached to the scaffolding provided by the story. The mathematical structure then is worked out further and provided with details ("sketch is done," "pick another arbitrary point as the origin"). Finally, the original goal of the lesson, the introduction of coordinates in a number plane, is achieved by using the narrative elements to motivate an objective description of an object as a point in a two-dimensional plane by two numbers representing normed steps in two independent, perpendicular directions ("we can now describe the point the fly is on," "by counting the amount of steps," "sideways," "upwards"). At last, at the very end of the whole process, and not earlier, the symbolic formalization of the new idea/knowledge is carried out ("think of a way we can write that down," "(steps sideway, steps up)"), paying regard to the didactic principle of advancing from "soft" illustrative concepts to "hard" abstract concepts of mathematics in school. The purpose of the narrative at last is picked up again by pointing out how a mathematical idea can help to describe und understand the natural world we live in as human beings together with others and other creatures ("perfectly tell someone else where that fly is located").

Of course, there is plenty of opportunity for criticism. The action taking part is rather poor, the amount of emotional involvement seems to be very low, the agents deal with no hard conflicts, and there is no high grade of suspense in the story. There is very little from everything narrative elements offer. However, the text the student produced was quite short, too, allowing only for a sketch of the idea of a story and a narrative approach. Writing a whole solid short story that offers suspense and all the elements of an anecdote would be a desirable goal for a teacher or a teacher student intending to use narrative didactics in his or her mathematics lesson, but such high expectations on a text to tell are, of course, far out of reach within a mathematics course in geometry focussing on mathematics and only offering additional methodical–-didactical scaffolding concepts such as the Interactive Mathematical Maps and a short introduction to the possibilities and elements of narrative didactics. However, the text of single case "Student 5" shows how such a brief encounter with those flanking and bridging teaching tools can result in a successful use of typical narrative elements in a correct way in order to provide a scaffolding for establishing sustainable learning processes of new mathematical ideas/knowledge by the principle of connection with other contexts.

Concerning the description of the historical evolvement of non-Euclidean geometry from Euclidean geometry by using information provided in the nodes or the timeline of the Interactive Mathematical Map (case B), we refrain from analysing another student's text as the use of narrative elements neither was demanded as strong as for case A nor were narrative elements (as described in Tables 1 and 2) used as intensively. Here lies the potential for future evaluation requiring a modification of the assignment design. We only want to cite an excerpt from Student 15's feedback, besides finding the Digital Interactive Mathematical Map *"helpful"* for solving the task, also explaining that he/she *"could easily have written an article ten times as long on the subject and enjoyed doing it"* (transl. by authors).

5 Conclusion and Opinions Obtained

The results of the two evaluation analysis parts show that on the one hand the technological development of the digital tool goes into the right direction and on the other hand an approach via methods from narrative didactics can lead to motivational narrative texts potentially, successfully usable in mathematics lessons. The two scaffoldings "Narrative Didactics" and "Digital Interactive Mathematical Map," belonging to the teaching strategy aspects of a supporting visualising technology and emotion addressing narrations, promise to be helpful to establish effective learning processes. Students' feedback leads to the conclusion that their impression of the maps is mainly positive concerning the ease of use and usefulness (Brandl et al., in press; Przybilla et al., 2022; Vinerean et al., in press). Thus, a future application by the students in their own mathematics lessons can be assumed to be probable as no technical obstacles prevent an application without much preparation. Besides, students built up further competencies in the area of digital media application for teaching mathematics.

Complementary to traditional ways of teaching mathematics as a seemingly disjointed sequence of static and isolated facts, the presented holistic teaching strategy also indicates to support favourable beliefs regarding the nature of mathematics (Vinerean et al., 2023, in press). Here the teaching strategy also benefits from the scaffolding of narrative didactics. By bringing a "human touch" back to mathematics, stories about the mathematicians involved, and their struggles, their triumphs, their failures, their successful solutions, or their errors can help to see mathematics as an emerging science connected by historical events. Additionally, an emotional engagement with the story allows the students to engage more deeply with the mathematical contents, too. The "recipes" from Norris et al. (2005) or Klassen (2006) may seem unusual and different to instructors used to teach mostly according to the needs of the "logico-scientific mode." However, as the extensively presented single case study of Student 5's narrative text tries to point out, even quite brief introductions to this kind of didactical–methodological procedure may pretty quickly lead to surprisingly good results. In the sense of an interdisciplinary and connecting science, technology, engineering, arts, and mathematics approach, our opinion is that the presented bridging holistic teaching strategy represents a fruitful combination of complementary aspects supporting the complex process of learning.

6 Summary and Outlook

By using a bridging teaching strategy which brings together mathematics itself, didactical methods, historical events, narrative elements, and technological aspects (Figure 7), we tried to address both the logico-scientific and the narrative mode of teaching (Table 2) in order to allow for more successful learning processes because of connections to various contexts serving as anchor points in a scaffolding structure. Therefore, we explained our interpretation of Wagenschein's "routed-in" knowledge (Figures 1–6), introduced an overview of the principles and elements of narrative didactics (Tables 1 and 2, Figure 8), presented the technological didactical tool in form of Digital Interactive Mathematical Maps (Figures 9–12), described the survey conducted, pointed out to results and findings of the first, mainly technology-oriented part of the evaluation analysis in Przybilla et al. (2022), Vinerean et al. (2023, in press) and Brandl et al. (in press), and decided for the second part of the evaluation analysis, on which we focused on here, on the method of a single case study for examining the use of narrative elements for a motivational piece of a student's mathematics lesson at school and showed a qualitative analysis of the text produced.

This first complete evaluation of our bridging teaching concept shows to be a promising way for a holistic design of mathematics lessons that brings together the areas of science and arts as well as the areas of rational thought and emotional feeling both taking part in a learning process.

We also see the restrictions of one single case study for a more general conclusion regarding the use of narrative didactics and already explained the still existing deficits of the examined text. Therefore, future research on this approach may consider a multiple case study taking into account all texts produced by the students in the assignments. Results from this may lead to an improvement of the teaching and examination style of the courses and represent a next iterative step of a design cycle in the design-based research of the learning environments (Datzmann et al., 2020; Przybilla et al., 2022). Still pending is the use in school classroom of the narrative piece in order to see and examine the influence of the bridging teaching technique on a school student's learning process in mathematics. Future research may focus on these methodological extension, solidification, and application both in the field of higher education in the context of teacher training and the field of mathematics education at schools.

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References

Baptist, P. (2008) (Hrsg.). *Alles ist Zahl*. Köln: Kölner Universitäts-Verlag.

Barnes, D. (1992). *From communication to curriculum* (2nd ed.). Portsmouth, NH: Boynton/Cook Publishers.

Brandl, M. (2009). The vibrating string – an initial problem for modern mathematics; historical and didactical aspects. In I. Witzke (Ed.), *Mathematical practice and development throughout history: Proceedings of the 18th Novembertagung on the History, Philosophy and Didactics of Mathematics* (pp. 95–114). Berlin: Logos Verlag.

Brandl, M. (2010). Narrative didactics in mathematical education: An innovative didactical concept. In T. Bianco & V. Ulm (Eds.), *Mathematics education with technology – Experiences in Europe* (pp. 103–110). Augsburg: University of Augsburg.

Brandl, M. (2016). Narrative Mathematik-Didaktik mittels Elementen bildender Kunst. In Institut für Mathematik und Informatik der Pädagogischen Hochschule Heidelberg (Ed.), *Beiträge zum Mathematikunterricht 2016* (pp. 1415–1418). Münster: WTM-Verlag.

Brandl, M. (2017). Narrative Didaktik als Vernetzungsinstrument: Die Schule von Athen. In T. Borys, M. Brandl, & A. Brinkmann (Hrsg.), *Mathe vernetzt – Anregungen und Materialien für einen vernetzenden Mathematikunterricht. Band 6* (pp. 7–20). Schriftenreihe des GDM-Arbeitskreises, Vernetzungen im

Mathematikunterricht' (A. Brinkmann, Ed.). Appelhülsen: Verlag Bücherbunt im MUED e.V.

Brandl, M., & Nordheimer, S. (2017). 'Verstehens-Shift' durch Vernetzung – exemplarische Darstellung anhand von Beispielen aus der Stochastik. In M. Brandl, A. Brinkmann, M. Bürker, & J. Maaß (Hrsg.), *Mathe vernetzt – Anregungen und Materialien für einen vernetzenden Mathematikunterricht. Band 4* (pp. 9–21). Schriftenreihe des GDM-Arbeitskreises, Vernetzungen im Mathematikunterricht' (A. Brinkmann, Ed.). Appelhülsen: Verlag Bücherbunt im MUED e.V.

Brandl, M., & Vinerean, M. (2022). Narrative didactics in mathematics education: Results from a university geometry course. In F. B. Soares, A. P. Lopes, C. Pinto, & J. Mendonça (Eds.), *Book of Abstracts of the First International Conference – Building Bridges in STEAM Education in the 21sr Century* (*BCC'22*) (p. 34). Porto Accounting and Business School & Porto School of Engineering. doi: 10.26537/20625.

Brandl, M., Kaiser T., Przybilla J., & Hackstein, U. (in press). Digitale Interaktive Mathematische Maps. In I. Brachmann, M. Dick, B. Heurich, B. Lukács, & J. Wölfl (Hrsg.), *Innovative Lehrkräftebildung, digitally enhanced. Medienintegrative Abschlusspublikation des Projekts SKILL.de*. Pressbooks OER.

Breithaupt, F. (2022). *Das narrative Gehirn* (*2. Aufl.*). Berlin: Suhrkamp.

Brinkmann, A., Maaß, J., Ossimitz, G., & Siller, H.-S. (2017). Vernetzungen und vernetztes Denken im Mathematikunterricht. In M. Brandl, A. Brinkmann, J. Maaß, & H.-S. Siller (Hrsg.), *Mathe vernetzt – Anregungen und Materialien für einen vernetzenden Mathematikunterricht. Band 1* (pp. 7–21). Schriftenreihe des GDM-Arbeitskreises, Vernetzungen im Mathematikunterricht' (A. Brinkmann, Ed.). Appelhülsen: Verlag Bücherbunt im MUED e.V.

Bruner, J. (1986). *Actual minds, possible worlds*. Cambridge (Mass.): Harvard University Press. https://archive.org/details/actualmindspossi00jero.

Cohen, I. E. (1993). A sense of history in science. *Science & Education*, *2*, 252–277.

Damasio, A. R. (1994). *Descartes' error: Emotion, reason, and the human brain*. New York: G.P. Putnam.

Datzmann, A., Przybilla, J., Brandl, M., & Kaiser, T. (2020). New Teaching Techniques aiming to connect School and University Mathematics in Geometry. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H.-G. Weigand (Eds.), *Mathematics Education in the Digital Age* (*MEDA*) *Proceedings 2020* (pp. 37–44). HAL.

Ebenezer, J. V., & Fraser, D. M. (2001). First year chemical engineering students' conceptions of energy in solution processes: Phenomenographic categories for common knowledge construction. *Science Education*, 85(5), 509–535. doi: 10.1002/sce.1021.

Egan, K. (1989a). The shape of the science text: A function of stories. In S. de Castell, A. Luke, & C. Luke, (Eds.), *Language, authority and criticism: Readings on the school textbook* (pp. 96–108). New York: The Falmer Press.

Egan, K. (1989b). Memory, imagination, and learning: Connected by the story. *Phi Delta Kappan*, *70*(6), 455–473.

Enzensberger, H. M. (1997). *Der Zahlenteufel*. München: Carl Hanser Verlag.

Felbrich, A., Müller, C., & Blömeke, S. (2008). Epistomological beliefs concerning the nature of mathematics among teacher educators and teacher education students in mathematics. *ZDM*, *40*, 763–776. doi: 10.1007/s11858-008-0153-5.

von Glasersfeld, E. (1990). Environment and communication. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: international perspectives* (pp. 30–38). Hillsdale, NJ: Lawrence Erlbaum Associates. doi: 10.4324/9780203052372.

Holton, G., Rutherford, J., & Watson, F. (1970). *The project physics course*. New York: Holt, Rinehart & Winston.

Howard, P. J. (2000). *The owner's manual for the brain: Everyday applications from mind-brain research* (2nd ed.). Austin, TX: Bard Press.

Jung, W. (1994). Toward preparing students for change: A critical discussion of the contribution of the history of physics in physics teaching. *Science & Education*, *3*, 99–130. doi: 10.1007/BF00486387.

Kipnis, N. (1996). The historical-investigative approach to teaching science. *Science & Education*, *5*, 271–292. doi: 10.1007/bf00414317.

Klassen, S. (2006). A theoretical framework for contextual science teaching. *Interchange*, *37*(1–2), 31–61. doi: 10.1007/s10780-006-8399-8.

Klassen, S. (2009). The construction and analysis of a science story: A proposed methodology. *Science & Education*, *18*, 401–423. doi: 10.1007/s11191-008-9141-y.

Koul, R., & Dana, R. (1997). Contextualized science for teaching science and technology. *Interchange*, *28*(2&3), 121–144. doi: 10.1023/a:1007348821292.

Krieger, D. J. (1996). *Einführung in die allgemeine Systemtheorie*. München: W. Fink Verlag.

Kubli, F. (1999). Historical aspects in physics teaching: Using Galileo's work in a new Swiss project. *Science & Education*, *8*(2), 137–150. doi: 10.1023/A:1008613706212.

Kubli, F. (2002). *Plädoyer für Erzählungen im Physikunterricht: Geschichte und Geschichten als Verstehenshilfen – Ergebnisse einer Untersuchung* (2nd ed.). Köln: Aulis Deubner.

Kubli, F. (2005a). Science teaching as a dialogue – Bakthin, vygotsky and some applications in the classroom. *Science & Education*, *14*, 501–534. doi: 10.1007/s11191-004-8046-7.

Kubli, F. (2005b). *Mit Geschichten und Erzählungen motivieren: Beispiele für den mathematisch-naturwissenschaftlichen Unterricht*. Köln: Aulis Deubner.

Moffett, J. (1983). *Teaching the universe of discourse*. Portsmouth, NH: Heinemann.

Mott, B. W., Callaway, C. B., Zettlemoyer, L. S., Lee, S. Y., & Lester, J. C. (1999). Towards narrative-centered learning environments. *Proceedings of the AAAI Fall Symposium on Narrative Intelligence*. https://www.aaai.org/Papers/Symposia/Fall/1999/FS-99-01/FS99-01-013.pdf.

Noddings, N., & Witherell, C. (1991). Epilogue: Themes remembered and foreseen. In C. Witherell & N. Noddings (Eds.), *Stories lives tell* (pp. 279–280). New York: Teachers College Press.

Norris, S., Guilbert, M., Smith, M., Shahram, H., & Phillips, L. (2005). A theoretical framework for narrative explanation in science. *Science Education*, *89*(4), 535–554. doi: 10.1002/sce.20063.

Phelan, J. (1996). *Narrative as rhetoric: Technique, audiences, ethics, ideology*. Columbus, OH: Ohio State University Press.

Przybilla, J., Brandl, M., Vinerean, M., & Liljekvist, Y. (2021). Interactive Mathematical Maps – A contextualized way of meaningful Learning. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, & F. Hreinsdóttir (Eds.), *Bringing Nordic mathematics education into the future. Preceedings of NORMA 20. The ninth Nordic Conference on Mathematics Education. Oslo, 2021* (pp. 209–216). SMDF. https://www.uv.uio.no/ils/english/about/events/2021/norma/proceedings/norma_20_preceedings.pdf.

Przybilla, J., Brandl, M., Vinerean, M., & Liljekvist, Y. (2022). Digital mathematical maps – results from iterative research cycles. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education* (*CERME12*) (pp. 4793–4800). Free University of Bozen-Bolzano and ERME. https://hal.archives-ouvertes.fr/hal-03754749/.

Resnick, L. B., & Resnick, D. P. (1992). Assessing the thinking curriculum: New tools for educational reform. In B. R. Gifford & M. C. O'Connor (Eds.), *Changing assessments: Alternative views of aptitude, achievement and instruction* (pp. 37–75). Boston: Kluwer. doi: 10.1007/978-94-011-2968-8_3.

Richardson, L. (1990). Narrative and sociology. *Journal of Contemporary Ethnography*, *19*(1), 116–135. doi: 10.1177/089124190019001.

Sousa, A. D. (2017). *How the brain learns* (5th ed.). Thousand Oaks, Calif: Corwin Press.

Vinerean, M., Brandl, M., & Liljekvist, Y. (2023). Promoting favourable beliefs of prospective math teachers concerning the nature of mathematics by using Interactive Mathematical Maps. In M. Trigueros, B. Barquero, R. Hochmuth, & J. Peters (Eds.), *Proceedings of the Fourth conference of the International Network for Didactic Research in University Mathematics* (*INDRUM 2022, 19-22 October 2022*) (pp. 574–575). Hannover: University of Hannover and INDRUM. https://hal.science/INDRUM2022/public/INDRUM2022_Proceedings.pdf.

Vinerean, M., Liljekvist, Y., Brandl, M., & Przybilla, J. (in press). Didactical usefulness of interactive mathematical maps-designing activities supporting student teachers' learning. In M. Misfeldt, U. T. Jankvist, E. Geraniou, I. H. Hojsted, A. L. Tamborg, R. J. E. Coopman, & B. E. Jessen (Eds.), *Nordic studies in mathematics education – Nordisk Matematikkdidaktikk, NOMAD – Thematic Issue Digital resources in mathematics education*. Göteborg: NCM.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. In M. Cole, V. Jolm-Steiner, S. Scribner, & E. Souberman (Eds.), Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and language*. Cambridge, MA: MIT Press. Orginal work published 1934.

Wagenschein, M. (1968). *Verstehen lernen* (*8. erg. Aufl., 1989*). Weinheim und Basel: Beltz Verlag.

Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem-solving. *Journal of Child Psychology and Psychiatry*, *17*, 89–100. doi: 10.1111/j.1469-7610.1976.tb00381.x.