J. Math. Anal. Appl. 495 (2021) 124702

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Uniqueness and stability with respect to parameters of solutions to a fluid-like driven system for active-passive pedestrian dynamics

T.K. Thoa Thieu [a,c,*], Matteo Colangeli [b], Adrian Muntean [a,d]

[a] Department of Mathematics and Computer Science, Karlstad University, Karlstad, Sweden
[b] Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, L'Aquila, Italy
[c] Department of Mathematics, Gran Sasso Science Institute, L'Aquila, Italy
[d] Center for Societal Risk Research (CSR), Karlstad University, Sweden

ARTICLE INFO

Article history:
Received 31 May 2020
Available online 22 October 2020
Submitted by H. Frankowska

Keywords:
Pedestrian flows
Nonlinear coupling
Forchheimer flows
Double nonlinear parabolic equation

ABSTRACT

We study a system of parabolic equations consisting of a double nonlinear parabolic equation of Forchheimer type coupled with a semilinear parabolic equation. The system describes a fluid-like driven system for active-passive pedestrian dynamics. The structure of the nonlinearity of the coupling allows us to prove the uniqueness of solutions. We provide also the stability estimate of solutions with respect to selected parameters.

1. Introduction

A recent result on the weak solvability of a mixed fluid-like driven system for active–passive pedestrians has been reported in [12], where the authors provided the existence of solutions to the problem (1) by using a Schauder's fixed point argument. This type of mixed pedestrian dynamics is originally proposed in [10] by considering their evacuation dynamics in a complex geometry in the presence of a fire as well as of a slowly spreading smoke curtain. From a stochastic processes perspective, various lattice gas models for active-passive pedestrian dynamics have been explored in [4,6]. Within the present framework, our model is embedded in a continuum scale and resembles the structure of Forchheimer flows in porous media [2]. The aim of this paper is to complete the proof of the well-posedness of the system (1) by showing the uniqueness and stability of solutions with respect to parameters. The nonlinear structure in the transport term where the Forchheimer polynomial appears, allows us to establish the wanted uniqueness and stability

* Corresponding author.
E-mail address: thoa.thieu@kau.se (T.K.T. Thieu).

https://doi.org/10.1016/j.jmaa.2020.124702

estimates. This work focuses on the structural stability of solutions with respect to initial and boundary data, nonlinear coupling coefficient, and to the diffusion coefficient from the semi-linear equation.

A number of relevant results are available on structural stability topics. In particular, standard nonlinear energy stability results have been presented in [11] for convection problems, where the author dealt with an integral inequality technique referred to as the energy method. The structural stability of solutions to generalized Forchheimer equations (introduced in [3]) has been provided in [1], where the authors investigated the uniqueness, the Lyapunov asymptotic stability together with the large time behavior features of the corresponding initial boundary value problems. A structural stability with respect to boundary data and the coefficients of Forchheimer is considered in [9]. In [14], a stability estimate is introduced by considering a nonlinear drag force term corresponding to the Forchheimer term in a Navier–Stokes type model of flow in non-homogeneous porous media. Such investigations on stability estimates not only contribute to the understanding of the well-posedness of model equations, but also can point out inherent delimitations of the parameters regions outside which it makes no sense to search for solutions, see e.g. [13].

This paper is organized as follows. In Section 2, the setting of the model equations is provided. In Section 3, preliminaries and assumptions are provided. Then, we recall available energy estimates in Section 4. In Section 5, we show the proof of the uniqueness of solutions to our system. Finally, the target stability estimate is established in Section 6.

2. Setting of the model equations

Let a bounded set $\Omega \neq \emptyset$, $\Omega \subseteq \mathbb{R}^2$ has C^1-boundary[1] $\partial\Omega$ such that $\partial\Omega = \Gamma^N \cup \Gamma^R$, $\Gamma^N \cap \Gamma^R = \emptyset$ with $\mathcal{H}(\Gamma^N) \neq \emptyset$ and $\mathcal{H}(\Gamma^R) \neq \emptyset$, where $\mathcal{H}$ denotes the surface measure on Γ^N, Γ^R and take $S = (0, T)$. We shall consider the following equations, where the pair of velocities is $(u = u(t, x), v = v(t, x))$ such that the mappings $u : S \times \Omega \longrightarrow \mathbb{R}^2$ and $v : S \times \Omega \longrightarrow \mathbb{R}^2$ satisfy

$$\begin{cases} \partial_t(u^\lambda) + \operatorname{div}(-K_1(|\nabla u|)\nabla u) = -b(u - v) \quad \text{in } S \times \Omega, \\ \partial_t v - K_2\Delta v = b(u - v) \quad \text{in } S \times \Omega, \\ -K_1(|\nabla u|)\nabla u \cdot \mathbf{n} = \varphi u^\lambda \quad \text{at } S \times \Gamma^R, \\ -K_1(|\nabla u|)\nabla u \cdot \mathbf{n} = 0 \quad \text{at } S \times \Gamma^N, \\ -K_2\nabla v \cdot \mathbf{n} = 0 \quad \text{at } S \times \partial\Omega, \\ u(t = 0, x) = u_0(x) \quad \text{for } x \in \bar{\Omega}, \\ v(t = 0, x) = v_0(x) \quad \text{for } x \in \bar{\Omega}. \end{cases} \tag{1}$$

In (1), $K_2 > 0$ and function K_1 stems from the derivation of a nonlinear version of the Darcy equation defined via a generalized polynomial with non-negative coefficients (e.g. [9], [1], [3]). The structure of K_1 in the first equation of (1) will be described in Section 2. In addition, $\lambda \in (0, 1]$ is a fixed number and $b(\cdot)$ is a sink/source term.

In model (1), the dynamics of interacting pedestrians involves the evolution of two distinct populations behaving very differently from each other. Seen at a microscopic level, the motion takes place in an heterogeneous domain - obstacles are obstructing the sight of the exit. The active pedestrians follow a predetermined velocity field (the map of the location is known), while the passive agents that have no preferred direction of motion. We assume that the size of the overall population is significantly large so that using macroscopic models makes sense. In this context, we consider that the active population of pedestrians follows velocity fields similar to a generalized Darcy flow, namely, a Forchheimer flow typically applicable for slightly compressible fluids in porous media, while the passive population is governed macroscopically by some averaged

[1] This boundary information is to guarantee the trace's inequality (8) (e.g. [3]).

diffusion equation. To build this model, we took inspiration from the field of reactive flows in porous media (see, e.g., [2]). It is worth pointing out that both reactive flows in porous media as well as pedestrian flows in heterogeneous domains are able to produce coherent flow patterns, manifestation of some sort of built-in self-organization mechanisms. Cf. e.g. [7,8], either uniform or not, pedestrian flows can form collective patterns of motion. For instance, one notices circulating flows at intersections, lane formation, local clogging due a complex geometry (typically caused by walls and obstacles under normal walking conditions or when the evacuation of pedestrians takes place during an emergency situation). If the pedestrian flow is composed of mixed active-passive populations, then we see that often groups of passive pedestrians block the motion of the active ones. This effect was pointed out by the numerical results reported in [4,10] and [5]. Of course, to get the needed trust in our model, equations (1) have to be approximated numerically and the corresponding numerical output has to be confronted with suitable statistics of experimental results. This will be one of our next steps in this investigation, which will be studied elsewhere.

3. Preliminaries and assumptions

We list in this section a couple of preliminary results (mostly inequalities and compactness results) as well as our assumptions on data and parameters.

Lemma 3.1. *Let $x, y \geq 0$. Then the following elementary inequalities hold:*

$$(x+y)^p \leq 2^p(x^p+y^p) \text{ for all } p>0, \tag{2}$$

$$(x+y)^p \leq x^p+y^p \text{ for all } 0<p\leq 1, \tag{3}$$

$$(x+y)^p \leq 2^{p-1}(x^p+y^p) \text{ for all } p\geq 1, \tag{4}$$

$$x^\beta \leq x^\alpha + x^\gamma \text{ for all } 0\leq \alpha\leq\beta\leq\gamma, \tag{5}$$

$$x^\beta \leq 1 + x^\gamma \text{ for all } 0\leq\beta\leq\gamma. \tag{6}$$

The proof is elementary and we omit it from here.

Lemma 3.2 *(Trace lemma). Let $\lambda \in (0,1]$, $\delta = 1-\lambda$, $a = \frac{\alpha_N}{\alpha_N+1} \in (0,1)$, $a>\delta$, $\alpha \geq 2-\delta$, $\alpha \leq 2$, $\mu_0 = \frac{a-\delta}{1-a}$, $\alpha_\star = \frac{n(a-\delta)}{2-a}$ and*

$$\theta = \theta_\alpha := \frac{1}{(1-a)(\alpha/\alpha_\star - 1)} \in (0,1). \tag{7}$$

Then it exists $C>0$ such that the following estimate holds

$$\int\limits_{\Gamma^R} |u|^\alpha d\sigma \leq 2\varepsilon \int\limits_{\Omega} |u|^{\alpha+\delta-2}|\nabla u|^{2-a}dx + C\|u\|^\alpha_{L^\alpha(\Omega)} + C\varepsilon^{-\frac{1}{1-a}}\|u\|^{\alpha+\mu_0}_{L^\alpha(\Omega)} + C\varepsilon^{-\mu_2}\|u\|^{\alpha+\mu_1}_{L^\alpha(\Omega)}, \tag{8}$$

where

$$\mu_1 = \mu_{1,\alpha} := \frac{\mu_0(1+\theta(1-a))}{1-\theta}, \tag{9}$$

$$\mu_2 = \mu_{2,\alpha} := \frac{1}{1-a} + \frac{\theta(2-a)}{(1-\theta)(1-a)}. \tag{10}$$

For the proof of Lemma 2.2, see Lemma 2.2 in [3].

3.1. Structure of K_1

In this section, we recall the definitions on the constructions based on the nonlinear Darcy equation and its monotonicity properties as they have been presented in [1]. First of all, we introduce the function $K_1 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined for $\xi \geq 0$ by $K_1(\xi) = \frac{1}{g(s(\xi))}$ which is supported to be the unique non-negative solution of the equation $sg(s) = \xi$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a polynomial with positive coefficients defined by

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N} \quad \text{for } s \geq 0, \tag{11}$$

where $\alpha_k \in \mathbb{R}_+$ with $k \in \{0, \ldots, N\}$.

The function g is taken to be independent of the spatial variable. Thus, we may have

$$G(|v|) = g(|v|)|v| = |\nabla p|, \tag{12}$$

where $G(s) = sg(s)$ for $s \geq 0$. From now on we use the following notation for the function G and its inverse G^{-1}, namely, $G(s) = sg(s) = \xi$ and $s = G^{-1}(\xi)$. To be successful with the analysis to follow, we impose the following condition on the polynomial g, referred to as (G).

(G_1) $g \in C([0, \infty)) \cap C^1((0, \infty))$ such that

$$g(0) > 0 \text{ and } g'(s) \geq 0 \text{ for all } s \geq 0.$$

(G_2) It exists $\theta > 0$ with $g \in C([0, \infty)) \cap C^1((0, \infty))$ such that

$$g(s) \geq \theta s g'(s) \text{ for all } s > 0. \tag{13}$$

To be able to ensure the uniqueness of solution to the system (1), we use the monotonicity properties of the function $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $F(y) = K_1(|y|)y$. This is related to the nonlinear Darcy structure (12). Furthermore, we recall the following basic essential ingredients:

Definition 3.1. Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a given mapping.

- F is monotone if

$$(F(y') - F(y)) \cdot (y' - y) \geq 0 \text{ for all } y', y \in \mathbb{R}^d. \tag{14}$$

- F is strictly monotone if there is $c > 0$, such that

$$(F(y') - F(y)) \cdot (y' - y) \geq c|y' - y|^2 \text{ for all } y', y \in \mathbb{R}^d. \tag{15}$$

- F is strictly monotone on bounded sets if for any $R > 0$, there is a positive number $c_R > 0$, such that

$$(F(y') - F(y)) \cdot (y' - y) \geq c_R|y' - y|^2 \text{ for all } |y'| \leq R, |y| \leq R. \tag{16}$$

See Definition III.3 in [1] for more details.

We introduce a useful formulation by defining the following function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ as follows

$$\Phi(y, y') = (K_1(|y'|)y' - K_1(|y|)y) \cdot (y' - y) \text{ for } y, y' \in \mathbb{R}^d. \tag{17}$$

Proposition 3.1. *Let g satisfy (G_1). Then $F(y) = K_1(|y|)y$ is monotone, hence $\Phi(y, y') \geq 0$ for all $y, y' \in \mathbb{R}^d$, where Φ is defined as in (17).*

For the proof of Proposition 3.1, see Proposition III.4 in [1].

Lemma 3.3. *Let g satisfy (G_1). The function $K_1(\cdot) = K_{1g}(\cdot) = \frac{1}{g(s(\cdot))}$, is well defined, belongs to $C^1([0,\infty))$, and is decreasing. Moreover, for any $\xi \geq 0$, let $s = G^{-1}(\xi)$, then one has*

$$K_1'(\xi) = -K_1(\xi)\frac{g'(s)}{\xi g'(s) + g^2(s)} \leq 0. \tag{18}$$

For the proof of Lemma 3.3, see Lemma III.2 in [1].

Proposition 3.2. *Let g satisfy (G_1) and (G_2). Then $F(y) = K_1(|y|)y$ is strictly monotone on bounded sets. More precisely,*

$$\Phi(y, y') \geq \frac{\lambda}{\lambda+1} K_1(\max\{|y|, |y'|\})|y' - y|^2 \text{ for all } y, y' \in \mathbb{R}^d. \tag{19}$$

For the proof 3.2, see Proposition III.6 in [1].

3.2. Assumptions

We make the following choices on the structure of the involved nonlinearities.

(A_1) The structure of $K_1(\xi)$ has the following properties hold $K_1 : [0,\infty) \longrightarrow (0, \frac{1}{a_0}]$ such that K_1 is decreasing and

$$\frac{d_1}{(1+\xi)^a} \leq K_1(\xi) \leq \frac{d_2}{(1+\xi)^a}; \tag{20}$$

$$d_3(\xi^{2-a} - 1) \leq K_1(\xi)\xi^2 \leq d_2\xi^{2-a} \text{ for all } \xi \in [0,\infty). \tag{21}$$

In (20), d_1, d_2, d_3 are strictly positive constants depending on $g(s)$ and $a \in (0,1)$.

(A_2) The function $b : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following structural condition: it exists $\hat{c} > 0$ such that $b(z) \leq \hat{c}|z|^\sigma$, with $\sigma \in (0,1)$.

(A_3) The source term $b : \mathbb{R} \longrightarrow \mathbb{R}$ is globally Lipschitz continuous.

(A_4) The boundary data satisfies $\varphi \in L^\infty(\Gamma^N)$.

Assumptions (A_1)-(A_4) are all technical. The choice of (A_1) was inspired by Theorem III.10 in [1].

We recall from [12] the following concept of solution to (1) fitting to the case $\alpha \in [1+\lambda, 2]$.

Definition 3.2. Find

$$(u, v) \in L^\alpha(S; L^\alpha(\Omega)) \cap L^{2-a}(S; W^{1,2-a}(\Omega)) \times L^2(S; W^{1,2}(\Omega))$$

satisfying the identities

$$\int_\Omega \partial_t(u^\lambda)\psi dx + \int_\Omega K_1(|\nabla u|)\nabla u \nabla\psi dx + \int_{\Gamma^R} \varphi u^\lambda \psi d\gamma = -\int_\Omega b(u - v)\psi dx \tag{22}$$

and

$$\int_\Omega \partial_t v\phi dx + \int_\Omega K_2\nabla v\nabla\phi dx = \int_\Omega b(u-v)\phi dx \tag{23}$$

for a.e. $t \in S$ and for all $(\psi,\phi) \in L^\alpha(\Omega)\times W^{1,2}(\Omega)$ with the initial data $(u_0,v_0) \in L^\alpha(\Omega)\times L^2(\Omega)$.

This weak formulation has been presented in [12].

Theorem 3.1. *Assume that* (A_1) *and* (A_2) *hold. Let* $\lambda \in (0,1]$, $\delta = 1-\lambda$, $a = \frac{\alpha_N}{\alpha_N+1} \in (0,1)$, $a > \delta$, $\alpha \geq 2-\delta$, $\alpha \leq 2$, $\sigma \leq \frac{\alpha}{2}$, $\sigma \in (0,1)$ *and* $u_0 \in L^\alpha(\Omega)$, $v_0 \in L^2(\Omega)$. *Then the problem* (1) *has at least a weak solution* $(u,v) \in L^\alpha(S;L^\alpha(\Omega)) \cap L^{2-a}(S;W^{1,2-a}(\Omega)) \times L^2(S;W^{1,2}(\Omega))$ *in the sense of Definition 3.2.*

For the proof of this result, see Theorem 3.4 in [12].

3.3. Statement of the main results

The main results of this paper are stated in Theorem 3.2 and Theorem 3.3. They correspond to the case $\lambda = 1, \delta = 0$.

Theorem 3.2. *Assume that* (A_1)*-*(A_4) *hold. Let* $a = \frac{\alpha_N}{\alpha_N+1} \in (0,1)$, $a > \delta$, $\alpha \geq 2-\delta$, $\alpha \leq 2$ *and* $\sigma \leq \frac{\alpha}{2}$, $\sigma \in (0,1)$. *Then, the problem* (1) *admits at most a weak solution in the sense of Definition 3.2.*

We look for the case when the coupling is linear, i.e. $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a given function such that $b(s) = rB(s)$, where $r \in (0,\infty)$. Here, B is fixed and B is taken such that (A_2) and (A_3) are satisfied. We call $S_1 = (0,T_1)$, $S_2 = (0,T_2)$ and $\mathcal{S} = (0,\min\{T_1,T_2\}) = (0,\tau)$. Let (u_i,v_i) be weak solutions to (1) corresponding to the choices of data $(D_i,\varphi_i,r_i,u_{0i},v_{0i}), i \in \{1,2\}$. We define a triplet $(u_i,v_i,\mathcal{D}_i)$, where $(u_i,v_i) \in \left(L^\alpha(S;L^\alpha(\Omega)) \cap L^{2-a}(S;W^{1,2-a}(\Omega))\right) \times L^2(S;W^{1,2}(\Omega))$ and $\mathcal{D}_i = (D_i,r_i,u_{0i},v_{0i}) \in (0,\infty)\times(0,\infty)\times L^\alpha(\Omega)\times L^2(\Omega)$. To avoid the use of multiple indices, we denote $D := K_2$, where $K_2 > 0$ is entering (1). We give stability estimates of the solutions with respect to initial and boundary data, nonlinear coupling coefficient r and the diffusion coefficient D.

Theorem 3.3. *Assume that* (A_1)*-*(A_4) *hold, where* (A_2) *and* (A_3) *hold for the function* $B(s)$. *For* $i \in \{1,2\}$, (D_i,r_i,u_{0i},v_{0i}) *belong to a fixed compact subset* $K \subset (0,\infty)\times(0,\infty)\times L^\alpha(\Omega)\times L^2(\Omega)$, $\lambda = 1$, $\bar{r} \geq |r_1-r_2| > 0$. *Then, the following stability estimate holds*

$$\begin{aligned}\|u_1-u_2\|^\alpha_{L^\alpha(\Omega)} + \|v_1-v_2\|^2_{L^2(\Omega)} \leq e^{C(\alpha,\lambda,\hat{c},\bar{r})|r_1-r_2|t}\Big[\|u_{01}-u_{02}\|^\alpha_{L^\alpha(\Omega)}\\ +\|v_{01}-v_{02}\|^2_{L^2(\Omega)} + Ct(|D_1-D_2| + |r_1-r_2| - \|\varphi_1-\varphi_2\|^2_{L^\infty(\Gamma^R)})\Big],\end{aligned} \tag{24}$$

for $t \in \mathcal{S}$.

The proofs of Theorem 3.2, Theorem 3.3 are given in Section 5 and Section 6, respectively.

4. Energy estimates

In this section, we recall the energy estimates available for the problem (1). In particular, Proposition 4.1 contains $L^\alpha - L^2$ estimates, while gradient and time derivative estimates are reported in Proposition 4.2.

Proposition 4.1. *Assume that* (A_1)-(A_5) *hold and let* $\lambda \in (0,1]$, $\delta = 1-\lambda$, $a = \frac{\alpha_N}{\alpha_N+1} \in (0,1)$, $a > \delta$, $\alpha \geq 2-\delta$, $\alpha \leq 2$, $\sigma \leq \frac{\alpha}{2}$, $\sigma \in (0,1)$ *and* $u_0 \in L^\alpha(\Omega)$, $v_0 \in L^2(\Omega)$. *Then, for any* $t \in S$, *the following estimates hold*

$$\frac{d}{dt}\int_\Omega |u|^\alpha dx + \int_\Omega |\nabla u|^{2-a}|u|^{\alpha+\delta-2}dx \leq C_1 + \Big(\frac{3}{2C_2}\hat{c} + \frac{d_3(\alpha-\lambda)}{C_2}\Big)\|u\|^\alpha_{L^\alpha(\Omega)} + \frac{\hat{c}}{2C_2}\|v\|^2_{L^2(\Omega)}. \tag{25}$$

$$\int_\Omega |u|^\alpha dx + \int_\Omega v^2 dx \leq e^{C_3 t}\left(1 + \|u_0\|^\alpha_{L^\alpha(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right), \tag{26}$$

$$\int_0^T\int_\Omega |u|^{\alpha+\delta-2}|\nabla u|^{2-a}dxdt + \int_0^T\int_\Omega |\nabla v|^2 dxdt \leq C_5 + C_6\left(\|u_0\|^\alpha_{L^\alpha(S;L^\alpha(\Omega))} + \|v_0\|^2_{L^2(S;L^2(\Omega))}\right), \tag{27}$$

where $C_1 := \frac{d_3(\alpha-\lambda)+\hat{c}}{C_5}|\Omega|$, $C_2 := \min\left\{\frac{\lambda}{\alpha}, d_3(\alpha-\lambda)\right\}$, $C_3 := \max\left\{\frac{5}{2}\underline{\tilde{c}}\hat{c}|\Omega|, 2\underline{\tilde{c}}\hat{c}\right\}$, $C_4 := \min\{\alpha-\lambda, K_2\}$, $C_5 := \frac{5T}{2C_4}\hat{c}|\Omega| + \frac{2T\hat{c}e^{C_3 t}}{C_4}$, *and* $C_6 := \frac{2\hat{c}e^{C_3 t}}{C_2}$ *with* $\underline{\tilde{c}} := \frac{1}{\min\{\frac{\lambda}{\alpha}, \frac{1}{2}\}}$ *and* $\hat{c}$ *is as in* (A_2), *respectively.*

For the proof of this result, see Proposition 4.1 in [12]. We consider the following function $H : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ given by

$$H(\xi) = \int_0^{\xi^2} K_1(\sqrt{s})ds \text{ for } \xi \in \mathbb{R}_+. \tag{28}$$

We admit a structural inequality between $H(\xi)$ and $K_1(\xi)\xi^2$ of the form:

$$K_1(\xi)\xi^2 \leq H(\xi) \leq 2K_1(\xi)\xi^2 \text{ for all } \xi \in \mathbb{R}_+. \tag{29}$$

By combining (20) and (29), we deduce also that

$$d_3(\xi^{2-a} - 1) \leq H(\xi) \leq 2d_2\xi^{2-a} \text{ for all } \xi \in \mathbb{R}_+. \tag{30}$$

In (30), d_2, d_3 and a are defined as in (A_1).

Proposition 4.2. *Assume that* (A_1) *and* (A_2) *hold. Let* $\lambda \in (0,1]$, $\delta = 1-\lambda$, $a = \frac{\alpha_N}{\alpha_N+1}$, $a > \delta$, $\alpha \geq 2-\delta$, $\alpha \leq 2$ *and* $\sigma \leq \frac{\alpha}{2}$. *Furthermore, suppose that* $\nabla u_0 \in L^\alpha(\Omega) \cap L^{2-a}(\Omega)$, $u_0 \in L^\alpha(\Omega)$, $v_0 \in H^1(\Omega)$ *and* $\varphi \in L^\infty(\Gamma^R)$. *Then, for any* $t \in S$, *the following estimates hold*

$$\begin{aligned}\int_\Omega |\nabla u|^{2-a}dx + \int_\Omega |\nabla v|^2 dx \leq{}& C(\hat{c},\lambda,a)\Bigg[\Lambda(0) + \int_0^t (1 + \|u\|^\alpha_{L^\alpha(\Omega)})^\beta ds \\ &+ \int_0^t \|v\|^2_{L^2(\Omega)}ds + \int_0^t\int_{\Gamma^R} |\varphi_t|^{\frac{\alpha}{\alpha-\lambda-1}}d\sigma ds\Bigg] + \int_\Omega |\nabla v_0|^2 dx \\ &+ \frac{\hat{c}^2}{C_2}|\Omega|t + \frac{\hat{c}^2}{2C_2}e^{C_3 t}\left(1 + \|u_0\|^\alpha_{L^\alpha(S,L^\alpha(\Omega))} + \|v_0\|^2_{L^2(S,L^2(\Omega))}\right).\end{aligned} \tag{31}$$

$$\int_\Omega |(u^\lambda)_t|^2 dx + \int_\Omega |v_t|^2 dx \leq C(\hat{c},\lambda,a)\Bigg[1 + (1 + \|u\|^\alpha_{L^\alpha(\Omega)})^\beta + \|v\|^2_{L^2(\Omega)}$$

$$+\int_{\Gamma^R} |\varphi_t|^{\frac{\alpha}{\alpha-\lambda-1}} d\sigma\Bigg] + \frac{\hat{c}^2}{C_2}|\Omega| + \frac{\hat{c}^2}{2C_2}e^{C_3 t}\left(1 + \|u_0\|^{\alpha}_{L^{\alpha}(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\right), \tag{32}$$

where $C(\hat{c}, \lambda, a) > 0$ is a constant and

$$\Lambda(0) := \frac{\lambda+1}{2}\int_{\Omega} H(|\nabla u_0|)dx + \int_{\Omega} |u_0|^{\alpha} dx.$$

For the proof of this result, see Proposition 4.2 in [12].

5. Proof of Theorem 3.2

Proof. To prove the uniqueness of solutions in the sense of Definition 3.2, we adapt the arguments by E. Aulisa et al. (cf. Section IV, [1]) to our setting. Essentially, we are using the monotonicity properties of the term $K_1(y)y$ as stated in Proposition 3.1 and Proposition 3.2.

Let $(u_i, v_i), i \in \{1, 2\}$ be two arbitrary weak solutions to problem (1) in the sense of Definition 3.2, where the initial data is take $u_i(t = 0, x) = u_{i0}(x)$ and $v_i(t = 0, x) = v_{i0}(x)$ for all $x \in \bar{\Omega}$. We denote $w = u_1 - u_2$ and $z = v_1 - v_2$. If we substitute the pair (w, z) into (22)-(23), we obtain

$$\int_{\Omega} \partial_t(u_1^{\lambda} - u_2^{\lambda})\psi dx + \int_{\Omega} \partial_t z\phi dx + \int_{\Omega} \left(K_1(|\nabla u_1|)\nabla u_1 - K_1(|\nabla u_2|)\nabla u_2\right)\nabla\psi dx +$$
$$+K_2\int_{\Omega} \nabla z\nabla\phi dx = -\int_{\Gamma^R} \varphi(u_1^{\lambda} - u_2^{\lambda})\psi d\gamma - \int_{\Omega}(b(u_1 - v_1) - b(u_2 - v_2))(\psi - \phi)dx \tag{33}$$

Now, choosing the test function

$$(\psi, \phi) := (|w|^{\alpha+\delta-1}, z) \in \left(\left(L^{\alpha}(\Omega) \cap W^{1,2-a}(\Omega)\right) \times W^{1,2}(\Omega)\right)$$

leads to

$$\frac{\lambda}{\alpha}\frac{d}{dt}\int_{\Omega} |w|^{\alpha} dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega} z^2 dx + \int_{\Omega}\Big(K_1(|\nabla u_1|)\nabla u_1 - K_1(|\nabla u_2|)\nabla u_2\Big)\nabla w|w|^{\alpha+\delta-2}dx + K_2\int_{\Omega}|\nabla z|^2 dx$$
$$+\int_{\Gamma^R}\varphi(u_1^{\lambda} - u_2^{\lambda})|w|^{\alpha+\delta-1}d\gamma = -\int_{\Omega}(b(u_1 - v_1) - b(u_2 - v_2))(|w|^{\alpha+\delta-1} - z)dx. \tag{34}$$

Using assumption (A_3) to handle the right hand side of (34), we have the following estimate

$$\frac{\lambda}{\alpha}\frac{d}{dt}\int_{\Omega} |w|^{\alpha} dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega} z^2 dx + \int_{\Omega}\Phi(\nabla u_1, \nabla u_2)|w|^{\alpha+\delta-2}dx + K_2\int_{\Omega}|\nabla z|^2 dx + \int_{\Gamma^R}|\varphi||u_1^{\lambda} - u_2^{\lambda}||w|^{\alpha+\delta-1}d\gamma$$
$$\leq \left|\int_{\Omega}(b(u_1 - v_1) - b(u_2 - v_2))(|w|^{\alpha+\delta-1} - z)dx\right|$$
$$\leq \left|\int_{\Omega}(|u_1 - u_2| + |v_1 - v_2|)(|w|^{\alpha+\delta-1} - z)dx\right|$$

$$\leq \left| \int_\Omega |w|^{\alpha+\delta} dx - \int_\Omega |w||z| dx - \int_\Omega |v_1 - v_2||w|^{\alpha+\delta-1} dx + \int_\Omega z^2 dx \right|$$
$$\leq \int_\Omega |w|^{\alpha+\delta} dx + \frac{1}{2} \int_\Omega |w|^{2(\alpha+\delta-1)} dx + \frac{1}{2} \int_\Omega |w|^2 dx + C \int_\Omega z^2 dx. \tag{35}$$

Since $\Phi(\nabla u_1, \nabla u_2) \geq 0$, (35) becomes

$$\frac{\lambda}{\alpha} \frac{d}{dt} \int_\Omega |w|^\alpha dx + \frac{1}{2} \frac{d}{dt} \int_\Omega z^2 dx \leq \int_\Omega |w|^{\alpha+\delta} dx + \frac{1}{2} \int_\Omega |w|^{2(\alpha+\delta-1)} dx + C \int_\Omega |w|^2 dx + C \int_\Omega |z|^2 dx. \tag{36}$$

We set $\delta = 0$ and use the inequality (5) to rewrite (36) as

$$\frac{d}{dt} \left(\int_\Omega |w|^\alpha dx + \int_\Omega z^2 dx \right) \leq C(\alpha, \lambda) + C(\alpha, \lambda) \left(\int_\Omega |w|^\alpha dx + \int_\Omega z^2 dx \right). \tag{37}$$

It is convenient to introduce the notation:

$$W(t) := \int_\Omega |w|^\alpha dx + \int_\Omega |z|^2 dx \text{ for } t \in S.$$

Hence, the inequality (37) becomes

$$\frac{d}{dt} W(t) \leq C(\alpha, \lambda) W(t), \tag{38}$$

for $t \in S$ with $W(0) = \int_\Omega |w_0|^\alpha dx + \int_\Omega |z_0|^2 dx$, where $w_0 := u_{01} - u_{02}$ and $z_0 := v_{01} - v_{02}$. Here we consider $u_{01}, u_{02} \in L^\alpha(\Omega)$ and $v_{01}, v_{02} \in L^2(\Omega)$.

By using Grönwall's inequality, (38) yields

$$W(t) \leq W(0) e^{tC(\alpha,\lambda)} \text{ for all } t \in S. \tag{39}$$

This also implies

$$\int_\Omega |w|^\alpha dx + \int_\Omega |z|^2 dx \leq (\|w_0\|^\alpha_{L^\alpha(\Omega)} + \|z_0\|^2_{L^2(\Omega)}) e^{tC(\alpha,\lambda)}. \tag{40}$$

Clearly, if $w_0 = z_0 = 0$, then the weak solution of (1) is unique. □

6. Proof of Theorem 3.3

Proof. Let us recall the weak formulation corresponding to the different choices of data: $(u_{0i}, v_{0i}, D_i, \varphi_i), i \in \{1, 2\}$. We denote $D = D_1 - D_2$, $\tilde{\varphi} = \varphi_1 - \varphi_2$, $\tilde{r} = r_1 - r_2$, $\tilde{u}_0 = u_{01} - u_{02}$ and $\tilde{v}_0 = v_{01} - v_{02}$. We denote also $w := u_1 - u_2$ and $z := v_1 - v_2$. Multiplying the first and the second equations of (1) with $\psi := |w|^{\alpha+\delta-1}, \phi := z$, respectively and iterating the result by parts over Ω together with combining the two equations, one gets

$$\int_\Omega \partial_t(u_1^\lambda - u_2^\lambda)\psi dx + \int_\Omega \partial_t(v_1 - v_2)\phi dx + \int_\Omega \Big(K_1(|\nabla u_1|)\nabla u_1$$
$$- K_1(|\nabla u_2|)\nabla u_2\Big)\nabla\psi dx + \int_\Omega (D_1\nabla v_1 - D_2\nabla v_2)\nabla\phi dx + \int_{\Gamma^R} \left(\varphi_1 u_1^\lambda - \varphi_2 u_2^\lambda\right)\psi d\gamma$$
$$= -\int_\Omega [r_1 B(u_1 - v_1)(\psi - \phi) - r_2 B(u_2 - v_2)(\psi - \phi)]\, dx. \tag{41}$$

Regarding (41), note that

$$\int_\Omega (D_1\nabla v_1 - D_2\nabla v_2)\nabla\phi dx = D_1\|\nabla\phi\|^2_{L^2(\Omega)} + (D_1 - D_2)\int_\Omega \nabla v_2\nabla\phi dx, \tag{42}$$

$$\int_{\Gamma^R} \left(\varphi_1 u_1^\lambda - \varphi_2 u_2^\lambda\right)\psi d\gamma = \int_{\Gamma^R} \varphi_1(u_1^\lambda - u_2^\lambda)\psi d\gamma + \int_{\Gamma^R} (\varphi_1 - \varphi_2)u_2^\lambda\psi d\gamma \tag{43}$$

and

$$\int_\Omega [r_1 B(u_1 - v_1)(\psi - \phi) - r_2 B(u_2 - v_2)(\psi - \phi)]\, dx$$
$$= \int_\Omega r_1\left(B(u_1 - v_1) - B(u_2 - v_2)\right)(\psi - \phi)dx + (r_1 - r_2)\int_\Omega B(u_2 - v_2)(\psi - \phi)dx. \tag{44}$$

Using now (42), (44), as well as Young's inequality applied to the last terms of (42), (43) together with the assumption (A_2) and (A_3), we use that (41) becomes

$$\frac{\lambda}{\alpha}\frac{d}{dt}\int_\Omega |u_1 - u_2|^\alpha dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \phi^2 dx + \int_\Omega \Phi(\nabla u_1, \nabla u_2)|u_1 - u_2|^{\alpha+\delta-2}dx +$$
$$+ D_1\int_\Omega |\nabla\phi|^2 dx + \int_{\Gamma^R} \varphi_1|u_1^\lambda - u_2^\lambda|\psi d\gamma \le C\varepsilon_1\int_\Omega |\nabla\phi|^2 dx +$$
$$+ C|D_1 - D_2|\int_\Omega |\nabla v_2|^2 dx - \frac{1}{2}\int_{\Gamma^R} |\varphi_1 - \varphi_2|^2|u_2|^{2\lambda}d\gamma$$
$$- \frac{1}{2}\int_{\Gamma^R} |u_1 - u_2|^{2(\alpha+\delta-1)}d\gamma + \int_\Omega r_1(|u_1 - u_2| + |v_1 - v_2|)(\psi - \phi)dx$$
$$+ |r_1 - r_2|\frac{\hat{c}}{\bar{r}}\int_\Omega |u_2 - v_2|^\sigma(\psi - \phi)dx. \tag{45}$$

Using the inequality (3), (45) receives the form

$$\frac{\lambda}{\alpha}\frac{d}{dt}\int_\Omega |u_1 - u_2|^\alpha dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \phi^2 dx + \int_\Omega \Phi(\nabla u_1, \nabla u_2)|u_1 - u_2|^{\alpha+\delta-2}dx +$$
$$+ D_1\int_\Omega |\nabla\phi|^2 dx + \int_{\Gamma^R} \varphi_1|u_1^\lambda - u_2^\lambda|\psi d\gamma \le C\varepsilon_1\int_\Omega |\nabla\phi|^2 dx +$$

$$
\begin{aligned}
&+ C|D_1 - D_2| \int_\Omega |\nabla v_2|^2 dx - \frac{1}{2} \int_{\Gamma^R} |\varphi_1 - \varphi_2|^2 |u_2|^{2\lambda} d\gamma \\
&- \frac{1}{2} \int_{\Gamma^R} |u_1 - u_2|^{2(\alpha+\delta-1)} d\gamma + \int_\Omega r_1 |u_1 - u_2| \psi dx - \int_\Omega r_1 |u_1 - u_2| \phi dx + \\
&\int_\Omega r_1 |v_1 - v_2| \psi dx - \int_\Omega r_1 |v_1 - v_2| \psi dx + |r_1 - r_2| \frac{\hat{c}}{\bar{r}} \int_\Omega (|u_2|^\sigma + |v_2|^\sigma)(\psi - \phi) dx.
\end{aligned} \tag{46}
$$

Applying the trace inequality (8) together with Cauchy-Schwarz's inequality, we obtain the following estimate

$$
\begin{aligned}
&\frac{\lambda}{\alpha} \frac{d}{dt} \int_\Omega |u_1 - u_2|^\alpha dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \phi^2 dx + \int_\Omega \Phi(\nabla u_1, \nabla u_2) |u_1 - u_2|^{\alpha+\delta-2} dx \\
&+ D_1 \int_\Omega |\nabla \phi|^2 dx + \int_{\Gamma^R} \varphi_1 |u_1^\lambda - u_2^\lambda| \psi d\gamma \le C\varepsilon_1 \int_\Omega |\nabla \phi|^2 dx + \\
&+ C|D_1 - D_2| \int_\Omega |\nabla v_2|^2 dx - \frac{1}{2} \int_{\Gamma^R} |\varphi_1 - \varphi_2|^2 |u_2|^{2\lambda} d\gamma \\
&- \frac{1}{2} \Big(2\varepsilon_2 \int_\Omega |\nabla u_1 - \nabla u_2|^{2-a} |u_1 - u_2|^{\alpha+\delta-2} dx + C \int_\Omega |u_1 - u_2|^\alpha dx \\
&+ C \int_\Omega |u_1 - u_2|^{\alpha+\mu_0} dx + C \int_\Omega |u_1 - u_2|^{\alpha+\mu_1} dx \Big) \\
&+ r_1 \int_\Omega |u_1 - u_2|^{\alpha+\delta} dx - \frac{r_1}{2} \int_\Omega |u_1 - u_2|^2 dx - \frac{r_1}{2} \int_\Omega |v_1 - v_2|^2 dx \\
&+ \frac{r_1}{2} \int_\Omega |v_1 - v_2|^2 dx + \frac{r_1}{2} \int_\Omega |u_1 - u_2|^{2(\alpha+\delta-1)} dx - r_1 \int_\Omega |v_1 - v_2|^2 dx \\
&+ |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |u_2|^{2\sigma} dx + |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |u_1 - u_2|^{2(\alpha+\delta-1)} dx \\
&- |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |u_2|^{2\sigma} dx - |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |v_1 - v_2|^2 dx \\
&+ |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |v_2|^{2\sigma} dx + |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |u_1 - u_2|^{2(\alpha+\delta-1)} dx \\
&- |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |v_2|^{2\sigma} dx - |r_1 - r_2| \frac{\hat{c}}{2\bar{r}} \int_\Omega |v_1 - v_2|^2 dx.
\end{aligned} \tag{47}
$$

Choosing $\varepsilon_1 = \frac{D_1}{C}$ and $\varepsilon_2 = 1$, we have

$$
\begin{aligned}
&\frac{\lambda}{\alpha} \frac{d}{dt} \int_\Omega |u_1 - u_2|^\alpha dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \phi^2 dx \le C|D_1 - D_2| \int_\Omega |\nabla v_2|^2 dx - \\
&\frac{1}{2} \int_{\Gamma^R} |\varphi_1 - \varphi_2|^2 |u_2|^{2\lambda} d\gamma + r_1 \int_\Omega |u_1 - u_2|^{\alpha+\delta} dx - \frac{r_1}{2} \int_\Omega |u_1 - u_2|^2 dx
\end{aligned}
$$

$$+\frac{r_1}{2}\int_\Omega |u_1-u_2|^{2(\alpha+\delta-1)}dx - r_1\int_\Omega |v_1-v_2|^2 dx$$
$$+|r_1-r_2|\frac{\hat{c}}{\bar{r}}\int_\Omega |u_1-u_2|^{2(\alpha+\delta-1)}dx - |r_1-r_2|\frac{\hat{c}}{\bar{r}}\int_\Omega |v_1-v_2|^2 dx. \tag{48}$$

Moreover, if we assume that $\delta = 0$, then the maximum allowed power of $\|w\|$ is α. As next step, we use the inequality (6) together with the energy estimates (26), (31) to deal with the terms $\int_\Omega |u_2|^\alpha dx$, $\int_\Omega |v_2|^2 dx$ and $\int_\Omega |\nabla u_2|^2 dx$. Furthermore, we use also the trace inequality (8). It yields

$$\frac{d}{dt}\Big(\int_\Omega |u_1-u_2|^\alpha dx + \int_\Omega |v_1-v_2|^2 dx\Big) \le C(\alpha,\lambda,\hat{c},\bar{r})(|D_1-D_2|+|r_1-r_2|$$
$$-\|\varphi_1-\varphi_2\|^2_{L^\infty}) + C(\alpha,\lambda,\hat{c},\bar{r})|r_1-r_2|\left(|u_1-u_2\|^\alpha_{L^\alpha(\Omega)} + \|v_1-v_2\|^2_{L^2(\Omega)}\right). \tag{49}$$

Denoting

$$Z(t) := \int_\Omega |u_1-u_2|^\alpha dx + \int_\Omega |v_1-v_2|^2 dx \text{ for any } t\in S. \tag{50}$$

The expansion (49) can be rewritten as follows

$$\frac{d}{dt}Z(t) \le C(\alpha,\lambda,\hat{c},\bar{r})(|D_1-D_2|+|r_1-r_2|-\|\varphi_1-\varphi_2\|^2_{L^\infty(\Gamma^R)}) + C(\alpha,\lambda,\hat{c},\bar{r})|r_1-r_2|Z(t), \tag{51}$$

for $t\in S$. It holds $Z(0) = \int_\Omega |u_{01}-u_{02}|^\alpha dx + \int_\Omega |v_{01}-v_{02}|^2 dx$.

Applying the Grönwall's inequality to (50), we obtain

$$Z(t) \le e^{\int_0^t C(\alpha,\lambda,\hat{c},\bar{r})|r_1-r_2|ds}\Big[Z(0) + \int_0^t C(\alpha,\lambda,\hat{c},\bar{r})(|D_1-D_2|+|r_1-r_2|-\|\varphi_1-\varphi_2\|^2_{L^\infty(\Gamma^R)})ds\Big]. \tag{52}$$

(52) implies

$$\|u_1-u_2\|^\alpha_{L^\alpha(\Omega)} + \|v_1-v_2\|^2_{L^2(\Omega)} \le e^{C(\alpha,\lambda,\hat{c},\bar{r})|r_1-r_2|t}\Big[\|u_{01}-u_{02}\|^\alpha_{L^\alpha(\Omega)}$$
$$+\|v_{01}-v_{02}\|^2_{L^2(\Omega)} + Ct(|D_1-D_2|+|r_1-r_2|-\|\varphi_1-\varphi_2\|^2_{L^\infty(\Gamma^R)})\Big], \tag{53}$$

which is precisely the kind of stability estimate with respect to data and parameters we are looking for. □

References

[1] E. Aulisa, L. Bloshanskaya, L. Hoang, A. Ibragimov, Analysis of generalized Forchheimer flows of compressible fluids in porous media, J. Math. Phys. 50 (2009) 103102.
[2] J. Bear, Dynamics of Fluids in Porous Media, vol. 1, American Elsevier Publishing Company, 1972.
[3] E. Celik, L. Hoang, T. Kieu, Generalized Forchheimer flows of isentropic gases, J. Math. Fluid Mech. 20 (2016) 83–115.
[4] E.N.M. Cirillo, M. Colangeli, A. Muntean, T.K.T. Thieu, A lattice model for active-passive pedestrian dynamics: a quest for drafting effects, Math. Biosci. Eng. 17 (2019) 460–477.
[5] E.N.M. Cirillo, M. Colangeli, A. Muntean, T.K.T. Thieu, When diffusion faces drift: consequences of exclusion processes for bi–directional pedestrian flows, Phys. D: Nonlinear Phenom. 413 (2020) 132651.
[6] M. Colangeli, A. Muntean, O. Richardson, T.K.T. Thieu, Modelling interactions between active and passive agents moving through heterogeneous environments, in: G. Libelli, N. Bellomo (Eds.), Crowd Dynamics, Vol. 1: Theory, Models and Safety Problems: Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, Springer, Boston, 2019.

[7] F.S.P.E. Cristiani, A. Tosin, Modeling rationality to control self-organization of crowds: an environmental approach, SIAM J. Appl. Math. 75 (2015) 605–629.
[8] L.B.D. Helbing, A. Johansson, T. Werner, Self-organized pedestrian crowd dynamics: experiments, simulations, and design solutions, Transp. Sci. 39 (2005) 1–24.
[9] L. Hoang, A. Ibragimov, Structural stability of generalized Forchheimer equations for compressible fluids in porous media, Nonlinearity 24 (2011) 1–41.
[10] O. Richardson, A. Jalba, A. Muntean, The effect of environment knowledge in evacuation scenarios involving fire and smoke – a multiscale modelling and simulation approach, Fire Technol. 55 (2019) 415–436.
[11] B. Straughan, The Energy Method, Stability, and Nonlinear Convection, Springer-Verlag, New York, 2014.
[12] T.K.T. Thieu, M. Colangeli, A. Muntean, Weak solvability of a fluid-like driven system for active-passive pedestrian dynamics, Nonlinear Stud. 26 (2019) 991–1006.
[13] A. Vromans, F.V.D. Ven, A. Muntean, Parameter delimitation of the weak solvability for a pseudo-parabolic system coupling chemical reactions, diffusion and momentum equations, Adv. Math. Sci. Appl. 28 (2019) 273–311.
[14] I. Wijaya, H. Notsu, Stability estimates and a Lagrange–Galerkin scheme for a Navier–Stokes type model of flow in non-homogeneous porous media, Discrete Contin. Dyn. Syst., Ser. S (2019), https://doi.org/10.3934/dcdss.2020234, in press.