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Homogenization of an elliptic transmission system modeling the flux of oxygen from blood vessels to tissues

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Abstract

Motivated by the study of the hypoxia problem in cancerous tissues, we propose a system of coupled partial differential equations defined on a heterogeneous, periodically perforated domain describing the flux of oxygen from blood vessels towards the tissue and the corresponding oxygen diffusion within the tissue. Using heuristics based on dimensional analysis, we rephrase the initially parabolic problem as a semi-linear elliptic transmission problem. Focusing on the elliptic case, we are able to define a microscopic ε-dependent problem that is the starting point of our mathematical analysis; here ε is linked to the scale of heterogeneity.

We study the well-posedness of the microscopic problem as well as the passage to the periodic homogenization limit. Additionally, we derive the strong formulation of the two-scale macroscopic limit problem. Finally, we prove a corrector estimate. This specific ingredient allows us to estimate, in an *a priori* way, the discrepancy between solutions to the microscopic and, respectively, macroscopic problem. Our working techniques include energy-type estimates, fixed-point type iterations, monotonicity arguments, as well as the two-scale convergence tool.

Keywords: oxygen, hypoxia, elliptic pde, homogenization, two-scale convergence, corrector estimate.

MSC Subject Classification (2020): 92-08, 92C17, 62J10, 35B27.

Contents

Introduction

The aim of this work is to introduce a mathematical model describing the flux of oxygen from blood vessels to tissues. The main application of the model we propose is the hypoxia problem. The presence of hypoxia, that is low concentration of oxygen in cells, can be related to other pathologies such as cancer [10], Alzheimer's disease [31] and diabetes [11]. Therefore, studying the hypoxia problem can help understanding and solving other correlated medical issues. For instance, the presence of hypoxia can affect the curability of solid tumors [33]. The presence of hypoxia corresponds to de-oxygenated human body tissues invaded by cancer. Indeed, solid tumors are less well-oxygenated than the normal tissues from which they arose [9]. For this reason, the model we propose and analyze can be directly adapted to the medical study of tumors.

The oxygen problem has been studied from the mathematical modelling and computational perspective by many authors; we refer for instance to [19], [30], [20], [23], [25], [3], [22].

In chapter one, we present the first model, which consists of a system of two parabolic partial differential equations, with Dirichlet, Neumann and Robin boundary conditions. Clearly, the two equations we introduce at the beginning describe the flux of oxygen in the blood vessels and tissues, respectively. Therefore, they will be defined in two different domains that communicate to each other through an interface. The unknowns of our problem are the concentrations of oxygen in the blood vessels and tissues, respectively. For what concerns the flux of oxygen in the tissues, we consider a non-linear term corresponding the the Michaelis-Menten type consumption rate [32]. Once the model is introduced, we proceed with a dimensional analysis of all the mathematical quantities involved in the model. This procedure will be the idea behind the passage from the parabolic problem to an elliptic problem. Since the latter is not the main aim of our discussion, we just present an heuristic procedure that can be made mathematically rigorous. The main interest of our work is to apply two-scale convergence and homogenization techniques to an elliptic microscopic model, that we present in sections 1.3 and 1.4. The microscopic problem is meant to take into account the micro-oscillations in the concentration of oxygen. This oscillating behavior can be applied to the study of the hypoxia problem. Indeed, as we mentioned before, we are dealing with de-oxygenated zones that can be mathematically represented through an heterogeneous concentration of oxygen.

In chapter two, we study the well-posedness of the microscopic problem. Namely, once we obtain the weak formulation of the microscopic problem, we prove the existence, uniqueness and energy estimates related to the solution of the microscopic problem. The main difficulty of this part relies on the fact that our model is non-linear. For instance, to prove the existence of solution for the microscopic problem, we rely on an iteration scheme.

In the third chapter, after introducing main results regarding homogenization and two-scale convergence, we proceed with the homogenization of our problem. The final goal is to obtain the macroscopic two-scale elliptic limit system related to our original problem. Clearly, we show that the solution of the macroscopic system exists and is unique. Finally, we will prove a corrector estimate, which is the most interesting mathematical result of the whole work. Indeed, the solution of the microscopic problem presents some oscillating features, while the two-scale limit problem is meant to average those oscillations. When it comes to the corrector estimate, the main question is *how much information have we lost after the averaging process is done?* In order to prove the corrector estimate, we will use both functional analysis results and geometrical arguments, that are correlated to the geometrical reasoning behind the formulation of the microscopic problem.

Finally, we present the last chapter ”Conclusion and Outlook”, in which we sum up all the most interesting results of this work together with some additional comments and remarks. Moreover, we also leave some questions open regarding how we can improve our work, also by touching other fields such as numerical analysis and simulations.

It is important to point out that the whole analysis of our work is in 2D, even though the oxygen problem lives in a 3D setting. We made this decision to simplify the whole discussion. However, it is possible to adapt the whole analysis to the 3D case. We do not set all the assumptions on data from the beginning, but we prefer to add extra-assumption on the moment we need to use them. This is meant to highlight those very assumptions in the moment we require them.

Chapter 1

Modelling

In this chapter, we propose a mathematical model to describe the concentration and flux of oxygen from the blood vessels to the tissue where cells are located. Namely, we start from a system of two parabolic partial differential equations with associated boundary conditions of Dirichlet, Neumann and Robin type. The first step is the nondimensionalization of the problem, with the goal of eliminating the time dependence in the two parabolic equations, to eventually obtain a system of two elliptic equations. When this step is done, the final aim of the modelling section is the formulation of the ε-problem, involving a perforated domain that will be defined in terms of the initial domain.

1.1 From a parabolic problem to the elliptic problem

We consider the following geometry:

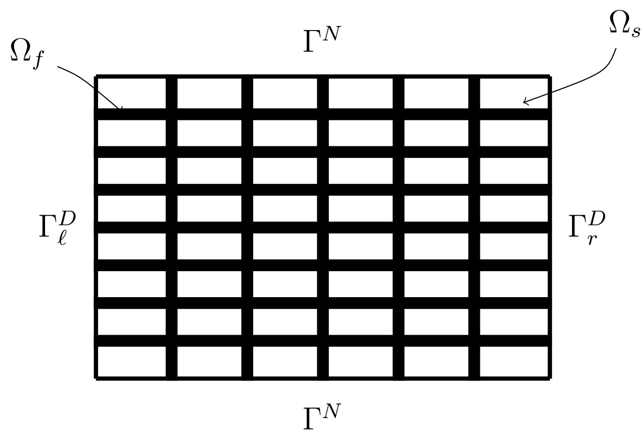

Figure 1.1.1: Description of the geometry of the problem

We denote our domain as Ω, with Lipschitz boundary Γ. We then consider a partition of Ω, namely $\Omega = \Omega_s \cup \Omega_f$, with $\Omega_s \cap \Omega_f = \emptyset$.

Ω_f is the zone corresponding to the blood vessels, while Ω_s is the region in which the oxygen enters from the blood vessels. Let v and u denote the concentration of oxygen in Ω_f and Ω_s respectively. Let $S = (0, T_{fin})$, with $T_{fin} > 0$. The following equation describes the oxygen flow through the blood vessels:

$$\partial_t v + \nabla \cdot (-D_v(x)\nabla v + Bv) = 0 \text{ in } \Omega_f \times S, \tag{1.1.1}$$

where $D_v = D_v(x)$ is the non-singular diffusion matrix corresponding to v. The right hand side of equation (1.1.1) is zero because we do not consider any source term related to the oxygen flow in the blood vessels.

Regarding the situation in Ω_s, we propose the following equation

$$\partial_t u + \nabla \cdot (-D_u(x)\nabla u) = -F(u) \text{ in } \Omega_s \times S, \tag{1.1.2}$$

where $D_u = D_u(x)$ is the non-singular diffusion matrix corresponding to u and $F : \mathbb{R} \to \mathbb{R}$, with

$$F(r) = \begin{cases} \dfrac{\alpha r}{\beta + r}, & \text{if } r \geq 0 \\ 0, & \text{if } r < 0 \end{cases} \tag{1.1.3}$$

with

- α maximum rate of oxygen consumption, $\alpha \in (0, \infty)$;
- β oxygen concentration at which $F(r) = \frac{1}{2}\alpha$, $\beta \in (0, \infty)$.

Expression (1.1.3) corresponds to the well-known Michaelis-Menten– type consumption rate (see [32]) . The term $-F(r)$ acts as a sink. This is due to the fact that the flow of oxygen coming from the blood vessels needs to be balanced, otherwise there would be an accumulation of oxygen in the domain Ω_s.

Considering now the external boundary Γ, we set Dirichlet boundary conditions on the left and right edges and Neumann conditions on the remaining two edges. Therefore, we rewrite $\Gamma = \Gamma_\ell^D \cup \Gamma_r^D \cup \Gamma^N$, and we suppose that the oxygen flows from the left edge Γ_ℓ^D to the right edge Γ_r^D.

We define $J_v := -D_v\nabla v + Bv$ and $J_u := -D_u\nabla u$ to be the fluxes related to v and u, respectively. In this scenario, we are able to formulate the boundary conditions on Γ:

$$u = \rho_\ell \text{ on } \Gamma_\ell^D \tag{1.1.4}$$
$$u = \rho_r \text{ on } \Gamma_r^D \tag{1.1.5}$$
$$J_u \cdot n = 0 \text{ on } \Gamma^N \setminus \partial\Omega_f \tag{1.1.6}$$
$$J_v \cdot n = 0 \text{ on } \Gamma \cap \partial\Omega_f, \tag{1.1.7}$$

with $\rho_r > \rho_\ell > 0$ and n is the outward normal vector. Notice that in the third boundary condition we had to exclude $\partial\Omega_f$ since we were considering J_u, that is the flux related to the domain Ω_s.

However, we still need boundary conditions on the set $\partial\Omega \setminus \Gamma$, that will allow us to describe how the oxygen flows from Ω_f to Ω_s. We indeed want that the flux exiting from Ω_f is conserved when entering Ω_s, we require:

$$J_v \cdot n = -J_u \cdot n \text{ on } \partial\Omega \setminus \Gamma. \tag{1.1.8}$$

Since there is mass exchange of oxygen through the boundary of the blood vessels, it is important to specify also a relation between u and v when the oxygen crosses the boundary $\partial\Omega \setminus \Gamma$. Let H be the Henry constant (see [5]) and let σ be a positive constant. We have the following relation

$$J_v \cdot n = \sigma(u - Hv) \text{ on } \partial\Omega \setminus \Gamma. \tag{1.1.9}$$

Finally, we need to set the initial conditions, namely

$$u(x,0) = u_0(x) \tag{1.1.10}$$
$$v(x,0) = v_0(x), \tag{1.1.11}$$

for all $x \in \bar{\Omega}$.

Summarizing, our hypoxia model is composed of equations (1.1.1), (1.1.2), (1.1.4), (1.1.5), (1.1.6), (1.1.7), (1.1.8), (1.1.9), (1.1.10) and (1.1.11). Now that the problem is set, we want to nondimensionalize it; in order to do so, we first provide a dimensional analysis of all the mathematical objects involved in the problem. We proceed as in [26].

- $[u] = [v] = ML^{-3}$;
- $[t] = T$, $[x] = L$;
- $[D_u] = [D_v] = L^2T^{-1}$, $[B] = LT^{-1}$, $[u_0] = [v_0] = ML^{-3}$, $[\rho_l] = [\rho_r] = ML^{-2}$, $[\sigma] = LT^{-1}$.

Notice that here we consider the problem as in a 3D setting, since we are referring to the physics of the problem, and not on the modelling. We take $x_{ref} := diam(\Omega)$ to be the characteristic length. The characteristic time scale t_{ref} will be chosen later. We choose

$$u_{ref} := \max\left\{\|u_0\|_{L^\infty(\Omega)}, \|\rho_l\|_{L^\infty((0,T)\times\Gamma_l^D)}\right\},$$

and

$$v_{ref} := \|v_0\|_{L^\infty(\Omega)}.$$

Let $\hat{\Omega}_s := \frac{1}{x_{ref}}\Omega_s$, $\hat{\Omega}_f := \frac{1}{x_{ref}}\Omega_f$, $\hat{\Gamma} := \frac{1}{x_{ref}}\Gamma$ and $\hat{T} = \frac{T_{fin}}{t_{ref}}$.

Let $\hat{F} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\hat{F}(r) = \begin{cases} \dfrac{\hat{\alpha} r}{\hat{\beta} + r}, & \text{if } r \geq 0 \\ 0, & \text{if } r < 0, \end{cases} \tag{1.1.12}$$

where $\hat{\alpha} = \alpha$ and $\hat{\beta} = \frac{\beta}{u_{ref}}$. We rewrite D_v as $D_v(x) := D_v\hat{D}_v(x)$, where $\hat{D}_v(x)$ is a dimensionless matrix and D_v is a positive diffusion constant. We substitute in (1.1.1) and (1.1.11) $t := t_{ref}\tau$, $x := x_{ref}z$, and $v := v_{ref}V$ and obtain

$$\frac{v_{ref}}{t_{ref}}\partial_\tau V + \nabla_z \cdot \left(-\frac{D_v\hat{D}_v(z)v_{ref}}{x_{ref}^2}\nabla_z V + \frac{Bv_{ref}}{x_{ref}}V\right) = 0 \text{ in } \hat{\Omega}_f \times (0,\hat{T}) \tag{1.1.13}$$

$$V(z,0) = \frac{v_0(z)}{v_{ref}} \text{ in } \hat{\Omega}_f. \tag{1.1.14}$$

We then do the same by setting $u := u_{ref}U$ and $D_u(x) := D_u\hat{D}_u(x)$ in (1.1.2), (1.1.10), (1.1.4), (1.1.5) to obtain

$$\frac{u_{ref}}{t_{ref}}\partial_\tau U + \nabla_z \cdot \left(-\frac{D_u\hat{D}_u(z)u_{ref}}{x_{ref}^2}\nabla_z U\right) = -\hat{F}(U) \text{ in } \hat{\Omega}_s \times (0,\hat{T}) \tag{1.1.15}$$

$$U(z,0) = \frac{u_0(z)}{u_{ref}} \text{ in } \hat{\Omega}_s \tag{1.1.16}$$

$$U(z,\tau) = \frac{\rho_\ell}{u_{ref}} \text{ on } \hat{\Gamma}_\ell^D \times (0,\hat{T}) \tag{1.1.17}$$

$$U(z,\tau) = \frac{\rho_r}{u_{ref}} \text{ on } \hat{\Gamma}_r^D \times (0,\hat{T}). \tag{1.1.18}$$

Setting $\sigma = \sigma_{ref}\tilde{\sigma}$, with $\sigma_{ref} = \dfrac{x_{ref}}{t_{ref}}$, and using the same reasoning for the boundary conditions (1.1.6), (1.1.7), (1.1.8) and (1.1.9), we end up with

$$\left(-\frac{D_u\hat{D}_u(z)u_{ref}}{x_{ref}}\nabla_z U\right)\cdot n = 0 \text{ on } \hat{\Gamma}^N \setminus \partial\hat{\Omega}_f, \tag{1.1.19}$$

$$\left(-\frac{D_v\hat{D}_v(z)v_{ref}}{x_{ref}}\nabla_z V + Bv_{ref}V\right)\cdot n = 0 \text{ on } \hat{\Gamma} \cap \partial\hat{\Omega}_f, \tag{1.1.20}$$

$$\left(-\frac{D_v\hat{D}_v(z)v_{ref}}{x_{ref}}\nabla_z V + Bv_{ref}V\right)\cdot n = \left(\frac{D_u\hat{D}_u(z)u_{ref}}{x_{ref}}\nabla_z U\right)\cdot n \text{ on } \partial\hat{\Omega} \setminus \hat{\Gamma}, \tag{1.1.21}$$

$$\left(-\frac{D_v\hat{D}_v(z)v_{ref}}{x_{ref}}\nabla_z V + Bv_{ref}V\right)\cdot n = \sigma_{ref}\tilde{\sigma}\left(u_{ref}U - Hv_{ref}V\right) \text{ on } \partial\hat{\Omega} \setminus \hat{\Gamma}. \tag{1.1.22}$$

The goal is to identify some parametric region where the hypoxia problem is a system of two coupled elliptic equations. We multiply (1.1.13) and (1.1.15) by $\dfrac{x_{ref}^2}{D_v v_{ref}}$ and $\dfrac{x_{ref}^2}{D_u u_{ref}}$, respectively. We obtain

$$\frac{x_{ref}^2}{D_v t_{ref}}\partial_\tau V + \nabla_z \cdot \left(-\hat{D}_v(z)\nabla_z V + \frac{Bx_{ref}}{D_v}V\right) = 0 \text{ in } \hat{\Omega}_f \times (0,\hat{T}) \tag{1.1.23}$$

$$\frac{x_{ref}^2}{D_u t_{ref}}\partial_\tau U + \nabla_z \cdot \left(-\hat{D}_u(z)\nabla_z U\right) = -\frac{x_{ref}^2}{D_u u_{ref}}\hat{F}(U) \text{ in } \hat{\Omega}_s \times (0,\hat{T}). \tag{1.1.24}$$

We consider the Robin boundary condition (1.1.22) and rewrite it as

$$\left(-\hat{D}_v(z)\nabla_z V + \frac{Bx_{ref}}{D_v}V\right) \cdot n = \frac{u_{ref}\sigma_{ref}\tilde{\sigma}x_{ref}}{D_v v_{ref}}\left(U - H\frac{v_{ref}}{u_{ref}}V\right) \quad \text{on } \partial\hat{\Omega} \setminus \hat{\Gamma}. \tag{1.1.25}$$

In order to obtain a couple of elliptic equations, we need to get rid of the terms involving the partial derivative with respect to time in (1.1.23) and (1.1.24). This translates into assuming that the terms $\frac{x_{ref}^2}{D_v t_{ref}}$ and $\frac{x_{ref}^2}{D_u t_{ref}}$ are sufficiently close to zero. Since one of our main interests is the analysis of the concentration of oxygen in the domain Ω_s, where the flux of oxygen is slower than the one in Ω_f, we can actually choose t_{ref} to be arbitrarily large in order to get a meaningful result. Therefore, we can set the problem such that the two quantities mentioned before are negligible. However, we still want to keep the information contained in (1.1.25). In order to do so, we observe that

$$\frac{x_{ref}^2}{D_v t_{ref}} = \frac{\sigma_{ref}x_{ref}}{D_v} \approx 0$$

and

$$\frac{x_{ref}^2}{D_u t_{ref}} = \frac{\sigma_{ref}x_{ref}}{D_u} \approx 0.$$

Since we want the coefficient $\frac{u_{ref}\sigma_{ref}\tilde{\sigma}x_{ref}}{D_v v_{ref}} \neq 0$, we just set $u_{ref} >> v_{ref}$ so that the ratio is balanced and the whole quantity does not go to zero. Another way to say this is that the concentration of oxygen in the tissue is bigger than the one in the blood vessels.

Eventually, we end up with the following elliptic problem

$$\nabla_z \cdot (-\hat{D}_v(z)\nabla_z V + \hat{B}V) = 0 \ \text{ in } \hat{\Omega}_f \tag{1.1.26}$$
$$\nabla_z \cdot (-\hat{D}_u(z)\nabla_z U) = -\Phi_1\hat{F}(U) \ \text{ in } \hat{\Omega}_s \tag{1.1.27}$$
$$(-\hat{D}_v(z)\nabla V + \hat{B}V) \cdot n = \Phi_2(\hat{D}_u(z)\nabla_z U) \cdot n \ \text{ on } \partial\hat{\Omega} \setminus \hat{\Gamma} \tag{1.1.28}$$
$$(-\hat{D}_v(z)\nabla_z V + \hat{B}V) \cdot n = \Phi_3\tilde{\sigma}\left(U - \hat{H}V\right) \ \text{ on } \partial\hat{\Omega} \setminus \hat{\Gamma} \tag{1.1.29}$$
$$(-\hat{D}_u(z)\nabla_z U) \cdot n = 0 \ \text{ on } \hat{\Gamma}^N \setminus \partial\hat{\Omega}_f \tag{1.1.30}$$
$$(-\hat{D}_v(z)\nabla_z V + \hat{B}V) \cdot n = 0 \ \text{ on } \hat{\Gamma} \cap \partial\hat{\Omega}_f \tag{1.1.31}$$
$$U(z) = \frac{\rho_\ell}{u_{ref}} \ \text{ on } \hat{\Gamma}_\ell^D \tag{1.1.32}$$
$$U(z) = \frac{\rho_r}{u_{ref}} \ \text{ on } \hat{\Gamma}_r^D, \tag{1.1.33}$$

where

- $\Phi_1 = \frac{x_{ref}^2}{D_u u_{ref}}$, $\Phi_2 = \frac{D_u u_{ref}}{D_v v_{ref}}$, $\Phi_3 = \frac{u_{ref}\sigma_{ref}x_{ref}}{D_v v_{ref}}$;
- $\hat{H} = H\frac{v_{ref}}{u_{ref}}$, $\hat{B} = \frac{Bx_{ref}}{D_v}$.

The heuristic procedure used here to reduce the parabolic hypoxia problem to an elliptic situation can be made mathematically rigorous. However, we do not do this here as we prefer to study in details the homogenization problem.

1.2 Formulation of the microscopic problem

We open this section with a question:

What does "microscopic problem" mean and why do we need to introduce it?

We remind that we want to be able to apply our model to the hypoxia problem. This means that the concentration of oxygen in the tissues appears to be heterogeneous and characterized by a lot of fluctuations. Introducing a microscopic model means to take into account all of those fluctuations in the solution of our problem and in the diffusion matrices as well. The first step in the formulation of the microscopic problem is to identify and separate the different scales, that can be referred to as macro-scale and micro-scale. The macro-scale takes into account the general behavior of our solution in the whole domain, while the micro-scale takes into account the fluctuations and micro-oscillation of the solution itself. What we said until now justifies the idea behind the mathematical reason why we should introduce the microscopic model as starting point our analysis. However, how do we actually take the micro-scale into account?

We notice that our geometry in Figure 1.1.1 presents a periodic structure. We consider the period of our geometry as a cell and we denote it with Y (see Figure 1.2.1). We can consider a partition of our cell, namely $Y = Y_f \cup Y_s \cup \Sigma$, with $Y_f \cap Y_S = \emptyset$, $\Sigma = \overline{Y}_f \cap \overline{Y}_s$. Clearly, Y_f corresponds to the part of the cell where the blood vessels are located, while Y_s corresponds to the tissue and Σ corresponds to the interface in between Y_s and Y_f.

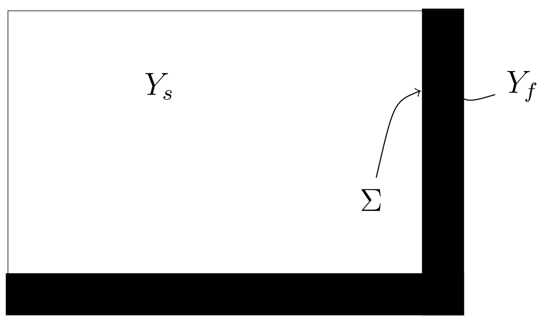

Figure 1.2.1: Periodic Cell Y

We now have to introduce a parameter, called ε, which is a real parameter taking values in a sequence of positive rational numbers tending to zero. It is important to underline that ε has a geometrical meaning. Indeed, one could take into account the number of times the periodic cell Y is contained in Ω. Mathematically, if we consider $L := diam(\Omega)$ and $l := diam(Y)$, we can define $\varepsilon := \dfrac{l}{L}$. This means that, when studying the microscopic model, we fix ε and we study the problem in an actual microscopic prospective, since we are interested in what happens in the single cell. Our final aim is to obtain the two-scale limit elliptic system as ε goes to zero. This means that, after studying the microscopic problem, we want to "eliminate" the oscillatory characteristic of the solution, considering a final averaged system, involving averaged quantities, that takes into account only the general behavior of the solution itself. When it comes to the applications, this procedure allows us to study complicated problems, such as the hypoxia problem, in a simplified setting. Moreover, starting from the microscopic model, we use two-scale convergence and homogenization techniques, that automatically guarantee the well-posedness of the limit problem.

We now formulate the ε-dependent microscopic problem (we also refer to [15]). Let $\chi_s(y)$ and $\chi_f(y)$ be the characteristic functions of Y_s and Y_f in Y, respectively.

We define the sets $\hat{\Omega}_s^\varepsilon := \left\{ z \in \hat{\Omega}_s; \chi_s\left(\frac{z}{\varepsilon}\right) = 1 \right\}$, $\hat{\Omega}_f^\varepsilon := \left\{ z \in \hat{\Omega}_f; \chi_f\left(\frac{z}{\varepsilon}\right) = 1 \right\}$, $\hat{\Omega}^\varepsilon := \hat{\Omega}_s^\varepsilon \cup \hat{\Omega}_f^\varepsilon$ and $\partial\hat{\Omega}_\varepsilon \setminus \Gamma := \{z \in \hat{\Omega}; \frac{z}{\varepsilon} \in \partial\hat{\Omega} \setminus \Gamma\}$.

We also define $\hat{B}_\varepsilon := \hat{B}\left(\frac{z}{\varepsilon}\right)$, $\hat{D}_{v\varepsilon}(z) := \hat{D}_v\left(\frac{z}{\varepsilon}\right)$ and $\hat{D}_{u\varepsilon}(z) := \hat{D}_u\left(\frac{z}{\varepsilon}\right)$. We suppose that

- $\hat{D}_v$ and $\hat{D}_u$ are Y_f-periodic and Y_s-periodic, respectively;

- there exist $m, M > 0$, $m \leq M$, such that

$$m|\xi|^2 \leq \sum_{i,j=1}^{2} \hat{D}_u^{ij}\xi_i\xi_j \leq M|\xi|^2, \tag{1.2.1}$$

$$m|\xi|^2 \leq \sum_{i,j=1}^{2} \hat{D}_v^{ij}\xi_i\xi_j \leq M|\xi|^2, \tag{1.2.2}$$

  for all $\xi \in \mathbb{R}^2$;

- $\hat{D}_u^{ij}$, $\hat{D}_v^{ij} \in L^\infty(Y)$ for all $i, j \in \{1, 2\}^2$;

- $\hat{D}_u$ and $\hat{D}_v$ are symmetric.

We now present the ε-problem: Find the couple $(u_\varepsilon, v_\varepsilon)$ satisfying the following system

$$\nabla_z \cdot (-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) = 0 \text{ in } \hat{\Omega}_f^\varepsilon \tag{1.2.3}$$

$$\nabla_z \cdot (-\hat{D}_{u\varepsilon}(z)\nabla_z u_\varepsilon) = -\Phi_1 \hat{F}(u_\varepsilon) \text{ in } \hat{\Omega}_s^\varepsilon \tag{1.2.4}$$

$$(-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) \cdot n = \Phi_2(\hat{D}_{u\varepsilon}(z)\nabla_z u_\varepsilon) \cdot n \text{ on } \partial\hat{\Omega}^\varepsilon \setminus \hat{\Gamma} \tag{1.2.5}$$

$$(-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) \cdot n = \varepsilon\Phi_3\tilde{\sigma}\left(u_\varepsilon - \hat{H}v_\varepsilon\right) \text{ on } \partial\hat{\Omega}^\varepsilon \setminus \hat{\Gamma} \tag{1.2.6}$$

$$(-\hat{D}_{u\varepsilon}(z)\nabla_z u_\varepsilon) \cdot n = 0 \text{ on } \hat{\Gamma}^N \setminus \partial\hat{\Omega}_f^\varepsilon \tag{1.2.7}$$

$$(-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) \cdot n = 0 \text{ on } \hat{\Gamma} \cap \partial\hat{\Omega}_f^\varepsilon \tag{1.2.8}$$

$$u_\varepsilon(z) = \frac{\rho_\ell}{u_{ref}} \text{ on } \hat{\Gamma}_\ell^D \tag{1.2.9}$$

$$u_\varepsilon(z) = \frac{\rho_r}{u_{ref}} \text{ on } \hat{\Gamma}_r^D, \tag{1.2.10}$$

where $u_\varepsilon(z) = u\left(z, \dfrac{z}{\varepsilon}\right)$ and $v_\varepsilon(z) = v\left(z, \dfrac{z}{\varepsilon}\right)$.

We want now to reformulate the microscopic problem such that we have homogeneous Dirichlet boundary conditions. Without loss of generality, we consider Ω such that $\overline{\Omega} = [0, L]^2 \subset \mathbb{R}^2$, with $L > 0$. Let us define a continuous function $g : \hat{\Omega}_s \to \mathbb{R}$,

$$g(z) := g(z_1, z_2) = \left(\frac{\rho_r}{u_{ref}} - \frac{\rho_\ell}{u_{ref}}\right)\frac{z_1}{L} + \frac{\rho_\ell}{u_{ref}}. \tag{1.2.11}$$

Since $z_1 = 0$ and $z_1 = L$ correspond to $\hat{\Gamma}_\ell^D$ and $\hat{\Gamma}_r^D$, respectively, we have

$$g(z) = \frac{\rho_\ell}{u_{ref}} \text{ on } \hat{\Gamma}_\ell^D,$$

$$g(z) = \frac{\rho_r}{u_{ref}} \text{ on } \hat{\Gamma}_r^D.$$

Moreover, we have that $\nabla_z g = \left(\left(\dfrac{\rho_r}{u_{ref}} - \dfrac{\rho_\ell}{u_{ref}}\right)\dfrac{1}{L}, 0\right) := \tilde{w}$. Therefore, we can reformulate problem (1.2.3)-(1.2.10) taking into account $\tilde{u}_\varepsilon(z) = u_\varepsilon(z) - g(z)$. Notice that, in order to define $\tilde{u}_\varepsilon$, we need to consider the function g restricted to the perforated domain Ω_s^ε.

Eventually, we end up with the following problem: Find the couple $(u_\varepsilon, v_\varepsilon)$ (we still write u_ε instead of $\tilde{u}_\varepsilon$ to simplify the notation) satisfying the following system

$$
\begin{aligned}
\nabla_z \cdot (-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) = 0 \ \text{ in } \hat{\Omega}_f^\varepsilon \\
\nabla_z \cdot (-\hat{D}_{u\varepsilon}(z)\nabla_z(u_\varepsilon + g)) = -\Phi_1 \hat{F}(u_\varepsilon + g) \ \text{ in } \hat{\Omega}_s^\varepsilon \\
(\hat{D}_{u\varepsilon}(z)\nabla_z(u_\varepsilon + g)) \cdot n = \varepsilon \frac{\Phi_3}{\Phi_2}\tilde{\sigma}\left(u_\varepsilon + g - \hat{H}v_\varepsilon\right) \ \text{ on } \partial\hat{\Omega}^\varepsilon \setminus \hat{\Gamma} \\
(-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) \cdot n = \varepsilon \Phi_3 \tilde{\sigma}\left(u_\varepsilon + g - \hat{H}v_\varepsilon\right) \ \text{ on } \partial\hat{\Omega}^\varepsilon \setminus \hat{\Gamma} \\
(-\hat{D}_{u\varepsilon}(z)\nabla_z(u_\varepsilon + g)) \cdot n = 0 \ \text{ on } \hat{\Gamma}^N \setminus \partial\hat{\Omega}_f^\varepsilon \\
(-\hat{D}_{v\varepsilon}(z)\nabla_z v_\varepsilon + \hat{B}_\varepsilon v_\varepsilon) \cdot n = 0 \ \text{ on } \hat{\Gamma} \cap \partial\hat{\Omega}_f^\varepsilon \\
u_\varepsilon(z) = 0 \ \text{ on } \hat{\Gamma}_\ell^D \\
u_\varepsilon(z) = 0 \ \text{ on } \hat{\Gamma}_r^D.
\end{aligned}
$$

Remark. *We notice that the function g defined in (1.2.11) is $C^\infty(\hat{\Omega}_s^\varepsilon)$. Therefore, the transformation $\tilde{u}_\varepsilon = u_\varepsilon - g$ is well defined. Indeed, when we will consider the weak formulation of our problem, we will ask H^1 regularity for $\tilde{u}$.*

1.3 Summary of equations of the microscopic model

For simplicity, we remove the ˆ notation, therefore from now on we will refer to the following microscopic problem: Find the couple $(u_\varepsilon, v_\varepsilon)$ satisfying the following system

$$\nabla_z \cdot (-D_{v\varepsilon}(z)\nabla_z v_\varepsilon + B_\varepsilon v_\varepsilon) = 0 \ \text{ in } \Omega_f^\varepsilon \tag{1.3.1}$$
$$\nabla_z \cdot (-D_{u\varepsilon}(z)\nabla_z(u_\varepsilon + g)) = -\Phi_1 F(u_\varepsilon + g) \ \text{ in } \Omega_s^\varepsilon \tag{1.3.2}$$
$$(D_{u\varepsilon}(z)\nabla_z(u_\varepsilon + g)) \cdot n = \varepsilon \frac{\Phi_3}{\Phi_2}\tilde{\sigma}\left(u_\varepsilon + g - Hv_\varepsilon\right) \ \text{ on } \partial\Omega^\varepsilon \setminus \Gamma \tag{1.3.3}$$
$$(-D_{v\varepsilon}(z)\nabla_z v_\varepsilon + B_\varepsilon v_\varepsilon) \cdot n = \varepsilon \Phi_3 \tilde{\sigma}\left(u_\varepsilon + g - Hv_\varepsilon\right) \ \text{ on } \partial\Omega^\varepsilon \setminus \Gamma \tag{1.3.4}$$
$$(-D_{u\varepsilon}(z)\nabla_z(u_\varepsilon + g)) \cdot n = 0 \ \text{ on } \Gamma^N \setminus \partial\Omega_f^\varepsilon \tag{1.3.5}$$
$$(-D_{v\varepsilon}(z)\nabla_z v_\varepsilon + B_\varepsilon v_\varepsilon) \cdot n = 0 \ \text{ on } \Gamma \cap \partial\Omega_f^\varepsilon \tag{1.3.6}$$
$$u_\varepsilon(z) = 0 \ \text{ on } \Gamma_\ell^D \tag{1.3.7}$$
$$u_\varepsilon(z) = 0 \ \text{ on } \Gamma_r^D. \tag{1.3.8}$$

Remark. *The set $\partial\Omega_\varepsilon \setminus \Gamma$ is the oscillating surface, that corresponds to the interface between Ω_s^ε and Ω_f^ε. When defining the microscopic problem (1.3.1)-(1.3.8), we need to consider ε multiplying the right hand side of (1.3.3) and (1.3.4). The reason why will be clearer when we will discuss the homogenization of our problem.*

Chapter 2

Well posedness of the microscopic model

The first aim of this chapter is to derive the weak formulation of the problem (1.3.1)-(1.3.8). After this step is done, we will use the weak formulation to prove the well posedness of the microscopic problem. Namely, we will prove the existence and uniqueness of weak solutions and an energy estimate. We list those results in section 2.2 and we reserve section 2.3 for the proofs.

2.1 Weak formulation

Let $H^1_\ell(\Omega^\varepsilon_s) := \{\phi \in H^1(\Omega^\varepsilon_s); \phi = 0 \text{ on } \Gamma^D_\ell\}$, $H^1_r(\Omega^\varepsilon_s) := \{\phi \in H^1(\Omega^\varepsilon_s); \phi = 0 \text{ on } \Gamma^D_r\}$ and $H^1_{\ell r}(\Omega^\varepsilon_s) := H^1_\ell \cap H^1_r$.

We consider the problem (1.3.1)-(1.3.8). Let $\varphi \in H^1(\Omega^\varepsilon_f)$ and let us multiply the left hand side of (1.3.1) by φ and integrate in Ω^ε_f; we get

$$\int_{\Omega^\varepsilon_f} \nabla \cdot (-D_{v\varepsilon}\nabla v_\varepsilon + B_\varepsilon v_\varepsilon)\varphi \, dz = \int_{\Omega^\varepsilon_f} (D_{v\varepsilon}\nabla v_\varepsilon - B_\varepsilon v_\varepsilon)\nabla\varphi \, dz + \varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} (u_\varepsilon + g - Hv_\varepsilon)\varphi \, d\sigma_z, \tag{2.1.1}$$

where we used the boundary conditions (1.3.4) and (1.3.6).

We then consider the left hand side of the equation (1.3.2), we multiply by $\phi \in H^1_{\ell r}(\Omega^\varepsilon_s)$ and integrate over Ω^ε_s.

$$\begin{aligned}
&\int_{\Omega^\varepsilon_s} \nabla \cdot (-D_{u\varepsilon}(\nabla u_\varepsilon + g))\phi \, dz \\
&= \int_{\Omega_s} D_{u\varepsilon}\nabla(u_\varepsilon + g)\nabla\phi \, dz - \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma}(u_\varepsilon + g - Hv_\varepsilon)\phi \, d\sigma_z \\
&= \int_{\Omega^\varepsilon_s} D_{u\varepsilon}\nabla u_\varepsilon\nabla\phi \, dz + \int_{\Omega^\varepsilon_s} w_\varepsilon\nabla\phi \, dz - \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma}(u_\varepsilon + g - Hv_\varepsilon)\phi \, d\sigma_z,
\end{aligned} \tag{2.1.2}$$

with $w_\varepsilon(z) := D_{u\varepsilon}(z)\tilde{w}$ and where we used (1.3.3), (1.3.4), (1.3.5), (1.3.7) and (1.3.8).

Definition 2.1.1. *A weak solution of the problem (1.3.1)-(1.3.8) is a couple*

$$(u_\varepsilon, v_\varepsilon) \in H^1_{\ell r}(\Omega^\varepsilon_s) \times H^1(\Omega^\varepsilon_f)$$

that satisfies the following system:

$$\int_{\Omega^\varepsilon_f} (D_{v\varepsilon}\nabla v_\varepsilon - B_\varepsilon v_\varepsilon)\nabla\varphi\, dz + \varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma}(u_\varepsilon + g - Hv_\varepsilon)\varphi\, d\sigma_z = 0 \tag{2.1.3}$$

$$\int_{\Omega^\varepsilon_s} D_{u\varepsilon}\nabla u_\varepsilon\nabla\phi\, dz - \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma}(u_\varepsilon + g - Hv_\varepsilon)\phi\, d\sigma_z = B(u_\varepsilon, \phi), \tag{2.1.4}$$

for all $\varphi \in H^1_0(\Omega^\varepsilon_f)$, $\phi \in H^1_{\ell r}(\Omega^\varepsilon_s)$, *with*

$$B(u_\varepsilon, \phi) = -\int_{\Omega^\varepsilon_s}\Phi_1 F(u_\varepsilon + g)\phi\, dz - \int_{\Omega^\varepsilon_s} w_\varepsilon\nabla\phi\, dz,$$

with F *defined in (1.1.12).*

2.2 Statement of the main results concerning the microscopic model

Lemma 2.2.1. *Let* $p \geq 1$. *Assume* Ω *is bounded and* $\partial\Omega$ *is Lipschitz. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(i) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$

and

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ *for each* $u \in W^{1,p}(\Omega)$, *with the constant* C *depending only on* p *and* Ω.

See [17] and chapter 5 of [14] for details on the proof of Lemma 2.2.1.

Lemma 2.2.2. *Let* $\varphi \in H^1(\Omega_\varepsilon)$. *Then there exists a constant* $\tilde{c} > 0$, *independent of* ε, *such that the following inequality holds:*

$$\varepsilon\|\varphi\|^2_{L^2(\partial\Omega_\varepsilon)} \leq \tilde{c}\|\varphi\|^2_{H^1(\Omega_\varepsilon)}. \tag{2.2.1}$$

See [27] for the proof.

Lemma 2.2.3. *Let* $p \geq 1$. *Assume* Ω *is open and bounded with* $\partial\Omega$ *Lipschitz. Let* $u \in W^{1,p}(\Omega)$. *Then there exists* $c = c(n, p)$ *such that*

$$\|u\|_{L^p(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)}. \tag{2.2.2}$$

We refer the reader to chapter 9 of [7] for details on Lemma 2.2.3.

Remark. *Notice that, as a consequence of (2.2.2), we have that, if* $u \in H^1(\Omega)$*, then the norms* $\|u\|_{H^1(\Omega)}$ *and* $\|\nabla u\|_{L^2(\Omega)}$ *are equivalent. Indeed*

$$\|\nabla u\|^2_{L^2(\Omega)} \leq \|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (1+c)\|\nabla u\|^2_{L^2(\Omega)}. \tag{2.2.3}$$

Proposition 2.2.1. *Let* $B_\varepsilon \in L^\infty(\Omega_f^\varepsilon)$ *with* $\|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}$ *arbitrarily small and let* $\tilde{\sigma} > 0$ *be arbitrarily small as well. Then the following inequality holds:*

$$\|v_\varepsilon\|^2_{H^1(\Omega_f^\varepsilon)} + \|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega_s^\varepsilon)} \leq \gamma\|g\|^2_{H^1(\Omega_s)} + \delta\|F\|_\infty, \tag{2.2.4}$$

where $\gamma, \delta > 0$ *depend on* $\tilde{\sigma}$*,*Φ_1*,* Φ_2 *and* Φ_3*, but are independent of* ε*.*

Proposition 2.2.2. *Assume the hypothesis of Proposition 2.2.1 to hold. If there exist a weak solution* $(u_\varepsilon, v_\varepsilon)$ *to the problem (1.3.1)-(1.3.8), then it is unique.*

We now want to show that there exists a weak solution $(u_\varepsilon, v_\varepsilon) \in H^1\ell r(\Omega_s^\varepsilon) \times H^1(\Omega_f^\varepsilon)$ of problem (2.1.3)-(2.1.4). In order to do so, we will construct an iteration scheme which will converge to the solution of our problem. We refer the reader to [18] for the application of the iteration scheme in a similar context. Let us refer to problem (1.3.1)-(1.3.8), we define the following iteration scheme:

Let $\{u_\varepsilon^k\}_{k\in\mathbb{N}}$ and $\{v_\varepsilon^k\}_{k\in\mathbb{N}}$ be two sequences of functions in $H^1_{\ell r}(\Omega_s^\varepsilon)$ and $H^1(\Omega_f^\varepsilon)$, respectively. Let $u_\varepsilon^0 = v_\varepsilon^0 = 0$. We have

$$\nabla_z \cdot (-D_{v\varepsilon}(z)\nabla_z v_\varepsilon^k + B_\varepsilon v_\varepsilon^k) = 0 \ \text{ in } \Omega_f^\varepsilon \tag{2.2.5}$$

$$\nabla_z \cdot (-D_{u\varepsilon}(z)\nabla_z (u_\varepsilon^k + g)) = -\Phi_1 F(u_\varepsilon^{k-1} + g) \ \text{ in } \Omega_s^\varepsilon \tag{2.2.6}$$

$$(D_{u\varepsilon}(z)\nabla_z (u_\varepsilon^k + g)) \cdot n = \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\left(u_\varepsilon^{k-1} + g - Hv_\varepsilon^{k-1}\right) \ \text{ on } \partial\Omega^\varepsilon \setminus \Gamma \tag{2.2.7}$$

$$(-D_{v\varepsilon}(z)\nabla_z v_\varepsilon^k + B_\varepsilon v_\varepsilon^k) \cdot n = \varepsilon\Phi_3\tilde{\sigma}\left(u_\varepsilon^{k-1} + g - Hv_\varepsilon^{k-1}\right) \ \text{ on } \partial\Omega^\varepsilon \setminus \Gamma \tag{2.2.8}$$

$$(-D_{u\varepsilon}(z)\nabla_z (u_\varepsilon^k + g)) \cdot n = 0 \ \text{ on } \Gamma^N \setminus \partial\Omega_f^\varepsilon \tag{2.2.9}$$

$$(-D_{v\varepsilon}(z)\nabla_z v_\varepsilon^k + B_\varepsilon v_\varepsilon^k) \cdot n = 0 \ \text{ on } \Gamma \cap \partial\Omega_f^\varepsilon \tag{2.2.10}$$

$$u_\varepsilon^k(z) = 0 \ \text{ on } \Gamma_\ell^D \tag{2.2.11}$$

$$u_\varepsilon^k(z) = 0 \ \text{ on } \Gamma_r^D. \tag{2.2.12}$$

The weak formulation related to (2.2.5)-(2.2.12) reads as follows:

Definition 2.2.1. *For all* $k \in \mathbb{N}$*, a weak solution of the problem (2.2.5)-(2.2.12) is a couple*

$$(u_\varepsilon^k, v_\varepsilon^k) \in H^1_{\ell r}(\Omega_s^\varepsilon) \times H^1(\Omega_f^\varepsilon)$$

that satisfies the following system:

$$\int_{\Omega_f^\varepsilon} (D_{v\varepsilon}\nabla v_\varepsilon^k - B_\varepsilon v_\varepsilon^k)\nabla\varphi \, dz + \varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} (u_\varepsilon^{k-1} + g - Hv_\varepsilon^{k-1})\varphi \, d\sigma_z = 0 \tag{2.2.13}$$

$$\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla u_\varepsilon^k \nabla\phi \, dz - \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} (u_\varepsilon^{k-1} + g - Hv_\varepsilon^{k-1})\phi \, d\sigma_z = B(u_\varepsilon^{k-1}, \phi), \tag{2.2.14}$$

for all $\varphi \in H_0^1(\Omega_f^\varepsilon)$, $\phi \in H_{\ell r}^1(\Omega_s^\varepsilon)$, *with*

$$B(u_\varepsilon^{k-1}, \phi) = -\int_{\Omega_s^\varepsilon} \Phi_1 F(u_\varepsilon^{k-1} + g)\phi \, dz - \int_{\Omega_s^\varepsilon} w_\varepsilon \cdot \nabla \phi \, dz,$$

with F *defined in (1.1.12).*

Lemma 2.2.4. *Let* H *be an Hilbert space and assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear form, for which there exist constants $m, M > 0$ *such that*

$$|B[u,v]| \leq M \|u\|_H \|v\|_H \ \ \forall u, v \in H, \tag{2.2.15}$$

and

$$m\|u\|_H^2 \leq B[u,u] \ \ \forall u \in H. \tag{2.2.16}$$

Let $f : H \to \mathbb{R}$ *be a bounded linear functional on* H*. Then there exists a unique element* $u \in H$ *such that*

$$B[u,v] = \langle f, v \rangle \,,$$

for all $v \in H$*.*

We refer the reader to chapter 6 of [14] for the proof of Lemma 2.2.4.

Lemma 2.2.5. *Let* $B_\varepsilon \in L^\infty(\Omega_f^\varepsilon)$ *with* $\|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}$ *arbitrarily small. There exists a unique solution of (2.2.13)-(2.2.14) for* $k = 1$*.*

Theorem 2.2.1. *Let* $B_\varepsilon \in L^\infty(\Omega_f^\varepsilon)$ *with* $\|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}$ *arbitrarily small and let* $\tilde{\sigma} > 0$ *be arbitrarily small as well. Then the sequences* $\{u_\varepsilon^k\}_{k\in\mathbb{N}}$ *and* $\{v_\varepsilon^k\}_{k\in\mathbb{N}}$ *are Cauchy in* $H_{\ell r}^1(\Omega_s^\varepsilon)$ *and* $H^1(\Omega_f^\varepsilon)$*, respectively. Moreover, there exists a constant* $\eta \in (0,1)$ *independent of* ε *such that*

$$\begin{aligned} &\|v_\varepsilon^{k+r} - v_\varepsilon^k\|_{H^1(\Omega_f^\varepsilon)}^2 + \|u_\varepsilon^{k+r} - u_\varepsilon^k\|_{H_{\ell r}^1(\Omega_s^\varepsilon)}^2 \\ &\leq C\frac{\eta^k(1-\eta^r)}{1-\eta}(\|v_\varepsilon^1\|_{H^1(\Omega_f^\varepsilon)}^2 + \|u_\varepsilon^1\|_{H_{\ell r}^1(\Omega_s^\varepsilon)}^2 + \|F\|_{L^\infty}^2), \end{aligned} \tag{2.2.17}$$

with $k, r \in \mathbb{N}$*,* c_1, c_2, C *positive constants independent of* ε*,* k *and* r*.*

Corollary 2.2.1. *Under the same assumptions of Theorem 2.2.1, there exists a weak solution of the problem (2.1.3)-(2.1.4). Moreover, the following estimate holds:*

$$\|v_\varepsilon - v_\varepsilon^k\|_{H^1(\Omega_f^\varepsilon)}^2 + \|u_\varepsilon - u_\varepsilon^k\|_{H_{\ell r}^1(\Omega_s^\varepsilon)}^2 \leq \frac{C\eta^k}{1-\eta}(\|v_\varepsilon^1\|_{H^1(\Omega_f^\varepsilon)}^2 + \|u_\varepsilon^1\|_{H_{\ell r}^1(\Omega_s^\varepsilon)}^2 + \|F\|_{L^\infty}^2), \tag{2.2.18}$$

with $\eta \in (0,1)$ *independent of* ε*,* $k \in \mathbb{N}$*,* C *positive constant independent of* ε *and* k*.*

2.3 Proofs of the main results concerning the microscopic model

Proof of Proposition 2.2.1

Combining (1.2.2) and (2.2.3), there exists a constant $a > 0$ such that

$$a\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} \leq \int_{\Omega^\varepsilon_f} D_{v\varepsilon}\nabla v_\varepsilon \nabla v_\varepsilon \, dz.$$

Therefore, considering $\varphi = v_\varepsilon$ in (2.1.3) and applying the triangular inequality, we get

$$\begin{aligned}
& a\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} \\
& \leq \varepsilon\Phi_3\tilde{\sigma}\left[\int_{\partial\Omega^\varepsilon\setminus\Gamma}|u_\varepsilon v_\varepsilon|\, d\sigma_z + \int_{\partial\Omega^\varepsilon\setminus\Gamma}|g v_\varepsilon|\, d\sigma_z + \int_{\partial\Omega^\varepsilon\setminus\Gamma}|H v_\varepsilon^2|\, d\sigma_z\right] + \|B_\varepsilon\|_\infty\|\nabla v_\varepsilon\|\|v_\varepsilon\| \\
& \leq \Phi_3\tilde{\sigma}\left[\frac{\varepsilon}{2}\|u_\varepsilon\|^2_{L^2(\partial\Omega^\varepsilon_s)} + \varepsilon(1+H)\|v_\varepsilon\|^2_{L^2(\partial\Omega^\varepsilon_f)} + \frac{\varepsilon}{2}\|g\|^2_{L^2(\partial\Omega^\varepsilon_s)}\right] + \frac{\|B_\varepsilon\|_\infty}{2}\left(\|\nabla v_\varepsilon\|^2 + \|v_\varepsilon\|^2\right) \\
& \leq \Phi_3\tilde{\sigma}\tilde{c}\left[\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} + (1+H)\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + \|g\|^2_{H^1(\Omega^\varepsilon_s)}\right] + \frac{\|B_\varepsilon\|_\infty}{2}\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}, \qquad (2.3.1)
\end{aligned}$$

where we used Young's inequality and (2.2.1). Bringing $\left[\Phi_3\tilde{\sigma}\tilde{c}(1+H) + \frac{\|B_\varepsilon\|_\infty}{2}\right]\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}$ to the left hand side of (2.3.1), and assuming that $\tilde{\sigma}$ and $\|B_\varepsilon\|_\infty$ are sufficiently small such that $\tilde{\tilde{c}} := a - \Phi_3\tilde{\sigma}\tilde{c}(1+H) - \frac{\|B_\varepsilon\|_\infty}{2} > 0$, we obtain

$$\tilde{\tilde{c}}\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} \leq \Phi_3\tilde{\sigma}\tilde{c}\left[\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} + \|g\|^2_{H^1(\Omega^\varepsilon_s)}\right]. \qquad (2.3.2)$$

We consider $\phi = u_\varepsilon$ in (2.1.4). Combining (1.2.1) and (2.2.3), there exists a constant $b > 0$ such that

$$\begin{aligned}
b\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} & \leq \int_{\Omega^\varepsilon_s} D_{u\varepsilon}\nabla u_\varepsilon \nabla u_\varepsilon \, dz \\
& \leq \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\left[\int_{\partial\Omega^\varepsilon\setminus\Gamma} u_\varepsilon^2\, d\sigma_z + \int_{\partial\Omega^\varepsilon\setminus\Gamma}|g u_\varepsilon|\, d\sigma_z + \int_{\partial\Omega^\varepsilon\setminus\Gamma}|H v_\varepsilon u_\varepsilon|\, d\sigma_z\right] \\
& + \int_{\Omega^\varepsilon_s}\Phi_1 F(u_\varepsilon + g)|u_\varepsilon|\, dz + \int_{\Omega^\varepsilon_s}|w_\varepsilon \nabla u_\varepsilon|\, dz. \qquad (2.3.3)
\end{aligned}$$

Let $\eta > 0$ be arbitrarily small. We notice that

$$\int_{\Omega^\varepsilon_s}\Phi_1 F(u_\varepsilon + g)|u_\varepsilon|\, dz \leq \frac{\Phi_1}{2}\left[\frac{1}{\eta}\|F\|^2 + \eta\|u_\varepsilon\|^2\right] \leq \frac{\Phi_1}{2}\left[\frac{1}{\eta}\|F\|^2 + \eta\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}\right],$$

and

$$\begin{aligned}
& \int_{\Omega^\varepsilon_s}|w_\varepsilon \nabla u_\varepsilon|\, dz \\
& \leq \left(\frac{1}{\eta}\|w_\varepsilon\|^2_{L^2(\Omega^\varepsilon_s)} + \eta\|\nabla u_\varepsilon\|^2_{L^2(\Omega^\varepsilon_s)}\right) \\
& \leq \left(\frac{1}{\eta}\|w_\varepsilon\|^2_{L^2(\Omega^\varepsilon_s)} + \eta\|u_\varepsilon\|^2_{H^1(\Omega^\varepsilon_s)}\right).
\end{aligned}$$

We apply (2.2.1) in (2.3.3) and we obtain

$$c_{\star\star}\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}\leq c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\left(\|g\|^2_{H^1(\Omega^\varepsilon_s)}+\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}\right)+\frac{\Phi_1}{2\eta}\|F\|_\infty+\frac{1}{\eta}\|w_\varepsilon\|^2_{L^2(\Omega^\varepsilon_s)}, \tag{2.3.4}$$

with $c_{\star\star}:=b-\frac{\Phi_1}{2}\eta-3c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}-\eta>0$. Summing up (2.3.2) and (2.3.4) we end up with

$$\begin{aligned}\tilde{\tilde{c}}\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}+c_{\star\star}\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}-\Phi_3\tilde{\sigma}\tilde{c}\|u_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}-c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}\\ \leq \Phi_3\tilde{\sigma}\tilde{c}\|g\|^2_{H^1(\Omega_s)}+c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\|g\|^2_{H^1(\Omega_s)}+\frac{\Phi_1}{2\eta}\|F\|_\infty+\frac{1}{\eta}\|w_\varepsilon\|^2_{L^2(\Omega^\varepsilon_s)},\end{aligned}$$

where we used

$$\|g\|_{H^1(\Omega^\varepsilon_s)}\leq\|g\|_{H^1(\Omega_s)}.$$

We adjust the coefficients in a proper way and choose $\tilde{\sigma}$ such that $c_1:=\tilde{\tilde{c}}-c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}>0$ and $c_2:=c_{\star\star}-c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}>0$. Moreover, defining $\gamma:=\left(\Phi_3\tilde{\sigma}\tilde{c}+c_\star\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\right)/\min\{c_1,c_2\}$ and $\delta:=\frac{\Phi_1}{2\eta\min\{c_1,c_2\}}$, we prove the result. □

Proof of Proposition 2.2.2

Assume that there exist $(u_\varepsilon,v_\varepsilon)$ and $(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)$ weak solutions of the problem (1.3.1)-(1.3.8). This means that both $(u_\varepsilon,v_\varepsilon)$ and $(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)$ satisfy (2.1.3) and (2.1.4). We then consider (2.1.3) in terms of $(u_\varepsilon,v_\varepsilon)$ and we subtract (2.1.3) written for $(\tilde{u}_\varepsilon,\tilde{v}_\varepsilon)$. We choose $\varphi:=v_\varepsilon-\tilde{v}_\varepsilon$ to obtain

$$\begin{aligned}a\|v_\varepsilon-\tilde{v}_\varepsilon\|_{H^1(\Omega^\varepsilon_f)}\leq\int_{\Omega^\varepsilon_f}D_{v\varepsilon}\nabla\left(v_\varepsilon-\tilde{v}_\varepsilon\right)\nabla\left(v_\varepsilon-\tilde{v}_\varepsilon\right)\,dz\leq\int_{\Omega^\varepsilon_f}B_\varepsilon(v_\varepsilon-\tilde{v}_\varepsilon)\nabla\left(v_\varepsilon-\tilde{v}_\varepsilon\right)\,dz+\\ -\varepsilon\Phi_3\tilde{\sigma}H\int_{\partial\Omega^\varepsilon\setminus\Gamma}\left(v_\varepsilon-\tilde{v}_\varepsilon\right)^2\,d\sigma_z+\varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma}|u_\varepsilon-\tilde{u}_\varepsilon|\,|v_\varepsilon-\tilde{v}_\varepsilon|\,d\sigma_z.\end{aligned} \tag{2.3.5}$$

We have that

$$\int_{\Omega^\varepsilon_f}|B_\varepsilon(v_\varepsilon-\tilde{v}_\varepsilon)\nabla\left(v_\varepsilon-\tilde{v}_\varepsilon\right)|\,dz\leq\frac{\|B_\varepsilon\|_\infty}{2}\|v_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}.$$

Since we can choose $\|B_\varepsilon\|_\infty$ arbitrarily small, we can then use the same argument as in the proof of Proposition 2.2.1.

Moreover, it holds

$$\varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma}|u_\varepsilon-\tilde{u}_\varepsilon|\,|v_\varepsilon-\tilde{v}_\varepsilon|\,d\sigma_z\leq\tilde{c}\Phi_3\tilde{\sigma}\left[\|u_\varepsilon-\tilde{u}_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}+\|v_\varepsilon-\tilde{v}_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}\right],$$

where we first used Young's inequality and then (2.2.1). Controlling $\tilde{\sigma}$ in an analogous way as we did in the proof of Proposition 2.2.1, (2.3.5) allows us to write

$$\tilde{a}\|v_\varepsilon-\tilde{v}_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)}\leq-\varepsilon\Phi_3\tilde{\sigma}H\int_{\partial\Omega^\varepsilon\setminus\Gamma}\left(v_\varepsilon-\tilde{v}_\varepsilon\right)^2\,d\sigma_z+\tilde{c}\Phi_3\tilde{\sigma}\|u_\varepsilon-\tilde{u}_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}, \tag{2.3.6}$$

with $\tilde{a} > 0$. Setting $\phi := u_\varepsilon - \tilde{u}_\varepsilon$ in (2.1.4), we get

$$
\begin{aligned}
b\|u_\varepsilon - \tilde{u}_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} &\leq \int_{\Omega^\varepsilon_s} D_{u\varepsilon}\nabla\left(u_\varepsilon - \tilde{u}_\varepsilon\right)\nabla\left(u_\varepsilon - \tilde{u}_\varepsilon\right)\, dz \\
\leq -\Phi_1 \int_{\Omega^\varepsilon_s} &\left(F(u_\varepsilon + g) - F(\tilde{u}_\varepsilon + g)\right)\left(u_\varepsilon - \tilde{u}_\varepsilon\right) dz \\
+ \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} &(u_\varepsilon - \tilde{u}_\varepsilon)^2\, d\sigma_z + H\varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} |u_\varepsilon - \tilde{u}_\varepsilon|\,|v_\varepsilon - \tilde{v}_\varepsilon|\, d\sigma_z
\end{aligned} \tag{2.3.7}
$$

Relying on similar arguments, (2.3.7) yields:

$$
\tilde{b}\|u_\varepsilon - \tilde{u}_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} \leq -\Phi_1\int_{\Omega^\varepsilon_s}\left(F(u_\varepsilon + g) - F(\tilde{u}_\varepsilon + g)\right)\left(u_\varepsilon - \tilde{u}_\varepsilon\right) dz + H\tilde{c}\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\|v_\varepsilon - \tilde{v}_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} \tag{2.3.8}
$$

We notice that $F'(r) = \dfrac{\alpha\beta}{(\beta + r)^2} \geq 0$, therefore $(F(r_1 + g) - F(r_2 + g)(r_1 - r_2) \geq 0$ $\forall r_1, r_2 \in \mathbb{R}$. Summing up (2.3.6) and (2.3.8), adjusting correspondingly the terms and choosing $\tilde{\sigma} > 0$ properly, we obtain

$$
\begin{aligned}
a_1\|v_\varepsilon - \tilde{v}_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + b_1\|u_\varepsilon - \tilde{u}_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} &\leq -\varepsilon\Phi_3\tilde{\sigma}H\int_{\partial\Omega^\varepsilon\setminus\Gamma}\left(v_\varepsilon - \tilde{v}_\varepsilon\right)^2\, d\sigma_z + \\
-\Phi_1\int_{\Omega^\varepsilon_s}\left(F(u_\varepsilon + g) - F(\tilde{u}_\varepsilon + g)\right)&\left(u_\varepsilon - \tilde{u}_\varepsilon\right) dz \leq 0,
\end{aligned} \tag{2.3.9}
$$

with $a_1, b_1 > 0$. Relation (2.3.9) directly implies $u_\varepsilon \equiv \tilde{u}_\varepsilon$ and $v_\varepsilon \equiv \tilde{v}_\varepsilon$ almost everywhere in Ω^ε_s, and respectively, in Ω^ε_f, which means that there exists an unique weak solution of the problem. □

Proof of Lemma 2.2.5 We observe that, for $k = 1$, (2.2.14) can be written in the following form:

$$
\begin{aligned}
&\int_{\Omega^\varepsilon_s}(\beta + g)D_{u\varepsilon}\nabla u^1_\varepsilon\nabla\phi\, dz \\
&= \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} g(\beta + g)\phi\, d\sigma_z - \int_{\Omega^\varepsilon_s}\Phi_1\alpha g\phi\, dz - \int_{\Omega^\varepsilon_s}(\beta + g)w_\varepsilon\nabla\phi\, dz.
\end{aligned} \tag{2.3.10}
$$

Indeed, we can just start from equations (2.2.6), (2.2.7), (2.2.9), substitute $u^0_\varepsilon = v^0_\varepsilon = 0$, multiply both sides by $(\beta + g)$ and then go on with the usual procedure to obtain the weak formulation. From the definitions, we have that β is a positive constant and, since $\rho_r > \rho_l > 0$, from (1.2.11) we have $g(z) \geq 0$ for all $z \in \Omega^\varepsilon$. This, together with property (1.2.1) of $D_{u\varepsilon}$, implies that the form

$$
\int_{\Omega^\varepsilon_s}(\beta + g)D_{u\varepsilon}\nabla u^1_\varepsilon\nabla\phi\, dz
$$

is bilinear, bounded and coercive. Moreover, from the linearity of g, we have that the right hand side of (2.3.10) is linear and bounded for all $\varepsilon > 0$. Therefore, using Lemma 2.2.4, we conclude about the existence and uniqueness of the solution $u^1_\varepsilon \in H^1_{\ell r}(\Omega^\varepsilon_s)$. Similarly, considering $k = 1$ in (2.2.13), we get

$$
\int_{\Omega^\varepsilon_f}(D_{v\varepsilon}\nabla v^1_\varepsilon - B_\varepsilon v^1_\varepsilon)\nabla\varphi\, dz = -\varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} g\varphi\, d\sigma_z. \tag{2.3.11}
$$

We have

$$(a - \|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)})\|v_\varepsilon^1\|^2_{H^1(\Omega_f^\varepsilon)} \leq \int_{\Omega_f^\varepsilon} (D_{v\varepsilon}\nabla v_\varepsilon^1 - B_\varepsilon v_\varepsilon^1)\nabla v_\varepsilon^1\, dz,$$

and we can choose $\|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}$ arbitrarily small so that $(a - \|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}) > 0$. Therefore, applying again Lemma 2.2.4, we can conclude about the existence and uniqueness of the weak solution $v_\varepsilon \in H^1(\Omega_f^\varepsilon)$ of problem (2.3.11).

□

Proof of Theorem 2.2.1

Let $w_{u_\varepsilon}^k := u_\varepsilon^k - u_\varepsilon^{k-1}$ and $w_{v_\varepsilon}^k := v_\varepsilon^k - v_\varepsilon^{k-1}$. We rewrite the respective weak formulations (2.2.13)-(2.2.14) for $(u_\varepsilon^k, v_\varepsilon^k)$, $(u_\varepsilon^{k-1}, v_\varepsilon^{k-1})$, we subtract them and we substitute $\phi = \tilde{\sigma} w_{u_\varepsilon}^k$ and $\varphi = w_{v_\varepsilon}^k$:

$$\int_{\Omega_f^\varepsilon} (D_{v\varepsilon}\nabla w_{v_\varepsilon}^k - B_\varepsilon w_{v_\varepsilon}^k)\nabla w_{v_\varepsilon}^k\, dz + \varepsilon\Phi_3\tilde{\sigma}\int_{\partial\Omega^\varepsilon\setminus\Gamma} (w_{u_\varepsilon}^{k-1} - H w_{v_\varepsilon}^{k-1}) w_{v_\varepsilon}^k\, d\sigma_z = 0, \quad (2.3.12)$$

$$\begin{aligned}\tilde{\sigma}\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla w_{u_\varepsilon}^k \nabla w_{u_\varepsilon}^k\, dz &= \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}^2\int_{\partial\Omega^\varepsilon\setminus\Gamma} (w_{u_\varepsilon}^{k-1} - H w_{v_\varepsilon}^{k-1}) w_{u_\varepsilon}^k\, d\sigma_z \\ &- \tilde{\sigma}\int_{\Omega_s^\varepsilon} \Phi_1 (F(u_\varepsilon^{k-1} + g) - F(u_\varepsilon^{k-2} + g) w_{u_\varepsilon}^k\, dz. \end{aligned} \quad (2.3.13)$$

From (2.3.12) we have

$$\begin{aligned}&(a - \|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)})\|w_{v_\varepsilon}^k\|^2_{H^1(\Omega_f^\varepsilon)} \\ &\leq \frac{\varepsilon\Phi_3\tilde{\sigma}}{2}\left(\|w_{u_\varepsilon}^{k-1}\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)} + (1+H)\|w_{v_\varepsilon}^k\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)} + H\|w_{v_\varepsilon}^{k-1}\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)}\right) \\ &\leq \tilde{c}\Phi_3\tilde{\sigma}\left(\|w_{u_\varepsilon}^{k-1}\|^2_{H^1(\Omega_f^\varepsilon)} + (1+H)\|w_{v_\varepsilon}^k\|^2_{H^1(\Omega_f^\varepsilon)} + H\|w_{v_\varepsilon}^{k-1}\|^2_{H^1(\Omega_f^\varepsilon)}\right),\end{aligned}$$

where we used Young's inequality and (2.2.1). Assuming $\|B_\varepsilon\|_{L^\infty(\Omega_f^\varepsilon)}$ and $\tilde{\sigma}$ to be arbitrarily small, there exists a constant $c_1 > 0$ such that

$$c_1\|w_{v_\varepsilon}^k\|^2_{H^1(\Omega_f^\varepsilon)} \leq \tilde{c}\Phi_3\tilde{\sigma}\left(\|w_{u_\varepsilon}^{k-1}\|^2_{H^1(\Omega_f^\varepsilon)} + H\|w_{v_\varepsilon}^{k-1}\|^2_{H^1(\Omega_f^\varepsilon)}\right). \quad (2.3.14)$$

Similarly, from (2.3.13) we obtain

$$\begin{aligned}b\tilde{\sigma}\|w_{u\varepsilon}^k\|^2_{H^1_{\ell r}(\Omega_s^\varepsilon)} &\leq \varepsilon\frac{\Phi_3}{2\Phi_2}\tilde{\sigma}^2\left[\|w_{u\varepsilon}^{k-1}\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)} + \|w_{u\varepsilon}^k\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)}\right] \\ &+ \varepsilon\frac{\Phi_3}{2\Phi_2}H\tilde{\sigma}^2\left[\|w_{v\varepsilon}^{k-1}\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)} + \|w_{u\varepsilon}^k\|^2_{L^2(\partial\Omega^\varepsilon\setminus\Gamma)}\right] \\ &+ \frac{\tilde{\sigma}\Phi_1}{2}\left[\frac{2}{\delta}\|F\|^2_{L^\infty} + \delta\|w_{u_\varepsilon}^k\|^2_{H^1_{\ell r}(\Omega_s^\varepsilon)}\right],\end{aligned} \quad (2.3.15)$$

where we used Young's Inequality and $\delta > 0$ is arbitrarily small. We use (2.2.1) in (2.3.15) to obtain

$$\begin{aligned}&\tilde{\sigma}\left(b - \frac{\tilde{c}\Phi_3\tilde{\sigma}}{\Phi_2}(1+H) - \frac{\Phi_1\delta}{2}\right)\|w_{u\varepsilon}^k\|^2_{H^1_{\ell r}(\Omega_s^\varepsilon)} \\ &\leq \frac{\tilde{c}\Phi_3}{\Phi_2}\tilde{\sigma}^2\left[\|w_{u\varepsilon}^{k-1}\|^2_{H^1_{\ell r}(\Omega_s^\varepsilon)} + H\|w_{v\varepsilon}^{k-1}\|^2_{H^1_{\ell r}(\Omega_s^\varepsilon)}\right] + \frac{\tilde{\sigma}\Phi_1}{\delta}\|F\|^2_{L^\infty},\end{aligned} \quad (2.3.16)$$

Assuming $\tilde{\sigma}$ and δ to be arbitrarily small so that $c_2 := \tilde{\sigma}\left(b - \frac{\tilde{c}\Phi_3\tilde{\sigma}}{\Phi_2}(1+H) - \frac{\Phi_1\delta}{2}\right) > 0$, we obtain

$$c_2\|w^k_{u_\varepsilon}\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} \leq \frac{\tilde{c}\Phi_3\tilde{\sigma}}{\Phi_2}\left(\|w^{k-1}_{u_\varepsilon}\|^2_{H^1(\Omega^\varepsilon_f)} + H\|w^{k-1}_{v_\varepsilon}\|^2_{H^1(\Omega^\varepsilon_f)}\right) + \frac{\tilde{\sigma}\Phi_1}{\delta}\|F\|^2_{L^\infty}. \tag{2.3.17}$$

Summing up (2.3.14) and (2.3.17), we obtain

$$\|w^k_{v_\varepsilon}\|^2_{H^1(\Omega^\varepsilon_f)} + \|w^k_{u_\varepsilon}\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} \leq \eta\left(\|w^{k-1}_{u_\varepsilon}\|^2_{H^1(\Omega^\varepsilon_f)} + \|w^{k-1}_{v_\varepsilon}\|^2_{H^1(\Omega^\varepsilon_f)} + \|F\|^2_{L^\infty}\right), \tag{2.3.18}$$

with $\eta := \max\left\{\tilde{c}\Phi_3\tilde{\sigma}, \tilde{c}\Phi_3\tilde{\sigma}H, \frac{\tilde{c}\Phi_3\tilde{\sigma}}{\Phi_2}, \frac{\tilde{c}\Phi_3\tilde{\sigma}H}{\Phi_2}, \frac{\tilde{\sigma}\Phi_1}{\delta}\right\}$.

We can still choose $\tilde{\sigma}$ arbitrarily small so that $\eta < 1$. Let $r \in \mathbb{N}$, we obtain

$$\begin{aligned}
&c_1\|v^{k+r}_\varepsilon - v^k_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + c_2\|u^{k+r}_\varepsilon - u^k_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} \\
&\leq C(\|v^{k+r}_\varepsilon - v^{k+r-1}_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + \dots + \|v^{k+1}_\varepsilon - v^k_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} \\
&+ \|u^{k+r}_\varepsilon - u^{k+r-1}_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} + \dots + \|u^{k+1}_\varepsilon - u^k_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)}) \\
&\leq C(\eta^{k+r-1} + \dots + \eta^k)(\|v^1_\varepsilon - v^0_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + \|u^1_\varepsilon - u^0_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} + \|F\|^2_{L^\infty}) \\
&\leq C\frac{\eta^k(1-\eta^r)}{1-\eta}(\|v^1_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + \|u^1_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} + \|F\|^2_{L^\infty}),
\end{aligned}$$

where C is a positive constant independent of ε, k and r. □

Proof of Corollary 2.2.1 From (2.2.17) we have that $\{u^k_\varepsilon\}_{k\in\mathbb{N}}$ and $\{v^k_\varepsilon\}_{k\in\mathbb{N}}$ are Cauchy sequences in $H^1_{\ell r}(\Omega^\varepsilon_s)$ and $H^1(\Omega^\varepsilon_f)$, respectively. Since $H^1_{\ell r}(\Omega^\varepsilon_s)$ and $H^1(\Omega^\varepsilon_f)$ are complete spaces, $\{u^k_\varepsilon\}_{k\in\mathbb{N}}$ and $\{v^k_\varepsilon\}_{k\in\mathbb{N}}$ converge strongly to u_ε and v_ε respectively. Taking $r \to +\infty$ in (2.2.17), we obtain

$$\|v_\varepsilon - v^k_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + \|u_\varepsilon - u^k_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} \leq \frac{C\eta^k}{1-\eta}(\|v^1_\varepsilon\|^2_{H^1(\Omega^\varepsilon_f)} + \|u^1_\varepsilon\|^2_{H^1_{\ell r}(\Omega^\varepsilon_s)} + \|F\|^2_{L^\infty}).$$

□

Remark. *The iteration technique we just applied to prove the existence of solutions gives us also the uniqueness. Namely, we showed two different ways to prove that the solution* $(u_\varepsilon, v_\varepsilon) \in H^1_{\ell r}(\Omega^\varepsilon_s) \times H^1(\Omega^\varepsilon_f)$ *of problem (2.1.3)-(2.1.4) is unique.*

Chapter 3

Passage to the homogenization limit

The aim of this chapter is to determine the strong formulation of the microscopic problem (1.3.1)-(1.3.8). Since we need to use two scale convergence and homogenization arguments, in the first section of this chapter we present some definitions and results of the Homogenization theory. We refer to chapter 9 of [12], chapter 3 of [29], [4] for the results related to homogenization theory and two-scale convergence; to [2], [16] when it comes to the application of those results.

3.1 The concept of two-scale convergence

Let Ω be a bounded open set in $\mathbb{R}^N$ and $Y := \prod_{i=1}^{N}[0, \ell_i]$, with ℓ_1,..., ℓ_N given positive numbers, be the reference cell.

Definition 3.1.1. *Let f_ε be a sequence of functions in $L^2(\Omega)$. One says that f_ε **two-scale converges** to $f_0 = w_0(x, y)$ with $w_0 \in L^2(\Omega \times Y)$ if for any function $\varphi = \varphi(x, y) \in D(\Omega, C^\infty_\#(Y))$ one has*

$$\lim_{\varepsilon \to 0} \int_\Omega f_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|Y|} \int_\Omega \int_Y f_0(x, y)\varphi(x, y)\, dydx. \tag{3.1.1}$$

We denote (3.1.1) by $f_\varepsilon \rightrightarrows f_0$, $\varepsilon \to 0$.

The following is a compactness result that is important for our discussions, it allows us to pass to the two-scale limit.

Theorem 3.1.1. *(i) Let f_ε be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence f'_ε and a function $f_0 \in L^2(\Omega \times Y)$ such that $f'_\varepsilon \rightrightarrows f_0$, $\varepsilon \to 0$.*

(ii) Let f_ε be a bounded sequence in $H^1(\Omega)$, which converges weakly to a limit function $f_0 \in H^1(\Omega \times Y)$. Then there exists $f_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that up to a subsequence f_ε two-scale converges to $f_0(x, y)$ and $\nabla f_\varepsilon \rightrightarrows \nabla_x f_0 + \nabla_y f_1$, $\varepsilon \to 0$.

Let Σ_ε be a ε-periodic surface.

Definition 3.1.2. *A sequence $f_\varepsilon \in L^2(\Sigma_\varepsilon)$ two scale converges $f_0(x,y) \in L^2(\Omega \times \Sigma)$ if for any $\varphi \in D(\Omega; C^\infty_\#(\Sigma)$ we have*

$$\lim_{\varepsilon \to 0} \int_{\Sigma_\varepsilon} f_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) d\sigma_x = \int_\Omega \int_\Sigma f_0(x,y)\varphi(x,y)\, d\sigma_y dx. \tag{3.1.2}$$

We denote (3.1.2) by $f_\varepsilon \overset{S}{\rightrightarrows} f_0$, $\varepsilon \to 0$.

We present another compactness result related to ε-periodic surfaces.

Theorem 3.1.2. *(i) From each bounded sequence $f_\varepsilon \in L^2(\Sigma_\varepsilon)$, one can extract a subsequence f_ε which two-scale converges to a function $f_0 \in L^2(\Omega \times \Sigma)$.*

(ii) If a sequence of functions f_ε is bounded in $L^\infty(\Sigma_\varepsilon)$, then f_ε two-scale converges to a function $f_0 \in L^\infty(\Omega \times \Sigma)$.

For the proof of (i) and (ii), we refer the reader to [28] and [24], respectively.

The final result of this section is an extension result in Sobolev spaces. We need it in order to be able to apply the two-scale compactness, since in Theorem 3.1.1 we consider the whole domain Ω, and not a perforated domain Ω^ε. For additional details on extensions, see [1].

Theorem 3.1.3. *Assume $\partial\Omega^\varepsilon$ to be Lipschitz. If $f_\varepsilon \in H^1(\Omega^\varepsilon)$, it exists an extension $\tilde{f}_\varepsilon \in H^1(\Omega)$ of f_ε, such that*

$$\|\tilde{f}_\varepsilon\|_{H^1(\Omega)} \leq c\|f_\varepsilon\|_{H^1(\Omega^\varepsilon)}, \tag{3.1.3}$$

where c is independent of ε.

3.2 Homogenization of the microscopic problem

Step 1: Extension

From Proposition 2.2.1 we have that u_ε and v_ε are uniformly bounded in $H^1_{\ell r}(\Omega^\varepsilon_s)$ and $H^1(\Omega^\varepsilon_f)$, respectively, namely there exist $C > 0$ independent of ε such that

$$\|u_\varepsilon\|_{H^1_{\ell r}(\Omega^\varepsilon_s)} \leq C,$$

$$\|v_\varepsilon\|_{H^1(\Omega^\varepsilon_f)} \leq C.$$

From (3.1.3) we obtain the existence of $\tilde{u}_\varepsilon \in H^1_{\ell r}(\Omega_s)$ and $\tilde{v}_\varepsilon \in H^1(\Omega_f)$ such that

$$\|\tilde{u}_\varepsilon\|_{H^1_{\ell r}(\Omega_s)} \leq \tilde{C}, \tag{3.2.1}$$

$$\|\tilde{v}_\varepsilon\|_{H^1(\Omega_f)} \leq \tilde{C}, \tag{3.2.2}$$

with $\tilde{C} > 0$ independent of ε. From now on, when it comes to the homogenization process, we will always refer to the extended functions. However, for simplicity, we get rid of the tilde ˜ notation.

Step 2: Compactness

Since u_ε is uniformly bounded in $H^1_{\ell r}(\Omega_s)$, applying Theorem 3.1.1, there exist

$$u_0 \in H^1_{\ell r}(\Omega_s; L^2(Y_s)),$$
$$u_1 \in L^2(\Omega_s; H^1_{\#}(Y_s)/\mathbb{R})$$

such that

$$u_\varepsilon \rightrightarrows u_0,$$
$$\nabla u_\varepsilon \rightrightarrows \nabla u_0 + \nabla_y u_1,$$

as $\varepsilon \to 0$. Similiarly, since v_ε is uniformly bounded in $H^1(\Omega_f)$, there exist

$$v_0 \in H^1(\Omega_f; L^2(Y_f)),$$
$$v_1 \in L^2(\Omega_f; H^1_{\#}(Y_f)/\mathbb{R})$$

such that

$$v_\varepsilon \rightrightarrows v_0,$$
$$\nabla v_\varepsilon \rightrightarrows \nabla v_0 + \nabla_y v_1,$$

as $\varepsilon \to 0$.

We now rely on the following homogenization ansatz:

$$u_\varepsilon(z) = \Big[u_0(z,y) + \varepsilon u_1(z,y) + \varepsilon^2 u_2(z,y) + h.o.t\Big]_{|y:=\frac{x}{\varepsilon}}, \tag{3.2.3}$$

$$v_\varepsilon(z) = \Big[v_0(z,y) + \varepsilon v_1(z,y) + \varepsilon^2 v_2(z,y) + h.o.t\Big]_{|y:=\frac{x}{\varepsilon}} \tag{3.2.4}$$

Proposition 3.2.1. *Assuming (3.2.3) and(3.2.4), we have $u_0 = u_0(z)$ and $v_0 = v_0(z)$.*

Proof. If $f = f\left(z, \frac{z}{\varepsilon}\right)$ is a sufficiently smooth function, then the following calculation rule applies:

$$\frac{d}{dz} f\left(z, \frac{z}{\varepsilon}\right) = \partial_z f\left(z, \frac{z}{\varepsilon}\right) + \frac{1}{\varepsilon}\partial_y \left(z, \frac{z}{\varepsilon}\right)_{|y:=\frac{z}{\varepsilon}}. \tag{3.2.5}$$

We substitute (3.2.3) in (1.3.2) to obtain

$$\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right) \cdot \left(-D_u(y)\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right)(u_0 + g + \varepsilon u_1 + \varepsilon^2 u_2 + h.o.t)\right) = -\Phi_1 F(u_\varepsilon + g), \tag{3.2.6}$$

with

$$F(u_\varepsilon + g) = \frac{\alpha(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2) + g)}{\beta + (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2) + g)}.$$

Multyplying both sides of (3.2.6) by $(\beta + u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2) + g)$, we get

$$\begin{aligned}&(\beta + u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2) + g)\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right) \cdot \\ &\left(-D_u(y)\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right)(u_0 + g + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2))\right) \\ &= -\Phi_1 \alpha(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^2) + g).\end{aligned} \tag{3.2.7}$$

We consider the part of (3.2.7) corresponding to the term of power ε^{-2}:

$$-(\beta + u_0 + g)\nabla_y \cdot (D_u(y)\nabla_y u_0) = 0 \text{ in } Y_s,$$

that is satisfied either if $u_0 = -\beta - g$ in Y_s, which implies $u_0 = u_0(z)$, or if

$$-\nabla_y \cdot (D_u(y)\nabla_y u_0) = 0 \text{ in } Y_s. \tag{3.2.8}$$

We substitute (3.2.3) in (1.3.3) to obtain

$$\begin{aligned} &D_u(y)\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right)((u_0 + g + \varepsilon u_1 + \varepsilon^2 u_2 + h.o.t)) \cdot n(y) \\ &= \frac{\Phi_3}{\Phi_2}\tilde{\sigma}(\varepsilon(u_0 + g - Hv_0) + \varepsilon^2(u_1 - Hv_1) + h.o.t) \text{ on } \partial Y_s. \end{aligned} \tag{3.2.9}$$

In (3.2.9), we just consider the terms corresponding to ε^{-1} and we get

$$-D_u(y)\nabla_y u_0 \cdot n(y) = 0 \text{ on } \partial Y_s. \tag{3.2.10}$$

Putting (3.2.8) and (3.2.10) together we obtain the following problem

$$\begin{cases} -\nabla_y \cdot (D_u(y)\nabla_y u_0) = 0 \text{ in } Y_s, \\ -D_u(y)\nabla_y u_0 \cdot n(y) = 0 \text{ on } \partial Y_s, \\ u_0(z, \cdot) \text{ is } Y_s - \text{periodic for each given } z \in \Omega_s. \end{cases} \tag{3.2.11}$$

(3.3.7) implies $u_0 = u_0(z)$, $z \in \Omega_s$.

We substitute (3.2.4) in (1.3.1) to obtain

$$\begin{aligned} &\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right) \cdot (-D_v(y)\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + h.o.t) \\ &+ B(y)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + h.o.t)) = 0. \end{aligned} \tag{3.2.12}$$

We consider the part of (3.2.12) corresponding to the term of power ε^{-2}:

$$-\nabla_y \cdot (D_v(y)\nabla_y v_0) = 0 \text{ in } Y_f. \tag{3.2.13}$$

We substitute (3.2.4) in (1.3.4) to obtain

$$\begin{aligned} &\left(-D_v(y)\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + h.o.t) + B(y)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + h.o.t)\right) \cdot n(y) \\ &= \Phi_3\tilde{\sigma}(\varepsilon(u_0 + g - Hv_0) + \varepsilon^2(u_1 - Hv_1) + h.o.t) \text{ on } \partial Y_f. \end{aligned} \tag{3.2.14}$$

In (3.2.14), we just consider the terms corresponding to ε^{-1} and we get

$$-D_v(y)\nabla_y v_0 \cdot n(y) = 0 \text{ on } \partial Y_f. \tag{3.2.15}$$

Putting (3.2.13) and (3.2.15) together we obtain the following problem

$$\begin{cases} -\nabla_y \cdot (D_v(y)\nabla_y v_0) = 0 \text{ in } Y_f, \\ -D_v(y)\nabla_y v_0 \cdot n(y) = 0 \text{ on } \partial Y_f, \\ v_0(z, \cdot) \text{ is } Y_f - \text{periodic for each given } z \in \Omega_f. \end{cases} \tag{3.2.16}$$

(3.3.8) implies $v_0 = v_0(z)$, $z \in \Omega_f$. $\square$

Remark. *If we were to develop the analysis in 3 dimensions, we could have proved Proposition 3.2.1 using a direct application of Rellich–Kondrachov theorem. We refer the reader to section 5.7 of [14]. Indeed, we can use the fact that, in 3 dimensions, $H^1(\cdot)$ is compactly embedded in $L^2(\cdot)$. Since u_ε is bounded in $H^1_{\ell r}(\Omega_s)$, it will converge strongly to a function $\hat{u}_0 = \hat{u}_0(x)$ in $L^2(\Omega_s)$. From Theorem 3.1.1, we know that u_ε converges weakly to a function $\overline{u}_0 = \overline{u}_0(x, y)$ in $L^2(\Omega_s \times Y_s)$. Therefore, from the uniqueness of the weak limit, we conclude*

$$\hat{u}_0(x) \equiv \overline{u}_0(x, y) \equiv u_0(x).$$

Step 3: Weak formulation of the limit two-scale problem

For simplicity of notation, we define $\Sigma_\varepsilon := \partial\Omega^\varepsilon \setminus \Gamma$. We also assume

$$|Y_s| = |Y_f| = 1. \tag{3.2.17}$$

We now consider (2.1.3) and we choose $\varphi = \psi(z) + \varepsilon\psi_1\left(z, \frac{z}{\varepsilon}\right)$, where $\psi \in C_0^\infty(\Omega_f)$ and $\psi_1 \in C_0^\infty(\Omega_f; C_\#^\infty(Y_f))$. We have

$$\begin{aligned}
&\int_{\Omega_f} (D_{v\varepsilon}(z)\nabla v_\varepsilon - B_\varepsilon v_\varepsilon)\left(\nabla\psi(z) + \varepsilon\nabla_x\psi_1\left(z, \frac{z}{\epsilon}\right) + \nabla_y\psi_1\left(z, \frac{z}{\epsilon}\right)\right) dz + \\
&+ \varepsilon\Phi_3\tilde{\sigma}\int_{\Sigma_\varepsilon} (u_\varepsilon + g - Hv_\varepsilon)\left(\psi(z) + \varepsilon\psi_1\left(z, \frac{z}{\varepsilon}\right)\right) d\sigma_\varepsilon = 0.
\end{aligned} \tag{3.2.18}$$

Rearranging the terms of (3.2.18), we obtain

$$\begin{aligned}
&\int_{\Omega_f} (D_{v\varepsilon}(z)\nabla v_\varepsilon - B_\varepsilon v_\varepsilon)\left(\nabla\psi(z) + \nabla_y\psi_1\left(z, \frac{z}{\epsilon}\right)\right) dz \\
&+ \varepsilon\int_{\Omega_f} (D_{v\varepsilon}(z)\nabla v_\varepsilon - B_\varepsilon v_\varepsilon)\nabla_z\psi_1\left(z, \frac{z}{\epsilon}\right) dz + \\
&+ \varepsilon\Phi_3\tilde{\sigma}\int_{\Sigma_\varepsilon} (u_\varepsilon + g - Hv_\varepsilon)\psi(z)\, d\sigma_\varepsilon + \varepsilon^2\Phi_3\tilde{\sigma}\int_{\Sigma_\varepsilon} (u_\varepsilon + g - Hv_\varepsilon)\psi_1\left(z, \frac{z}{\varepsilon}\right) d\sigma_\varepsilon = 0
\end{aligned} \tag{3.2.19}$$

We now pass to the two scale limit for $\varepsilon \to 0$. We recall Definition 3.1.1 and 3.1.2 and the compactness step; notice that the second and fourth integrals on the left hand side of (3.2.19) go to zero when $\varepsilon \to 0$, we end up with

$$\begin{aligned}
&\int_{\Omega_f}\int_{Y_f} \left[D_v(y)\left(\nabla v_0 + \nabla_y v_1\right) - B(y)v_0\right]\left(\nabla\psi(z) + \nabla_y\psi_1\left(z, y\right)\right) dz dy \\
&= -\Phi_3\tilde{\sigma}\int_{\Omega_f}\int_{\Sigma} (u_0 + g - Hv_0)\psi(z)\, dz d\sigma_y.
\end{aligned}$$

We now consider (2.1.4) and we choose $\phi = \theta(z) + \varepsilon\theta_1\left(z, \frac{z}{\varepsilon}\right)$, where $\theta \in C_0^\infty(\Omega_s)$ and $\theta_1 \in C_0^\infty(\Omega_s; C_\#^\infty(Y_s))$. We have

$$
\begin{aligned}
&\int_{\Omega_s^\varepsilon} D_{u\varepsilon}(z)\nabla u_\varepsilon \left(\nabla\theta(z) + \varepsilon\nabla_z\theta_1\left(z,\frac{z}{\epsilon}\right) + \nabla_y\theta_1\left(z,\frac{z}{\epsilon}\right)\right)\, dz \\
&- \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Sigma\varepsilon}(u_\varepsilon + g - Hv_\varepsilon)\left(\theta(z) + \varepsilon\theta_1\left(z,\frac{z}{\varepsilon}\right)\right)\, d\sigma_\varepsilon \\
&= -\int_{\Omega_s^\varepsilon}\Phi_1 F(u_\varepsilon + g)\left(\theta(z) + \varepsilon\theta_1\left(z,\frac{z}{\varepsilon}\right)\right)\, dz \\
&- \int_{\Omega_s^\varepsilon} w_\varepsilon\left(\nabla\theta(z) + \varepsilon\nabla_z\theta_1\left(z,\frac{z}{\epsilon}\right) + \nabla_y\theta_1\left(z,\frac{z}{\epsilon}\right)\right)\, dz. \qquad (3.2.20)
\end{aligned}
$$

Rearranging the terms of (3.2.20), we obtain

$$
\begin{aligned}
&\int_{\Omega_s^\varepsilon} D_{u\varepsilon}(z)\nabla u_\varepsilon\left(\nabla\theta(z) + \nabla_y\theta_1\left(z,\frac{z}{\epsilon}\right)\right)\, dz + \varepsilon\int_{\Omega_s^\varepsilon} D_{u\varepsilon}(z)\nabla u_\varepsilon\nabla_x\theta_1\left(z,\frac{z}{\epsilon}\right)\, dz \\
&- \varepsilon\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Sigma\varepsilon}(u_\varepsilon + g - Hv_\varepsilon)\theta(z)\, d\sigma_z - \varepsilon^2\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Sigma\varepsilon}(u_\varepsilon + g - Hv_\varepsilon)\theta_1\left(z,\frac{z}{\varepsilon}\right)\, d\sigma_\varepsilon \\
&= -\int_{\Omega_s^\varepsilon}\Phi_1 F(u_\varepsilon + g)\theta(z)\, dz - \varepsilon\int_{\Omega_s^\varepsilon}\Phi_1 F(u_\varepsilon + g)\theta_1\left(z,\frac{z}{\varepsilon}\right)\, dz \\
&- \int_{\Omega_s^\varepsilon} w_\varepsilon\left(\nabla\theta(z) + \nabla_y\theta_1\left(z,\frac{z}{\epsilon}\right)\right)\, dz - \varepsilon\int_{\Omega_s^\varepsilon} w_\varepsilon\nabla_z\theta_1\left(z,\frac{z}{\epsilon}\right)\, dzdy. \qquad (3.2.21)
\end{aligned}
$$

Passing to the two-scale limit for $\varepsilon \to 0$ in (3.2.21), we obtain

$$
\begin{aligned}
&\int_{\Omega_s}\int_{Y_s} D_u(y)(\nabla u_0 + \nabla_y u_1)\left(\nabla\theta(z) + \nabla_y\theta_1(z,y)\right)\, dz - \frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Omega_s}\int_{\Sigma}(u_\varepsilon + g - Hv_\varepsilon)\theta(z)\, dzd\sigma_y \\
&= -\int_{\Omega_s}\int_{Y_s}\Phi_1 F(u_0 + g)\theta(z)\, dzdy - \int_{\Omega_s}\int_{Y_s} w(y)\left(\nabla\theta(z) + \nabla_y\theta_1(z,y)\right)\, dzdy, \qquad (3.2.22)
\end{aligned}
$$

with $w(y) := D_u(y)\tilde{w}$. Notice that we used the continuity of F in order to pass to the limit.

Definition 3.2.1. *The weak formulation of the two scale limit problem reads as follows:*

Find

$$(u_0, v_0, u_1, v_1) \in H^1_{\ell r}(\Omega_s) \times H^1(\Omega_f) \times L^2(\Omega_s; H^1_\#(Y_s)/\mathbb{R}) \times L^2(\Omega_f; H^1_\#(Y_f)/\mathbb{R})$$

that satisfies the following system:

$$
\begin{aligned}
&\int_{\Omega_f}\int_{Y_f}\left[D_v(y)\left(\nabla v_0 + \nabla_y v_1\right) - B(y)v_0\right]\left(\nabla\psi(z) + \nabla_y\psi_1(z,y)\right)\, dzdy \\
&= -\Phi_3\tilde{\sigma}\int_{\Omega_f}\int_{\Sigma}(u_0 + g - Hv_0)\psi(z)\, dzd\sigma_y, \qquad (3.2.23)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Omega_s}\int_{Y_s} D_u(y)(\nabla u_0 + \nabla_y u_1)\left(\nabla\theta(z) + \nabla_y\theta_1(z,y)\right)\, dz - \frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Omega_s}\int_{\Sigma}(u_0 + g - Hv_0)\theta(z)\, dzd\sigma_y \\
&= -\int_{\Omega_s}\int_{Y_s}\Phi_1 F(u_0 + g)\theta(z)\, dzdy - \int_{\Omega_s}\int_{Y_s} w(y)\left(\nabla\theta(z) + \nabla_y\theta_1(z,y)\right)\, dzdy, \qquad (3.2.24)
\end{aligned}
$$

for all $\psi \in C_0^\infty(\Omega_f)$, $\psi_1 \in C_0^\infty(\Omega_f; C_\#^\infty(Y_f))$, $\theta \in C_0^\infty(\Omega_s)$, $\theta_1 \in C_0^\infty(\Omega_s; C_\#^\infty(Y_s))$.

Step 4: Strong formulation of the two-scale limit problem Integrating (3.2.23) by parts, we end up with

$$\begin{aligned}
&\int_{\Omega_f} -\nabla_z \cdot \left(\int_{Y_f} D_v(y)\,(\nabla v_0 + \nabla_y v_1) - B(y) v_0 \, dy \right) \psi(z)\, dz \\
&+ \int_{\Omega_f} \int_{Y_f} -\nabla_y \cdot (D_v(y)(\nabla v_0 + \nabla_y v_1) - B(y) v_0) \psi_1 (z, y) \; dzdy \\
&= -\Phi_3 \tilde{\sigma} \int_{\Omega_f} \int_{\Sigma} (u_0 + g - H v_0) \psi(z)\, dz d\sigma_y. \qquad (3.2.25)
\end{aligned}$$

Integrating (3.2.24) by parts, we end up with

$$\begin{aligned}
&\int_{\Omega_s} -\nabla_z \cdot \left(\int_{Y_s} D_u(y)(\nabla u_0 + \nabla_y u_1)\, dy \right) \theta(z)\, dz \\
&+ \int_{\Omega_s} \int_{Y_s} -\nabla_y \cdot (D_u(y)(\nabla u_0 + \nabla_y u_1)) \theta_1 (z, y) \; dzdy \\
&= \frac{\Phi_3}{\Phi_2} \tilde{\sigma} \int_{\Omega_s} \int_{\Sigma} (u_\varepsilon + g - H v_\varepsilon) \theta(z)\, dz d\sigma_y \\
&- \int_{\Omega_s} \int_{Y_s} \Phi_1 F(u_0 + g) \theta(z)\, dzdy + \int_{\Omega_s} \nabla_z \cdot \left(\int_{Y_s} w(y)\, dy \right) \theta(z)\, dz \\
&+ \int_{\Omega_s} \int_{Y_s} \nabla_y \cdot (w(y)) \theta_1 (z, y) \; dzdy \qquad (3.2.26)
\end{aligned}$$

Choosing $\psi = 0$ in (3.2.25), we obtain

$$-\nabla_y \cdot (D_v(y)(\nabla v_0 + \nabla_y v_1) - B(y) v_0) = 0 \;\text{ in } \Omega_f \times Y_f. \qquad (3.2.27)$$

Choosing $\psi_1 = 0$ in (3.2.25), we obtain

$$-\nabla_z \cdot \left(\int_{Y_f} D_v(y)(\nabla v_0 + \nabla_y v_1) - B(y) v_0 \, dy \right) = -\Phi_3 \tilde{\sigma} |\Sigma| (u_0 + g - H v_0) \;\text{ in } \Omega_f. \qquad (3.2.28)$$

Choosing $\theta = 0$ in (3.2.26), we obtain

$$-\nabla_y \cdot (D_u(y)(\nabla u_0 + \tilde{w} + \nabla_y u_1)) = 0 \;\text{ in } \Omega_s \times Y_s. \qquad (3.2.29)$$

Choosing $\theta_1 = 0$ in (3.2.26), we obtain

$$\begin{aligned}
&-\nabla_z \cdot \left(\int_{Y_s} D_u(y)(\nabla u_0 + \nabla_y u_1)\, dy \right) \\
&= \frac{\Phi_3}{\Phi_2} \tilde{\sigma} |\Sigma| (u_0 + g - H v_0) - \Phi_1 F(u_0 + g) \;\text{ in } \Omega_s, \qquad (3.2.30)
\end{aligned}$$

where we used that $\nabla_z \cdot \left(\int_{Y_s} w(y)\, dy \right) = 0$.

Lemma 3.2.1. *Let $\nabla_y \cdot B(y) = 0$. We can write*

$$u_1(z, y) = -\sum_{j=1}^{2} w_j(y) \partial_{z_j} (u_0 + g) + \tilde{u}_1(z) \qquad (3.2.31)$$

and

$$v_1(z,y) = -\sum_{j=1}^{2} \hat{w}_j(y)\partial_{z_j} v_0 + \tilde{v}_1(z), \tag{3.2.32}$$

where the cell functions w_j are solutions of the cell problems

$$\begin{cases} -\nabla_y \cdot (D_u(y)\nabla_y w_j) = -\sum_{i=1}^{2} \partial_{y_i} D_u^{ij}(y) \ \text{ in } Y_s, \\ \int_{Y_s} w_j(y)\,dy = 0 \\ w_j \text{ is } Y_s - \text{periodic for all } j \in \{1,2\}, \end{cases} \tag{3.2.33}$$

and where the cell functions $\hat{w}_j$ are solutions of the cell problems

$$\begin{cases} -\nabla_y \cdot (D_v(y)\nabla_y \hat{w}_j) = -\sum_{i=1}^{2} \partial_{y_i} D_v^{ij}(y) \ \text{ in } Y_f, \\ \int_{Y_f} \hat{w}_j(y)\,dy = 0 \\ \hat{w}_j \text{ is } Y_f - \text{periodic for all } j \in \{1,2\}. \end{cases} \tag{3.2.34}$$

Proof. We substitute (3.2.31) in (3.2.29) to obtain

$$-\nabla_y \cdot (D_u(y)\nabla_z(u_0(z)+g)) = -\nabla_y \cdot \left(D_u(y) \sum_{j=1}^{2} \nabla_y w_j(y) \partial_{z_j}(u_0+g) \right),$$

which implies (3.2.33).

We substitute (3.2.32) in (3.2.27) to obtain

$$-\nabla_y \cdot (D_v(y)\nabla_z v_0(z) - B(y)v_0(z)) = -\nabla_y \cdot \left(D_v(y) \sum_{j=1}^{2} \nabla_y \hat{w}_j(y) \partial_{z_j} v_0 \right).$$

Noticing that $\nabla_y \cdot (B(y)v_0) = v_0 \nabla_y \cdot (B(y)) = 0$, we obtain (3.2.34). □

Assuming B to be divergence free and using Lemma 3.2.1, we are able to derive the strong form of the two-scale limit problem.

Proposition 3.2.2. *The strong formulation of the two-scale limit problem reads as follows*

$$\begin{cases} -\sum_{i,k=1}^{2} \left[\boldsymbol{D}_v^{ik} \partial^2_{z_i z_k} v_0 - \boldsymbol{B}_i \partial_{z_i} v_0 \right] = -\Phi_3 \tilde{\sigma} |\Sigma| (u_0 + g - Hv_0) \ \text{ in } \Omega_f, \\ -\sum_{i,k=1}^{2} \left[\boldsymbol{D}_u^{ik} \partial^2_{z_i z_k} u_0 \right] = \dfrac{\Phi_3}{\Phi_2} \tilde{\sigma} |\Sigma| (u_0 + g - Hv_0) - \Phi_1 F(u_0+g) \ \text{ in } \Omega_s, \\ (-\boldsymbol{D}_u \nabla_z u_0) \cdot n = 0 \ \text{ on } \Gamma^N \setminus \partial\Omega_f \\ (-\boldsymbol{D}_v \nabla_z v_0 + \boldsymbol{B} v_0) \cdot n = 0 \ \text{ on } \Gamma \cap \partial\Omega_f \\ u_0(z) = 0 \ \ z \in \Gamma^D_\ell \\ u_0(z) = 0 \ \ z \in \Gamma^D_r. \end{cases} \tag{3.2.35}$$

where

$$\boldsymbol{D}_v^{ik} := \sum_{j=1}^{2} \int_{Y_f} \left(D_v^{ik} - D_v^{ij} \partial_{y_j} \hat{w}_k \right) dy,$$

$$\boldsymbol{B}_i = \int_{Y_f} B_i(y)\, dy,$$

$$\boldsymbol{D}_u^{ik} := \sum_{j=1}^{2} \int_{Y_s} \left(D_u^{ik} - D_u^{ij} \partial_{y_j} w_k \right) \, dy.$$

Proof. From Lemma 3.2.1, we substitute (3.2.31) in (3.2.30) and (3.2.32) in (3.2.28), respectively. □

Remark. *If one removes the assumption (3.2.17), the final strong formulation of the two-scale limit problem reads as follows*

$$\begin{cases} -\sum_{i,k=1}^{2} \left[\boldsymbol{D}_v^{ik} \partial^2_{z_i z_k} v_0 - \boldsymbol{B}_i \partial_{z_i} v_0 \right] = -\Phi_3 \tilde{\sigma} |\Sigma||Y_f|(u_0 + g - H v_0) \ \text{ in } \Omega_f, \\ -\sum_{i,k=1}^{2} \left[\boldsymbol{D}_u^{ik} \partial^2_{z_i z_k} u_0 \right] = |Y_s| \left(\dfrac{\Phi_3}{\Phi_2} \tilde{\sigma} |\Sigma| (u_0 + g - H v_0) - \Phi_1 F(u_0 + g) \right) \ \text{ in } \Omega_s, \\ (-\boldsymbol{D}_u \nabla_z u_0) \cdot n = 0 \ \text{ on } \Gamma^N \setminus \partial\Omega_f \\ (-\boldsymbol{D}_v \nabla_z v_0 + \boldsymbol{B} v_0) \cdot n = 0 \ \text{ on } \Gamma \cap \partial\Omega_f \\ u_0(z) = 0 \ \ z \in \Gamma^D_\ell \\ u_0(z) = 0 \ \ z \in \Gamma^D_r. \end{cases}$$

where

$$\boldsymbol{D}_v^{ik} := \sum_{j=1}^{2} \int_{Y_f} \left(D_v^{ik} - D_v^{ij} \partial_{y_j} \hat{w}_k \right) \, dy,$$

$$\boldsymbol{B}_i = \int_{Y_f} B_i(y)\, dy,$$

$$\boldsymbol{D}_u^{ik} := \sum_{j=1}^{2} \int_{Y_s} \left(D_u^{ik} - D_u^{ij} \partial_{y_j} w_k \right) \, dy.$$

Step 5: Well posedness of the two-scale limit problem

The existence of a solution of the strong formulation of the two-scale limit problem is a direct consequence of the existence of the two-scale limit. Therefore, we just present the proof of the uniqueness. For details on the following result, we refer the reader to section 2.3 of [8].

Lemma 3.2.2. *The operators $\boldsymbol{D}_v^{ik}$ and $\boldsymbol{D}_u^{ik}$ satisfy the following coercivity conditions: There exists $\gamma > 0$ such that*

$$\sum_{i,k=1}^{2} \boldsymbol{D}_v^{ik}\xi_i\xi_j \geq \alpha|\xi|^2, \tag{3.2.36}$$

$$\sum_{i,k=1}^{2} \boldsymbol{D}_u^{ik}\xi_i\xi_j \geq \alpha|\xi|^2, \tag{3.2.37}$$

$\forall \xi \in \mathbb{R}^2$.

Proof.

$$\mathbf{D}_v^{ik} = \sum_{j=1}^{2} \int_{Y_f} \left(D_v^{ik} - D_v^{ij}\partial_{y_j}\hat{w}_k\right)\, dy =$$

$$= \int_{Y_f} D_v(y)\left(-\nabla_y \hat{w}_k + e_k\right)\cdot e_i\, dy = \int_{Y_f} \left(D_v(y)\left(e_i - \nabla_y \hat{w}_i\right)\right)\cdot\left(e_k - \nabla_y \hat{w}_k\right)\, dy,$$

so from the coercivity of $D_v(y)$ we obtain (3.2.36). The procedure to obtain (3.2.37) is analogous.

Proposition 3.2.3. *Let $B \in L^\infty(\Omega_f)$ with $\|B\|_{L^\infty(\Omega_f)}$ be arbitrarily small and let $\tilde{\sigma} > 0$ be arbitrarily small as well. There exists a unique solution $(u_0, v_0) \in H^1_{\ell r}(\Omega_s) \times H^1(\Omega_f)$ of the problem (3.2.23)-(3.2.24).*

Proof. Let us assume there exist two different solutions (u_0, v_0) and $(\overline{u}_0, \overline{v}_0)$ of our problem in the sense of Definition 2.1.1. Then both (u_0, v_0) and $(\overline{u}_0, \overline{v}_0)$ satisfy (3.2.23), so we subtract the corresponding weak formulations, use (3.2.32), and choose $\psi_1 = 0$, $\psi = v_0 - \overline{v}_0$ to obtain

$$\int_{\Omega_f}\int_{Y_f} D_v(y)(I - \hat{W})\nabla(v_0 - \overline{v}_0)\nabla(v_0 - \overline{v}_0)\, dzdy = \int_{\Omega_f}\int_{Y_f} B(y)(v_0 - \overline{v}_0)\nabla(v_0 - \overline{v}_0)\, dzdy$$
$$- \Phi_3\tilde{\sigma}\int_{\Omega_f}\int_{\Sigma}(u_0 - \overline{u}_0)(v_0 - \overline{v}_0)\, dzd\sigma_y + \Phi_3\tilde{\sigma}H\int_{\Omega_f}\int_{\Sigma}(v_0 - \overline{v}_0)^2\, dzd\sigma_y, \tag{3.2.38}$$

where I is the identity matrix and

$$\hat{W} = \begin{pmatrix} \partial_{y_1}\hat{w}_1 & \partial_{y_1}\hat{w}_2 \\ \partial_{y_2}\hat{w}_1 & \partial_{y_2}\hat{w}_2 \end{pmatrix}.$$

Similiarly, both (u_0, v_0) and $(\overline{u}_0, \overline{v}_0)$ satisfy (3.2.24), so we subtract the corresponding weak formulations, use (3.2.31), and choose $\theta_1 = 0$, $\theta = u_0 - \overline{u}_0$ to obtain

$$\int_{\Omega_s}\int_{Y_s} D_u(y)(I - W)\nabla(u_0 - \overline{u}_0)\nabla(u_0 - \overline{u}_0)\, dzdy = \frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Omega_s}\int_{\Sigma}(u_0 - \overline{u}_0)^2\, dzd\sigma_y$$
$$- H\frac{\Phi_3}{\Phi_2}\tilde{\sigma}\int_{\Omega_s}\int_{\Sigma}(v_0 - \overline{v}_0)(u_0 - \overline{u}_0)\, dzd\sigma_y - \Phi_1\int_{\Omega_s}\int_{Y_s}(F(u_0 + g) - F(\overline{u}_0 + g))(u_0 - \overline{u}_0)\, dzdy, \tag{3.2.39}$$

with

$$W = \begin{pmatrix} \partial_{y_1} w_1 & \partial_{y_1} w_2 \\ \partial_{y_2} w_1 & \partial_{y_2} w_2 \end{pmatrix}.$$

Using Lemma 3.2.2 we get

$$a\|v_0 - \hat{v}_0\|^2_{H^1(\Omega_f)} \leq \int_{\Omega_f} \int_{Y_f} D_v(y)(I + \hat{W})\nabla(v_0 - \overline{v}_0)\nabla(v_0 - \overline{v}_0)\, dzdy$$

and

$$b\|u_0 - \hat{u}_0\|^2_{H^1(\Omega_f)} \leq \int_{\Omega_s} \int_{Y_s} D_u(y)(I + \hat{W})\nabla(u_0 - \overline{u}_0)\nabla(u_0 - \overline{u}_0)\, dzdy.$$

From this point on, the proof is analogous to the one of Proposition 2.2.2.

3.3 Corrector Estimates

We saw in section 3.2 that we can write

$$u_\varepsilon(z) = u_0(z) + \varepsilon u_1\left(z, \frac{z}{\varepsilon}\right) + \varepsilon^2 u_2\left(z, \frac{z}{\varepsilon}\right) + h.o.t.,$$

where $u_j(z, \cdot)$ are periodic functions in Y_s for each $j \in \mathbb{N}$, $z \in \Omega$. It is possible to show that we can write

$$u_1\left(z, \frac{z}{\varepsilon}\right) = -\sum_{k=1}^{2} \chi_1^k\left(\frac{z}{\varepsilon}\right) \frac{\partial u_0}{\partial z_k}(z) \tag{3.3.1}$$

and

$$u_2\left(z, \frac{z}{\varepsilon}\right) = \sum_{k,\ell=1}^{2} \gamma_1^{k\ell}\left(\frac{z}{\varepsilon}\right) \frac{\partial^2 u_0}{\partial z_k \partial z_l}(z), \tag{3.3.2}$$

with $\chi_1^k \in H^1(Y_s)$ solving

$$\begin{cases} -\nabla_y \cdot (D_u(y)\chi_1^k(y)) = \sum_{i=1}^2 \partial_{y_i} D_u^{ij}(y) \text{ in } Y_s, \\ -D_u(y)\chi_1^k(y) \cdot n(y) = D_u^{ij} n_i \text{ on } \partial Y_s, \\ \chi_1^k \text{ is } Y_s - \text{periodic}, \end{cases} \tag{3.3.3}$$

and $\gamma_1^{k\ell} \in H^1(Y_s)$ solving

$$\begin{cases} -\sum_{i=1}^2 \frac{\partial}{\partial y_i}\left[D_u^{ij}(y)\left(\frac{\partial \gamma_1^{k\ell}}{\partial y_j} - \delta_{\ell j}\chi_1^k(y)\right)\right] = -\sum_{\ell,j=1}^2 \frac{\partial(\chi_1^k - y_k)}{\partial y_j} - \frac{|Y_s|}{|Y|}\widehat{q_1^{ik}} \text{ in } Y_s, \\ \sum_{i,j=1}^2 D_u^{ij}(y)\left(\frac{\partial \gamma_1^{k\ell}}{\partial y_j} - \delta_{\ell j}\chi_1^k(y)\right) n_i = 0 \text{ on } \partial Y_s, \\ \gamma_1^{k\ell} \text{ is } Y_s - \text{periodic}, \end{cases} \tag{3.3.4}$$

where

$$\widehat{q_1^{ik}} = \frac{1}{|Y_s|}\left(\int_{Y_s} D_u^{ik}(y)\, dy - \int_{Y_s} D_u^{ij}(y) \frac{\partial \chi_1^k(y)}{\partial y_j}\, dy\right).$$

Similarly, we can write

$$v_\varepsilon(z) = v_0(z) + \varepsilon v_1\left(z, \frac{z}{\varepsilon}\right) + \varepsilon^2 v_2\left(z, \frac{z}{\varepsilon}\right) + h.o.t.$$

It is possible to show that we can write

$$v_1\left(z, \frac{z}{\varepsilon}\right) = -\sum_{k=1}^{2} \chi_2^k\left(\frac{z}{\varepsilon}\right)\frac{\partial v_0}{\partial z_k}(z) \tag{3.3.5}$$

and

$$v_2\left(z, \frac{z}{\varepsilon}\right) = \sum_{k,\ell=1}^{2} \gamma_2^{k\ell}\left(\frac{z}{\varepsilon}\right)\frac{\partial^2 v_0}{\partial z_k \partial z_l}(z), \tag{3.3.6}$$

with $\chi_2^k \in H^1(Y_f)$ solving

$$\begin{cases} -\nabla_y \cdot (D_v(y)\chi_2^k(y)) = \sum_{i=1}^2 \partial_{y_i} D_v^{ij}(y) \ \text{ in } Y_f, \\ -D_v(y)\chi_2^k(y) \cdot n(y) = D_v^{ij} n_i \ \text{ on } \partial Y_f, \\ \chi_2^k \text{ is } Y_f - \text{periodic}, \end{cases} \tag{3.3.7}$$

and $\gamma_2^{k\ell} \in H^1(Y_f)$ solving

$$\begin{cases} -\sum_{i=1}^2 \dfrac{\partial}{\partial y_i}\left[D_v^{ij}(y)\left(\dfrac{\partial \gamma_2^{k\ell}}{\partial y_j} - \delta_{\ell j}\chi_2^k(y)\right)\right] = -\sum_{\ell,j=1}^2 \dfrac{\partial(\chi_2^k - y_k)}{\partial y_j} - \dfrac{|Y_f|}{|Y|}\widehat{q_2^{ik}} \ \text{ in } Y_f, \\ \sum_{i,j=1}^2 D_v^{ij}(y)\left(\dfrac{\partial \gamma_2^{k\ell}}{\partial y_j} - \delta_{\ell j}\chi_2^k(y)\right) n_i = 0 \ \text{ on } \partial Y_f, \\ \gamma_2^{k\ell} \text{ is } Y_f - \text{periodic}, \end{cases} \tag{3.3.8}$$

where

$$\widehat{q_2^{ik}} = \frac{1}{|Y_f|}\left(\int_{Y_f} D_v^{ik}(y)\, dy - \int_{Y_f} D_v^{ij}(y)\frac{\partial \chi_2^k(y)}{\partial y_j}\, dy\right).$$

We refer the reader to chapter 2 of [13] for a complete discussion of the results that we just mentioned, i.e the structure of (3.3.5)-(3.3.8).

From now on, we denote with C a generic constant independent of ε. Let us define two functions $m_{u\varepsilon}$ and $m_{v\varepsilon}$ such that

- $m_{u\varepsilon} \in \mathcal{D}(\Omega_s)$,
- $m_{u\varepsilon} = 0$ if $dist(z, \partial\Omega_s) \leq \varepsilon$,
- $m_{u\varepsilon} = 1$ if $dist(z, \partial\Omega_s) \geq 2\varepsilon$,
- $\varepsilon\left|\dfrac{\partial m_{u\varepsilon}}{\partial z_i}\right| \leq C,\ i = 1, 2,$

- $m_{v\varepsilon} \in \mathcal{D}(\Omega_f)$,
- $m_{v\varepsilon} = 0$ if $dist(z, \partial\Omega_f) \leq \varepsilon$,
- $m_{v\varepsilon} = 1$ if $dist(z, \partial\Omega_f) \geq 2\varepsilon$,
- $\varepsilon \left| \dfrac{\partial m_{v\varepsilon}}{\partial z_i} \right| \leq C$, $i = 1, 2$.

Theorem 3.3.1. *Let $B(\cdot)$ be divergence free. Let $B_\varepsilon \in L^\infty(\Omega_f^\varepsilon)$. Let us assume there exists a constant γ such that*

$$\|B_\varepsilon - B\|_{L^\infty(\Omega_f^\varepsilon)} \leq C\varepsilon^\gamma, \tag{3.3.9}$$

with C independent of ε. The following estimate holds:

$$\|u_\varepsilon - u_0 - m_{u\varepsilon}(\varepsilon u_1 + \varepsilon^2 u_2)\|_{H^1_{\ell r}(\Omega_s^\varepsilon)} + \|v_\varepsilon - v_0 - m_{v\varepsilon}(\varepsilon v_1 + \varepsilon^2 v_2)\|_{H^1(\Omega_f^\varepsilon)} \leq c_1 \varepsilon^{1/2} + c_2 \varepsilon^\gamma, \tag{3.3.10}$$

with c_1 and c_2 independent of ε.

Proof. We have to find an estimate for the function

$$\Phi_{u\varepsilon} := u_\varepsilon - u_0 - m_{u\varepsilon}(\varepsilon u_1 + \varepsilon^2 u_2) = \phi_\varepsilon + (1 - m_{u\varepsilon})(\varepsilon u_1 + \varepsilon^2 u_2),$$

where $\phi_\varepsilon := u_\varepsilon - (u_0 + \varepsilon u_1 + \varepsilon^2 u_2)$.

We introduce the following operators:

$$\begin{aligned} U_0 &= -\nabla_y \cdot (D_u(y)\nabla_y), \\ U_1 &= -\nabla_z \cdot (D_u(y)\nabla_y) - \nabla_y \cdot (D_u(y)\nabla_z), \\ U_2 &= -\nabla_z \cdot (D_u(y)\nabla_z). \end{aligned} \tag{3.3.11}$$

Let us introduce the following notation:

$$\frac{\partial f}{\partial \nu_{U_0}} := D_u(y)\nabla_y f \cdot n.$$

Equating the coefficients corresponding to the powers ε^{-2}, ε^{-1} and ε^0 in (3.2.7) and the ones corresponding to the powers ε^{-1}, ε^0 and ε^1 in (3.2.9), respectively, we obtain:

$$\begin{cases} U_0 u_0 = 0 \;\text{ in } Y_s, \\ \dfrac{\partial u_0}{\partial \nu_{U_0}} = 0 \;\text{ on } \partial Y_s, \\ u_0(z, \cdot) \text{ is } Y_s - \text{periodic for each given } z \in \Omega_s, \end{cases} \tag{3.3.12}$$

$$\begin{cases} U_0 u_1 = -U_1(u_0 + g) \;\text{ in } Y_s, \\ \dfrac{\partial u_1}{\partial \nu_{U_0}} = -D_u(y)\nabla_z(u_0 + g) \cdot n(y) \;\text{ on } \partial Y_s, \\ u_1(z, \cdot) \text{ is } Y_s - \text{periodic for each given } z \in \Omega_s, \end{cases} \tag{3.3.13}$$

$$
\begin{cases}
U_0 u_2 = -\dfrac{u_1}{\beta + u_0 + g}\left[U_1(u_0+g) + U_0 u_1\right] - \dfrac{\Phi_1 \alpha(u_0+g)}{\beta + u_0 + g} - U_1 u_1 - U_2(u_0+g) \ \text{ in } Y_s,\\
\dfrac{\partial u_2}{\partial \nu_{U_0}} = \dfrac{\Phi_3}{\Phi_2}\tilde{\sigma}(u_0 + g - Hv_0) - D_u(y)\nabla_z u_1 \cdot n(y) \ \text{ on } \partial Y_s,\\
u_2(z,\cdot) \text{ is } Y_s - \text{periodic for each given } z \in \Omega_s.
\end{cases}
\tag{3.3.14}
$$

We can show that, if we consider a sufficiently smooth function $f_\varepsilon(z) = f\left(z, \dfrac{z}{\varepsilon}\right)$, the following chain rule of differentiation holds:

$$
-\nabla_z \cdot (D_{u\varepsilon}\nabla_z f_\varepsilon) = [(\varepsilon^{-2}U_0 + \varepsilon^{-1}U_1 + U_2)f]\left(z, \frac{z}{\varepsilon}\right). \tag{3.3.15}
$$

We have that on Ω_s^ε, the function ϕ_ε satisfies

$$
\begin{aligned}
&-\nabla_z \cdot (D_{u\varepsilon}\nabla_z \phi_\varepsilon) = [(\varepsilon^{-2}U_0 + \varepsilon^{-1}U_1 + U_2)\phi]\left(z, \frac{z}{\varepsilon}\right)\\
&= -\nabla_z \cdot (D_{u\varepsilon}\nabla_z u_\varepsilon) - \varepsilon^{-2}U_0 u_0 - \varepsilon^{-1}(U_0 u_1 + U_1 u_0)\\
&- (U_0 u_2 + U_1 u_1 + U_2 u_0) - \varepsilon(U_1 u_2 + U_2 u_1) - \varepsilon^2 U_2 u_2\\
&= -\Phi_1 F(u_\varepsilon + g) + \frac{\Phi_1 \alpha(u_0+g)}{\beta + u_0 + g} - \varepsilon(U_1 u_2 + U_2 u_1) - \varepsilon^2 U_2 u_2,
\end{aligned}
$$

where we used (3.3.12), (3.3.13), (3.3.14), and (1.3.2). Notice that we also used

$$
\nabla_z \cdot (D_{u\varepsilon}\tilde{w})\left(z, \frac{z}{\varepsilon}\right) = \left[\left(\nabla_z + \frac{1}{\varepsilon}\nabla_y\right) \cdot (D_{u\varepsilon}\tilde{w})\right]_{|y=\frac{z}{\varepsilon}}.
$$

Similarly, using (1.3.3), (3.2.9), we obtain that on Σ^ε, ϕ_ε satisfies the relation:

$$
D_{u\varepsilon}\nabla_z \phi_\varepsilon \cdot n = \varepsilon \frac{\Phi_3}{\Phi_2}\tilde{\sigma}(u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) - \varepsilon^2 D_{u\varepsilon}\nabla_z u_2 \cdot n.
$$

Therefore, ϕ_ε satisfies the following system:

$$
\begin{cases}
-\nabla_z \cdot (D_{u\varepsilon}\nabla_z \phi_\varepsilon) = g_\varepsilon \ \text{ in } \Omega_s^\varepsilon,\\
D_{u\varepsilon}\nabla_z \phi_\varepsilon \cdot n = \varepsilon \dfrac{\Phi_3}{\Phi_2}\tilde{\sigma}(u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) - \varepsilon^2 D_{u\varepsilon}\nabla_z u_2 \cdot n \ \text{ on } \Sigma^\varepsilon,\\
\phi_\varepsilon = -\varepsilon u_1 - \varepsilon^2 u_2 \ \text{ on } \Gamma_\ell^D \cup \Gamma_r^D,\\
D_{u\varepsilon}\nabla_z \phi_\varepsilon \cdot n = -\varepsilon^2 D_{u\varepsilon}\nabla_z u_2 \cdot n \ \text{ on } \Gamma^N \setminus \partial\Omega_f^\varepsilon,
\end{cases}
\tag{3.3.16}
$$

where

$$
g_\varepsilon := -\Phi_1 F(u_\varepsilon + g) + \frac{\Phi_1 \alpha(u_0+g)}{\beta + u_0 + g} - \varepsilon(U_1 u_2 + U_2 u_1) - \varepsilon^2 U_2 u_2. \tag{3.3.17}
$$

Let $\phi \in H^1_{\ell r}(\Omega_s^\varepsilon)$ be taken arbitrarily. We have

$$
\begin{aligned}
&\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla_z \phi_\varepsilon \nabla_z \phi \, dz = \int_{\Omega_s^\varepsilon} g_\varepsilon \phi \, dz\\
&+ \int_{\Sigma^\varepsilon}\left(-\varepsilon \frac{\Phi_3}{\Phi_2}\tilde{\sigma}(u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) + \varepsilon^2 D_{u\varepsilon}\nabla_z u_2 \cdot n\right)\phi \, d\sigma\\
&+ \int_{\Gamma^N \setminus \partial\Omega_f^\varepsilon} \varepsilon^2 D_{u\varepsilon}\nabla_z u_2 \cdot n\phi \, d\sigma.
\end{aligned}
\tag{3.3.18}
$$

Noticing that $F(u_0+g) = \dfrac{\Phi_1 \alpha (u_0+g)}{\beta + u_0 + g}$, we can use the Taylor expansion of $F(\cdot)$ in a neighborhood of u_0+g, and we conclude that

$$|F(u_\varepsilon + g) - F(u_0+g)| \leq \varepsilon |F'(u_0+g)|. \tag{3.3.19}$$

Using (3.3.19), we can conclude that

$$\|g_\varepsilon\|_{L^2(\Omega_s^\varepsilon)} \leq \varepsilon C, \tag{3.3.20}$$

with C independent of ε. Indeed, we notice that using (3.3.5), (3.3.6) and (3.3.11), we are able to express (3.3.38) in terms of

- the derivatives of u_0 up to the fourth order, which are in $L^\infty(\Omega_s)$;
- χ_1^k and $\gamma_1^{k\ell}$, that are in $H^1(Y_s)$, $k, l \in \{1,2\}$.

Let us define

$$N_\varepsilon := \frac{|\Omega|}{|\varepsilon Y|} \approx \varepsilon^{-2} \frac{|\Omega|}{|Y|},$$

we have

$$\begin{aligned}
\left(\int_{\Sigma_\varepsilon} |D_{u\varepsilon} \nabla_z u_2 \cdot n|^2 \, d\sigma_z \right)^{1/2} &\leq \left(C \sum_{k,\ell=1}^{2} \int_{\Sigma_\varepsilon} \left| \gamma_1^{k\ell} \left(\frac{x}{\varepsilon} \right) \right|^2 d\sigma_z \right)^{1/2} \\
= \left(C\varepsilon \sum_{k,\ell=1}^{2} \sum_{i=1}^{N_\varepsilon} \int_{\Sigma} \left| \gamma_1^{k\ell}(y) \right|^2 d\sigma_y \right)^{1/2} &\approx \left(C\varepsilon^{-1} \frac{|\Omega|}{|Y|} \sum_{k,\ell=1}^{2} \int_{\Sigma} \left| \gamma_1^{k\ell}(y) \right|^2 d\sigma_y \right)^{1/2} = \tilde{C}\varepsilon^{-1/2},
\end{aligned} \tag{3.3.21}$$

where we used the periodicity of $\gamma_1^{k\ell}$.

From (3.3.20) we obtain

$$\left| \int_{\Omega_s^\varepsilon} g_\varepsilon \phi \, dz \right| \leq C\varepsilon \|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)} \tag{3.3.22}$$

while from (3.3.21) we obtain

$$\left| \int_{\Sigma^\varepsilon} \varepsilon^2 D_{u\varepsilon} \nabla_z u_2 \cdot n \phi \, d\sigma \right| \leq C\varepsilon^{3/2} \|\phi\|_{L^2(\Sigma^\varepsilon)} \leq C\varepsilon \|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}, \tag{3.3.23}$$

where we used (2.2.1). Recalling Proposition 2.2.1, and using again (2.2.1) and the regularity of u_0 and v_0, we get

$$\left| \int_{\Sigma^\varepsilon} -\varepsilon \frac{\Phi_3}{\Phi_2} \tilde{\sigma} (u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) \phi \, d\sigma \right| \leq C\varepsilon \|\phi\|_{L^2(\Sigma^\varepsilon)} \leq C\varepsilon^{1/2} \|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}. \tag{3.3.24}$$

Recalling Lemma 2.2.1, we get

$$\left| \int_{\Gamma^N \setminus \partial\Omega_f^\varepsilon} \varepsilon^2 D_{u\varepsilon} \nabla_z u_2 \cdot n \phi \, d\sigma \right| \leq C\varepsilon^2 \|\phi\|_{L^2(\Gamma^N \setminus \partial\Omega_f^\varepsilon)} \leq C\varepsilon^2 \|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}. \tag{3.3.25}$$

Combining (3.3.22), (3.3.23), (3.3.24) and (3.3.25) we get

$$\left|\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla_z\phi_\varepsilon\nabla_z\phi\,dz\right| \le C\varepsilon^2\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}+C\varepsilon\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}+C\varepsilon^{1/2}\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}\le C\varepsilon^{1/2}\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}. \tag{3.3.26}$$

We observe that the following upper bound holds:

$$\begin{aligned}&\left|\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla_z((1-m_{u\varepsilon})(\varepsilon u_1+\varepsilon^2 u_2))\nabla_z\phi\,dz\right|\\ &\le C\|\nabla_z((1-m_{u\varepsilon})(\varepsilon u_1+\varepsilon^2 u_2))\|_{L^2(\Omega_s^\varepsilon)}\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}.\end{aligned} \tag{3.3.27}$$

and

$$\begin{aligned}&\|\nabla_z((1-m_{u\varepsilon})(\varepsilon u_1+\varepsilon^2 u_2))\|_{L^2(\Omega_s^\varepsilon)}\\ &\le C\varepsilon\|\nabla_z((1-m_{u\varepsilon})\|_{L^2(\Omega_s^\varepsilon)}\|(1-m_{u\varepsilon})\nabla_z(\varepsilon u_1+\varepsilon^2 u_2))\|_{L^2(\Omega_s^\varepsilon)},\end{aligned} \tag{3.3.28}$$

where we used the regularity of u_1 and u_2. Recalling again the properties of $m_{u\varepsilon}$, we have

$$\begin{aligned}&\|\nabla_z((1-m_{u\varepsilon})\|^2_{L^2(\Omega_s^\varepsilon)}\le \int_{\Omega_s^\varepsilon\cap\{z|dist(z,\partial\Omega_s^\varepsilon)\le 2\varepsilon\}}|\nabla_z m_{u\varepsilon}|^2\,dz\\ &\le \frac{1}{\varepsilon^2}\,|\Omega_s^\varepsilon\cap\{z|dist(z,\partial\Omega_s^\varepsilon)\le 2\varepsilon\}|\le\frac{C\varepsilon}{\varepsilon^2}=C\varepsilon^{-1}.\end{aligned} \tag{3.3.29}$$

Similiarly,

$$\begin{aligned}&\|(1-m_{u\varepsilon})\nabla_z(\varepsilon u_1+\varepsilon^2 u_2))\|^2_{L^2(\Omega_s^\varepsilon)}\\ &\le\left[\int_{\Omega_s^\varepsilon}|\nabla_z(\varepsilon u_1+\varepsilon^2 u_2)|^2\,dz\right]\left[\int_{\Omega_s^\varepsilon}(1-m_{u\varepsilon})^2\,dz\right]\\ &\le C\varepsilon^2\int_{\Omega_s^\varepsilon\cap\{z|dist(z,\partial\Omega_s^\varepsilon)\le 2\varepsilon\}}(1-m_{u\varepsilon})^2\,dz\\ &\le C\varepsilon^2\,|\Omega_s^\varepsilon\cap\{z|dist(z,\partial\Omega_s^\varepsilon)\le 2\varepsilon\}|\le C\varepsilon^3.\end{aligned} \tag{3.3.30}$$

Using (3.3.28), (3.3.29), (3.3.30) we obtain

$$\|\nabla_z((1-m_{u\varepsilon})(\varepsilon u_1+\varepsilon^2 u_2))\|_{L^2(\Omega_s^\varepsilon)}\le C\varepsilon^{1/2}+C\varepsilon^{3/2}\le C\varepsilon^{1/2},$$

so that (3.3.27) becomes

$$\left|\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla_z((1-m_{u\varepsilon})(\varepsilon u_1+\varepsilon^2 u_2))\nabla_z\phi\,dz\right|\le C\varepsilon^{1/2}\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}. \tag{3.3.31}$$

Using (3.3.26) and (3.3.31), we obtain

$$\left|\int_{\Omega_s^\varepsilon} D_{u\varepsilon}\nabla_z\Phi_{u\varepsilon}\nabla_z\phi\,dz\right|\le C\varepsilon^{1/2}\|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}. \tag{3.3.32}$$

Since $\Phi_{u\varepsilon}\in H^1_{\ell r}(\Omega_s^\varepsilon)$, we can choose $\phi=\Phi_{u\varepsilon}$ and, using the uniform ellipticity of $D_{u\varepsilon}$ we obtain

$$\|\Phi_{u\varepsilon}\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}\le C\varepsilon^{1/2}.$$

We now have to find an estimate for the function

$$\Phi_{v\varepsilon} := v_\varepsilon - v_0 - m_{v\varepsilon}(\varepsilon v_1 + \varepsilon^2 v_2) = \varphi_\varepsilon + (1 - m_{v\varepsilon})(\varepsilon v_1 + \varepsilon^2 v_2),$$

where $\varphi_\varepsilon := v_\varepsilon - (v_0 + \varepsilon v_1 + \varepsilon^2 v_2)$.

We introduce the following operators:

$$\begin{aligned} V_0 &= -\nabla_y \cdot (D_v(y)\nabla_y)\,, \\ V_1 &= -\nabla_z \cdot (D_v(y)\nabla_y) - \nabla_y \cdot (D_v(y)\nabla_z)\,, \\ V_2 &= -\nabla_z \cdot (D_v(y)\nabla_z)\,. \end{aligned} \tag{3.3.33}$$

Equating the coefficients corresponding to the powers ε^{-2}, ε^{-1} and ε^0 in (3.2.12) and the ones corresponding to the powers ε^{-1}, ε^0 and ε^1 in (3.2.14), respectively, we obtain:

$$\begin{cases} V_0 v_0 = 0 \ \text{ in } Y_f, \\ \dfrac{\partial v_0}{\partial \nu_{V_0}} = 0 \ \text{ on } \partial Y_f, \\ v_0(z,\cdot) \text{ is } Y_f - \text{periodic for each given } z \in \Omega_f, \end{cases} \tag{3.3.34}$$

$$\begin{cases} V_0 v_1 = -V_1 v_0 \ \text{ in } Y_f, \\ \dfrac{\partial v_1}{\partial \nu_{V_0}} = -D_v(y)\nabla_z v_0 \cdot n(y) + B(y) v_0 \cdot n(y) \ \text{ on } \partial Y_f, \\ v_1(z,\cdot) \text{ is } Y_f - \text{periodic for each given } z \in \Omega_f, \end{cases} \tag{3.3.35}$$

$$\begin{cases} V_0 v_2 = -V_1 v_1 - V_2 v_0 - B(y)(\nabla_z v_0 + \nabla_y v_1) \ \text{ in } Y_f, \\ \dfrac{\partial v_2}{\partial \nu_{V_0}} = -\Phi_3 \tilde{\sigma}(u_0 + g - H v_0) - D_v(y)\nabla_z v_1 \cdot n(y) + B(y) v_1 \ \text{ on } \partial Y_f, \\ v_2(z,\cdot) \text{ is } Y_f - \text{periodic for each given } z \in \Omega_f. \end{cases} \tag{3.3.36}$$

We have that on Ω_f^ε, the function φ_ε satisfies

$$\begin{aligned} &- \nabla_z \cdot (D_{v\varepsilon}\nabla_z \varphi_\varepsilon) = [(\varepsilon^{-2} V_0 + \varepsilon^{-1} V_1 + V_2)\phi]\left(z, \frac{z}{\varepsilon}\right) \\ &= -\nabla_z \cdot (D_{v\varepsilon}\nabla_z v_\varepsilon) - \varepsilon^2 V_0 v_0 - \varepsilon^{-1}(V_0 v_1 + V_1 v_0) \\ &- (V_0 v_2 + V_1 v_1 + V_2 v_0) - \varepsilon(V_1 v_2 + V_2 v_1) - \varepsilon^2 V_2 v_2 \\ &= -B\left(\frac{z}{\varepsilon}\right)\nabla_z v_\varepsilon + B(y)(\nabla_z v_0 + \nabla_y v_1) - \varepsilon(V_1 v_2 + V_2 v_1) - \varepsilon^2 V_2 v_2, \end{aligned}$$

where we used (3.3.34), (3.3.35), (3.3.36), and (1.3.1). Similarly, using (1.3.4), (3.2.14), we obtain that on Σ^ε, φ_ε satisfies

$$D_{v\varepsilon}\nabla_z \varphi_\varepsilon \cdot n = \left(B\left(\frac{z}{\varepsilon}\right) v_\varepsilon - B(y) v_0\right)\cdot n - \varepsilon\Phi_3\tilde{\sigma}(u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) - \varepsilon B(y) v_1 \cdot n - \varepsilon^2 D_{v\varepsilon}\nabla_z v_2.$$

Therefore, φ_ε satisfies the following system:

$$\begin{cases} -\nabla_z \cdot (D_{v\varepsilon} \nabla_z \varphi_\varepsilon) = h_\varepsilon \text{ in } \Omega_f^\varepsilon, \\ D_{v\varepsilon} \nabla_z \varphi_\varepsilon \cdot n = (B_\varepsilon v_\varepsilon - B(y) v_0) \cdot n \\ -\varepsilon \Phi_3 \tilde{\sigma}(u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) - \varepsilon B(y) v_1 \cdot n - \varepsilon^2 D_{v\varepsilon} \nabla_z v_2 \text{ on } \Sigma^\varepsilon, \\ D_{v\varepsilon} \nabla_z \varphi_\varepsilon \cdot n = [B_\varepsilon v_\varepsilon - B(y)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2)] \cdot n \text{ on } \Gamma \cap \partial \Omega_f^\varepsilon, \end{cases} \tag{3.3.37}$$

where

$$h_\varepsilon := -B_\varepsilon \nabla_z v_\varepsilon + B(y)(\nabla_z v_0 + \nabla_y v_1) - \varepsilon(V_1 v_2 + V_2 v_1) - \varepsilon^2 V_2 v_2. \tag{3.3.38}$$

Let $\varphi \in H^1(\Omega_f^\varepsilon)$, we have

$$\begin{aligned} \int_{\Omega_f^\varepsilon} D_{v\varepsilon} \nabla_z \varphi_\varepsilon \nabla_z \varphi \, dz &= \int_{\Omega_f^\varepsilon} h_\varepsilon \varphi \, dz \\ &+ \int_{\Sigma^\varepsilon} (-(B_\varepsilon v_\varepsilon - B(y) v_0) \cdot n \\ &+ \varepsilon \Phi_3 \tilde{\sigma}(u_\varepsilon - u_0 - H(v_\varepsilon - v_0)) + \varepsilon B(y) v_1 \cdot n + \varepsilon^2 D_{v\varepsilon} \nabla_z v_2) \varphi \, d\sigma \\ &+ \int_{\Gamma \cap \partial \Omega_f^\varepsilon} (-B_\varepsilon v_\varepsilon + B(y)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2)) \cdot n \varphi \, d\sigma. \end{aligned} \tag{3.3.39}$$

We notice that

$$|B_\varepsilon v_\varepsilon - B(y) v_0| = |(B_\varepsilon - B(y)) v_0 + \varepsilon B_\varepsilon v_1 + \varepsilon^2 B_\varepsilon v_2 + h.o.t.| \tag{3.3.40}$$

and

$$\begin{aligned} &|B_\varepsilon \nabla_z v_\varepsilon - B(y)(\nabla_z v_0 + \nabla_y v_1)| \\ &= |(B_\varepsilon - B(y))(\nabla_z v_0 + \nabla_y v_1) + \varepsilon(\nabla_z v_1 + \nabla_y v_2) + \varepsilon^2 \nabla_z v_2 + h.o.t.|. \end{aligned} \tag{3.3.41}$$

Using (3.3.9), (3.3.40), (3.3.41) and the same arguments we mentioned in the first part of this proof, we obtain

$$\left| \int_{\Omega_s^\varepsilon} D_{v\varepsilon} \nabla_z \varphi_\varepsilon \nabla_z \varphi \, dz \right| \leq C \varepsilon^\gamma \|\phi\|_{H^1_{\ell r}(\Omega_s^\varepsilon)}. \tag{3.3.42}$$

When it comes to the term $(1 - m_{v\varepsilon})(\varepsilon v_1 + \varepsilon^2 v_2)$, the discussion is analogous. Therefore, we obtain

$$\left| \int_{\Omega_f^\varepsilon} D_{v\varepsilon} \nabla_z \Phi_{v\varepsilon} \nabla_z \varphi \, dz \right| \leq C \varepsilon^\gamma \|\phi\|_{H^1(\Omega_f^\varepsilon)}. \tag{3.3.43}$$

Since $\Phi_{v\varepsilon} \in H^1(\Omega_f^\varepsilon)$, we can choose $\varphi = \Phi_{v\varepsilon}$ and, using the uniform ellipticity of $D_{v\varepsilon}$ we obtain

$$\|\Phi_{v\varepsilon}\|_{H^1(\Omega_f^\varepsilon)} \leq C \varepsilon^\gamma.$$

□

Corollary 3.3.1. *Under the same assumptions of Theorem 3.3.1, the following estimate holds:*

$$\|u_\varepsilon - u_0\|_{H^1_{\ell r}(\Omega_s^\varepsilon)} + \|v_\varepsilon - v_0\|_{H^1(\Omega_f^\varepsilon)} \leq c_1 \varepsilon^{1/2} + c_2 \varepsilon^\gamma, \tag{3.3.44}$$

where c_1 and c_2 are constants independent of ε.

3.4 Summary of homogenization results

Starting from the microscopic ε-dependent problem (1.3.1)-(1.3.8), we were able to apply homogenization and two-scale convergence results in order to obtain the following final strong form of the two-scale limit problem:

$$\begin{cases} -\sum_{i,k=1}^{2}\left[\mathbf{D}_v^{ik}\partial^2_{z_i z_k}v_0 - \mathbf{B}_i\partial_{z_i}v_0\right] = -\Phi_3\tilde{\sigma}|\Sigma||Y_f|(u_0+g-Hv_0) \ \text{ in } \Omega_f, \\ -\sum_{i,k=1}^{2}\left[\mathbf{D}_u^{ik}\partial^2_{z_i z_k}u_0\right] = |Y_s|\left(\dfrac{\Phi_3}{\Phi_2}\tilde{\sigma}|\Sigma|(u_0+g-Hv_0) - \Phi_1 F(u_0+g)\right) \ \text{ in } \Omega_s, \\ (-\mathbf{D}_u\nabla_z u_0)\cdot n = 0 \ \text{ on } \Gamma^N\setminus\partial\Omega_f \\ (-\mathbf{D}_v\nabla_z v_0 + \mathbf{B}v_0)\cdot n = 0 \ \text{ on } \Gamma\cap\partial\Omega_f \\ u_0(z) = 0 \ \ z\in\Gamma^D_\ell \\ u_0(z) = 0 \ \ z\in\Gamma^D_r. \end{cases}$$

where

$$\mathbf{D}_v^{ik} := \sum_{j=1}^{2}\int_{Y_f}\left(D_v^{ik} - D_v^{ij}\partial_{y_j}\hat{w}_k\right)dy,$$

$$\mathbf{B}_i = \int_{Y_f} B_i(y)\,dy,$$

$$\mathbf{D}_u^{ik} := \sum_{j=1}^{2}\int_{Y_s}\left(D_u^{ik} - D_u^{ij}\partial_{y_j}w_k\right)dy.$$

Namely, we were able to determine a system of macroscopic partial differential equations with constant coefficients. On the homogenization point of view, the matrices $\mathbf{D}_v$ and $\mathbf{D}_u$ are the averaged matrices and they are defined in terms of the oscillating matrices D_v and D_u. This gives our final problem a simpler and nicer structure than the one of the microscopic problem (1.3.1)-(1.3.8), where we are dealing with the micro-oscillation of both solution and diffusion matrices. Once the macroscopic two-scale limit system is determined, we prove the following corrector estimate:

$$\|u_\varepsilon - u_0\|_{H^1_{\ell r}(\Omega^\varepsilon_s)} + \|v_\varepsilon - v_0\|_{H^1(\Omega^\varepsilon_f)} \leq c_1\varepsilon^{1/2} + c_2\varepsilon^{\gamma}.$$

The latter is the most important result of this work, since we are able to understand the quantity of information we loose via averaging, in term of powers of ε. We want to stress the fact that the hypothesis (3.3.9) is determinant to let the corrector estimate work. It is indeed curious how the drift vector $B(\cdot)$ needs to be controlled in order to let the whole discussion work. The smaller γ is, the better our estimate is.

Chapter 4

Conclusion and outlook

In this work, we were able to formulate a mathematical model describing the flux of oxygen from the blood vessels to the tissues. Using homogenization techniques, we studied the well-posedness of the problem and finally we obtained the corrector estimate in section 3.3. Corollary 3.3.1 is the most important result of this work, because it closes the whole circle, showing how the solution $(u_\varepsilon, v_\varepsilon)$ of the microscopic problem and (u_0, v_0) of the strong two-scale limit problem are linked to each other. It is worth pointing out that $(u_\varepsilon, v_\varepsilon) \in H^1_{\ell r}(\Omega_s^\varepsilon) \times H^1(\Omega_f^\varepsilon)$ and $(u_0, v_0) \in H^1_{\ell r}(\Omega_s) \times H^1(\Omega_f)$. Therefore, when we write (3.3.44), we are implicitely saying that we consider u_0 and v_0 to be projected in the spaces $H^1_{\ell r}(\Omega_s^\varepsilon)$ and $H^1(\Omega_f^\varepsilon)$, respectively. It would have also been possible to consider the norms $H^1_{\ell r}(\Omega_s)$ and $H^1(\Omega_f)$, in this case we would have considered the extensions of u_ε and v_ε considered in section 3.1. Another aspect that needs to be highlighted is the importance of the drift vector $B(y)$ (or equivalently $B_\varepsilon(z)$ for the microscopic model) and the constant $\tilde{\sigma}$. Indeed, in order to prove the well-posedness of the problem, we had to control both $\|B\|_{L^\infty}$ and $\tilde{\sigma}$. When it comes to the physical interpretation of the problem, it makes sense that $\|B\|_{L^\infty}$ needs to be arbitrarily small, since the drift vector B controls the flux of the oxygen in the blood vessels. Having no control on B means having no control of the oxygen flow in Ω_f. Similarly, $\tilde{\sigma}$ is correlated to the Robin-type boundary condition (1.1.29). Physically, the latter regulates the difference of the concentration of oxygen from blood vessels to tissues on the interface, therefore we also need to control it.

For what concerns the mathematical prospective, the discussion we have done in this master thesis could be expanded. For instance, one could ask to consider the sequences $\{u_\varepsilon^k\}_k$ and $\{v_\varepsilon^k\}_k$ defined as in Definition 2.2.1 and try to find an ε and k-dependent bound of the following quantity:

$$\|u_\varepsilon^k - u_0\|_{H^1_{\ell r}(\Omega_s^\varepsilon)} + \|v_\varepsilon^k - v_0\|_{H^1(\Omega_f^\varepsilon)}. \tag{4.0.1}$$

Finding such a bound would lead to ad interesting discussion, since we have already determined (3.3.44) and (2.2.18). Notice that, to determine (4.0.1), one could implement a FEniCS code using finite element methods [21]. In this way, we would be able to move the oxygen problem to a numerical analysis prospective. Indeed, using FEniCS we are able to solve PDEs on a python environment and, when it comes to the formulation of the microscopic problem, FEniCS allows us to define perforated domains with relatively simple commands. It would be interesting to choose different values of ε and k and compute (4.0.1) to check how the error changes. Still from the computational perspective, we could actually check how different values of $\tilde{\sigma}$ and $\|B(\cdot)\|_{L^\infty}$ affect the well-posedness of the problem. Moreover, we could try to determine critical values of those two parameters. Another mathematical question that may arise is the following:

What happens if $|Y_f|$ goes to zero?

Indeed, it makes sense to imagine a situation in which we just consider Ω_s as domain, since the tissues occupies a volume that is way bigger than the blood vessels. We refer the reader to [6] for a similar setting.

Appendix

Function spaces In this section we clarify the notation used for the function spaces.

Let $\Omega \subset \mathbb{R}^N$ open subset, with $N \in \mathbb{N}$. Let Y be a periodic cell, in the sense of section 3.1.

- $C^n(\Omega)$, $n \in \mathbb{N}$ space of continuous functions defined on Ω which derivatives are continuous up to the n-th order;
- $\mathcal{D}(\Omega)$ space of test function defined on Ω, i.e. set of $C_0^\infty(\Omega)$ functions with compact support in Ω;
- $L^p(\Omega) := \{u$ measurable on Ω and $\int_\Omega |u|^p \, dx < \infty\}$, $p \in [1, +\infty)$;
- $L^\infty(\Omega) := \{u$ measurable on Ω and there exists C such that $|u(x)| \leq C$ a.e. on $\Omega\}$;
- $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1, +\infty]$, space of functions $u : \Omega \to \mathbb{R}$ locally integrable such that for all α multindex, with $|\alpha| \leq k$, there exists the weak derivative $D^\alpha u \in L^p(\Omega)$. In particular, we have $W^{k,p}(\Omega) := H^k(\Omega)$;
- $C_\#(Y)$ space of continuous functions that are Y-periodic.

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List of Figures

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