# Homogenization of a poroelasticity model for fiber-reinforced hydrogels 

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#### Abstract

In this paper, the analysis and homogenization of a poroelastic model for the hydro-mechanical response of fiber-reinforced hydrogels are considered. Here, the medium in question is considered to be a highly heterogeneous two-component media composed of a connected fiber-scaffold with periodically distributed inclusions of hydrogel. While the fibers are assumed to be elastic, the hydromechanical response of hydrogel is modeled via Biot's poroelasticity.

We show that the resulting mathematical problem admits a unique weak solution and investigate the limit behavior (in the sense of two-scale convergence) of the solutions with respect to a scale parameter, $\varepsilon$, characterizing the heterogeneity of the medium. While doing $\varepsilon \rightarrow 0$, we arrive at an effective model where the micro variations of the pore pressure give rise to a micro stress correction at the macro scale.


## KEYWORDS

homogenization, poroelasticity, system of elliptic and parabolic equations, tissue engineering, two-scale models, well-posedness

## MSC CLASSIFICATION

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## 1 | INTRODUCTION

Fiber-reinforced hydrogels (FIHs) are composites of a synthetic hydrogel* reinforced by a scaffold of microfibers, see Figure 1. They are used in tissue engineering (e.g., cartilage, tendon and ligament tissue, and vascular tissue ${ }^{1}$ ) where the FIH is used as a surrogate framework for in vitro growth. We refer to previous studies ${ }^{1-3}$ for the biochemical details regarding this process, but in short: host stem cells are seeded in the FIH where they are able to grow in an lab environment into fully functional tissue which can then be transplanted back into the host. With the highly hydrated polymer network of hydrogels, FIHs mimic the environment of natural extracellular matrices (ECMs), while the fiber scaffold improves the mechanical properties, see Castilho et al ${ }^{2}$ and references therein. Without this reinforcement, it is very difficult to get mechanical strength and structural resilience comparable with its native biological counterpart. ${ }^{1,4}$
This kind of in vitro tissue engineering is a relatively new approach and has some important advantages when compared to alternative treatments: it does not involve donor cells, which removes the danger of adverse immune response, and it has the prospect of enabling therapies that are "cost-effective, time-efficient, and single procedure." ${ }^{1}$

[^0]FIGURE 1 The periodic fiber scaffold empty (left) and saturated with hydrogel (right). This figure is taken from Castilho et $\mathrm{al}^{2}$ under a Creative Commons license (see: https://creativecommons.org/licen se/by/4.0/) [Colour figure can be viewed at wileyonlinelibrary.com]


In practice, the filament spacing of the scaffold is usually in the range of $\mu \mathrm{m}$ while the overall size of an FIH is in the range of mm to cm (Figure 1). Due to this scale heterogeneity, the effective hydromechanical properties of FIHs are not yet fully understood, and as a consequence, there is an interest in describing, modeling, and calculating their effective properties based on the underlying microstructure, cf. literature. ${ }^{2,5}$

In this work, we present a rigorously derived effective model for the hydromechanical properties of an FIHs based on a microstructure model describing the interplay between hydrogel and fiber structure. This micromodel assumes the fiber scaffold to be elastic and the hydrogel to be linearly poroelastic (Biot's poroelasticity). After showing that this micromodel has a unique solution (well-posedness), a limit process in the context of mathematical homogenization is employed to arrive at the effective model. In particular, this method also gives us effective material parameters like the elastic modulus. To our knowledge, this is first mathematical work rigorously treating this particular problem.

In Castilho et al, ${ }^{2}$ a two-scale finite element computational model is proposed, where the hydrogel is assumed to behave like a Neo Hookean solid (hyperelastic). More closely related to our work, in Chen et al, ${ }^{5}$ an effective model is derived from a two-phase elastic-poroelastic microproblem via formal asymptotic expansion.

It is worth noting that similar models to the micromodel considered in this work with some of the same features emerge in different applications as well. In Showalter and Momken, ${ }^{6}$ a general mathematical analysis of Biot's poroelascticity model is presented. In ensuing work, mathematical homogenization scenarios in the context of double poroelasticity were explored, see, for example, previous works. ${ }^{7-9}$ For further examples in the context of the related thermoelasticity, we refer to previous studies. ${ }^{10-12}$ For a general overview of homogenization problems related to elasticity, we refer to Oleinik et al. ${ }^{13}$

Regarding the structure of the article, in Section 2, we introduce the setting and the model as well as present the main results. Sections 3 and 4 are dedicated to the study of the microproblem and the proof of the homogenization result, respectively.

## 2 | SETTING OF THE PROBLEM AND MAIN RESULTS

In this section, we provide the detailed geometric setup as well as the mathematical model that we are considering. In addition, we present our main results, namely, Theorems 2.4 and 2.5 , which are proved in the ensuing sections.

In the following, let $\Omega \subset \mathbb{R}^{n}$ be a bounded $C^{1,1}$-domain representing the overall system and let $S=(0, T), T>0$, represent the time interval of interest. In addition, we denote the outer normal vector of $\Omega$ with $v=v(x)$. Let $Y=(0,1)^{d}$ be the open unit cell in $\mathbb{R}^{d}, d=2,3$. Take $Y^{f}, Y^{g} \subset Y$ two disjoint open sets, such that $Y^{f}$ is connected, such that $\Gamma:=\overline{Y^{f}} \cap \overline{Y^{g}}=\partial Y^{g}, \overline{Y^{g}} \subset Y$, and $Y=Y^{f} \cup Y^{g} \cup \Gamma$, see Figure 2. With $n_{\Gamma}=n_{\Gamma}(y), y \in \Gamma$, we denote the normal vector of $\Gamma$ pointing outwards of $Y^{g}$.

For $\varepsilon>0$, we introduce the $\varepsilon Y$-periodic initial domains $\Omega_{\varepsilon}^{f}, \Omega_{\varepsilon}^{g}$ and the interface $\Gamma_{\varepsilon}$ representing the fiber domain, gel domain and the boundary between fiber and gel, respectively. By $\partial \Omega$ we denote the outer boundary of $\Omega$. Via $(i=f, g)$

$$
\Omega_{\varepsilon}^{i}=\Omega \cap\left(\bigcup_{k \in \mathbb{Z}^{d}} \varepsilon\left(Y^{i}+k\right)\right), \quad \Gamma_{\varepsilon}=\Omega \cap\left(\bigcup_{k \in \mathbb{Z}^{d}} \varepsilon(\Gamma+k)\right) .
$$

In the following, $\chi_{\varepsilon}^{i}: \Omega \rightarrow\{0,1\}(i=f, g)$ denotes the characteristic functions corresponding to $\Omega_{\varepsilon}^{i}$.
Remark 2.1. By this design, $\Omega_{\varepsilon}^{f}$ is connected and $\Omega_{\varepsilon}^{g}$ is disconnected, which does not match the scaffold depicted in Figure 1 where holes between cells are clearly visible. As in the similar work done in Chen et al, ${ }^{5}$ however, we assume


FIGURE 2 The periodicity cell $Y$ and the geometrical setup of the periodic domain
the fiber scaffold to be closed. In particular, two separate neighboring gel cells can only interact via the fiber scaffold. In the case where both domains are connected, additional problems arise in the homogenization part; for example, certain uniform extension operators are needed. This can be done for certain well-behaved geometries. ${ }^{14}$ With our geometry, we are primarily thinking about a two-dimensional cross section of the FIH geometry (cf. Figure 1 and Figure 2). As our results are valid in $3 d$ as well, we have formulated them for the more general case.

Furthermore, in reality, the fiber scaffold is very thin in relation to the size of an individual cell, see Figure 1. As a consequence, it might make sense to introduce an additional scale parameter measuring this thickness. This would introduce another layer of complexity leading to a different homogenization problem in the context of thin structures. ${ }^{15}$

Now, let $w_{\varepsilon}: \Omega_{\varepsilon}^{f} \rightarrow \mathbb{R}^{d}, d=2,3$, represent the deformation in the fiber part. Assuming that the mechanical response of the fiber scaffold is governed by quasi-stationary elasticity, we then have

$$
\begin{equation*}
-\nabla \cdot\left(C e\left(w_{\varepsilon}\right)\right)=f \text { in } S \times \Omega_{\varepsilon}^{f} \tag{2.1a}
\end{equation*}
$$

Here, $e\left(w_{\varepsilon}\right)=1 / 2\left(\nabla w_{\varepsilon}+\nabla w_{\varepsilon}^{T}\right)$ denotes the linearized strain tensor, $\mathcal{C}$ the elasticity tensor, and $f$ possible volume forces. The hydrogel is itself a composite (polymer chains saturated with water) with complex mechanical properties. In this work, as in many others, for example, previous studies, ${ }^{5,16-18}$ we model it as a linear poroelastic material. There are, of course, other approaches like mixture theory, we refer to, for example, Hui and Muralidharan, ${ }^{19}$ but experimentally Biot's law seems to be quite accurate. ${ }^{20}$ To the knowledge of the authors, there does not yet exist a mathematical investigation of the effective material properties of hydrogels. Now, let $u_{\varepsilon}: \Omega_{\varepsilon}^{g} \rightarrow \mathbb{R}^{d}$ denote the solid deformation in the gel part and let $p_{\varepsilon}: \Omega_{\varepsilon}^{g} \rightarrow \mathbb{R}$ denote the pore pressure. The model of Biot's linear poroelasticity is given by

$$
\begin{gather*}
-\nabla \cdot\left(\mathcal{D e}\left(u_{\varepsilon}\right)-\alpha p_{\varepsilon} \mathbb{I}\right)=g \text { in } S \times \Omega_{\varepsilon}^{g},  \tag{2.1b}\\
\partial_{t}\left(c p_{\varepsilon}+\alpha \nabla \cdot u_{\varepsilon}\right)-\nabla \cdot\left(\varepsilon^{2} K \nabla p_{\varepsilon}\right)=h \text { in } S \times \Omega_{\varepsilon}^{g} . \tag{2.1c}
\end{gather*}
$$

Here, $\mathcal{D}$ is the elasticity tensor, $c$ is the Biot modulus, $\alpha$ is the Biot-Willis parameter, and $\varepsilon^{2} K$ is the permeability. $\mathbb{I}$ denotes the unit matrix, and, again, $g$ and $h$ are possible volume sources. Please note that this particular $\varepsilon^{2}$-scaling for the permeability in the gel part is the typical choice for these kinds of two-scale problems, cf. literature. ${ }^{9,21-23}$ The volume forces $f, g, h$ correspond to the forces acting on the system like gravity. However, we do not have a specific scenario in mind here, as our goal is to describe the general effective mechanical behavior of FIH's independent of the concrete forces.

Regarding the interaction of the two different phases, we assume both deformations and forces to be continuous across the interface $\Gamma_{\varepsilon}$. That is,

$$
\begin{gather*}
\left(\mathcal{D e}\left(u_{\varepsilon}\right)-\alpha p_{\varepsilon} \mathbb{I}\right) n_{\varepsilon}=\left(C e\left(w_{\varepsilon}\right)\right) n_{\varepsilon} \text { on } S \times \Gamma_{\varepsilon},  \tag{2.1d}\\
u_{\varepsilon}=w_{\varepsilon} \text { on } S \times \Gamma_{\varepsilon}, \tag{2.1e}
\end{gather*}
$$

Please note that these natural boundary conditions for the mechanical interplay of a general poroelastic and an elastic medium can be justified via mathematical homogenization ${ }^{24}$ in the case where the poroelastic medium on the microscale is given as a porous medium.

In addition, we assume no gel flux between the different cells

$$
\begin{equation*}
-\varepsilon^{2} K \nabla p_{\varepsilon} \cdot n_{\varepsilon}=0 \text { on } S \times \Gamma_{\varepsilon} \tag{2.1f}
\end{equation*}
$$

Finally, we close the model with homogeneous outer boundary and initial conditions:

$$
\begin{gather*}
u_{\varepsilon}=0 \text { on } S \times \partial \Omega  \tag{2.1g}\\
p_{\varepsilon}(0)=0 \text { in } \Omega_{\varepsilon}^{g} \tag{2.1h}
\end{gather*}
$$

We assume all scalar coefficients to be positive constants and the permeability matrix $K$ to be constant, symmetric, and positive definite. Also, we assume the tensor coefficients $\mathcal{C}$ and $\mathcal{D}$ to be constant, symmetric, and positive on the space of symmetric matrices.

Remark 2.2. We want to emphasize that our model, as given by (2.1a)-(2.1h), already is a strong simplification of the complex hydromechanical structure of FIHs, both in terms of geometry (see Remark 2.1) and as well as with respect to the model equations. It should be considered as a first step of a better understanding of this relatively new structure; thereby opening the way for the treatment of more complex problems. We also point out that similar investigations up to date work within the same framework. ${ }^{5}$

We note that, due to interface condition (2.1e), we can expect $U_{\varepsilon} \in H_{0}^{1}(\Omega)$ where

$$
U_{\varepsilon}:= \begin{cases}w_{\varepsilon} & \text { in } S \times \Omega_{\varepsilon}^{f} \\ u_{\varepsilon} & \text { in } S \times \Omega_{\varepsilon}^{g}\end{cases}
$$

In the following, we will denote the zero extension of any function $\psi$ defined on $\Omega_{\varepsilon}^{f}$ or $\Omega_{\varepsilon}^{g}$ to the whole of $\Omega$ by $\hat{\psi}$; with that we have $U_{\varepsilon}=\widehat{w_{\varepsilon}}+\widehat{u_{\varepsilon}}$.

Setting $\mathcal{A}_{\varepsilon}=\chi_{\Omega_{\varepsilon}^{f}} \mathcal{C}+\chi_{\Omega_{\varepsilon}^{g}} \mathcal{D}$, a corresponding weak form of the full model is then given by:
Find a pair of functions $\left(U_{\varepsilon}, p_{\varepsilon}\right) \in L^{2}\left(S ; H_{0}^{1}(\Omega)^{d} \times H^{1}\left(\Omega_{\varepsilon}^{g}\right)\right)$ where $\partial_{t} p_{\varepsilon} \in L^{2}(S \times \Omega)$ and $\partial_{t} \nabla \cdot U_{\varepsilon} \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$ that satisfy

$$
\begin{gather*}
\int_{\Omega} \mathcal{A}_{\varepsilon} e\left(U_{\varepsilon}\right): e(v) \mathrm{d} x-\int_{\Omega_{\varepsilon}^{g}} \alpha p_{\varepsilon} \nabla \cdot v \mathrm{~d} x=\int_{\Omega_{\varepsilon}^{f}} f \cdot v \mathrm{~d} x+\int_{\Omega_{\varepsilon}^{g}} g \cdot v \mathrm{~d} x  \tag{2.2a}\\
\int_{\Omega_{\varepsilon}^{g}} \partial_{t}\left(c p_{\varepsilon}+\alpha \nabla \cdot U_{\varepsilon}\right) \varphi \mathrm{d} x+\int_{\Omega_{\varepsilon}^{g}} \varepsilon^{2} K \nabla p_{\varepsilon} \cdot \nabla \varphi \mathrm{d} x=\int_{\Omega_{\varepsilon}^{g}} h \varphi \mathrm{~d} x \tag{2.2b}
\end{gather*}
$$

for all $(v, \phi) \in H_{0}^{1}(\Omega)^{d} \times H^{1}\left(\Omega_{\varepsilon}^{g}\right)$ and for almost all $t \in S$ as well as the initial condition $p_{\varepsilon}(0)=0$.
Remark 2.3. A more restrictive notion of a weak solution only has to satisfy $\partial_{t}\left(c p_{\varepsilon}+\alpha \nabla \cdot U_{\varepsilon}\right) \in L^{2}\left(S ; H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*}\right)$. This however is not enough to justify the homogenization procedure.

We assume that $f \in C^{1}\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{f}\right)\right)^{d}, g \in C^{1}\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)^{d}$, and $h \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$. The higher time-regularity in the mechanical parts is needed for our particular approach to the coupling. Also, for any Banach space $V, V^{*}$ denotes its topological dual and the bracket $\langle\cdot, \cdot\rangle_{V}$ indicates the corresponding dual pairing.

## 2.1 | Summary of the assumptions

We provide a short summary of the main regularity assumptions of our system. In the following, $\varepsilon>0$ is a small parameter and $d=2,3$ the spatial dimension, and $S=(0, T), T>0$, the time interval of interest.

- Geometry: $\Omega \subset \mathbb{R}^{d}$ bounded $C^{1,1}$-domain and $\varepsilon$-periodic domains $\Omega_{\varepsilon}^{f}, \Omega_{\varepsilon}^{g} \subset \Omega$ where $\Omega_{\varepsilon}^{f}$ is connected and $\Omega_{\varepsilon}^{g}$ is disconnected.
- Scalar coefficients: Biot modulus $c>0$, Biot-Willis parameter $\alpha>0$.
- Matrix coefficients: Positive definite, symmetric permeability matrix $K \in \mathbb{R}^{d \times d}$.
- Tensor coefficients: Elasticity tensors $\mathcal{C}, \mathcal{D} \in \mathbb{R}^{d \times d \times d \times d}$ are symmetric and positive on the space of the symmetric matrices and $\mathcal{A}_{\varepsilon}=\chi_{\Omega_{\varepsilon}^{f}} \mathcal{C}+\chi_{\Omega_{\varepsilon}^{g}} \mathcal{D}$
- Data: Right hand sides $f \in C^{1}\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{f}\right)\right)^{d}, g \in C^{1}\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)^{d}$, and $h \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$
- Solution regularity: $\left(U_{\varepsilon}, p_{\varepsilon}\right) \in L^{2}\left(S ; H_{0}^{1}(\Omega)^{d} \times H^{1}\left(\Omega_{\varepsilon}^{g}\right)\right)$ with $\partial_{t} p_{\varepsilon} \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$ and $\partial_{t} \nabla \cdot U_{\varepsilon} \in L^{2}(S \times \Omega)^{d}$


## 2.2 | Main results

In the following, we present our main results. Namely, the existence result and $\varepsilon$-controlled estimates for the $\varepsilon$-problem, see Theorem 2.4, and the final homogenization result, see Theorem 2.5. The detailed proofs of these results can be found in Sections 3 and 4, respectively.

Theorem 2.4. There is a unique solution $\left(U_{\varepsilon}, p_{\varepsilon}\right) \in L^{2}\left(S ; H_{0}^{1}(\Omega)^{d} \times H^{1}\left(\Omega_{\varepsilon}^{g}\right)\right)$ with $\partial_{t} p_{\varepsilon} \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$ and $\partial_{t} \nabla \cdot U_{\varepsilon} \in$ $L^{2}(S \times \Omega)^{d}$ satisfying equations (2.2a) and (2.2b) for all test functions $(v, \phi) \in H_{0}^{1}(\Omega)^{d} \times H^{1}\left(\Omega_{\varepsilon}^{g}\right)$ and for almost all $t \in S$. In addition,

$$
\sup _{\varepsilon>0}\left(\left\|p_{\varepsilon}\right\|_{L^{\infty}\left(S ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)}^{2}+\left\|U_{\varepsilon}\right\|_{L^{\infty}\left(S ; H_{0}^{1}(\Omega)^{d}\right)}^{2}+\varepsilon^{2}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)}^{2}\right)<\infty .
$$

Proof. This theorem is proved in Section 3. For the existence of a unique solution, see Lemma 3.5. The estimates are provided in Lemma 3.6.

Theorem 2.5 (Homogenization result). There are functions $U \in L^{2}\left(S ; H_{0}^{1}(\Omega)\right)^{d}$ and $p \in L^{2}\left(S \times \Omega ; H_{p e r}^{1}(Y)\right)$ satisfying $\partial_{t} p \in L^{2}\left(S \times \Omega ; H^{1}(Y)^{*}\right)$ such that $U_{\varepsilon} \stackrel{2}{\rightharpoonup} U$ in $L^{2}\left(S ; H_{0}^{1}(\Omega)\right)^{d}$ as well as $\widehat{p_{\varepsilon}} \rightharpoonup[p]_{y}$, at least up to a subsequence.

Together with the additional micro deformation fields $\tilde{u}_{f} \in L^{2}\left(S \times \Omega ; H_{\#}^{1}\left(Y^{f}\right)\right)^{d}$ as well as $\tilde{u}_{g} \in L^{2}\left(S \times \Omega ; H_{\#}^{1}\left(Y^{g}\right)\right)^{d}$ ( $\tilde{u}=\widehat{u_{f}}+\widehat{\hat{u}_{g}}$ is related to $U$ via (4.8) and is a result of the micro variations of the pore pressure), they solve the following homogenized system

$$
\begin{array}{rlrl}
-\nabla \cdot\left(\mathcal{A}^{h} e(U)+\mathcal{A}\left[e_{y}(\tilde{u})\right]_{Y}-\alpha[p]_{Y^{g}}\right) & =F & & \text { in } \Omega, \\
U & =0 & & \text { on } \partial \Omega, \\
-\nabla \cdot\left(\mathcal{A}^{h} e\left(U_{0}\right)\right) & =F(0) & & \text { in } \Omega, \\
U_{0} & =0 & & \text { on } \partial \Omega, \\
-\nabla_{y} \cdot\left(\mathcal{C} e_{y}\left(\tilde{u}_{f}\right)\right) & =0 & & \text { in } Y^{f}, \\
-\left(\mathcal{D} e_{y}\left(\tilde{u}_{g}\right)-\alpha p \mathbb{I}_{3}\right) n_{\Gamma} & \left.\left.=-C e_{y}\left(\tilde{u}_{f}\right) n_{\Gamma}\right)-\alpha p \mathbb{I}_{3}\right) & =0 & \\
\tilde{u}_{g} & =\tilde{u}_{f} & & \text { on } \Gamma, \\
y & \mapsto \tilde{u} & \text { on } \Gamma, \\
\partial_{t}\left(c p+\alpha^{h}: \nabla U+\alpha \nabla_{y} \cdot \tilde{u}\right)-\nabla_{y} \cdot\left(\mathcal{K} \nabla_{y} p\right) & =h & & Y-p e r i o d i c, \\
-K \nabla p \cdot n_{\Gamma} & =0 & & \text { in } S \times Y^{g} \\
{\left[c p+\alpha^{h}: \nabla U+\alpha \nabla_{y} \cdot \tilde{u}\right](0)} & =\alpha^{h}: \nabla U_{0} & & \text { on } S \times \Gamma, \\
& & \text { on } Y^{f} . \tag{2.3l}
\end{array}
$$

in the weak sense given by (4.10a)-(4.10c).
Here, $[\psi]_{A}$ denotes integration of a function $\psi$ over a domain $A$.

Proof. This theorem is proved in Section 4. For the convergence results, see Lemma 4.1. The homogenized system is then deduced in starting with (4.1). For the definitions of the effective coefficients like $\mathcal{A}^{h}$, see (4.7a)-(4.7d)

This homogenized model exhibits several interesting features. First and foremost, we have the additional micro deformations and stresses given via $\tilde{u}$ that arises via the micro variations of the pore pressure $p$ governed by (2.3e)-(2.3i).

First, the micro pore pressure problem given by Equations (2.3j)-(2.31) is almost the standard micro system for the chosen geometrical setup and $\varepsilon$-scaling (see, e.g., previous works ${ }^{7,9}$ for similar homogenization results) where the isotropy of the pressure deformation coupling is $\operatorname{lost}\left(\alpha \operatorname{div} U_{\varepsilon}\right.$ vs. $\left.\alpha^{h}: \nabla U\right)$. The main difference is the additional micro dissipation term given by $\alpha \operatorname{div}_{y} \tilde{u}$ : due to the micro variations of the pore pressure and the resulting micro deformations and stresses, a purely macroscopic term like $\alpha^{h}: \nabla U$ is not sufficient to capture the mechanical dissipation.
For the same reasons, the macroscopical momentum problem ((2.3a) and (2.3b)) includes an additional averaged micro stress contribution (namely, $\left.\mathcal{A}\left[e_{y}(\tilde{u})\right]_{Y}\right)$ accounting for the stresses due to the deformations at the micro scale. Those are governed by equations $(2.3 \mathrm{e})-(2.3 \mathrm{~g})$ and are solely a consequence of the micro variations of the pore pressures.

In addition, the initial condition (2.3l) of the homogenized pore pressure equation (2.3j) involves the function $U_{0}$ which is itself the solution of a elliptic problem, namely, (2.3c) and (2.3d). This initial condition basically corrects for the deformation due to the initial volume force density $F(0)$ without any pressure.

## 3 | ANALYSIS OF THE $\varepsilon$-PROBLEM

In this section, we tend to the analysis of the weak form given by (2.2a) and (2.2b). To this end, we will introduce an equivalent abstract linear operator formulation of the problem, see (3.3), and establish some important properties of the involved operators. We want to point out that the treatment of the coupling is strongly motivated by the approach outlined in Showalter and Momken. ${ }^{6}$ An alternative approach for tackling this kind of problem is the Galerkin method; we refer to Marciniak-Czochra and Mikelić ${ }^{25}$ where this is done in the context of the homoegenization of a poroelasticity problem.

We start with the momentum balance equation and observe that, for every $\psi \in H^{-1}(\Omega)^{d}$, there is a unique $U_{\varepsilon} \in H_{0}^{1}(\Omega)^{d}$ such that

$$
\int_{\Omega} \mathcal{A}_{\varepsilon} e\left(U_{\varepsilon}\right): e(v) \mathrm{d} x=\langle\psi, v\rangle_{H_{0}^{1}(\Omega)^{d}}
$$

for all $v \in H_{0}^{1}(\Omega)^{d}$. Since $\mathcal{A}$ is positive definite, this follows by the Lemma of Lax-Milgram by using Korns inequality. Also, the induced operator $E_{\varepsilon}: H_{0}^{1}(\Omega)^{d} \rightarrow H^{-1}(\Omega)$ given via

$$
\left\langle E_{\varepsilon} u, v\right\rangle_{H_{0}^{1}(\Omega)}=\int_{\Omega} \mathcal{A}_{\varepsilon} e(u): e(v) \mathrm{d} x
$$

is a homeomorphism. This operator $E_{\varepsilon}$ is additionally bounded uniformly with respect to $\varepsilon$ due to $\mathcal{A}_{\varepsilon}=\chi_{\Omega_{\varepsilon}^{f}} \mathcal{C}+\chi_{\Omega_{\varepsilon}^{g}} \mathcal{D}$ and

$$
\begin{equation*}
\left|\left\langle E_{\varepsilon} u, v\right\rangle_{H_{0}^{1}(\Omega)}\right| \leq \max \{|\mathcal{C}|,|\mathcal{D}|\}\|\nabla u\|_{L^{2}(S \times \Omega)^{d \times d}}\|\nabla v\|_{L^{2}(S \times \Omega)^{d \times d}} . \tag{3.1}
\end{equation*}
$$

We introduce the $\varepsilon$-divergence operator

$$
\nabla_{\varepsilon} \cdot: H_{0}^{1}(\Omega)^{d} \rightarrow L^{2}\left(\Omega_{\varepsilon}^{g}\right) \operatorname{via}\left(\nabla_{\varepsilon} \cdot v, p\right)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}=\int_{\Omega_{\varepsilon}^{g}} \alpha p \nabla \cdot v \mathrm{~d} x
$$

as well as the dual $\varepsilon$-gradient operator $\nabla_{\varepsilon}{ }^{\dagger}{ }^{\dagger}$

$$
\nabla_{\varepsilon}:=-\left(\nabla_{\varepsilon} \cdot\right)^{*}: L^{2}\left(\Omega_{\varepsilon}^{g}\right) \rightarrow H^{-1}(\Omega)^{d} \operatorname{via}\left\langle\nabla_{\varepsilon} p, v\right\rangle_{H_{0}^{1}(\Omega)^{d}}=-\int_{\Omega_{\varepsilon}^{g}} \alpha p \nabla \cdot v \mathrm{~d} x
$$

[^1]With these operators in mind, we see that (2.2a) is equivalent to

$$
E_{\varepsilon} U_{\varepsilon}(t)+\nabla_{\varepsilon} p_{\varepsilon}(t)=\mathcal{F}_{\varepsilon}(t) \text { in } H^{-1}(\Omega)^{d}
$$

where

$$
\mathcal{F}_{\varepsilon}(t)= \begin{cases}f(t) & \text { in } \Omega_{\varepsilon}^{f} \\ g(t) & \text { in } \Omega_{\varepsilon}^{g}\end{cases}
$$

For any given $p_{\varepsilon} \in H^{1}\left(\Omega_{g}^{\varepsilon}\right)$, this leads to the solution

$$
\begin{equation*}
U_{\varepsilon}(t)=-E_{\varepsilon}^{-1} \nabla_{\varepsilon} p_{\varepsilon}+E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}(t) \text { in } H_{0}^{1}(\Omega) \tag{3.2}
\end{equation*}
$$

We go on introducing the linear operator

$$
\mathcal{K}_{\varepsilon}: H^{1}\left(\Omega_{\varepsilon}^{g}\right) \rightarrow H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*} \operatorname{via}\left\langle\mathcal{K}_{\varepsilon} p, \varphi\right\rangle_{H^{1}\left(\Omega_{\varepsilon}^{g}\right)}=\varepsilon^{2}(K \nabla p, \nabla \varphi)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)} .
$$

For (2.2b), this leads to

$$
\begin{equation*}
\partial_{t}\left(c p_{\varepsilon}-\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \nabla_{\varepsilon} p_{\varepsilon}\right)+\mathcal{K}_{\varepsilon} p_{\varepsilon}=\mathcal{H}_{\varepsilon} \text { in } H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*} \tag{3.3}
\end{equation*}
$$

where $\mathcal{H}_{\varepsilon}$ is given by

$$
\mathcal{H}_{\varepsilon}=h+\partial_{t} \nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}
$$

Remark 3.1. Please note that the operator $\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \nabla_{\varepsilon}$, although involving differential operators, is not itself a differential operator. Formally, both $\nabla_{\varepsilon}$ and $\nabla_{\varepsilon}$. are differential operators of order 1 and $E_{\varepsilon}^{-1}$, being the inverse of an elliptic operator, will lift the function for two derivatives. As a consequence, $\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \nabla_{\varepsilon} \operatorname{maps} L^{2}\left(\Omega_{\varepsilon}^{g}\right)$ into $L^{2}\left(\Omega_{\varepsilon}^{g}\right)$ as well as $H^{1}\left(\Omega_{\varepsilon}^{g}\right)$ to $H^{1}\left(\Omega_{\varepsilon}^{g}\right)$. Also, for $\mathcal{F}_{\varepsilon} \in C^{1}\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right), \partial_{t} \nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}$ is well defined as $\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1}$ is linear, bounded, and time independent.

We set

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}:=\operatorname{cId}-\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \nabla_{\varepsilon}: L^{2}\left(\Omega_{\varepsilon}^{g}\right) \rightarrow L^{2}\left(\Omega_{\varepsilon}^{g}\right) \tag{3.4}
\end{equation*}
$$

where Id denotes the identity operator.
Lemma 3.2. The operator $\mathcal{B}_{\varepsilon}$ is linear, continuous, positive-definite, and self-adjoint.
Proof. Linearity and continuity are clear since $\mathcal{B}_{\varepsilon}$ is composed of linear and continuous operators. For the positivity, we observe $\left(p \in L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)$

$$
\begin{aligned}
\left(\mathcal{B}_{\varepsilon} p, p\right)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)} & =c\|p\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}-\left(\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \nabla_{\varepsilon} p, p\right)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)} \\
& =c\|p\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}+\left\langle\nabla_{\varepsilon} p, E_{\varepsilon}^{-1} \nabla_{\varepsilon} p\right\rangle_{H_{0}^{1}(\Omega)^{d}} \\
& \geq c\|p\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}+c_{\mathcal{A}}^{-1}\left\|\nabla_{\varepsilon} p\right\|_{H^{-1}(\Omega)^{d}}^{2} .
\end{aligned}
$$

Here, we have used the duality between the $\varepsilon$-gradient and $\varepsilon$-divergence operators. The positivity of $E_{\varepsilon}^{-1}$ thus implies positivity of $\mathcal{B}_{\varepsilon}$. Similarly, as $E_{\varepsilon}^{-1}$ is self-adjoint, we find that

$$
\begin{aligned}
\left(\mathcal{B}_{\varepsilon} p, q\right)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)} & =c(p, q)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}-\left(\nabla_{\varepsilon} \cdot E_{\varepsilon}^{-1} \nabla_{\varepsilon} p, q\right)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)} \\
& =c(p, q)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}+\left\langle\nabla_{\varepsilon} q, E_{\varepsilon}^{-1} \nabla_{\varepsilon} p\right\rangle_{H_{0}^{1}(\Omega)^{d}} \\
& =\left(p, \mathcal{B}_{\varepsilon} q\right)_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}
\end{aligned}
$$

showing that $B_{\varepsilon}$ is indeed self-adjoint.

Remark 3.3. Lemma 3.2 implies that $\mathcal{B}_{\varepsilon}$ is invertible as it is linear, continuous, positive definite, and self-adjoint. As a consequence, $\lim _{n \rightarrow \infty} \mathcal{B}_{\varepsilon} p_{n}=q$ implies the convergence of $p_{n}$ via $p_{n} \rightarrow \mathcal{B}_{\varepsilon}{ }^{-1} q$. Moreover, $\mathcal{B}_{\varepsilon}^{-1}$ is also continuous, positive definite, and self-adjoint.

The following technical lemma allows us to argue the existence of the time derivative of solutions $p_{\varepsilon}$ based on the existence for $\mathcal{B}_{\varepsilon} p_{\varepsilon}$.

Lemma 3.4. Let $p \in L^{2}\left(S ; H^{1}\left(\Omega_{\varepsilon}^{g}\right)\right)$ such that $\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right) \in L^{2}\left(S ; H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*}\right)$. Then, $\partial_{t} p \in L^{2}\left(S ; H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*}\right)$. In the case of higher regularity $\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right) \in L^{2}(S \times \Omega)$, it even holds $\partial_{t} p=\mathcal{B}_{\varepsilon}^{-1} \partial_{t}(B p)$ as well as $\partial_{t}\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right)=\mathcal{B} \partial_{t} p_{\varepsilon}$.

Proof. By definition, we have

$$
\int_{S}\left\langle\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right), \phi\right\rangle_{H^{1}\left(\Omega_{\varepsilon}^{g}\right.} w(t) \mathrm{d} t=-\int_{S}\left(\mathcal{B}_{\varepsilon} p, \phi\right)_{\Omega_{\varepsilon}^{g_{e}}} \omega^{\prime}(t) \mathrm{d} t
$$

for all $\phi \in H^{1}\left(\Omega_{\varepsilon}^{g}\right)$ and $w \in C_{0}(\bar{S})$.
We start with the second statement for higher regularity, i.e., $\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right) \in L^{2}(S \times \Omega)$. In this case, we have (note that both $\mathcal{B}_{\varepsilon}$ and $\mathcal{B}_{\varepsilon}^{-1}$ are self-adjoint)

$$
\int_{S}\left(\mathcal{B}_{\varepsilon}^{-1} \partial_{t}\left(\mathcal{B}_{\varepsilon} p\right), \phi\right)_{\Omega_{\varepsilon}^{g}} w(t) \mathrm{d} t=-\int_{S}\left(\mathcal{B}_{\varepsilon} p, \mathcal{B}_{\varepsilon}^{-1} \phi\right)_{\Omega_{\varepsilon}^{s}} w^{\prime}(t) \mathrm{d} t=-\int_{S}(p, \phi)_{\Omega_{\varepsilon}^{g}} W^{\prime}(t) \mathrm{d} t
$$

which implies $\partial_{t} p_{\varepsilon}=\mathcal{B}_{\varepsilon}^{-1} \partial_{t}\left(\mathcal{B}_{\varepsilon} p\right)$ concluding the second part.
In the case of lower regularity, i.e., $\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right) \in L^{2}\left(S ; H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*}\right)$ it still holds that

$$
\begin{align*}
-\int_{S}(p, \phi)_{\Omega_{\varepsilon}^{g}} w^{\prime}(t) \mathrm{d} t & =-\int_{S}\left(\mathcal{B}_{\varepsilon} p, \mathcal{B}_{\varepsilon}^{-1} \phi\right)_{\Omega_{\varepsilon}^{g}} w^{\prime}(t) \mathrm{d} t \\
& =\int_{S}\left\langle\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right), \mathcal{B}_{\varepsilon}^{-1} \phi\right\rangle_{H^{1}\left(\Omega_{\varepsilon}^{g}\right)} w(t) \mathrm{d} t \tag{3.5}
\end{align*}
$$

for all $\phi \in L^{2}\left(\Omega_{\varepsilon}^{g}\right)$ and $w \in C_{0}(\bar{S})$ but we are not able to use the self-adjointedness of $\mathcal{B}_{\varepsilon}^{-1}$ to get an explicit form for the time derivative of $p$. However, the function $q: H^{1}\left(\Omega_{\varepsilon}^{g}\right) \rightarrow \mathbb{R}$ via

$$
q(\phi)=\left\langle\partial_{t}\left(\mathcal{B}_{\varepsilon} p\right), \mathcal{B}_{\varepsilon}^{-1} \phi\right\rangle_{H^{1}\left(\Omega_{\varepsilon}^{g}\right)}
$$

is both linear and continuous, that is, $q \in H^{1}\left(\Omega_{\varepsilon}^{g}\right)$, which implies $q=\partial_{t} p$ due to (3.5).
Lemma 3.5 (Existence of a unique solution). Let $\mathcal{F}_{\varepsilon} \in C^{1}\left(\bar{S} ; L^{2}(\Omega)\right)$ and $h \in L^{2}(S \times \Omega)$.. There is a unique $\left(U_{\varepsilon}, p_{\varepsilon}\right) \in$ $L^{2}\left(S ; H_{0}^{1}(\Omega)^{d} \times H^{1}\left(\Omega_{\varepsilon}^{g}\right)\right)$ satisfying $p_{\varepsilon}(0)=0$ and $\partial_{t} p_{\varepsilon} \in L^{2}(S \times \Omega)$ solving the operator problem

$$
\begin{gather*}
E_{\varepsilon} U_{\varepsilon}+\nabla_{\varepsilon} p_{\varepsilon}=\mathcal{F}_{\varepsilon} \text { in } H^{-1}(\Omega),  \tag{3.6}\\
\partial_{t}\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right)+\mathcal{K}_{\varepsilon} p_{\varepsilon}=\mathcal{H}_{\varepsilon} \text { in } H^{1}\left(\Omega_{\varepsilon}^{g}\right)^{*} . \tag{3.7}
\end{gather*}
$$

Moreover, we have $\partial_{t} U_{\varepsilon} \in L^{2}\left(S ; H_{0}^{1}(\Omega)^{d}\right)$.
Proof. Since the operator $\mathcal{B}_{\varepsilon}$ is linear, continuous, positive, and self-adjoint and $\mathcal{K}_{\varepsilon}$ is positive definite, there is a unique solution $p_{\varepsilon} \in L^{2}\left(S ; H^{1}\left(\Omega_{\varepsilon}^{g}\right)\right)$ of (3.7) such that $\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right)(0)=0$ and $\partial_{t}\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right) \in L^{2}\left(S ; H^{1}(\Omega)^{*}\right)$, see Showalter $^{26}$, Chapter III.3, Proposition 3.2 and Theorem 5.1. In addition, via Equation (3.2), we get the corresponding $U_{\varepsilon} \in L^{2}\left(S ; H_{0}^{1}(\Omega)^{d}\right)$ satisfying (3.6).

At this point, it is not clear that time derivatives of $p_{\varepsilon}$ or $U_{\varepsilon}$ exist (we only have $\partial_{t}\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right) \in L^{2}\left(S ; H^{1}(\Omega)^{*}\right)$ ). However, from $B_{\varepsilon} p_{\varepsilon} \in C\left(\bar{S} ; L^{2}(\Omega)\right)$ and $\lim _{t \rightarrow 0} B_{\varepsilon} p_{\varepsilon}(t)=0$, we know that $p_{\varepsilon}(t) \rightarrow 0$, see Remark 3.3. With the same argument, we also find that $p_{\varepsilon} \in C\left(\bar{S} ; L^{2}(\Omega)\right)$.

As a next step, we argue that $\partial_{t}\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right) \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$ as this would ensure $\partial_{t} p_{\varepsilon} \in L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)$ via Lemma 3.4. For that, we note that, by assumption, $\mathcal{H}_{\varepsilon} \in L^{2}(S \times \Omega)$ and that the elliptic operator $\mathcal{K}_{\varepsilon}$ is basically a scaled Laplace operator and therefore regularizing. As a consequence, it follows that $\partial_{t}\left(\mathcal{B}_{\varepsilon} p_{\varepsilon}\right) \in L^{2}(S \times \Omega)$ via parabolic regularity. ${ }^{*}$ Alternatively, this can also be shown by utilizing a priori estimations with difference quotients in time, we refer to


Next, we define the function $W_{\varepsilon}(t) \in H_{0}^{1}(\Omega)^{d}$ as

$$
W_{\varepsilon}(t)=-E_{\varepsilon}^{-1} \nabla_{\varepsilon} \partial_{t} p_{\varepsilon}+E_{\varepsilon}^{-1} \partial_{t} \mathcal{F}_{\varepsilon},
$$

which is our candidate function for the time derivative of $U_{\varepsilon}$. This implies

$$
W_{\varepsilon}(t)=\partial_{t}\left(-E_{\varepsilon}^{-1} \nabla_{\varepsilon} p_{\varepsilon}+E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}\right)
$$

as all involved operators are independent of time, linear, and bounded:

$$
W_{\varepsilon}(t)=-E_{\varepsilon}^{-1} \nabla_{\varepsilon}\left(\lim _{h \rightarrow 0} \frac{\left.p_{\varepsilon}(t+h)-p_{\varepsilon}(t)\right)}{h}\right)+E_{\varepsilon}^{-1}\left(\lim _{h \rightarrow 0} \frac{\left.\mathcal{F}_{\varepsilon}(t+h)-\mathcal{F}_{\varepsilon}(t)\right)}{h}\right)
$$

Since $U_{\varepsilon}=-E_{\varepsilon}^{-1} \nabla_{\varepsilon} p_{\varepsilon}+E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}$, we have $\partial_{t} U_{\varepsilon}=W_{\varepsilon} \in L^{2}\left(S ; H_{0}^{1}(\Omega)^{d}\right)$.
With this result regarding the existence of a unique solution, we now turn to $\varepsilon$-controlled energy estimates. Those are extremely important in the homogenization context as they will be used to facilitate the limit analysis $\varepsilon \rightarrow 0$.

Lemma 3.6 (Estimates). The solutions $\left(U_{\varepsilon}, p_{\varepsilon}\right)$ satisfy

$$
\begin{equation*}
\sup _{\varepsilon>0}\left(\left\|p_{\varepsilon}\right\|_{L^{\infty}\left(S ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)}^{2}+\left\|U_{\varepsilon}\right\|_{L^{\infty}\left(S ; H_{0}^{1}(\Omega)\right)}^{2}+\varepsilon^{2}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)}^{2}\right)<\infty . \tag{3.8}
\end{equation*}
$$

Proof. We test the weak formulation with $\left(\partial_{t} U_{\varepsilon}, p_{\varepsilon}\right)^{\S}$ and get

$$
\begin{aligned}
& \int_{\Omega}\left(\chi_{\Omega_{\varepsilon}^{J}} \mathcal{C}+\chi_{\Omega_{\varepsilon}^{g_{e}^{g}}} \mathcal{D}\right) e\left(U_{\varepsilon}\right): e\left(\partial_{t} U_{\varepsilon}\right) \mathrm{d} x-\int_{\Omega_{\varepsilon}^{g}} \alpha p_{\varepsilon} \nabla \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \\
= & \int_{\Omega_{\varepsilon}^{f}} f \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x+\int_{\Omega_{\varepsilon}^{g}} g \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x, \int_{\Omega_{\varepsilon}^{g}} \partial_{t}\left(c p_{\varepsilon}+\alpha \nabla \cdot u_{\varepsilon}\right) p_{\varepsilon} \mathrm{d} x+\int_{\Omega_{\varepsilon}^{\varepsilon}} \varepsilon^{2} K \nabla p_{\varepsilon} \cdot \nabla p_{\varepsilon} \mathrm{d} x \\
= & \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x .
\end{aligned}
$$

Since $K$ is positive definite (i.e., there is $c_{K}>0$ such that $K x \cdot x \geq c_{K}|x|^{2}$ for all $x \in \mathbb{R}^{d}$ ), we infer that

$$
\int_{\Omega_{\varepsilon}^{g}} \partial_{t}\left(c p_{\varepsilon}+\alpha \nabla \cdot u_{\varepsilon}\right) p_{\varepsilon} \mathrm{d} x+\varepsilon^{2} c_{K}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \leq \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x .
$$

Integrating both equations with respect to time for some $t>0$, we get

$$
\begin{gathered}
\int_{0}^{t} \int_{\Omega}\left(\chi_{\Omega_{\varepsilon}^{f}} \mathcal{C}+\chi_{\Omega_{\varepsilon}^{g}} \mathcal{D}\right) e\left(U_{\varepsilon}\right): e\left(\partial_{t} U_{\varepsilon}\right) \mathrm{d} x \mathrm{~d} \tau-\int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} \alpha p_{\varepsilon} \nabla \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau=\int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} f \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau+\int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} g \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau \\
\int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} \partial_{t}\left(c p_{\varepsilon}+\alpha \nabla \cdot u_{\varepsilon}\right) p_{\varepsilon} \mathrm{d} x+\int_{0}^{t} \varepsilon^{2} c_{K}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \mathrm{~d} \tau=\int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau
\end{gathered}
$$

[^2]Adding both equations leads to

$$
\begin{aligned}
& c \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} \partial_{t} p_{\varepsilon} p_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau+\int_{0}^{t} \int_{\Omega}\left(\chi_{\Omega_{\varepsilon}^{f}} \mathcal{C}+\chi_{\Omega_{\varepsilon}^{g}} \mathcal{D}\right) e\left(U_{\varepsilon}\right): e\left(\partial_{t} U_{\varepsilon}\right) \mathrm{d} x \mathrm{~d} \tau+\varepsilon^{2} c_{K} \int_{0}^{t}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \mathrm{~d} \tau \\
\leq & \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau+\int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} f \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau+\int_{0}^{t} \int_{\Omega_{\varepsilon}^{\mathrm{g}}} g \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau
\end{aligned}
$$

Further estimating the terms with time derivative gives us (here, $c_{\mathcal{A}}$ denotes the minimum of the positivity constants of $\mathcal{C}$ and $\mathcal{D}$ )

$$
\begin{aligned}
& c\left\|p_{\varepsilon}(t)\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}+c_{\mathcal{A}}\left\|e\left(U_{\varepsilon}\right)(t)\right\|_{L^{2}(\Omega)}^{2}+2 \varepsilon^{2} c_{K} \int_{0}^{t}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \mathrm{~d} \tau \\
\leq & c\left\|p_{\varepsilon}(0)\right\|_{L^{2}\left(\Omega_{\varepsilon}^{f}\right)}^{2}+c_{\mathcal{A}}\left\|e\left(U_{\varepsilon}\right)(0)\right\|_{L^{2}(\Omega)}^{2}+2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau \\
& +2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} f \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau+2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} g \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau .
\end{aligned}
$$

Integration by parts with respect to time gives

$$
\int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} f \cdot \partial_{t} U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau=-\int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} \partial_{t} f \cdot U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau+\left[\int_{\Omega_{\varepsilon}^{f}} f \cdot U_{\varepsilon} \mathrm{d} x\right]_{0}^{t}
$$

As a consequence, we are led to (using also $p_{\varepsilon}(0)=0$ and $U_{\varepsilon}(0)=-E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}(0)$ )

$$
\begin{aligned}
& c\left\|p_{\varepsilon}(t)\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}+c_{\mathcal{A}}\left\|e\left(U_{\varepsilon}\right)(t)\right\|_{L^{2}(\Omega)}^{2}+2 \varepsilon^{2} c_{K} \int_{0}^{t}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \mathrm{~d} \tau \\
& \leq c_{\mathcal{A}} C_{E}\left(\|f(0)\|_{L^{2}\left(\Omega_{\varepsilon}^{f}\right)}^{2}+\|g(0)\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}\right)+2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau \\
& \quad-2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} \partial_{t} f \cdot U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau-2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} \partial_{t} g \cdot U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau \\
& \quad+\left[\int_{\Omega_{\varepsilon}^{f}} f \cdot U_{\varepsilon} \mathrm{d} x+\int_{\Omega_{\varepsilon}^{g}} g \cdot U_{\varepsilon} \mathrm{d} x\right]_{0}^{t}
\end{aligned}
$$

The continuity constant of the operator $E_{\varepsilon}$ can be estimated uniformly with respect to $\varepsilon$ with some constant $C_{E}>0$, see (3.1). With $c_{K o}>0$ denoting the Korn's inequality constant, we then have

$$
\begin{aligned}
c\left\|p_{\varepsilon}(t)\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} & +c_{K o} c_{\mathcal{A}}\left\|U_{\varepsilon}(t)\right\|_{H_{0}^{1}(\Omega)}^{2}+2 \varepsilon^{2} c_{K} \int_{0}^{t}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \mathrm{~d} \tau \\
\leq & c_{\mathcal{A}} C_{E}\left(\|f(0)\|_{L^{2}\left(\Omega_{\varepsilon}^{f}\right)}^{2}+\|g(0)\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}\right)+2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} h p_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau \\
& -2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{f}} \partial_{t} f \cdot U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau-2 \int_{0}^{t} \int_{\Omega_{\varepsilon}^{g}} \partial_{t} g \cdot U_{\varepsilon} \mathrm{d} x \mathrm{~d} \tau \\
& +\left[\int_{\Omega_{\varepsilon}^{f}} f \cdot U_{\varepsilon} \mathrm{d} x+\int_{\Omega_{\varepsilon}^{g}} g \cdot U_{\varepsilon} \mathrm{d} x\right]_{0}^{t}
\end{aligned}
$$

Then, applying Young's inequality and setting $\tilde{c}=c_{K o} c_{\mathcal{A}}$, we are led to

$$
\begin{aligned}
c\left\|p_{\varepsilon}(t)\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} & +\frac{\tilde{c}}{2}\left\|U_{\varepsilon}(t)\right\|_{H_{0}^{1}(\Omega)}^{2}+2 \varepsilon^{2} c_{K} \int_{0}^{t}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2} \mathrm{~d} \tau \\
\leq & \left(c_{\mathcal{A}} C_{E}+\frac{C_{E}}{2}+\frac{1}{2}\right)\left(\|f(0)\|_{L^{2}\left(\Omega_{\varepsilon}^{f}\right)}^{2}+\|g(0)\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}\right) \\
& +2 \int_{0}^{t}\left(\|h\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}+\left\|\partial_{t} f\right\|_{L^{2}\left(\Omega_{\varepsilon}^{f}\right)}^{2}+\left\|\partial_{t} g\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}\right) \mathrm{d} \tau \\
& +2 \int_{0}^{t}\left(\left\|U_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2}+\left\|p_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{g}\right)}^{2}\right) \mathrm{d} \tau+\frac{1}{2 \tilde{c}}\left(\|f\|_{C\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{f}\right)\right)}^{2}+\|g\|_{C\left(\overline{\mathrm{~S}} ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)}^{2}\right)
\end{aligned}
$$

Finally, using Gronwall's inequality, we conclude that there is $C>0$, which is independent of the choice of $\varepsilon$, such that

$$
\left\|p_{\varepsilon}\right\|_{L^{\infty}\left(S ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right)}^{2}+\left\|U_{\varepsilon}(t)\right\|_{L^{\infty}\left(S ; H_{0}^{1}(\Omega)\right)}^{2}+\varepsilon^{2}\left\|\nabla p_{\varepsilon}\right\|_{L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)}^{2} \leq C\left(\|f\|_{C^{1}\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{f}\right)\right.}^{2}+\|g\|_{C^{1}\left(\left(\bar{S} ; L^{2}\left(\Omega_{\varepsilon}^{g}\right)\right.\right.}^{2}+\|h\|_{L^{2}\left(S \times \Omega_{\varepsilon}^{g}\right)}^{2}\right) .
$$

## 4 | HOMOGENIZATION

In this section, we are considering the limit process $\varepsilon \rightarrow 0$ in the context of the two-scale convergence technique. For the convenience of the reader, we shortly recall the definition and present the main results used here in the appendix.
We notice that the two-scale convergence is defined for the fixed domain and as the solution $u_{\varepsilon}, w_{\varepsilon}$ and $p_{\varepsilon}$ are defined on the domains $\Omega_{\varepsilon}^{g}, \Omega_{\varepsilon}^{f}$ and $\Omega_{\varepsilon}^{g}$, respectively, in order to apply the definition and the results of two-scale convergence, we need to define the solution on the whole domain $\Omega$. Generally speaking, in a nonlinear setting, this would require the use of so-called extension operators (see, e.g., previous studies ${ }^{28-30}$ ), since we are working with a linear problem simply extending by zero is sufficient.

In the following, for every function $\psi$ defined on either $\Omega_{\varepsilon}^{f}$ or $\Omega_{\varepsilon}^{g}, \hat{\psi}$ will denote the zero extension the the whole of $\Omega$. With that, we can discuss the two scale limit of $u_{\varepsilon}, w_{\varepsilon}, p_{\varepsilon}$ and their derivatives. This is addressed in the following lemma:

Lemma 4.1. There exist functions $U \in L^{2}\left(S ; H^{1}(\Omega)\right)^{d}, p \in L^{2}\left(S \times \Omega ; H_{\#}^{1}(Y)\right)^{d}$, and a unique $U^{1} \in L^{2}\left(S \times \Omega ; H_{\#}^{1}(Y)\right)^{d}$ such that

$$
\text { i) } U_{\varepsilon} \stackrel{2}{\rightharpoonup} U \text {, ii) } e\left(U_{\varepsilon}\right) \stackrel{2}{\rightharpoonup} e(U)+e_{y}\left(U^{1}\right), \text { iii) } \widehat{p_{\varepsilon}} \stackrel{2}{\rightharpoonup} p \text {, iv) } \varepsilon \nabla \widehat{p_{\varepsilon}} \stackrel{2}{\rightharpoonup} \nabla_{y} p .
$$

Proof. The convergences (i), (iii), and (iv) follow from the a priori estimates given by Lemma 3.6 and Lemmas 5.4,5.8, and 5.9. Moreover, there is a function $U^{1} \in L^{2}\left(S \times \Omega ; H^{1}(Y)\right)^{d}$ such that

$$
\nabla U_{\varepsilon} \stackrel{2}{\rightharpoonup} \nabla U+\nabla_{y} U^{1}
$$

The rest of the proof of (ii) follows from the two-scale convergence of $\nabla U_{\varepsilon}$.
Our goal is to pass the two-scale limit in each equation of the model (2.1a)-(2.1h) using the limits given in Lemma 4.1. We will first pass the two-scale limit in the momentum equation. To that end, let $v_{0} \in C_{0}^{\infty}(\Omega)^{d}$ and $v_{1} \in C_{0}^{\infty}\left(\Omega ; C_{\#}^{\infty}(Y)\right)^{d}$ such that we choose the test function as $v_{\varepsilon}(x)=v_{0}(x)+\varepsilon v_{1}\left(x, \frac{x}{\varepsilon}\right)$ in (2.2a), that is,

$$
\int_{\Omega} \mathcal{A} e\left(U_{\varepsilon}(x)\right): e\left(v_{\varepsilon}\right) \mathrm{d} x-\int_{\Omega} \alpha \widehat{p_{\varepsilon}} \nabla \cdot v_{\varepsilon} \mathrm{d} x=\int_{\Omega}(\hat{f}+\hat{g}) \cdot v_{\varepsilon} \mathrm{d} x .
$$

For the sake of notations, from here and on, we omit the arguments $x$ and $\left(x, \frac{x}{\varepsilon}\right)$ below as it is obvious in the calculation. Now, due to $\nabla v_{\varepsilon}=\nabla v_{0}+\varepsilon \nabla_{x} v_{1}+\nabla_{y} \nu_{1}$ (here, $x$ and $y$ denote the differentiation with respect to $x \in \Omega$ and $y \in Y$,
respectively), this leads to

$$
\begin{equation*}
\int_{\Omega} \mathcal{A}_{\varepsilon} e\left(U_{\varepsilon}\right):\left(e\left(v_{0}\right)+\varepsilon e_{x}\left(v_{1}\right)+e_{y}\left(v_{1}\right)\right) \mathrm{d} x-\int_{\Omega} \alpha \hat{p_{\varepsilon}}\left(\nabla \cdot v_{0}+\varepsilon \nabla_{x} \cdot v_{1}+\nabla_{y} \cdot v_{1}\right) \mathrm{d} x=\int_{\Omega}(\hat{f}+\hat{g}) \cdot\left(v_{0}+\varepsilon v_{1}\right) \mathrm{d} x \tag{4.1}
\end{equation*}
$$

As the $L^{2}$ norms of $e\left(U_{\varepsilon}\right), \widehat{p_{\varepsilon}}, \hat{f}$, and $\hat{g}$ are bounded with respect to $\varepsilon$, we find that the integrals

$$
\varepsilon \int_{\Omega} \mathcal{A}_{\varepsilon} e\left(U_{\varepsilon}\right): e_{x}\left(v_{1}\right) \mathrm{d} x, \varepsilon \int_{\Omega} \alpha \widehat{p_{\varepsilon}} \nabla_{x} \cdot v_{1} \mathrm{~d} x, \varepsilon \int_{\Omega}(\hat{f}+\hat{g}) \cdot v_{1} \mathrm{~d} x
$$

converge to 0 for $\varepsilon \rightarrow 0$.
Passing to the two-scale limits as $\varepsilon \rightarrow 0$ in (4.1), we are therefore led to

$$
\begin{equation*}
\int_{\Omega \times Y} \mathcal{A}\left(e(U)+e_{y}\left(U^{1}\right)\right):\left(e\left(v_{0}\right)+e_{y}\left(v_{1}\right)\right) \mathrm{d}(x, y)-\int_{\Omega \times Y} \chi_{g} \alpha p\left(\nabla \cdot v_{0}+\nabla_{y} \cdot v_{1}\right) \mathrm{d}(x, y)=\int_{\Omega \times Y}\left(\chi_{f} f+\chi_{g} g\right) \cdot v_{0} \mathrm{~d}(x, y) \tag{4.2}
\end{equation*}
$$

Here, $\chi_{f}$ and $\chi_{g}$ denote the characteristic function of $Y_{f}$ and $Y_{g}$, respectively, and $\mathcal{A}=\chi_{f} \mathcal{C}+\chi_{g} \mathcal{D}$. By density arguments, (4.2) also holds true for all $\left(v_{0}, v_{1}\right) \in W_{0}^{1,2}(\Omega)^{d} \times L^{2}\left(\Omega ; H_{\#}^{1}(Y)\right)^{d}$.

Now, when choosing $v_{0}=0$, we arrive at the problem

$$
\int_{\Omega \times Y} \mathcal{A}\left(e(U)+e_{y}\left(U^{1}\right)\right) e_{y}\left(v_{1}\right) \mathrm{d}(x, y)=\int_{\Omega \times Y} \chi_{g} \alpha p \nabla_{y} \cdot v_{1} \mathrm{~d}(x, y)
$$

for all $v_{1} \in L^{2}\left(\Omega ; H_{\#}^{1}(Y)\right)^{d}$, which can be localized to

$$
\int_{Y} \mathcal{A}\left(e(U)+e_{y}\left(U^{1}\right)\right) e_{y}\left(v_{1}\right) \mathrm{d} y=\int_{Y} \chi_{g} \alpha p \nabla_{y} \cdot v_{1} \mathrm{~d} y\left(v_{1} \in H_{\#}^{1}(Y)^{d}, \text { a.e. in } \Omega\right) .
$$

Similarly, choosing $v_{1}=0$, we get

$$
\begin{equation*}
\int_{\Omega \times Y} \mathcal{A}\left(e(U)+e_{y}\left(U^{1}\right)\right): e\left(v_{0}\right) \mathrm{d}(x, y)-\int_{\Omega \times Y} \chi_{g} \alpha p \nabla \cdot v_{0} \mathrm{~d}(x, y)=\int_{\Omega \times Y}\left(\chi_{f} f+\chi_{g} g\right) \cdot v_{0} \mathrm{~d}(x, y) \tag{4.3}
\end{equation*}
$$

for all $v_{0} \in H_{0}^{1}(\Omega)^{d}$.

Next, we will pass the two-scale limit in (2.2b). Let $\phi \in H^{1,2}\left(S ; L^{2}\left(\Omega ; C_{\#}(Y)\right)\right)$ and $\phi_{\varepsilon}(t, x):=\phi\left(t, x, \frac{x}{\varepsilon}\right)$ such that $\phi_{\varepsilon}(T, x)=0$. Then,

$$
\begin{align*}
& -\int_{S} \int_{\Omega} \chi_{\Omega_{\varepsilon}^{g}}\left(c \widehat{p_{\varepsilon}}+\alpha \nabla \cdot U_{\varepsilon}\right) \partial_{t} \phi_{\varepsilon} \mathrm{d} x \mathrm{~d} t+\int_{\Omega} \chi_{\Omega_{\varepsilon}^{g}} \alpha \nabla \cdot U_{\varepsilon}(0) \phi_{\varepsilon}(0) \mathrm{d} x+\varepsilon^{2} \int_{S} \int_{\Omega} \chi_{\Omega_{\varepsilon}^{g}} \mathcal{\mathcal { L }} \widehat{p_{\varepsilon}} \cdot\left(\nabla \phi_{\varepsilon}+\varepsilon^{-1} \nabla_{y} \phi_{\varepsilon}\right) \mathrm{d} x \mathrm{~d} t \\
= & \int_{S} \int_{\Omega} \chi_{\Omega_{\varepsilon}^{g}} \hat{h} \phi_{\varepsilon} \mathrm{d} x \mathrm{~d} t \tag{4.4}
\end{align*}
$$

where we have used that $p_{\varepsilon}(0)=0$. Now, taking a closer look at the initial condition term, we use (3.2) to see that $U_{\varepsilon}(0)=E_{\varepsilon}^{-1} \mathcal{F}_{\varepsilon}(0)$ or, more concretely, $U_{\varepsilon, 0}:=U_{\varepsilon}(0) \in H_{0}^{1}(\Omega)^{d}$ is the unique solution to the elliptic problem given via

$$
\int_{\Omega} \mathcal{A}_{\varepsilon} e\left(U_{\varepsilon, 0}\right): e(v) \mathrm{d} x=\int_{\Omega}(\hat{f}(0)+\hat{g}(0)) \cdot v \mathrm{~d} x
$$

for all $v \in H_{0}^{1}(\Omega)^{d}$. Above problem of course admits standard elliptic a priori estimates, and, as a consequence, we can conclude the existence of two-scale limits $U_{0} \in H^{1}(\Omega)^{d}$ and $U_{0}^{1} \in L^{2}\left(\Omega ; H_{\#}^{1}(Y)^{d}\right)$ such that $U_{\varepsilon, 0} \stackrel{2}{\rightharpoonup} U_{0}$ as well as $e\left(U_{\varepsilon, 0}\right) \stackrel{2}{\rightharpoonup}$ $e\left(U_{0}\right)+e_{y}\left(U_{0}^{1}\right)$ and where $U_{0}$ and $U_{0}^{1}$ solve the variational problem

$$
\begin{aligned}
\int_{Y} \mathcal{A}\left(e\left(U_{0}\right)+e_{y}\left(U_{0}^{1}\right)\right): e_{y}\left(v_{1}\right) \mathrm{d} y=0, \\
\int_{\Omega \times Y} \mathcal{A}\left(e\left(U_{0}\right)+e_{y}\left(U_{0}^{1}\right)\right): e\left(v_{0}\right) \mathrm{d}(x, y)=\int_{\Omega \times Y}\left(\chi_{f} f(0)+\chi_{g} g(0)\right) \cdot v_{0} \mathrm{~d}(x, y)
\end{aligned}
$$

for all $\left(v_{0}, v_{1}\right) \in H_{0}^{1}(\Omega)^{d} \times H_{\#}^{1}(Y)^{d}$. Similarly,

$$
\int_{\Omega} \chi_{\Omega_{\varepsilon}^{g}} \alpha \nabla \cdot U_{\varepsilon}(0) \phi_{\varepsilon}(0) \mathrm{d} x \rightarrow \int_{\Omega} \int_{Y} \chi_{g} \alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right) \phi(0) \mathrm{d}(x, y) .
$$

We pass to the two-scale limit in (4.4) as $\varepsilon \rightarrow 0$ and get

$$
\begin{align*}
-\int_{S} \int_{\Omega \times Y} \chi_{g}\left(c p+\alpha\left(\nabla \cdot U+\nabla_{y} \cdot U_{1}\right)\right) \partial_{t} \phi \mathrm{~d}(x, y) \mathrm{d} t & +\int_{\Omega} \int_{Y} \chi_{g} \alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right) \phi(0) \mathrm{d}(x, y) \\
& \quad+\int_{S} \int_{\Omega \times Y} \chi_{g} \mathcal{K} \nabla_{y} p \cdot \nabla \phi \mathrm{~d}(x, y) \mathrm{d} t=\int_{S} \int_{\Omega \times Y} \chi_{g} h \phi \mathrm{~d}(x, y) \mathrm{d} t, \tag{4.5}
\end{align*}
$$

which is by density true for all $\phi \in L^{2}\left(S \times \Omega ; H_{\#}^{1}(Y)\right)$ such that $\partial_{t} \phi \in L^{2}\left(S \times \Omega ; H_{\#}^{1}(Y)^{*}\right)$. As all terms are restricted to $Y_{g}$, we can restrict to $\phi \in L^{2}\left(S \times \Omega ; H^{1}\left(Y^{g}\right)\right.$ ) (note that the periodicity property disappears since $Y_{g}$ does not touch $\Gamma$ ). Moreover, we can localize in $x \in \Omega$ and arrive at

$$
\begin{aligned}
& -\int_{S} \int_{Y \mathrm{Y}}\left(c p+\alpha\left(\nabla \cdot U+\nabla_{y} \cdot U^{1}\right)\right) \partial_{t} \phi \mathrm{~d} y \mathrm{~d} t \\
& +\int_{Y g} \alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right) \phi(0) \mathrm{d} y+\int_{S} \int_{Y g} \mathcal{K} \nabla_{y} p \cdot \nabla_{y} \phi \mathrm{~d} y \mathrm{~d} t \\
= & \int_{S} \int_{Y g} h \phi \mathrm{~d} y \mathrm{~d} t .
\end{aligned}
$$

which holds true for all $\phi \in L^{2}\left(S ; H_{\#}^{1}(Y)\right)$ with $\partial_{t} \phi \in L^{2}(S \times Y)$ and $\phi(T)=0$ almost everywhere in $\Omega$. From here, we have

$$
\int_{Y g}\left[c p+\alpha\left(\nabla \cdot U+\nabla_{y} \cdot U^{1}\right)\right](0) \phi(0)=\int_{Y g} \alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right) \phi(0) \mathrm{d} y
$$

and, as a consequence, $\left[c p+\alpha\left(\nabla \cdot U+\nabla_{y} \cdot U^{1}\right)\right](0)=\alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right)$ a.e. in $\Omega \times Y^{g}$. ${ }^{\mathbb{I}}$ Note that the regularity of the involved functions is not sufficient to say something about the initial state of $p, \nabla \cdot U$, and $\nabla \cdot U_{1}$ (see also Remark 3.4). Summarizing this limit process, we obtain the following system of variational equalities given via

$$
\begin{gather*}
\int_{Y} \mathcal{A}\left(e(U)+e_{y}\left(U^{1}\right)\right): e_{y}\left(v_{1}\right) \mathrm{d} y=\int_{Y g} \alpha p \nabla_{y} \cdot v_{1} \mathrm{~d} y,  \tag{4.6a}\\
\int_{\Omega \times Y} \mathcal{A}\left(e(U)+e_{y}\left(U^{1}\right)\right): e\left(v_{0}\right) \mathrm{d}(x, y)-\int_{\Omega \times Y \mathrm{~g}} \alpha p \nabla \cdot v_{0} \mathrm{~d}(x, y)=\int_{\Omega \times Y}\left(\chi_{f} f+\chi_{g} g\right) \cdot v_{0} \mathrm{~d}(x, y),  \tag{4.6b}\\
\int_{Y} \mathcal{A}\left(e\left(U_{0}\right)+e_{y}\left(U_{0}^{1}\right)\right): e_{y}\left(v_{1}\right) \mathrm{d} y=0,  \tag{4.6c}\\
\int_{\Omega \times Y} \mathcal{A}\left(e\left(U_{0}\right)+e_{y}\left(U_{0}^{1}\right)\right): e\left(v_{0}\right) \mathrm{d}(x, y)=\int_{\Omega \times Y}\left(\chi_{f} f(0)+\chi_{g} g(0)\right) \cdot v_{0} \mathrm{~d}(x, y),  \tag{4.6d}\\
-\int_{S} \int_{Y g}\left(c p+\alpha\left(\nabla \cdot U+\nabla_{y} \cdot U_{1}\right)\right) \partial_{t} \phi \mathrm{~d} y \mathrm{~d} t+\int_{Y g} \alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right) \phi(0) \mathrm{d} y  \tag{4.6e}\\
+\int_{S} \int_{Y g} \mathcal{K} \nabla_{y} p \cdot \nabla \phi \mathrm{~d} y \mathrm{~d} t=\int_{S} \int_{Y g} h \phi \mathrm{~d} y \mathrm{~d} t
\end{gather*}
$$

for all $\left(v_{0}, v_{1}, \phi\right) \in H_{0}^{1}(\Omega)^{d} \times H_{\#}^{1}(Y)^{d} \times L^{2}\left(S ; H^{1}\left(Y^{g}\right)\right)$ where $\partial_{t} \phi \in L^{2}(S \times Y)$ and $\phi(T)=0$.
We go on by introducing cell problems and effective quantities in order to get a more accessible form of the homogenization limit. For $j, k \in\{1,2,3\}$, almost every $t \in S$, and $x \in \Omega$ let $\tau_{j k} \in H_{\#}^{1}(Y)^{d}$ be solutions of

$$
\begin{equation*}
0=\int_{Y} \mathcal{A} e_{y}\left(\tau_{j k}+y_{j} e_{k}\right): e_{y}\left(v_{1}\right) \mathrm{d} y . \tag{4.7a}
\end{equation*}
$$

for all $\left.v_{1} \in H_{\#}^{1}(Y)\right)^{d}$. Here, via the linearity of $e_{k}$ denotes the $k$ th unit vector and $y_{j}$ the $j$ th coordinate of $y \in Y$.
Remark 4.2. Solutions of the variational problems of (4.7a) are unique up to constants. Utilizing first Korn inequality (cf. Oleinik et al. ${ }^{13}$ ), this can be shown via the Lemma of Lax-Milgram with respect to the Banach space of functions with zero average $\left\{u \in H^{1}(Y)^{d}: \int_{Y} u \mathrm{~d} y=0\right\}$.
We introduce the constant effective positive (for symmetric matrices) elasticity tensor $\mathcal{A}^{h} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$, the constant effective Biot-Willis parameter $\alpha^{h}: Y \rightarrow \in \mathbb{R}^{3 \times 3}$, as well as the averaged volume force densities $F: S \times \Omega \rightarrow \mathbb{R}^{d}$ via

$$
\begin{gather*}
\left(\mathcal{A}^{h}\right)_{i j k l}=\int_{Y} \mathcal{A} e_{y}\left(\tau_{i j}+y_{i} e_{j}\right): e_{y}\left(\tau_{k l}+y_{k} e_{l}\right) \mathrm{d} y,  \tag{4.7b}\\
\alpha_{j k}^{h}(y)=\alpha\left(1+\nabla_{y} \cdot \tau_{j k}(y)\right),  \tag{4.7c}\\
F(t, x)=\int_{Y f} f \mathrm{~d} y+\int_{Y \mathrm{~g}} g \mathrm{~d} y . \tag{4.7d}
\end{gather*}
$$

[^3]We introduce the function $\tilde{u}: S \times \Omega \times Y \rightarrow \mathbb{R}^{d}$ such that the linearity of (4.6a) allows us to decompose $U^{1}$ as

$$
\begin{equation*}
U^{1}(t, x, y)=\sum_{j, k=1}^{d} \tau_{j k}(y)(e(U)(t, x))_{j k}-\tilde{u}(t, x, y) . \tag{4.8}
\end{equation*}
$$

Similarly, via the linearity of (4.6d), we see that $U_{0}^{1}$ can be decomposed like

$$
U_{0}^{1}(t, x, y)=\sum_{j, k=1}^{d} \tau_{j k}(y)\left(e\left(U_{0}\right)(t, x)\right)_{j k}
$$

Remark 3.3. In the standard mechanical case, without the pressure, i.e. when $p_{\varepsilon}=0$, from (4.6a) we would have $\tilde{u}=0$ (as $U^{1}$ can then be represented as a linear combination of derivatives of $U$ and the solution of the cell problem (4.7a). When $p$ is constant over $Y$, then from (4.6a) and (4.8), we have $\tilde{u}=\tau p$, where $\tau \in H_{\#}^{1}(Y)$ is a solution of the cell problem

$$
0=\int_{Y} \mathcal{A} e_{y}(\tau): e_{y}\left(v_{1}\right) \mathrm{d} y-\int_{Y 8} \alpha \nabla \cdot v_{1} \mathrm{~d} y .
$$

Expressing $U^{1}$ in terms of $U$ and $\tilde{u}$ and inserting it into the variational equality (4.6a), we calculate using the cell problem (4.7a) that

$$
\begin{equation*}
\int_{Y} \mathcal{A} e_{y}(\tilde{u}): e_{y}\left(v_{1}\right) \mathrm{d} y=\int_{Y g} \alpha p \nabla_{y} \cdot v_{1} \mathrm{~d} y . \tag{4.9}
\end{equation*}
$$

In its localized form, this corresponds to the $\operatorname{PDE}\left(\tilde{u}=\widehat{\widehat{u}_{f}}+\widehat{\hat{u}_{g}}\right)$

$$
\begin{aligned}
-\nabla_{y} \cdot\left(\mathcal{C} e_{y}\left(\tilde{u}_{f}\right)\right) & =0 & & \text { in } Y^{f}, \\
-\nabla_{y} \cdot\left(\mathcal{D} e_{y}\left(\tilde{u}_{g}\right)-\alpha p \mathbb{I}_{3}\right) & =0 & & \text { in } Y^{g}, \\
-\left(\mathcal{D} e_{y}\left(\tilde{u}_{g}\right)-\alpha p \mathbb{I}_{3}\right) n_{\Gamma} & =-\mathcal{C} e_{y}\left(\tilde{u}_{f}\right) n_{\Gamma} & & \text { on } \Gamma, \\
\tilde{u}_{g} & =\tilde{u}_{f} & & \text { on } \Gamma, \\
y & \mapsto \tilde{u} & & Y \text {-periodic. }
\end{aligned}
$$

With the effective elasticity tensor $A^{h}$ and the effective Biot-Willis matrix $\alpha^{h}$, the system given by equations (4.6a), (4.6b), (4.6d), and (4.6e) for ( $U, U^{1}, U_{0}, U_{0}^{1}, p$ ) can equivalently be written as a problem for $\left(U, \tilde{u}, U_{0}, p\right)$ :

$$
\begin{array}{r}
\int_{Y} \mathcal{A} e_{y}(\tilde{u}): e_{y}\left(v_{1}\right) \mathrm{d} y=\int_{Y s} \alpha p \nabla_{y} \cdot v_{1} \mathrm{~d} y, \\
\int_{\Omega} \mathcal{A}^{h} e(U): e\left(v_{0}\right) \mathrm{d} x+\int_{\Omega} \int_{Y} \mathcal{A} e_{y}(\tilde{u}) \mathrm{d} y: e\left(v_{0}\right) \mathrm{d} x-\int_{\Omega} \alpha \int_{Y_{g}} p \mathrm{~d} y \nabla \cdot v_{0} \mathrm{~d} x=\int_{\Omega} F \cdot v_{0} \mathrm{~d} x, \\
\int_{\Omega} \mathcal{A}^{h} e\left(U_{0}\right): e\left(v_{0}\right) \mathrm{d}(x, y)=\int_{\Omega \times Y} F(0) \cdot v_{0} \mathrm{~d}(x, y), \\
-\int_{S} \int_{Y^{g}}\left(c p+\alpha^{h}: \nabla U+\alpha \nabla_{y} \cdot \tilde{u}\right) \partial_{t} \phi \mathrm{~d} y \mathrm{~d} t+\int_{Y g} \alpha^{h}: \nabla U_{0} \phi(0) \mathrm{d} y+\int_{S} \int_{Y s} \mathcal{K} \nabla_{y} p \cdot \nabla_{y} \phi \mathrm{~d} y \mathrm{~d} t=\int_{S} \int_{Y g} h \phi \mathrm{~d} y \mathrm{~d} t \tag{4.10d}
\end{array}
$$

for all $\left(v_{0}, v_{1}, \phi\right) \in H_{0}^{1}(\Omega)^{d} \times H_{\#}^{1}(Y)^{d} \times L^{2}\left(S ; H^{1}\left(Y^{g}\right)\right)$.

Remark 3.4. Again, with the regularity given via the limit process, we can only conclude that $\left[c p+\alpha^{h}: \nabla U+\alpha \nabla_{y} \cdot \tilde{u}\right](0)=\alpha^{h}: \nabla U_{0}$ in $L^{2}\left(S \times Y^{g}\right)$ for the initial condition. To get a stronger result like $p(0)=0$, we would need either stronger a priori estimates for the $\varepsilon$-problem (controlling the time derivative in $L^{2}$ w.r.t. $\varepsilon$ ) or some additional regularity analysis on the homogenization problem would have to be employed.

The system given by (4.10a), (4.10b), and (4.10d) corresponds to the following system of PDEs:

Effective, macroscopic mechanics

$$
\begin{align*}
-\nabla \cdot\left(\mathcal{A}^{h} e(U)+\mathcal{A}\left[e_{y}(\tilde{u})\right]_{Y}-\alpha[p]_{Y g}\right) & =F \text { in } \Omega,  \tag{4.11a}\\
U & =0 \text { on } \partial \Omega . \tag{4.11b}
\end{align*}
$$

Initial fluid storage

$$
\begin{align*}
-\nabla \cdot\left(\mathcal{A}^{h} e\left(U_{0}\right)\right) & =F(0) \text { in } \Omega,  \tag{4.12a}\\
U_{0} & =0 \text { on } \partial \Omega . \tag{4.12b}
\end{align*}
$$

Micro mechanical correction, $\tilde{u}=\widehat{\tilde{u}_{f}}+\widehat{\hat{u}}_{g}$

$$
\begin{align*}
-\nabla_{y} \cdot\left(\mathcal{C} e_{y}\left(\tilde{u}_{f}\right)\right) & =0 \text { in } Y^{f},  \tag{4.13a}\\
-\nabla_{y} \cdot\left(\mathcal{D} e_{y}\left(\tilde{u}_{g}\right)-\alpha p \mathbb{I}_{3}\right) & =0 \text { in } Y^{g},  \tag{4.13b}\\
-\left(\mathcal{D} e_{y}\left(\tilde{u}_{g}\right)-\alpha p \mathbb{I}_{3}\right) n_{\Gamma} & =-\mathcal{C} e_{y}\left(\tilde{u}_{f}\right) n_{\Gamma} \text { on } \Gamma,  \tag{4.13c}\\
\tilde{u}_{g} & =\tilde{u}_{f} \text { on } \Gamma,  \tag{4.13d}\\
y & \mapsto \tilde{u} Y \text {-periodic. } \tag{4.13e}
\end{align*}
$$

Micro pore pressure

$$
\begin{gather*}
\partial_{t}\left(c p+\alpha^{h}: \nabla U+\alpha \nabla_{y} \cdot \tilde{u}\right)-\nabla \cdot\left(\mathcal{K} \nabla_{y} p\right)=h \text { in } S \times Y^{g}  \tag{4.14a}\\
-K \nabla p \cdot n_{\Gamma}=0 \text { on } S \times \Gamma,  \tag{4.14b}\\
{\left[c p+\alpha^{h}: \nabla U+\alpha \nabla_{y} \cdot \tilde{u}\right](0)=\alpha^{h}: \nabla U_{0} \text { on } Y^{g} .} \tag{4.14c}
\end{gather*}
$$

Here, $[\psi]_{A}$ denotes integration of a function $\psi$ over a domain $A$.

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## CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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## APPENDIXA

## A. 1 | Existence theorem for implicit parbolic PDEs

For the convenience of the reader, we state the existence theorem for implicit parabolic PDEs used in this work as given in Showalter. ${ }^{26,}$ Chapter III.3, Proposition 3.2

In the following, let $V, W$ be separable Hilbert spaces with a continuous and dense embedding $V \hookrightarrow W$. We say that a family of linear, continuous operators $\mathcal{B}(t): W \rightarrow V^{*}(0 \leq t \leq T)$ is regular if, for each $u, v \in H^{1}((0, T)$; $V)$, the function $t \mapsto\langle\mathcal{B}(t) u(t), v(t)\rangle_{V}$ is absolutely continuous on $[0, T]$ and there is a constant $K \in L^{1}(0, T)$ satisfying

$$
\left|\frac{\mathrm{d}}{\mathrm{~d} t}\langle\mathcal{B}(t) u, v\rangle\right|_{W} \leq K(t)\|u\|_{W}\|v\|_{W} .
$$

Theorem 5.1 (Existence Theorem). Let $\mathcal{A}(t), \mathcal{B}(t): V \rightarrow V^{*}$ be linear, continuous operators for $0 \leq t \leq T, u_{0} \in W$, and $f \in L^{2}\left(S ; V^{*}\right)$. In addition, we assume that $\mathcal{B}(t)$ is a regular family of self-adjoint operators where $\mathcal{B}(0)$ is monotone and where there are numbers $c>0$ and $\lambda$ such that

$$
2\langle\mathcal{A}(t) v, v\rangle_{V}+\lambda\langle\mathcal{B} v, v\rangle_{V}+\left\langle\mathcal{B}^{\prime}(t) v, v\right\rangle_{V} \geq c\|v\|_{V}^{2}
$$

for all $v \in V$ and $0 \leq t \leq T$. Then, there exists a solution $u \in L^{2}(S ; V)$ such that $\partial_{t} u \in L^{2}\left(S ; V^{*}\right)$ satisfying

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}(\mathcal{B}(t) u(t))+\mathcal{A}(t) u(t) & =f(t) \text { in } V^{*} \\
B(0) u(0) & =u_{0} \in W
\end{aligned}
$$

Remark 5.2. Please note that in our case both families of operators $\mathcal{A}, \mathcal{B}$ are independent of time, making them both regular. Uniqueness is given when, additionally, the family $\mathcal{A}$ is also regular (see the remark after Showalter ${ }^{26, \text { Chapter III.3, Proposition 3.2). }}$

## A. 2 | Two-scale convergence

We have used the two-scale convergence method for the homogenisation. Since our problem here deals with homogenization with respect to space variable only but not the time, we have modified the definitions to allow for homogenization with a parameter which we denote by $t$. We will state some theorems on two-scale convergence. The proofs of all these theorems can be found in previous studies ${ }^{31-34}$ with a minor generalization with $t$. Let $1 \leq p, q,<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$.

Definition 5.3. Let $\varepsilon$ be a sequence of positive real numbers converging to 0 . A sequence of functions $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ in $L^{p}(S \times \Omega)$ is said to be two-scale convergent to a limit $u \in L^{p}(S \times \Omega \times Y)$ if

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{S \times \Omega} u_{\varepsilon}(t, x) \phi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d} x=\int_{S \times \Omega} \int_{Y} u(t, x, y) \phi(t, x, y) \mathrm{d} x \mathrm{~d} y, \tag{A1}
\end{equation*}
$$

for all $\phi \in L^{q}\left(S \times \Omega ; C_{\#}(Y)\right)$. ${ }^{\#}$
If $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ is two-scale convergent to $u$ then we write $u_{\varepsilon} \stackrel{2}{\rightharpoonup} u$. The above definition is followed from the following theorem:

Lemma 5.4. For every bounded sequence, $\left(u_{\varepsilon}\right)_{\varepsilon>0}$, in $L^{p}(S \times \Omega)$ there exist a subsequence $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ (still denoted by same symbol) and a $u \in L^{p}(S \times \Omega \times Y)$ such that $u_{\varepsilon} \stackrel{2}{\rightharpoonup} u$.

In Definition 5.3, the space of test functions is chosen as $L^{q}\left(S \times \Omega ; C_{\#}(Y)\right.$ ), but we can replace the space of test functions by $C_{0}^{\infty}\left(S \times \Omega ; C_{\#}^{\infty}(Y)\right.$ ), if $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ satisfies certain condition which is given in the following theorem:

Lemma 5.5. Let $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ be bounded in $L^{p}(S \times \Omega)$ such that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{S \times \Omega} u_{\varepsilon}(t, x) \phi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d} x=\int_{S \times \Omega} \int_{Y} u(t, x, y) \phi(t, x, y) \mathrm{d} x \mathrm{~d} y . \tag{A2}
\end{equation*}
$$

for all $\phi \in C_{0}^{\infty}\left(S \times \Omega ; C_{\#}^{\infty}(Y)\right)$. Then, $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ is two-scale convergent to $u$.
Lemma 5.6. Let $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ be strongly convergent to $u \in L^{p}(S \times \Omega)$, then $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ is two-scale convergent to $u_{1}(t, x, y)=u(t, x)$.
Lemma 5.7. Let $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ be two-scale convergent to $u$ in $L^{p}(S \times \Omega \times Y)$, then $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ is weakly convergent to $\int_{Y} u(x, y) \mathrm{d} y$ in $L^{p}(S \times \Omega)$ and $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ is bounded.

Lemma 5.8. Let $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ be a sequence in $L^{p}\left(S ; H^{1, p}(\Omega)\right)$ such that $u_{\varepsilon} \rightarrow u$ in $L^{p}\left(S ; H^{1, p}(\Omega)\right)$. Then, $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ two-scale converges to $u$ and there exists a unique function $u_{1} \in L^{p}\left(S \times \Omega ; H_{\#}^{1, p}(Y)\right)$ such that up to a subsequence, still denoted by same symbol, $\nabla_{x} u_{\varepsilon} \stackrel{2}{\rightharpoonup} \nabla u+\nabla_{y} u_{1}$.

Lemma 5.9. Let $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ and $\left(\varepsilon \nabla_{x} u_{\varepsilon}\right)_{\varepsilon>0}$ be bounded in $L^{p}(S \times \Omega)$ and $L^{p}\left(S ;\left[L^{p}(\Omega)\right]^{n}\right)$ respectively. Then, there exists $a$ $u \in L^{p}\left(\Omega ; H_{p e r}^{1, p}(Y)\right)$ such that, up to a subsequence, still denoted by $\varepsilon$, we have

$$
\begin{array}{r}
u_{\varepsilon} \stackrel{2}{\rightharpoonup} u \\
\text { and } \varepsilon \nabla_{x} u_{\varepsilon} \stackrel{2}{\rightharpoonup} \nabla_{y} u
\end{array}
$$

as $\varepsilon \rightarrow 0$.

[^4]
[^0]:    *A hydrogel is a network of hydrophilic polymer chains; think edible jelly for an every day life example.
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[^1]:    ${ }^{\dagger}$ Here, we have identified the Hilbert space $L^{2}\left(\Omega_{\varepsilon}^{g}\right)$ with its dual. Moreover, * denotes the dual operator.

[^2]:    ${ }^{\ddagger}$ For a similar situation, see Showalter and Momken. ${ }^{6}$, Theorem 2
    ${ }^{\S}$ This is justified with the regularity provided via Lemma 3.5.

[^3]:    ${ }^{\mathbb{I}}$ In the sense of $\lim _{t \rightarrow 0}\left(c p+\alpha\left(\nabla \cdot U+\nabla_{y} \cdot U^{1}\right)\right)=\alpha\left(\nabla \cdot U_{0}+\nabla_{y} \cdot U_{0}^{1}\right)$ in $L^{2}\left(\Omega \times Y^{g}\right)$.

[^4]:    ${ }^{\#} C_{\#}(Y)$ denotes the space of $Y$-periodic continuous functions in $y \in Y$.

