Asymptotic analysis of an $\varepsilon$-Stokes problem with Dirichlet boundary conditions

Asymptotisk analys av ett $\varepsilon$-Stokes problem med Dirichlet randvillkor

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Abstract

In this thesis, we propose an $\varepsilon$-Stokes problem connecting the Stokes problem and the corresponding pressure-Poisson equation using one parameter $\varepsilon > 0$. We prove that the solution to the $\varepsilon$-Stokes problem, converges as $\varepsilon$ tends to 0 or $\infty$ to the Stokes and pressure-Poisson problem, respectively. Most of these results are new. The precise statements of the new results are given in Proposition 3.5, Theorem 4.1, Theorem 5.2, and Theorem 5.3. Numerical results illustrating our mathematical results are also presented.

Keywords: Stokes problem, Pressure-Poisson equation, Asymptotic analysis, Weak solutions, FEM approximations

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1 Introduction

Navier–Stokes and Stokes equations as proposed in fluid mechanics are used as for instance to simulate flow in blood vessels, pipelines or in different hydraulic systems involving pumps, etc. It is important to find numerical solutions for the problems. Marker and cell (MAC), simplified MAC (SMAC), and projection methods are efficient tools used for solving the Navier–Stokes equations (cf. [1, 2, 3, 4, 5, 6, 7, 8], e.g.). In these cases, a pressure-Poisson problem is used instead of the incompressibility equation of the flow. In this thesis, we study the Stokes problem and the corresponding approximation problem using an \(\varepsilon\)-Stokes problem and its connection to a corresponding pressure-Poisson problem.

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n(n \geq 2, n \in \mathbb{N})\) with Lipschitz continuous boundary \(\Gamma\) and let \(F \in L^2(\Omega)^n, u_b \in H^{1/2}(\Gamma)^n\) satisfy \(\int_{\Gamma} u_b \cdot \nu = 0\), where \(\nu\) is the unit outward normal vector on \(\Gamma\). The weak form of the Stokes problem is:

\[
\begin{align*}
-u_S + \nabla p &= F \quad \text{in } H^{-1}(\Omega)^n, \\
\text{div } u_S &= 0 \quad \text{in } L^2(\Omega), \\
\quad u_S &= u_b \quad \text{on } H^{1/2}(\Gamma)^n.
\end{align*}
\]

(S)

We refer to [9] for the details on the Stokes problem, (i.e. more physical background and corresponding mathematical analysis). Taking the divergence of the first equation, we are led to

\[
\text{div } F = \text{div}(u_S + \nabla p) = -\Delta \text{div } u_S + \Delta p = \Delta p
\]

in distribution sense. This is often called a pressure-Poisson equation and is used in MAC, SMAC, and the projection method. Based on the above, we consider a similar problem:

Find \(u_{PP} \in H^1(\Omega)^n\) and \(p_{PP} \in H^1(\Omega)\) satisfying

\[
\begin{align*}
-\Delta u_{PP} + \nabla p_{PP} &= F \quad \text{in } H^{-1}(\Omega)^n, \\
-\Delta p_{PP} &= -\text{div } F \quad \text{in } H^{-1}(\Omega), \\
\quad u_{PP} &= u_b \quad \text{on } H^{1/2}(\Gamma)^n, \\
\quad p_{PP} &= p_b \quad \text{on } H^{1/2}(\Gamma).
\end{align*}
\]

(PP)

with \(p_b \in H^{1/2}(\Gamma)\). Let this problem be called a pressure-Poisson problem. This idea using (1.1) instead of \(\text{div } u_S = 0\) is useful to calculate the pressure numerically in the Navier–Stokes equation. For example, the idea is used in MAC, SMAC, and the projection methods. The Dirichlet boundary condition for the pressure is used in an outflow boundary \([10, 11]\). See also \([12, 13, 14]\).

In this thesis, we propose an “interpolation” between these problems (S) and (PP), i.e. we introduce an intermediate problem:
For $\varepsilon > 0$, find $u_\varepsilon \in H^1(\Omega)^n$ and $p_\varepsilon \in H^1(\Omega)$ which satisfy
\[
\begin{cases}
-\Delta u_\varepsilon + \nabla p_\varepsilon = F & \text{in } H^{-1}(\Omega)^n, \\
-\varepsilon \Delta p_\varepsilon + \text{div } u_\varepsilon = -\varepsilon \text{div } F & \text{in } H^{-1}(\Omega), \\
u_\varepsilon = u_b & \text{on } H^{1/2}(\Gamma)^n, \\
p_\varepsilon = p_b & \text{on } H^{1/2}(\Gamma). 
\end{cases}
\]

(ES)

Let this problem be called an $\varepsilon-$Stokes problem. In [15, 16, 17], the authors treat the problem (ES) as an approximation of the Stokes problem to avoid numerical instabilities. The $\varepsilon$-Stokes problem (ES) formally approximates the Stokes problem (S) as $\varepsilon \to 0$ and the pressure-Poisson problem (PP) as $\varepsilon \to \infty$ (Figure 1). We show in this thesis that (ES) is a natural link between (S) and (PP) in Proposition 3.5. The main target of this work is to give a precise convergence rate for the asymptotic behavior of the solutions to (ES) when $\varepsilon$ tends to zero or $\infty$.

The organization of this thesis is as follows. In Section 3 we introduce the notation used in this work and, prove the well-posedness of the problems (PP) and (ES), and show some of their properties. In Section 4 we show that the solution to (ES) converges to the solution to (PP) in the strong topology as $\varepsilon \to \infty$. Section 5 is devoted to proving that the solution for (ES) converges to the solution for (S) in the weak and strong topology as $\varepsilon \to 0$. Finally, in Section 6, we show several numerical illustrations of the solution to these problems using the P2/P1 finite element method.

### 2 Notation. Technical preliminaries

We remind the standard notation used in functional analysis [18]. For a Banach space $E$ with norm $\|\cdot\|_E$, we denote by $E^*$ the dual space of $E$, that is, the space of all continuous linear functionals on $E$. The norm on $E^*$ is defined by
\[
\|f\|_{E^*} := \sup_{x \in E, \|x\|_E \leq 1} \langle f, x \rangle.
\]

Given $f \in E^*$ and $x \in E$, we write $\langle f, x \rangle$ instead of $f(x)$.

The space $E$ is said to be reflexive if a mapping $J : E \to (E^*)^*$ is surjective, where $J : E \ni x \mapsto \xi \in (E^*)^*$ is defined by $\langle \xi, f \rangle = \langle f, x \rangle$ for all $f \in E^*$.  

![Figure 1: Sketch of the connections between the problems (S), (PP), and (ES).](image)
We define the strong and weak convergence. Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(E\) and let \(x \in E\). If \(\|x_n - x\|_E \to 0\) as \(n \to \infty\), then we write that

\[ x_n \to x \text{ strongly in } E \text{ as } n \to \infty. \]

If \((f, x_n) \to (f, x)\) as \(n \to \infty\) for all \(f \in E^*\), then we write

\[ x_n \rightharpoonup x \text{ weakly in } E \text{ as } n \to \infty. \]

We define function spaces,

\[ C^\infty(\Omega) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is infinitely differentiable on } \Omega \text{ and can be continuously extended with all its derivatives to the closure } \overline{\Omega} \text{ of } \Omega. \right\}, \]

\[ C^\infty_0(\Omega) := \left\{ f \in C^\infty(\Omega) \mid \text{supp}(f) \text{ is compact subset in } \Omega \right\}, \]

\[ L^2(\Omega) := \left\{ f : \Omega \to \mathbb{R} : \text{measurable} \mid \int_\Omega |f|^2 < \infty \right\}, \]

\[ H^1(\Omega) := \left\{ f \in L^2(\Omega) \mid \frac{\partial f}{\partial x_i} \in L^2(\Omega) \text{ for all } i = 1, \ldots, n \right\}, \]

\[ H^1_0(\Omega) := \left\{ f \in C^\infty_0(\Omega) \mid \int_\Omega f = 0 \right\}, \]

\[ H^{1/2}(\Gamma) := \left\{ \eta \in L^2(\Gamma) \mid \|\eta\|_{L^2(\Gamma)}^2 + \int_\Gamma \int_\Gamma \frac{|\eta(x) - \eta(y)|^2}{|x - y|^3} dxdy < \infty \right\}, \]

where \(\text{supp}(f)\) means the support of the function \(f\).

The spaces \(L^2(\Omega)^m (m \in \mathbb{N}), H^1(\Omega), \text{ and } H^1(\Omega)^n\) are Hilbert spaces with the scalar products

\[ (f, g)_{L^2(\Omega)} := \int_\Omega f \cdot g \quad \text{for } f = (f_1, \ldots, f_m), \quad g = (g_1, \ldots, g_m) \in L^2(\Omega)^m, \]

\[ (p, q)_{H^1(\Omega)} := \int_\Omega pq + \int_\Omega \nabla p \cdot \nabla q \quad \text{for } p, q \in H^1(\Omega), \]

\[ (u, v)_{H^1(\Omega)^n} := \int_\Omega u \cdot v + \int_\Omega \nabla u : \nabla v \quad \text{for } u, v \in H^1(\Omega)^n \]

where

\[ \nabla p := \left( \frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n} \right)^T, \]

\[ f \cdot g := \sum_{i=1}^n f_i g_i, \]

\[ \nabla u : \nabla v := \sum_{i=1}^n \sum_{j=1}^n \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}. \]
respectively. Here, $A^T$ denotes the transpose of the matrix $A$.

For $m = 1$ or $m = n$, the dual space $H^{-1}(\Omega)^m = (H^1_0(\Omega)^m)^*$ is equipped with the norm

$$
\|f\|_{H^{-1}(\Omega)^m} := \sup_{\varphi \in S_m} \langle f, \varphi \rangle
$$

for $f \in H^{-1}(\Omega)^m$, where

$$
S_m := \{ \varphi \in H^1(\Omega)^m \mid \|\nabla \varphi\|_{L^2(\Omega)^{n \times m}} = 1 \}.
$$

We define the following objects:

$$
[f] := f - \frac{1}{|\Omega|} \int_{\Omega} f, \\
\|f\|_{L^2(\Omega)} := \sqrt{(f, f)_{L^2(\Omega)}} \\
\|F\|_{L^2(\Omega)^n} := \sqrt{(F, F)_{L^2(\Omega)^n}} \\
\|p\|_{H^1(\Omega)} := \sqrt{(p, p)_{H^1(\Omega)}} \\
\|u\|_{H^1(\Omega)^n} := \sqrt{(u, u)_{H^1(\Omega)^n}} \\
\|\nabla u\|_{L^2(\Omega)^{n \times n}} := \sqrt{(u, u)_{H^1(\Omega)^n} - (u, u)_{L^2(\Omega)}} \\
\|p\|_{H^1(\Omega)/H^1_0(\Omega)} := \inf_{\psi \in H^1_0(\Omega)} \|p + \psi\|_{H^1(\Omega)} \\
\|f\|_{L^2(\Omega)/\mathbb{R}} := \inf_{a \in \mathbb{R}} \|f - a\|_{L^2(\Omega)^n} = \|[f]\|_{L^2(\Omega)} , \\
\langle \nabla f, \varphi \rangle := -\int_{\Omega} f \text{div} \varphi, \\
\|\eta\|_{H^{1/2}(\Gamma)} := \left( \|\eta\|^2_{L^2(\Gamma)} + \int_{\Gamma} \int_{\Gamma} |\eta(x) - \eta(y)|^2 \frac{\text{d}x\text{d}y}{|x - y|^3} \right)^{1/2}
$$

for all $f \in L^2(\Omega), F \in L^2(\Omega)^n, p \in H^1(\Omega), u \in H^1(\Omega)^n, \varphi \in H^1_0(\Omega)$, and $\eta \in H^{1/2}(\Gamma)$, where $|\Omega|$ is the volume of $\Omega$. We use the following lemmas and theorems. For their proofs, we refer the reader for instance to [9, 18, 19].

**Lemma 2.1** (Cauchy–Schwarz inequality). For $f, g \in L^2(\Omega)$, we have the following inequality:

$$
\left| \int_{\Omega} fg \right| \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.
$$

**Lemma 2.2** (Poincaré inequality). There exists a constant $c > 0$ such that

$$
\|f\|_{L^2(\Omega)} \leq c \|\nabla f\|_{L^2(\Omega)^n}
$$

for all $f \in H^1_0(\Omega)$. 

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**Theorem 2.3.** Assume that $E$ is a reflexive Banach space and let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in $E$. Then there exist $x \in E$ and a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$x_{n_k} \rightharpoonup x \text{ weakly in } E \text{ as } k \to \infty.$$ 

**Lemma 2.4.** Assume that $E$ is a reflexive Banach space and let $M \subset E$ be a closed linear subspace of $E$. Then $M$ is reflexive.

**Theorem 2.5** (Lax–Milgram). Assume that $a(\cdot, \cdot) : H \times H \to \mathbb{R}$ is a continuous bilinear form on a Hilbert space $H$. If the form $a$ is coercive, i.e. there exists a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha\|v\|^2_H \text{ for all } v \in H,$$

then for any $f \in H^*$ there exists a unique element $u \in H$ such that

$$a(u, v) = \langle f, v \rangle$$

for all $v \in H$.

**Theorem 2.6** (Rellich–Kondrachov). The space $H^1_0(\Omega)$ is compactly embedded in $L^2(\Omega)$.

Let $u \in C^\infty(\overline{\Omega})$ and let us denote its boundary value by $u|_\Gamma$. The following trace theorem extends the domain $C^\infty(\Omega)$ of the mapping $u \mapsto u|_\Gamma$ to $H^1(\Omega)$.

**Theorem 2.7.** [19, Theorem 1.5] The mapping $u \mapsto u|_\Gamma$ defined on $C^\infty(\overline{\Omega})$ has a unique linear continuous extension $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$. In particular, we have

$$\gamma_0 u = u|_\Gamma \text{ for all } u \in C^\infty(\Omega).$$

Moreover, the operator $\gamma_0$ is surjective and its kernel satisfies that

$$\ker(\gamma_0) := \{u \in H^1(\Omega) \mid \gamma_0 u = 0\} = H^1_0(\Omega).$$

Let $\nu$ be the unit outward normal on $\Gamma$. Since $\nu$ is a unit vector, $H^1(\Omega) \ni u \mapsto \gamma_\nu u := (\gamma_0 u) \cdot \nu \in H^{1/2}(\Gamma)$ is a linear continuous operator. The following Gauss divergence formula holds:

**Lemma 2.8.** [19, Lemma 1.4] For $u \in H^1(\Omega)^n$ and $\omega \in H^1(\Omega)$, we have the following equation:

$$\int_\Omega u \cdot \nabla \omega + \int_\Omega (\text{div } u) \omega = \int_\Gamma \gamma_\nu u \gamma_0 \omega.$$ 

An important tool is the following proposition:
Proposition 2.9. If \( p \in H^1(\Omega) \) satisfies 
\[
\int_{\Omega} \nabla p \cdot \nabla \psi = 0 \quad \text{for all} \quad \psi \in H^1_0(\Omega),
\]
then there exists a constant \( c \) such that 
\[
\| \nabla p \|_{L^2(\Omega)^n} \leq c \| \gamma_0 p \|_{H^{1/2}(\Gamma)}.
\]

Proof. We pick up \( p_0 \in H^1(\Omega) \) such that \( p_0 - p \in H^1_0(\Omega) \). We obtain 
\[
\begin{align*}
\| \nabla p \|_{L^2(\Omega)^n}^2 &= \int_{\Omega} \nabla p \cdot \nabla (p - p_0) + \int_{\Omega} \nabla p \cdot \nabla p_0 \\
&\leq \| \nabla p \|_{L^2(\Omega)^n} \| \nabla p_0 \|_{L^2(\Omega)^n}.
\end{align*}
\]

Hence, 
\[
\| \nabla p \|_{L^2(\Omega)^n} \leq \| \nabla p_0 \|_{L^2(\Omega)^n}
\]
for all \( p_0 \in H^1(\Omega) \) satisfying \( p_0 - p \in H^1_0(\Omega) \). By Theorem 2.7, there exists a constant \( c > 0 \) such that 
\[
\| q \|_{H^1(\Omega)/H^1_0(\Omega)} \leq c \| \gamma_0 q \|_{H^{1/2}(\Gamma)}
\]
for all \( q \in H^1(\Omega) \). Hence, we obtain 
\[
\begin{align*}
\| \nabla p \|_{L^2(\Omega)^n} &\leq \inf_{\psi \in H^1_0(\Omega)} \| \nabla (p + \psi) \|_{L^2(\Omega)^n} \\
&\leq \inf_{\psi \in H^1_0(\Omega)} \| p + \psi \|_{H^1(\Omega)} \\
&= \| p \|_{H^1(\Omega)/H^1_0(\Omega)} \\
&\leq c \| \gamma_0 p \|_{H^{1/2}(\Gamma)}.
\end{align*}
\]

We recall the following Theorem 2.10 that plays an important role in the proof of the existence of pressure solution of Stokes problem; see [20, Lemma 7.1] and [21, Theorem 3.2 and Remark 3.1] for the proof.

Theorem 2.10. There exists a constant \( c > 0 \) such that 
\[
\| f \|_{L^2(\Omega)} \leq c (\| f \|_{H^{-1}(\Omega)} + \| \nabla f \|_{H^{-1}(\Omega)})
\]
for all \( f \in L^2(\Omega) \).

The following two results follow from Theorem 2.10.

Theorem 2.11. [19, Corollary 2.1, 2°] There exists a constant \( c > 0 \) such that 
\[
\| f \|_{L^2(\Omega)/\mathbb{R}} \leq c \| \nabla f \|_{H^{-1}(\Omega)^n}
\]
for all \( f \in L^2(\Omega) \).

Theorem 2.12. [19, Corollary 2.4, 2°] The operator \( \text{div} : H^1_0(\Omega)^n \to L^2(\Omega)/\mathbb{R} \) is surjective.
3 Well-posedness for the Stokes, pressure-Poisson, and \( \varepsilon \)-Stokes problems

It is well-known that the Stokes problem has a unique solution:

**Theorem 3.1.** For \( F \in L^2(\Omega)^n \) and \( u_b \in H^{1/2}(\Gamma)^n \), there exists a unique pair of functions \((u_S, p_S) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})\) satisfying (S).

See [9, Theorem 2.4 and Remark 2.5] for the proof.

It is easy to check that there exists a unique solution to the pressure-Poisson problem:

**Theorem 3.2.** For \( F \in L^2(\Omega)^n \), \( u_b \in H^{1/2}(\Gamma)^n \), and \( p_b \in H^{1/2}(\Gamma)^n \), there exists a unique pair of functions \((u_{PP}, p_{PP}) \in H^1(\Omega)^n \times H^1(\Omega)\) satisfying (PP).

**Proof.** From the second and fourth equations of (PP), \( p_{PP} \in H^1(\Omega) \) is uniquely determined. Then \( u_{PP} \in H^1(\Omega)^n \) is also uniquely determined from the first and third equations. \( \square \)

**Corollary 3.3.** If the solution \((u_{PP}, p_{PP}) \in H^1(\Omega)^n \times H^1(\Omega)\) of (PP) satisfies \( \text{div} \, u_{PP} = 0 \), by Theorem 3.1, \( u_S = u_{PP} \) and \( p_S = [p_{PP}] \) hold.

We establish the well-posedness of the \( \varepsilon \)-Stokes problem in the following theorem.

**Theorem 3.4.** For \( \varepsilon > 0, F \in L^2(\Omega)^n, u_b \in H^{1/2}(\Gamma)^n \), and \( p_b \in H^{1/2}(\Gamma) \), there exists a unique pair of functions \((u_\varepsilon, p_\varepsilon) \in H^1(\Omega)^n \times H^1(\Omega)\) satisfying the problem (ES).

**Proof.** We pick \( u_1 \in H^1(\Omega)^n \) and \( p_0 \in H^1(\Omega) \) with \( \gamma_0 u_1 = u_b, \gamma_0 p_0 = p_b \). By Theorem 2.12, there exists \( u_2 \in H^1_0(\Omega)^n \) such that \( \text{div} \, u_2 = \text{div} \, u_1 \).

We put \( u_0 := u_1 - u_2 \), and then \( \gamma_0 u_0 = u_b \) and \( \text{div} \, u_0 = 0 \).

To simplify the notation, we set \( u := u_\varepsilon - u_0 \in H^1_0(\Omega)^n, p := p_\varepsilon - p_0 \in H^1_0(\Omega) \), \( f \in H^{-1}(\Omega)^n \), and \( g \in H^{-1}(\Omega) \) such that

\[
\langle f, v \rangle = \int_\Omega Fv - \int_\Omega \nabla u_0 : \nabla v - \int_\Omega (\nabla p_0) \cdot v \quad \text{for} \quad v \in H^1_0(\Omega)^n,
\]

\[
\langle g, q \rangle = \int_\Omega F \cdot \nabla q - \int_\Omega \nabla p_0 \cdot \nabla q \quad \text{for} \quad q \in H^1_0(\Omega).
\]

Then we have

\[
\begin{aligned}
\int_\Omega \nabla u : \nabla \varphi + \int_\Omega (\nabla p) \cdot \varphi &= \langle f, \varphi \rangle \quad \text{for all} \quad \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_\Omega \nabla p \cdot \nabla \psi + \int_\Omega (\text{div} \, u) \psi &= \varepsilon \langle g, \psi \rangle \quad \text{for all} \quad \psi \in H^1_0(\Omega).
\end{aligned}
\]
Adding the equations in (3.2), we get
\[
\left( \begin{array}{c} u \\ p \end{array} \right), \left( \begin{array}{c} \varphi \\ \psi \end{array} \right) = \langle f, \varphi \rangle + \epsilon \langle g, \psi \rangle.
\]

Here, we denote
\[
\left( \begin{array}{c} u \\ p \end{array} \right), \left( \begin{array}{c} \varphi \\ \psi \end{array} \right) := \int_{\Omega} \nabla u : \nabla \varphi + \epsilon \int_{\Omega} \nabla p \cdot \nabla \varphi + \int_{\Omega} (\nabla p) \cdot \varphi + \int_{\Omega} (\nabla u) \psi.
\]

We check that \((*, *)\) is a continuous coercive bilinear form on \(H_0^1(\Omega)^n \times H_0^1(\Omega)\). The bilinearity and continuity of \((*, *)\) are obvious.

The coercivity of \((*, *)\) is obtained in the following way: Let \(\left( \begin{array}{c} u \\ p \end{array} \right) \in H_0^1(\Omega)^n \times H_0^1(\Omega)\). We have the following sequence of inequalities;
\[
\left( \begin{array}{c} u \\ p \end{array} \right), \left( \begin{array}{c} u \\ p \end{array} \right) = \int_{\Omega} \nabla u : \nabla u + \epsilon \int_{\Omega} \nabla p \cdot \nabla p + \int_{\Gamma} \gamma_{\nu} u \gamma_{\nu} p \\
\geq \min\{1, \epsilon\} (\|\nabla u\|_{L^2(\Omega)^n}^2 + \|\nabla p\|_{L^2(\Omega)^n}^2) \\
\geq c \min\{1, \epsilon\} (\|u\|_{H^1(\Omega)^n}^2 + p^2_{H^1(\Omega)})
\]
by the Poincaré inequality. Summarizing, \((*, *)\) is a continuous coercive bilinear form and \(H_0^1(\Omega)^{n+1}\) is a Hilbert space. Therefore, the conclusion of Theorem 3.4 follows based on the Lax-Milgram Theorem.

From now on, let the solutions of (S), (PP), and (ES) be denoted by \((u_S, p_S)\), \((u_{PP}, p_{PP})\), and \((u_\epsilon, p_\epsilon)\), respectively. We show their property in connection with a pressure error on the boundary \(\Gamma\).

**Proposition 3.5.** Suppose that \(p_S \in H^1(\Omega)\). Then there exists a constant \(c > 0\) independent of \(\epsilon\) such that
\[
\|u_S - u_{PP}\|_{H^1(\Omega)^n} \leq c\|\gamma_0 p_S - p_\epsilon\|_{H^{1/2}(\Gamma)} , \quad \|u_S - u_\epsilon\|_{H^1(\Omega)^n} \leq c\|\gamma_0 p_S - p_\epsilon\|_{H^{1/2}(\Gamma)}.
\]
In particular, if \(\gamma_0 p_S = p_\epsilon\), then \(p_{PP} = p_\epsilon = p_S\) hold for all \(\epsilon > 0\).

**Proof.** From (S) and (PP), we have
\[
\begin{cases}
\int_{\Omega} \nabla (u_S - u_{PP}) : \nabla \varphi = -\int_{\Omega} (\nabla (p_S - p_{PP})) \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\
\int_{\Omega} \nabla (p_S - p_{PP}) \cdot \nabla \psi = 0 & \text{for all } \psi \in H_0^1(\Omega).
\end{cases}
\]
Putting \(\varphi := u_S - u_{PP} \in H_0^1(\Omega)^n\) in (3.3), we get
\[
\|\nabla (u_S - u_{PP})\|_{L^2(\Omega)^{n \times n}}^2 = -\int_{\Omega} (\nabla (p_S - p_{PP})) \cdot (u_S - u_{PP}) \\
\leq \|\nabla (p_S - p_{PP})\|_{L^2(\Omega)^n} \|u_S - u_{PP}\|_{L^2(\Omega)^n},
\]

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and then \( \| u_\delta - u_{PP} \|_{H^1(\Omega)^n} \leq c_1 \| \nabla (p_\delta - p_{PP}) \|_{L^2(\Omega)^n} \) follows. From the second equation of (3.3) and Proposition 2.9, there exists a constant \( c_2 > 0 \) such that

\[
\| \nabla (p_\delta - p_{PP}) \|_{L^2(\Omega)^n} \leq c_2 \| \gamma_0 p_\delta - p_b \|_{H^{1/2}(\Gamma)}.
\]

Therefore we obtain

\[
\| u_\delta - u_{PP} \|_{H^1(\Omega)^n} \leq c_1 c_2 \| \gamma_0 p_\delta - p_b \|_{H^{1/2}(\Gamma)}.
\]

Let \( w_\varepsilon := u_\delta - u_\varepsilon \in H^1_0(\Omega)^n, r_\varepsilon := p_{PP} - p_\varepsilon \in H^1_0(\Omega) \). By (S), (PP), and (ES), we have

\[
\begin{align*}
\int_\Omega \nabla w_\varepsilon : \nabla \varphi + \int_\Omega (\nabla r_\varepsilon) \cdot \varphi &= - \int_\Omega (\nabla (p_\delta - p_{PP})) \cdot \varphi \quad \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\int_\Omega \varepsilon \nabla r_\varepsilon \cdot \nabla \psi + \int_\Omega (\text{div } w_\varepsilon) \psi &= 0 \quad \text{for all } \psi \in H^1_0(\Omega).
\end{align*}
\]

Putting \( \varphi := w_\varepsilon \) and \( \psi := r_\varepsilon \), and adding the two equations of (3.4), we get

\[
\| \nabla w_\varepsilon \|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \| \nabla r_\varepsilon \|_{L^2(\Omega)^n}^2 = \int_\Omega (\nabla (p_\delta - p_{PP})) \cdot w_\varepsilon \leq \| \nabla (p_\delta - p_{PP}) \|_{L^2(\Omega)^n} \| w_\varepsilon \|_{L^2(\Omega)^n}
\]

from \( \int_\Omega (\nabla r_\varepsilon) \cdot w_\varepsilon = - \int_\Omega (\text{div } w_\varepsilon) r_\varepsilon \). Thus it leads to

\[
\| w_\varepsilon \|_{H^1(\Omega)^n} \leq c_2 \| \nabla (p_\delta - p_{PP}) \|_{L^2(\Omega)^n}.
\]

Hence we obtain \( \| u_\delta - u_\varepsilon \|_{H^1(\Omega)^n} = \| w_\varepsilon \|_{H^1(\Omega)^n} \leq c_2 c_3 \| \gamma_0 p_\delta - p_b \|_{H^{1/2}(\Gamma)} \). 

In order to show that if the boundary data \( p_b \) is not equal to the boundary value of \( p_\delta \) up to a constant then \( \| \nabla p_\varepsilon - \nabla p_\delta \|_{L^2(\Omega)^n} \to 0 \) as \( \varepsilon \to 0 \), we prove the following proposition.

**Proposition 3.6.** Under the hypotheses of Proposition 3.5, if \( \bar{p} \in H^1(\Omega) \) satisfies \( \gamma_0 \bar{p} = p_b \), then we have

\[
\| \nabla (\bar{p} - p_{PP}) \|_{L^2(\Omega)^n} \leq \| \nabla (\bar{p} - p_\delta) \|_{L^2(\Omega)^n}.
\]

**Proof.** By the second equation of (3.3) and \( \bar{p} - p_{PP} \in H^1_0(\Omega) \), it yields

\[
\| \nabla (\bar{p} - p_{PP}) \|_{L^2(\Omega)^n} = \int_\Omega (\nabla \bar{p} - p_\delta + p_\delta - p_{PP}) \cdot (\nabla \bar{p} - p_{PP}) \leq \| \nabla (\bar{p} - p_\delta) \|_{L^2(\Omega)^n} \| \nabla (\bar{p} - p_{PP}) \|_{L^2(\Omega)^n}.
\]

Hence we obtain \( \| \nabla (\bar{p} - p_{PP}) \|_{L^2(\Omega)^n} \leq \| \nabla (\bar{p} - p_\delta) \|_{L^2(\Omega)^n} \). 

**Remark 3.7.** If \( p_\delta \in H^1(\Omega) \), then we have

\[
\| \nabla (p_\varepsilon - p_{PP}) \|_{L^2(\Omega)^n} \leq \| \nabla (p_\varepsilon - p_\delta) \|_{L^2(\Omega)^n}
\]

for all \( \varepsilon > 0 \), (from Proposition 3.6). Hence, if \( \| \nabla p_\varepsilon - \nabla p_\delta \|_{L^2(\Omega)^n} \to 0 \) as \( \varepsilon \to 0 \), then \( u_{PP} = u_\delta \) and \( [p_{PP}] = p_S \), which imply that there exists a constant \( c \in \mathbb{R} \) such that \( \gamma_0 p_S = p_b + c \). In other words, if \( p_\delta \in H^1(\Omega) \) satisfies \( \gamma_0 p_S \neq p_b + c \) for all \( c \in \mathbb{R} \), then \( \| \nabla p_\varepsilon - \nabla p_S \|_{L^2(\Omega)^n} \to 0 \) as \( \varepsilon \to 0 \).
4 Links between the $\varepsilon$-Stokes and pressure-Poisson problems

The main result of this section gives the strong convergence in $H^1(\Omega)^n \times H^1(\Omega)$ of the solution for (ES) to the solution for (PP) and the convergence rate.

**Theorem 4.1.** There exists a constant $c > 0$ independent of $\varepsilon$ satisfying

$$\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon} \|\text{div } u_{PP}\|_{H^{-1}(\Omega)}, \quad \|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon} \|\text{div } u_{PP}\|_{H^{-1}(\Omega)}.$$ 

for all $\varepsilon > 0$. In particular, we have

$$\|u_\varepsilon - u_{PP}\|_{H^1(\Omega)^n} \to 0, \quad \|p_\varepsilon - p_{PP}\|_{H^1(\Omega)} \to 0 \quad \text{as } \varepsilon \to \infty.$$ 

**Proof.** From (PP) and (ES), we have

$$\begin{cases}
\int_\Omega \nabla(u_\varepsilon - u_{PP}) : \nabla \varphi + \int_\Omega (\nabla(p_\varepsilon - p_{PP})) \cdot \varphi = 0, \\
\varepsilon \int_\Omega \nabla(p_\varepsilon - p_{PP}) : \nabla \psi + \int_\Omega (\text{div}(u_\varepsilon - u_{PP})) \psi = - \int_\Omega (\text{div } u_{PP}) \psi 
\end{cases} \quad (4.5)$$

for all $\varphi \in H_0^1(\Omega)^n$ and $\psi \in H_0^1(\Omega)$. Putting $\varphi := u_\varepsilon - u_{PP} \in H_0^1(\Omega)^n$ and $\psi := p_\varepsilon - p_{PP} \in H_0^1(\Omega)$, and adding the two equations of (4.5), we obtain

$$\|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^{n \times n}}^2 + \varepsilon \|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n}^2 \leq \|\text{div } u_{PP}\|_{H^{-1}(\Omega)} \|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n},$$

where we have used $\int_\Omega (\nabla(p_\varepsilon - p_{PP})) : (u_\varepsilon - u_{PP}) = - \int_\Omega (\text{div}(u_\varepsilon - u_{PP}))(p_\varepsilon - p_{PP})$. Thus

$$\|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n} \leq \frac{1}{\varepsilon} \|\text{div } u_{PP}\|_{H^{-1}(\Omega)}$$

follows. In addition, by (4.5) and the Poincaré inequality, we have

$$\|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^n}^2 = - \int_\Omega (\nabla(p_\varepsilon - p_{PP})) : (u_\varepsilon - u_{PP}) \leq \|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n} \|u_\varepsilon - u_{PP}\|_{L^2(\Omega)^n} \leq c \|\nabla(p_\varepsilon - p_{PP})\|_{L^2(\Omega)^n} \|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^{n \times n}},$$

and then

$$\|\nabla(u_\varepsilon - u_{PP})\|_{L^2(\Omega)^n} \leq \frac{c}{\varepsilon} \|\text{div } u_{PP}\|_{H^{-1}(\Omega)}$$

follows.

By Theorem 4.1, the following corollary immediately holds.

**Corollary 4.2.** If $u_{PP}$ satisfies $\text{div } u_{PP} = 0$, then $u_\varepsilon = u_{PP}$ and $p_\varepsilon = p_{PP}$ hold for all $\varepsilon > 0$. Furthermore, $u_S = u_\varepsilon = u_{PP}$ and $p_S = [p_\varepsilon] = [p_{PP}]$ hold for all $\varepsilon > 0$. 

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5 Links between the $\varepsilon$-Stokes and Stokes problems

In this section, we prove the weak and strong convergence in $H^1_0(\Omega)^n \times L^2(\Omega)$ of the solution for (ES) to the solution for (S). First, we establish a lemma.

**Lemma 5.1.** If $v \in H^1(\Omega)^n$, $q \in L^2(\Omega)$, and $f \in H^{-1}(\Omega)^n$ satisfy
\[
\int_{\Omega} \nabla v : \nabla \varphi + \langle \nabla q, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H^1_0(\Omega)^n,
\]
then there exists a constant $c > 0$ such that
\[
\|q\|_{L^2(\Omega)/\mathbb{R}} \leq c(\|\nabla v\|_{L^2(\Omega)^n} + \|f\|_{H^{-1}(\Omega)^n}).
\]

**Proof.** Let $c$ be the constant arising in Theorem 2.11. Then we have
\[
\|q\|_{L^2(\Omega)/\mathbb{R}} \leq c\|\nabla q\|_{H^{-1}(\Omega)^n} = c \sup_{\varphi \in S_n} |\langle \nabla q, \varphi \rangle|
\leq c \sup_{\varphi \in S_n} \left( \left\| \int_{\Omega} \nabla v : \nabla \varphi \right\| + \|f\|_{H^{-1}(\Omega)^n} \right)
\leq c(\|\nabla v\|_{L^2(\Omega)^n} + \|f\|_{H^{-1}(\Omega)^n}).
\]

We show the strong convergence of the solution to (ES) without an assumption in the following theorem.

**Theorem 5.2.** There exists a constant $c > 0$ independent of $\varepsilon$ such that
\[
\|u_\varepsilon\|_{H^1(\Omega)^n} \leq c, \quad \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq c \quad \text{for all } \varepsilon > 0.
\]

Furthermore, we have
\[
u_\varepsilon - u_S \rightharpoonup 0 \text{ weakly in } H^1_0(\Omega)^n, \quad [p_\varepsilon] - p_S \rightharpoonup 0 \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.
\]

**Proof.** We use the notations $u_0 \in H^1(\Omega)^n$, $p_0 \in H^1(\Omega)$, $f \in H^{-1}(\Omega)^n$, and $g \in H^{-1}(\Omega)$ in Theorem 3.4. We put $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in H^1_0(\Omega)^n$, $\tilde{p}_\varepsilon := p_\varepsilon - p_0 \in H^1_0(\Omega)$. Then we have
\[
\left\{ \begin{array}{l}
\int_{\Omega} \nabla \tilde{u}_\varepsilon : \nabla \varphi + \varepsilon \int_{\Omega} \nabla \tilde{p}_\varepsilon : \nabla \varphi = \langle f, \varphi \rangle \quad \text{for all } \varphi \in H^1_0(\Omega)^n, \\
\varepsilon \int_{\Omega} \nabla \tilde{p}_\varepsilon : \nabla \psi + \int_{\Omega} (\text{div } \tilde{u}_\varepsilon) \psi = \varepsilon \langle g, \psi \rangle \quad \text{for all } \psi \in H^1_0(\Omega).
\end{array} \right. \tag{5.6}
\]

Putting $\varphi := \tilde{u}_\varepsilon$, $\psi := \tilde{p}_\varepsilon$ and adding the two equations of (5.6), we get
\[
\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)^n}^2 + \varepsilon \|\nabla \tilde{p}_\varepsilon\|_{L^2(\Omega)}^2 \leq \|f\|_{H^{-1}(\Omega)^n} \|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)^n} + \varepsilon \|g\|_{H^{-1}(\Omega)} \|\nabla \tilde{p}_\varepsilon\|_{L^2(\Omega)}^2
\]
since $\int_{\Omega} (\text{div } \tilde{u}_\varepsilon) \tilde{u}_\varepsilon = -\int_{\Omega} (\text{div } \tilde{u}_\varepsilon) \tilde{p}_\varepsilon$. It leads to that
\[
\|\tilde{u}_\varepsilon\|_{H^1(\Omega)^n} \quad \text{and} \quad \|\sqrt{\varepsilon} \tilde{p}_\varepsilon\|_{H^{1}(\Omega)} \quad \text{are bounded.}
\]
In addition, 
\[ \|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq c(\|\nabla \tilde{u}_\varepsilon\|_{L^2(\Omega)^n} + \|f\|_{H^{-1}(\Omega)^n}) \]
by Lemma 5.1, which implies that \((\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{0<\varepsilon<1}\) is bounded. By Theorem 4.1, the sequences 
\[ (\|u_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon \geq 1} \text{ and } (\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon \geq 1} \]
are bounded, and then 
\[ (\|u_\varepsilon\|_{H^1(\Omega)^n})_{\varepsilon > 0} \text{ and } (\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}})_{\varepsilon > 0} \]
are bounded.

The spaces \(H^1_0(\Omega)^n\) and \(L^2(\Omega)/\mathbb{R}\) are closed in \(H^1(\Omega)^n\) and \(L^2(\Omega)\), respectively. By Lemma 2.4, the spaces \(H^1_0(\Omega)^n\) and \(L^2(\Omega)/\mathbb{R}\) are reflexive in \(H^1(\Omega)^n\) and \(L^2(\Omega)\), respectively. We have that sequences \((\tilde{u}_\varepsilon)_{0<\varepsilon<1}\) and \((\tilde{p}_\varepsilon)_{0<\varepsilon<1}\) are bounded in \(H^1_0(\Omega)^n\) and \(L^2(\Omega)/\mathbb{R}\), respectively. By Theorem 2.3, there exist \((u, p) \in H^1_0(\Omega)^n \times (L^2(\Omega)/\mathbb{R})\) and a subsequence of pair \((\tilde{u}_{\varepsilon_k}, \tilde{p}_{\varepsilon_k})_{k \in \mathbb{N}} \subset H^1_0(\Omega)^n \times H^1_0(\Omega)\) such that
\[ \tilde{u}_{\varepsilon_k} \to u \text{ weakly in } H^1_0(\Omega)^n, \quad [\tilde{p}_{\varepsilon_k}] \to p \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } k \to \infty. \]

Hence, from (5.6) with \(\varepsilon := \varepsilon_k\), taking \(k \to \infty\), we obtain
\[
\begin{dcases}
\int_\Omega \nabla u : \nabla \varphi + \langle \nabla p, \varphi \rangle = \langle f, \varphi \rangle & \text{ for all } \varphi \in H^1_0(\Omega)^n \\
\int_\Omega (\text{div } u) \psi = 0 & \text{ for all } \psi \in H^1_0(\Omega),
\end{dcases}
\] (5.7)
where
\[ |\varepsilon_k \int_\Omega \nabla \tilde{p}_{\varepsilon_k} \cdot \nabla \psi| \leq \sqrt{\varepsilon_k} \|\tilde{\psi}\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)} \to 0, \]
\[ \int_\Omega \nabla \tilde{p}_{\varepsilon_k} \cdot \varphi = -\int_\Omega [\tilde{p}_{\varepsilon_k}] \text{div } \varphi \to -\int_\Omega p \text{div } \varphi = \langle \nabla p, \varphi \rangle \]
as \(k \to \infty\). The first equation of (5.7) implies that
\[ -\Delta (u + u_0) + \nabla (p + p_0) = F \quad \text{in } H^{-1}(\Omega)^n. \]

From the second equation of (5.7), we see that \(\text{div}(u + u_0) = 0\) follows. Hence, we obtain \(u_S = u + u_0\) and \(p_S = p + [p_0]\). Then we have
\[ u_{\varepsilon_k} - u_S = u_{\varepsilon_k} - u - u_0 = \tilde{u}_{\varepsilon_k} - u_S \to 0 \text{ weakly in } H^1_0(\Omega)^n, \]
\[ [p_{\varepsilon_k}] - p_S = [p_{\varepsilon_k} - p - p_0] = [\tilde{p}_{\varepsilon_k}] - p \to 0 \text{ weakly in } L^2(\Omega)/\mathbb{R}. \]
An arbitrarily chosen subsequence of \(((u_\varepsilon, [p_\varepsilon]))_{0<\varepsilon<1}\) has a subsequence which converges to \((u_S, p_S)\), so we can conclude the proof.

If we assume regularity for the pressure solution to (S), then the strong convergence holds.

**Theorem 5.3.** Suppose that \(p_S \in H^1(\Omega)\). Then we have
\[ u_\varepsilon - u_S \to 0 \text{ strongly in } H^1_0(\Omega)^n, \quad [p_\varepsilon] - p_S \to 0 \text{ strongly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0. \]
Proof. We have

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\int_{Ω} \nabla (u_ε - u_S) : \nabla φ + \int_{Ω} (\nabla (p_ε - p_S)) \cdot φ = 0 & \text{for all } φ \in H^1_0(Ω)^n, \\
ε \int_{Ω} \nabla (p_ε - p_S) \cdot \nabla ψ + \int_{Ω} (\text{div } u_ε) ψ = 0 & \text{for all } ψ \in H^1_0(Ω),
\end{array} \right.
\end{align*}
\]

(5.8)

We use the notations \( p_0 \in H^1(Ω) \) in Theorem 3.4. Putting \( φ := u_ε - u_S \in H^1_0(Ω)^n, \psi := p_ε - p_0 \in H^1_0(Ω) \) and \( \hat{p}_S := p_S - p_0 \in H^1(Ω) \), we get

\[
\|\nabla (u_ε - u_S)\|_{L^2(Ω)^{n \times n}}^2 + ε\|\nabla (p_ε - p_S)\|_{L^2(Ω)}^2 \\
= \int_{Ω} (\nabla \hat{p}_S) \cdot (u_ε - u_S) - ε \int_{Ω} \nabla (p_ε - p_S) \cdot \nabla \hat{p}_S \\
\leq \|\nabla \hat{p}_S\|_{L^2(Ω)} \|u_ε - u_S\|_{L^2(Ω)^n} + ε\|\nabla (p_ε - p_S)\|_{L^2(Ω)} \|\nabla \hat{p}_S\|_{L^2(Ω)}.
\]

By Corollary 5.2 and the Rellich–Kondrachov Theorem, there exists a sequence \((ε_k)_{k \in \mathbb{N}} \subset \mathbb{R}\) such that

\[ u_{ε_k} \to u_S \text{ strongly in } L^2(Ω)^n \text{ as } k \to ∞. \]

So, we can write that

\[
\|\nabla (u_{ε_k} - u_S)\|_{L^2(Ω)^{n \times n}}^2 \\
\leq \|\nabla \hat{p}_S\|_{L^2(Ω)} \|u_{ε_k} - u_S\|_{L^2(Ω)^n} + ε\|\nabla (p_{ε_k} - p_S)\|_{L^2(Ω)} \|\nabla \hat{p}_S\|_{L^2(Ω)} \\
\to 0
\]

as \( k \to ∞ \). It implies that

\[
\|p_{ε_k} - p_S\|_{L^2(Ω)} \leq \|p_{ε_k} - p_S\|_{L^2(Ω)} \leq c\|\nabla (u_{ε_k} - u_S)\|_{L^2(Ω)^{n \times n}} \to 0 \text{ as } k \to ∞
\]

by Lemma 5.1. An arbitrarily chosen subsequence of \((u_{ε}, [p_{ε}])_{0 < ε < 1}\) has a subsequence which converges to \((u_S, p_S)\), so we can conclude the proof.

6 Numerical examples

In this section, we illustrate numerically the behavior of the weak solutions to (ES) as \( ε \to ∞ \) and as \( ε \to 0 \).

For our simulations, we consider \( Ω = (0, 1) \times (0, 1) \). We take the following boundary conditions:

\[ u_b = (x(x - 1), y(y - 1))^T, \quad p_b = 2x + 2y - 2 \cdot ν \]

on \( Γ \). The exact solutions for (PP) are \( u_{PP} = (x(x - 1), y(y - 1))^T \) and \( p_{PP} = 2x + 2y - 2 \). We solve the problems (PP), (ES) and (S) numerically by using the finite element method with P2/P1 elements by the software FreeFem++ [22]. The numerical solutions \((u_{PP}, p_{PP}), (u_ε, p_ε) \ (ε = 1, 10^{-3} \text{ or } 10^{-4})\), and \((u_S, p_S)\) to the
problems (PP₁), (ES₁), and (S'), respectively, are illustrated in Figure 2, 3, 4, 5, and 6. From these pictures we observe that \((u_\varepsilon, p_\varepsilon)\) converges to \((u_{PP}, p_{PP})\) as \(\varepsilon \to \infty\) and to \((u_S, p_S)\) as \(\varepsilon \to 0\) (as expected from Theorem 4.1 and Theorem 5.2).

The structure of the code is as follows.

```plaintext
int n = 3; //mesh size: 10*n
real a = 1.0, b = 1.0; //square size (Omega = (0,a)*(0,b))
real eps = 1.0e-4; //epsilon

func exau = x*(x-1); func exav = y*(y-1); func exp = 2.0*(x+y-1); //boundary data
func fx = 0; func fy = 0; //F = (fx, fy)

real pc;

border Gamma1(t = 0, a) { x = t; y = 0; }
border Gamma2(t = 0, b) { x = a; y = t; }
border Gamma3(t = a, 0) { x = t; y = b; }
border Gamma4(t = b, 0) { x = 0; y = t; }

mesh Th=buildmesh(Gamma1(10*n)+Gamma2(10*n)+Gamma3(10*n)+Gamma4(10*n));

fespace Uh(Th, P2); Uh us, vs, ue, ve, u2, v2;
//P2-mesh
fespace Ph(Th, P1); Ph p, ps, pe, p1, up, vp, u1, v1; //P1-mesh

solve pressure(p,p1) = //a solver of the pressure-Poisson problem
int2d(Th)(dx(p)*dx(p1) + dy(p)*dy(p1)) - int2d(Th)(fx*dx(p1) + fy*dy(p1))
+ on(1, 2, 3, 4, p = exap);
solve velocity([up,vp],[u1,v1]) =
int2d(Th)(dx(up)*dx(u1) + dy(up)*dy(u1) + dx(vp)*dx(v1) + dy(vp)*dy(v1))
- int2d(Th)(fx*u1 + fy*v1 - dx(p)*u1 - dy(p)*v1)
+ on(1, 2, 3, 4, up = exau,vp = exav);

solve estokes([ue,ve,pe],[u2,v2,p1]) = //a solver of the e-Stokes problem
int2d(Th)(dx(ue)*dx(u2) + dy(ue)*dy(u2) + dx(ve)*dx(v2) + dy(ve)*dy(v2))
+ dx(ue)*u2 + dy(ue)*v2 + p1*(dx(ue)+dy(ve))
+ eps*(dx(ue)*dx(u1) + dy(ue)*dy(v1))
+ on(1, 2, 3, 4, ue=exau,ve=exav, pe=exap);

solve stokes([us,vs,ps],[u2,v2,p1]) = //a solver of the Stokes problem
int2d(Th)(dx(us)*dx(u2) + dy(us)*dy(u2) + dx(vs)*dx(v2) + dy(vs)*dy(v2))
+ dx(ps)*u2 + dy(ps)*v2 + p1*(dx(us)+dy(vs))
+ on(1, 2, 3, 4, us = exau, vs = exav);

pc = int2d(Th)(ps); ps = ps - pc/(a*b); //the average of ps = 0
```

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(a) $p_{PP}$: numerical solution is given by green “×” on each node of the mesh and exact solution is given by magenta lines.

(b) $u_{PP}$: rainbow colors indicate the length of $|u_{PP}(\xi)|$ at each node $\xi$.

Figure 2: $p_{PP}$ (left) and $u_{PP}$ (right).

(a) $p_1$: numerical solution is given by green “×” on each node of the mesh and exact solution $p_{PP}$ is given by magenta lines.

(b) $u_1$: rainbow colors indicate the length of $|u_1(\xi)|$ at each node $\xi$.

Figure 3: $p_1$ (left) and $u_1$ (right).

(a) $p_{10^{-3}}$: numerical solution is given by green “×” on each node of the mesh and exact solution $p_{PP}$ is given by magenta lines.

(b) $u_{10^{-3}}$: rainbow colors indicate the length of $|u_{10^{-3}}(\xi)|$ at each node $\xi$.

Figure 4: $p_{10^{-3}}$ (left) and $u_{10^{-3}}$ (right).
Figure 5: $p_{10^{-4}}$ (left) and $u_{10^{-4}}$ (right).

Figure 6: $p_S$ (left) and $u_S$ (right).

7 Conclusion

We introduced the $\varepsilon$-Stokes problem (ES) connecting the classical Stokes problem (S) and corresponding pressure-Poisson problem (PP). For any fixed $\varepsilon > 0$, the $\varepsilon$-Stokes problem has a unique solution $(u_\varepsilon, p_\varepsilon)$ (Theorem 3.4) and $u_\varepsilon$ is a good approximation to the solution to (S) while the solution to (S) and (PP) are close in the sense that

$$
\|u_S - u_{PP}\|_{H^1(\Omega)^n} \leq C \|\gamma_0 p_S - p_0\|_{H^{1/2}(\Gamma)}, \quad \|u_S - u_\varepsilon\|_{H^1(\Omega)^n} \leq C \|\gamma_0 p_S - p_0\|_{H^{1/2}(\Gamma)};
$$

see Proposition 3.5 for details.

We proved in Theorem 4.1 that a sequence $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ converges strongly in $H^1(\Omega)^n \times H^1(\Omega)$ to the solution to (PP) as $\varepsilon \to \infty$. The convergence rate is of
order $O(1/\varepsilon)$. We also proved in Theorem 5.2 that $((u_\varepsilon, p_\varepsilon))_{\varepsilon>0}$ converges weakly in $H^1_0(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to the solution $(u_S, p_S)$ to (S) as $\varepsilon \to 0$. If we add an assumption $p_S \in H^1(\Omega)$, then strong convergence holds by Theorem 5.3. By the numerical examples, we observed the expected convergences as $\varepsilon \to \infty$ or $\varepsilon \to 0$.

We summarize our results as follows:

- We introduce the $\varepsilon$-Stokes problem (ES) as an interpolation between the Stokes problem (S) and the pressure-Poisson problem (PP).

- The solution $(u_\varepsilon, p_\varepsilon)$ of (ES) strongly converges in $H^1(\Omega)^n \times H^1(\Omega)$ to the solution $(u_{PP}, p_{PP})$ of (PP) as $\varepsilon \to \infty$. The convergence rate is $1/\varepsilon$.

- The solution $(u_\varepsilon, p_\varepsilon)$ of (ES) weakly converges in $H^1_0(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ to the solution $(u_S, p_S)$ of (S) as $\varepsilon \to 0$. If $p_S \in H^1(\Omega)$, then the strong convergence of $(u_\varepsilon, p_\varepsilon)$ to $(u_S, p_S)$ holds.

The main results of this thesis have been published in the journal Advances in Mathematical Sciences and Applications [23].

8 Future work

In this thesis, the domain of the numerical examples is in $\mathbb{R}^2$. Numerical comparison of (ES), (PP), and (S) in 3D is one of our interesting future works, for example the convergence rates and numerical instability. As another extension of our research, generalization of our results to the Navier–Stokes problem and other boundary conditions such as a stress boundary condition;

$$\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - p\delta_{ij}\right) v_j = 0$$

and a boundary condition introduced in [12];

$$u \times \nu = 0 \text{ and } p = p_b,$$

is important but is still completely open.
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