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Studying concept elements as a way to trace students' conceptual understanding

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The understanding of mathematical concepts has been described in terms of *concept definition* and *concept image*. We suggest an elaboration of these constructs, the *concept element*, to find a way to theoretically describe students' understanding. The concept element construct was tested in a setting with students working with linear functions at the secondary school level. Our empirical findings reveal traces of students' concept elements regarding linear functions. Some concept elements appeared early in the process while others appeared after a cognitive conflict (e.g. evoked by the task construction and setting). The detailed grid on which concept elements are defined was a useful tool, yielding new insights into students' knowledge and understanding.

Conceptual understanding is central to mathematical proficiency (Niss, 2006) and our scholarly knowledge of students' conceptual understanding has developed over the years (cf. Kilpatrick, 2001; Star & Stylianides, 2013). Understanding can be regarded as intertwined conceptual and procedural knowledge with which students construct and refine connections between old and newly acquired concepts (cf. Baroody, Feil & Johnson, 2007; Hiebert & Carpenter, 1992; Kilpatrick, 2001; Niss, 2006). Hiebert and Carpenter (1992) demonstrated that students connect internal representations (e.g. internal thoughts and internal images) with external representations (e.g. symbols, images and practical material). Understanding can be perceived as a network of connections between internal and external representations, where external representations can be shared with others. This implies that the breadth and depth of understanding correlates to the number and strength of the connections. For instance, students who have built a network with strong connections can be more

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innovative and create new connections that expand their network (their conceptual understanding).

When a student creates, modifies, or rejects connections that do not work, the level of conceptual understanding advances (Hiebert & Carpenter, 1992; Jonsson, Norqvist, Liljekvist & Lithner, 2014; Norqvist, 2018). To understand a "new thing" (e.g. a concept, an idea, etc.), students need to connect it to their existing internal network of connections. A cognitive conflict occurs if a student tries to connect a new concept or idea that does not fit into the student's existing network. Such conflict may be both an obstacle and a learning opportunity (cf. Granberg & Olsson, 2015). Students need to modify or reject existing connections to expand their conceptual understanding (cf. Norqvist, 2018; Pettersson, 2016). However, students cannot develop their conceptual understanding by themselves. According to Tall (2017), the role of the teacher is to "encourage the learner to seek to develop more powerful techniques that support long-time learning" (p. 59). Long-time learning is important, he continues, both in relation to the progression within mathematics and in how students as individuals interpret it.

A key question is then how to plan lessons to support long-time learning and to assess students' conceptual understanding. Education should also be based on research as well as on teachers' professional knowledge (SFS 2010:800). Components of students' conceptual understanding can provide crucial input during the planning of lessons and tasks (Carlson, Oehrtman & Thompson, 2007). The aim of this study is therefore to develop knowledge useful for both teachers (e.g. in planning lessons and assessing student knowledge) and researchers (e.g. in probing student understanding). The research question guiding our study is: How can we theoretically describe students' conceptual understanding in a productive way?

In this paper, we suggest a theoretical construct, the *concept element*, to expand the Tall and Vinner framework (1981) along with its terms *concept image* and *concept definition*. We will use empirical findings regarding students working with linear functions in the form $y = mx + c$ in a digital environment (see e.g. Pettersson, 2016) to reveal the underpinnings of our theoretical proposal. In this case, the students needed to create a network of connections linking the concept variable and the specific different meaning of each of the symbols y, $=$, m, x and c, in order to expand their conceptual understanding.

Theoretical frame: concept definition and concept image

In the Tall and Vinner (1981) framework, mathematical concepts per se, and the cognitive processes needed to understand them, are

theoretically described in terms of *concept definition* and *concept image*. A concept definition is straightforward: it is a written or spoken definition of a certain concept. Tall and Vinner (1981) described two kinds of concept definitions: 1) a personal concept definition that the person formulates from his or her own understanding and 2) a formal concept definition shared within a larger group of people. However, a concept image is a more complex theoretical construct. It is the sum of the cognitive structures that the student associates with the concept (Tall & Vinner, 1981). It contains the wordings, images, thoughts, procedures, etc., that a student has connected to the concept, that is, in Hiebert and Carpenter's (1992) terms, all the internal and external representations the student uses, as well as their connections. The concept image is the persona of the concept as created by the student. However, the concept image is dynamic and changes as the student becomes more acquainted with the concept and more aware of how to handle it: the student connects more elements to the network of connections that constitutes the understanding of the concept.

In the Nordic context, Viirman, Attorps and Tossavainen (2010) studied concept images of the function concept among engineering students and teaching students. They captured the students' concept images using mind maps and demonstrated that most students had an operational conception of functions. Juter (2009) also used concept images when she studied students' understanding of various concepts (e.g. functions) before and after a course in analysis. As in our study, Juter used tasks and interviews to capture traces of concept images. Both Juter and Viirman et al. studied connections – links – between concepts: Viirman et al. used the number of links as a measure of depth of understanding, whereas Juter developed a tool for classifying links in three main categories: valid, invalid and irrelevant. In a more recent study (Breen, Larsson, O'Shea & Pettersson, 2017) using data from interviews and questionnaires, the idea of evoked concept images regarding inverse functions was used.

Concept images can be used by teachers to make students aware of their own conceptual understandings, as well as to explore their students' conceptual understandings. De Bock, Neyens and Van Dooren (2017) explored students' concept images by examining a student's ability to connect functions to their corresponding properties. Their study shows the importance of function properties in students' concept images and highlights the value of making this explicit to students. Furthermore, Kontorovich (2018) shows the potential of using concept images as tools when planning lessons. Specifically, Kontorovich states the following when approaching cross-curricular concepts (e.g. tangent line, angel between two lines, graph):

> [I]n the landscape of student's mathematical education, some concepts are reconsidered in different domains. The domains can be rooted in different axiomatic systems and contain different or new objects. Accordingly, a domanial shift of these *cross-curricular* concepts is often accompanied by a substantial change in familiar dimensions (definitions, properties, procedures and connections with other concepts). The domanial shift and the substantial change are potential sources for students' difficulties and mistakes.
>
> (Kontorowich, 2017, p. 6. Italics in original).

Since concept image has proven to be a useful construct for analysing students' conceptual understanding, we now want to consider the possibility of expanding the theory to make students' concept images more accessible and useful for teaching. Although complex and impossible to describe fully (Tall & Vinner, 1981), a student's concept image comprises various parts, which can be systematically described.

Observing students working with linear functions

Several researchers have pointed out that the concept of functions is difficult for students to grasp (see e.g. Knuth, 2000; Watson, Ayalon & Lerman, 2017; Zaslavsky, Sela & Leronet, 2002). Although it is important to be able to shift between representations, for instance between an algebraic representation and a graphical representation (Janvier, 1987), students struggle with this skill (Bloch, 2003; Moschkovich, Schoenfeld & Arcavi, 1993). The graphical shift seems to be particularly difficult (Schwarz, Dreyfus & Bruckheimer, 1990). Markovits, Eylon and Bruckheimer (1986) studied the shift between algebraic and graphical representations and found that the shift from graph to algebra was more difficult than the other way around. When students are given the opportunity, they avoid the graphical representation (Artigue, 1992; Knuth, 2000). Tasks in class expose students more often to the shift from algebra to graph then to the shift from graph to algebra (Gagatsis & Shiakalli, 2004).

In linear functions of the form $y = mx + c$, the symbols m and c (parameters in this case) exist in four dimensions: symbolic, graphic, numerical and contextual (Bardini & Stacey, 2006). The symbolic dimension is tied to the algebraic expression, the graphical dimension to the graph, the numerical dimension to counting and determining values and the contextual dimension is tied to real-life examples. The different dimensions contain similarities and differences that make them difficult for students to handle. For instance, in the symbolic dimension, m is a coefficient and c is a constant. In the graphical dimension, c can be determined at the intersection (y-axis, line), but m cannot be determined just by reading

coordinates in the coordinate system. Students often show difficulties when being asked to explain relations between models and representations (De Bock, Van Dooren & Verschaffel, 2015). Bardini and Stacy (2006) conclude that m is more complex to understand than c in all four dimensions. However, when function properties are addressed explicitly, students perform better (De Bock et al., 2017). Still, De Bock et al. find that despite the student knowing and recognizing the properties of the functions, errors are made (e.g. regarding proportionality) depending on the representational mode used.

Other studies show that some students understand a change in c-value as a movement of the graph along the x-axis (Goldenberg, 1988; Moschkovich, 1990) and that students look for values of m and c at the intersection with the x- and y-axis, respectively (Schoenfeld, Smith & Arcavi, 1993). Working specifically with the parameters m and c could develop students' symbol sense (Drijvers, 2003). Symbol sense, according to Drijvers, incorporates the ability to make sense of symbols, expressions and formulae, and to understand their structure. The specific structure of a function can be clarified when using its parameters. Varying the value of the parameters will give a description of a family of functions (Drijvers, 2003) which, in a learning situation, will give students the opportunity to work with functions as objects per se, not just as distinct lines (cf. Yerushalmy, 1991). This seems to be important in order to make the function properties explicit for the students (De Bock et al., 2017).

Expanding the framework: concept element

A student's concept image (e.g. of linear functions) comprises various parts and connections that form the conceptual understanding at a specific moment. Each of these parts, or concept elements, is discernible. For example, a student can have concept elements such as "slope" or will use phrases such as "crosses the y-axis". The student expresses the graphical representation of a linear function as a "straight line" and as the "intersection with the y-axis is the c-value" (see figure 1).

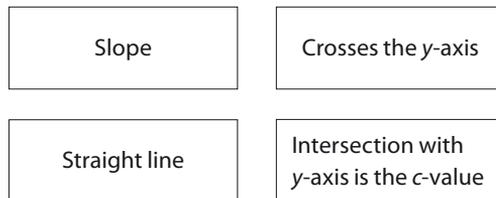| Slope	Crosses the *y*-axis
Straight line	Intersection with *y*-axis is the *c*-value

Figure 1. *A concept image: linear functions containing four concept elements*

The student's concept image can be evoked on different occasions (see Breen et al., 2017; Hansson, 2006; Juter, 2006; Vinner, 1983), such as when a student is working on a mathematical task. Consider a linear function task in which it is possible to notice that the c-value can be determined by looking at the intersection with the y-axis versus one in which the representation task is different, in which one cannot see where the function intersects the y-axis. For a student addressing the concept element related to the y-intercept ("crosses the y-axis"), a contradictive concept element is evoked and a cognitive conflict may occur. A student's concept image might not make sense and the new element cannot be linked to the existing network of connections. Here, the concept element that the c-value cannot always be determined just by looking to see where the line intersects the y-axis needs to be internalised within the concept image to create a conceptual understanding without contradictions.

Concept images are mental constructions and cannot be observed or studied directly. However, many studies have demonstrated that traces of concept images can be studied (e.g. Hansson, 2006; Juter, 2009; Vinner, 1983). Breen et al. (2017) state that Tall and Vinners' (1981) concept image framework was appropriate to explore students' concept images of the notion of inverse functions.

Accordingly, we argue that traces of students' concept elements can be studied to learn more about their conceptual understanding. In the following sections, we describe a study of upper secondary students working with linear functions in order to illustrate how the theoretical construct concept element can be useful.

Method

The study was designed to evoke traces of the students' concept elements. In the study, the tasks and the students' activities were the centre of attention. In this section, we outline the principles guiding the intervention (e.g. the task design) and the data collection.

Participants and setting

The empirical data were collected within a qualitative study. The setting was students working in pairs on mathematical tasks in a dynamic software environment (GeoGebra). Video recordings, screen recordings (around four hours total) and stimulated recall interviews (around two hours total) of six students working on designed tasks were analysed. The students were enrolled in the second-level mathematics course in upper secondary school. All students in the course (48) were asked if

they wanted to participate of which six students volunteered. This paper focusses on the work of four students (Maria, Alma, Lisa, Hanna) as their work, provided us with a variety of examples to illustrate the concept elements construct. The first author conducted all parts of the study. The teacher merely mediated contact with the students. The interviews were conducted at a school, but not during regular lessons. The regional ethical committee approved the design. Written informed consent was obtained from each participant individually. All names used in the text are pseudonyms.

The students had worked with linear functions $y = mx + c$ in previous courses and at the time of the study the students were working on this topic in their current course. A pre-test was conducted to investigate each of the student's pre-knowledge about how the parameters m and c change linear functions[1]. Connecting representations, both from graphical to algebraic and vice versa (as mentioned in De Bock et al., 2015, and by Janvier, 1987) was only visible in the pre-test for two of the students, Alma and Hanna. Maria and Lisa did not show that they could connect the graphical and algebraic representations (Pettersson, 2016).

Design and procedure

In order to make the function's properties explicit for the students, as suggested by De Bock et al. (2017), the study contained five tasks incorporating the four aspects that affect the graphical representation of linear functions: parameters m and c, the domain of the function and the scale of the coordinate axes. In each task, three aspects were held constant and the fourth was varied (see table 1). In this paper, we focus on tasks A and B. In tasks A and B (figure 2), the students were asked to transfer the images (presented on paper) onto the computer screen by writing functions in the forms $y = mx$ (task A) and $y = mx + c$ (task B) in GeoGebra's algebra window.

The students worked in pairs, as this configuration is beneficial when studying learning (Granberg & Olsson, 2015), and were encouraged to comment aloud while working on the task and to use the mouse cursor

Table 1. *Constant and varied aspects of the tasks*

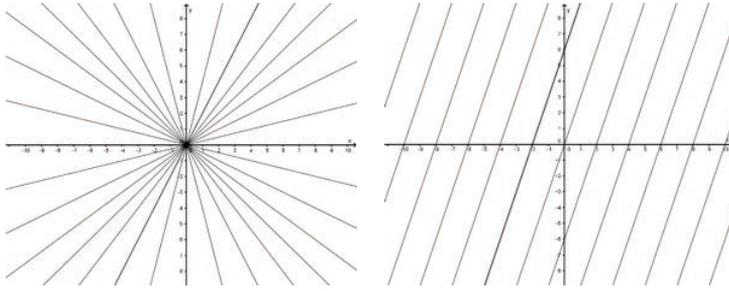| Task/aspect	Parameter m	Parameter c	The domain	Scale on the x-axis
A	Varied	Constant	Constant	Constant
B	Constant	Varied	Constant	Constant
D	Constant	Constant	Varied	Constant
C and E	Constant	Constant	Constant	Varied

Figure 2. *Task A (left) and task B (right). Adapted from Magidson (1992, p.65)*

to indicate what they were referring to on the computer screen. The students were provided with pencils and paper along with the outlined task. The students' activity was video recorded and the screen activity was logged, allowing us to detect traces of concept elements in their dialogues and actions. A stimulated recall interview was conducted with the students after each intervention to evoke further explanations (De Bock et al., 2015). The recorded student activities were categorised regarding two aspects: traces of concept elements (e.g. oral comments, calculating $\Delta y/\Delta x$ and counting squares) and the actual actions taken to solve the task (writing a function in the algebra window). Video sequences and interview sequences identified as containing traces of concept elements were selected for transcription and further analysis.

The selected sequences were analysed and labelled. For example, students' utterances such as "downwards slope" were labelled using mathematical terms such as "negative slope". Traces of concept elements appeared at different stages of the task solving. Moreover, cognitive conflicts were evident in the students' working process, such as when the feedback from the computer screen contradicted their concept images, or when the outlined solving process did not lead to a solution of the task. The following two categories therefore evolved: 1) concept elements the students displayed initially and 2) concept elements the students displayed after a cognitive conflict (e.g. between a previously used concept element and feedback from the screen). These categories each have two levels: successful (leading to a solution, but not necessarily a mathematically correct one) and unsuccessful (not leading to a solution, but could be mathematically correct). Concept elements are written in italics.

Given the setting of tasks A and B, one example of an initial and successful concept element is *counting steps* (note that in the Swedish context the "rise over run" method where run = 1 is common; in this paper we call this counting steps). Another example is the concept element that *the c-value can be read at the line's intersection with the y-axis*. A concept element which is labelled unsuccessful and that appears after a

cognitive conflict is *if the c-value is decreased, the line moves to the right on the x-axis*. This is mathematically correct but will not help the student solve the task. The concept element the *c-value is the initial value* does not help the student when a line's graphical representation does not show the intersection with the *y*-axis.

Analysis and results

In this section, we first present an overview of the results and then refine the description using quotations from the interviews and the video recordings.

Successful and unsuccessful concept elements

The first task (A) was to create a specific pattern in a digital environment, using the linear function $y = mx$. In this task, the *m*-parameter was varied. In the second task (B), the pattern could be created by varying the *c*-parameter. The concept elements the students displayed were categorised and labelled according to when in the solving process the concept element became evident (initially or after a cognitive conflict) and whether or not it was successful as shown in table 2.

Table 2. *Examples of categorisation of concept elements*

Concept elements the student displayed initially	Concept elements the student displayed after a cognitive conflict between earlier concept images and feedback from the computer or worksheet
Successful	
Counting steps, one step to the right and m steps upwards/downwards.	The *m*-value can be determined from one step to the left and *m* steps upwards if the *m*-value is a negative number.
The *m*-value can be calculated using the difference quotient, $\Delta y / \Delta x$.	The *c*-value can be calculated from the line's intersection with the *x*-axis and a formula.
The *c*-value can be read from the line's intersection with the *y*-axis.	If the lines are parallel and equidistant, you can use the distance in the *y*-direction to calculate the *m*-values.
Not successful	
The line starts at the origin.	
The *c*-value is the initial value, which is where the line starts farthest left.	
If the *c*-value is decreased, the line moves to the right on the *x*-axis.	

Concept elements regarding the origin

We will now describe how the students Maria, Alma, Lisa and Hanna displayed traces of concept elements by citing excerpts from their solving process.

Initially, Maria entered the function $y = x$, but when the line appeared on the screen she asked: "Did we get the whole [line]?" Her reaction was categorised as the concept element *the line starts at the origin*. In the stimulated recall interview, Maria explained that it was the feedback from the computer screen (addressing explicit properties of a function, De Bock et al., 2017) that changed her concept element to *the line crosses the origin*:

Maria: Well, we hadn't expected that to happen, but when we saw it, it became, like, obvious.

Concept elements regarding the m-value

The students displayed two concept elements regarding how to determine the *m*-value: *go one step to the right and then m steps up* (i.e. counting steps) and by computing $\Delta y/\Delta x$ (i.e. the difference quotient). They did not think of the *m*-value as the intersection with the *x*-axis as suggested by Schoenfeld et al. (1993).

Alma: But if you think like this (.) it's a slope. If you go one step to the right you have to go one, two, three, four [steps] upwards to come to the line.

Hanna: Four, three – isn't that four divided by three then?

Alma displayed a modification of the concept element *counting steps* when she explained in writing how the *m*-value and the graphical representation of the linear function are connected.

Alma: You jump one step to the right or to the left depending on whether it is a positive number, then you see how many steps it takes before it intersects the line.

What Alma displayed is the concept element *the m-value can be determined from one step to the left and m steps upwards if the m-value is a negative number*. When the students handled the graphical representations of functions with negative *m*-values (changing the value of parameters and thereby obtaining a family of functions) (Drijvers, 2003), they displayed variations of the concept element lines with negative slopes have negative *m*-values.

Johanna: We can take the negative ones on this side [points at the line $y = -0.25x$ in the fourth quadrant], so counting the *m*-value here is probably the same except for a minus sign in front.

Concept elements regarding the c-value

Alma and Lisa worked on task B. Alma began with the function $y = 3x + 36$. She determined, in her words, "the initial value"[2] of the line. She saw the point (-11, 3) as "the start" (figure 3). She then pointed out the y-value, $y = 3$, as "the initial value", and not the intersection with the y-axis (as suggested by Schoenfeld et al., 1993). The displayed concept element is *the initial value is where the line begins farthest to the left.*

Alma: Well, the initial value is where it begins, so if you begin here it is three [points at the function $y = 3x + 36$ on the paper, looks at where the line appears farthest to the left in the graph and then follows $y = 3$ to the y-axis].
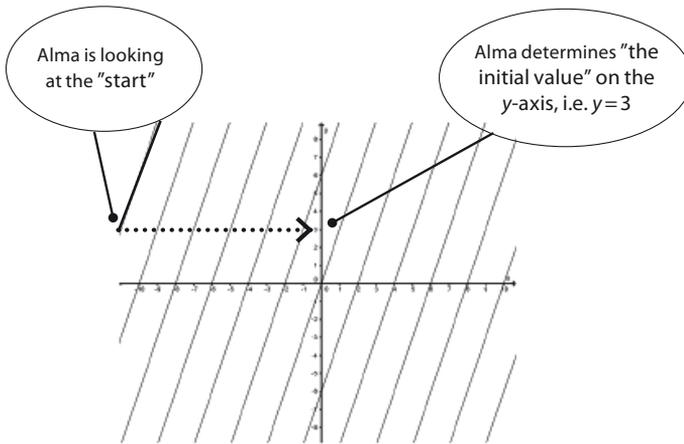

Figure 3. *Alma looks at the start of the line*

During the stimulated recall interview, our analysis of Alma's concept element was confirmed.

Alma: The slope was $3x$ and since I didn't know where it crossed the y-axis, I thought it was three [points with her pencil on the paper at the arrow in figure 3].

Alma then decided to work with another function, $y = 3x + 30$. She studied the line on the worksheet: "It cannot start because it does not intersect the line [the y-axis]". Alma instead went to the x-intercept, that is, displaying the concept element *c-value is the line's intersection with the x-axis* (see figure 4):

Alma: No negative (.) or (.) this is negative ten [pointing at the line $y = 3x + 30$, where $x = -10$].

[...]

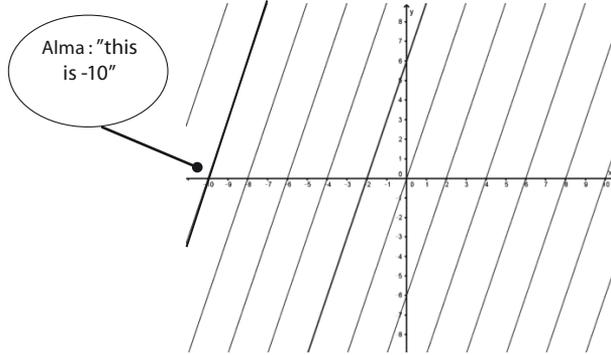Alma: If it is (.) this is one, two, three, $y = 3x - 10$ [writes $y = 3x - 10$ in the algebra window].

Figure 4. *Alma is reading the line's intersection with the x-axis*

When Alma looked at the feedback from the screen, she erased the function.

Alma: But it doesn't cross the y-axis; it is the x-axis it crosses.

By then Alma knew that they were looking for the line's intersection with the y-axis, but their line did not have one. The representational mode contributed to this error (De Bock et al., 2017). She changed the function again, this time to $y = 3x$. After analysing the line on the worksheet, she again changed the line to $y = 3x - 6$, which she thought was easier and for which she could determine the y-intercept (i.e. $y = -6$). Alma managed all the other lines with a visible y-intercept. Alma called this point the "initial value".

Alma: We can take this one [points at the function $y = 3x - 6$]. For this [other] one we have no initial value [points at the function $y = 3x - 12$]. In this [third] one we have an initial value [points at the function $y = 3x$].

Alma then continued to determine the c-value of functions that do not visibly cross the y-axis. These lines created an obstacle.

Alma: But how can you determine this one that is not [points with a sweeping gesture across the lines crossing the x-axis on the negative side], is this negative x times x or what? Because it is on the x-axis (.) because the initial value you can only see on the y-axis. These ones are on the x-axis (.) [points at the x-axis].

Lisa: Isn't it possible to pick any place? You do it like this? (.) [points one step to the right and three steps up on the line $y = 3x + 36$]

Alma: The initial value as well, we don't have an initial value [points at the line $y = 3x + 36$].

Lisa: Let's do these last.

Alma responded to the Lisa's request to wait with the function $y = 3x + 36$ and entered $y = 3x - 10$ in the algebra window without any comments. She then, without saying anything, entered $y = 3x - 12$ in the algebra window and stated that it was correct. She explained her thinking to Lisa.

Alma: If you consider that 12 and 4 is 16 in difference (.) ... But 4 times 3 is 12. How much is 6 times 3? 18?

Lisa: Uhuh.

Alma: If we take $y = 3x - 18$ [writes $y = 3x - 18$ in the algebra window]

Lisa did not understand Alma, so she asked again.

Lisa: I don't understand what you are doing.

Alma: Well (.) I think (.) I tried a bit and then it was like, we don't know where it crosses here, do we?

Lisa: No.

Alma: And then, I kind of made a combination of 4 times 3, that's 12, but it is negative 12 and then it is correct, and then I took 6 times 3 is 18 but -18. Look, if you move (.) [moves the coordinate system on the screen so that the y-intercept at $y = -18$ is visible] it occurs at -18 [moves the coordinate system back to the original position so the intercept at $y = -18$ is invisible again].

Alma searched for a pattern for how to calculate the c-value with help of the x-intercept. Alma finally discovered a relationship: $c = -(the\ x\text{-}intercept\ times\ 3)$. Alma referred to the minus sign in her formula as follows: "But it is on the negative side [of the x-axis]". Alma did not display any traces of concept elements regarding why there should be a minus sign. She just changed it to make it fit. One important thing to notice here is that Alma moved the coordinate system on the screen to visually verify her calculation; she then moved it back. She could have done the same (moving the coordinate system) to solve the rest of the lines, but she did not. Instead, she continued to calculate the c-value using the x-intercepts. Finally, Alma suggested a conclusion.

Alma: Hey, the number of the x-axis times the slope. Because then it is, it is three upwards all the time.

Alma's concept element can now be labelled $c = -x\text{-}intercept\ times\ the\ slope$.

Concept elements regarding the slope

Next we describe how Hanna and Maria worked on the task. Maria looked at the worksheet and entered $y = 3x$ in the algebra window. She continued to look at the worksheet at the line $y = 3x - 6$ and determined the c-value to be -6 by looking at the line's x-intercept. She entered the function $y = 3x - 6$ in the algebra window, compared the picture on the screen with the worksheet, and concluded that they were the same. Hanna and Maria then started to work with the function $y = 3x - 12$. Hanna was unable to find where the function "started".

Hanna: Where does it start?

Maria: Oh, what a shame [Maria notices that the line does not have any visible y-intercept].

Hanna: But it is three steps, for each.

Maria: Uhuh.

Hanna: Well then, we should be able to go downwards.

Maria: No.

Hanna: [points at (0, -6)] No, it is six steps.

The students recognized the pattern of equal distance between the lines and by decreasing and increasing the c-value by six steps they could work through all the functions without any obstacles causing a cognitive conflict.

We have now illustrated how traces of concept elements can be detected in students' work on mathematical tasks. In the next section, we illustrate how the identified concept elements occurred both in the students' initial work and in their evolving understanding of linear functions.

Concept elements before and after a cognitive conflict

Traces of concept elements became evident at various times: some of the concept elements were displayed from the outset and some only after a cognitive conflict. These conflicts occurred, for instance, between the students' interpretation of the task (on the worksheet) and the feedback from the computer screen. They could also occur between the students' interpretation of the task and their previous concept elements. In this section, we focus on describing the traces of concept elements displayed just before and after a cognitive conflict.

While Maria and Hanna were working on the task, a cognitive conflict became evident when Maria stated that "Did we get the whole [line]?". Her concept element *the line starts at the origin* changed in response to feedback from the computer screen to *the line crosses the origin*. In figure 5, we describe the concept element before and after the cognitive conflict.

Hanna and Maria did not display any other cognitive conflicts when working on this task. They finished task A after ten minutes.
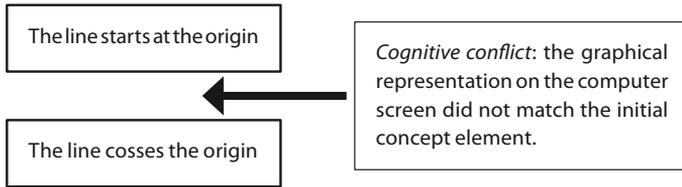
| The line starts at the origin |
| The line cosses the origin |

Cognitive conflict: the graphical representation on the computer screen did not match the initial concept element.

Figure 5. *Cognitive conflict in task A, Maria and Hanna*

While working on task B, Maria and Hanna displayed the concept element *the c-value can be read from the lines intersection with the y-axis.* This concept element was successful for certain lines, but a conflict emerged for lines that did not have a visible y-intercept. The students resolved this conflict after they realised that the c-value for these lines could be calculated from the distances between the lines in the y-direction (figure 6). They ended their work on task B after about seven minutes.

| The c-value can be read at the line's intersection with the y-axis. |
| If the lines are parallel and equidistant, you can use the distance in the y-direction to calculate the m-values. |

Cognitive conflict: the line in focus did not have a visible y-intercept.
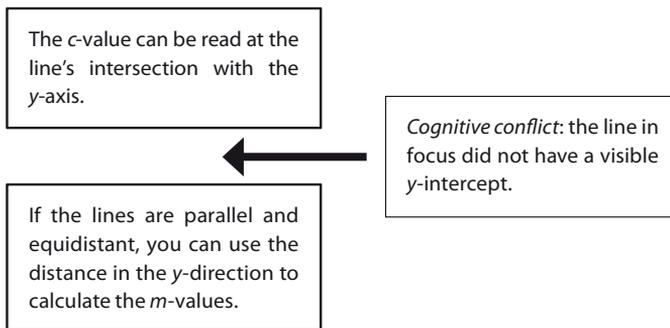
Figure 6. *Cognitive conflict in task B, Maria and Hanna*

If we now look into the working process of Alma and Lisa, we can trace three cognitive conflicts (see figure 7). At the beginning, a cognitive conflict occurred when the students were to determine the line's intersection with the y-axis, as the line they chose did not have a visible intersection. Alma then determined an initial value where the line began on the lefthand side. This led to the second cognitive conflict, as the feedback from the computer screen was inconsistent with the picture on the worksheet. The third cognitive conflict occurred when Alma determined the c-value to be the line's x-intercept.
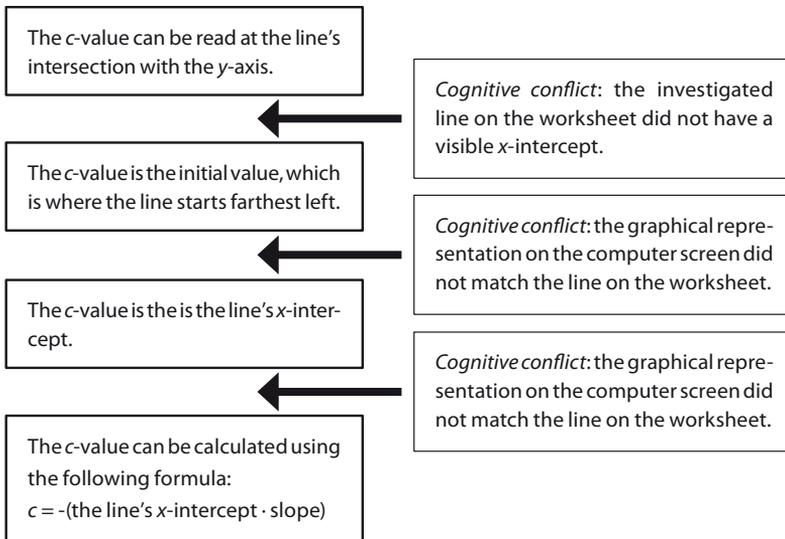
Figure 7. *Cognitive conflict in task B, Alma and Lisa*

Summary

In the "Result and analysis" section, we illustrated how to use the theoretical construct *concept element* in describing the traces of the students' initial and evolving conceptual understanding evident when they were engaging in the tasks. We could see both successful and unsuccessful concept elements. Furthermore, we could describe the students' working process, that is, how concept elements changed when a cognitive conflict occurred. In the next section, we discuss the implications of using the concept element as a tool for understanding the evolution of the students' conceptual understanding.

Discussion

In this paper, we suggested a theoretical construct, the *concept element*, as an elaboration of Tall and Vinner's (1981) framework. We did this to find a way to theoretically describe the evolution of students' conceptual understanding that is useful for both teachers and researchers. We illustrated how students' activities could be analysed and understood as conceptual processes, since traces of concept elements were evident in the students' working process. We also illustrated how cognitive conflicts (e.g. evoked by the task construction and setting) could change existing concept elements. As mentioned before, Tall and Vinner (1981) defined

concept images to describe the students' internal and external representations of a concept and its connections. The elaboration into concept elements suggested in this paper provides a detailed framework with which it is possible to analyse students' conceptual understanding.

A student's concept image comprises various parts that can be described in several systematic ways. For instance, Juter (2009) and Viirman et al. (2010) studied the connections, or links, between concepts. Viirman et al. used the number of links to measure depth of understanding, whereas Juter developed a tool for classifying links into three main categories: valid, invalid and irrelevant. In our study, the connections *per se* are not emphasised. We instead focus on the concept elements building the concept image, in order to follow the traces of evolving student understanding. In this way, our emphasis is similar to that of Greefrath et al. (2016), who inquired into aspects of students' "Grundvorstellungen" as meaning-making and, thus, as the interpretation of a mathematical concept (cf. Vom Hofe & Blum, 2016):

> A concept image may contain several individual *Grundvorstellungen* that conceptualize different perspectives on that concept. Individual *Grundvorstellungen* are central components of a valid concept image. (Greefrath et al., 2016, p. 103)

Grundvorstellungen as a pedagogical construct can roughly be translated as "basic idea" or "basic concept" (see e.g. Roos, 2017) and, like concept images, it must be divided into parts ("aspects") when inquiring into students' conceptual understanding. One may ask why it is important to extend the theory by adding another level. The answer is not just about providing a tool to theorise empirical findings; it is also a matter of supporting teachers' professional approach to teaching and learning.

Teachers need tools to systematically develop lessons that promote long-term learning (Tall, 2017), founded on aspects that can uncover the "increasing sophistication of mathematics" (Tall, 2017, p. 56). The theoretical description of conceptual understanding through, for instance, concept images, is difficult to implement in class (Bingolbali & Monaghan, 2008). Concept elements, however, may be useful in both identifying and implementing lessons supporting mathematical proficiency, as described by Niss (2006). Breen et al. (2017) discussed the connection between questions asked and the evoked concept image and found that the "components" differed. This indicates a relation between students' conceptual understanding and (some) instruction, pinpointing the need for a more fine-grained tool. De Bock et al. (2017) discuss the role of explicitly addressed function properties when explaining students' results in tests. Their findings raise questions about the inconsistencies

in the students' concept images. They argue that the properties may have different statuses in a student's concept image, some easily assessable and other assigned first after careful considerations. De Bock et al. (2017) therefore call for more "ecologically valid studies on students' ability to make use of and to discriminate between different function properties" (De Bock et al., 2017 p. 953). Our study demonstrated that students' conceptual understanding of linear functions can be described using concept elements, and we suggest that concept elements can serve as a tool for identification, description and implementation in other topic areas as well. This is in line with the work by Kontorovich (2018). He states the need for models that teachers can use both when planning lessons and analysing students' ways of thinking concerning cross-curricular concepts. Hence, concept elements may be useful for mathematics teachers. For example, it seems to be promising as a tool for detecting traces of students' conceptual understanding, both initial and evolving (e.g. through tasks provoking a cognitive conflict). In this respect, its use recalls the detection of critical aspects of an object of learning in variation theory and the connections between these aspects (e.g. van Bommel, 2014). Here the focus is on the students' process and their own connections. The scholarly knowledge of students' conceptual understanding, regarding evoked concept images (e.g. Breen el al., 2017; Juter, 2009), and using symbols and representations (Bardini & Stacey, 2006; Drijvers, 2003), can inform the practice and, by using concept elements as a theoretical tool, the teaching can be research-based. We can therefore see concept elements as an expansion of Tall and Vinner's (1981) framework that is useful for both teachers' and researchers' work.

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References

Artigue, M. (1992). Functions from an algebraic and graphic point of view: cognitive difficulties and teaching practices. In E. Dubinsky & G. Harel (Eds.), *The concept of function: aspects of epistemology and pedagogy* (pp. 109–132). Washington: Mathematical Association of America.

Bardini, C. & Stacey, K. (2006). Students' conceptions of m and c: how to tune a linear function. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Prag: Charles University.

Baroody, A. J., Feil, Y. & Johnson, A. R. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, 38 (2), 115–131.

Bingolbali, E. & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, 68 (1), 19–35.

Bloch, I. (2003). Teaching functions in a graphic milieu: What forms of knowledge enable students to conjecture and prove? *Educational Studies in Mathematics*, 52 (1), 3–28.

Bommel, J. van (2014) The teaching of mathematical knowledge for teaching – a learning study of primary school teacher education. *Nordic Studies in Mathematics Education*, 19 (3/4), 185–201.

Breen, S., Larson, N., O'Shea, A. & Pettersson, K. (2017). A study of students' concept images of inverse functions in Ireland and Sweden. *Nordic Studies in Mathematics Education*, 22 (4), 85–102.

Carlson, M. P., Oehrtman, M. C. & Thompson, P. W. (2007). Key aspects of knowing and learning the concept of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: research and practice in undergraduate mathematics* (pp. 150–171). Washington: Mathematical Association of America.

De Bock, D., Neyens, D. & Van Dooren, W. (2017). Students' ability to connect function properties to different types of elementary functions: an empirical study on the role of external representations. *International Journal of Science and Mathematics Education*, 15, 939–955.

De Bock, D., Van Dooren, W. & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine functions: two studies on the role of experimental representations. *International Journal of Science and Mathematics Education*, 13 (S1), 47–69.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: design research on the understanding of the concept of parameter* (Doctoral thesis). University of Utrecht.

Gagatsis, A. & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, 24 (5), 645–657.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *The Journal of Mathematical Behavior*, 7 (2), 135–173.

Granberg, C. & Olsson, J. (2015). ICT-supported problem solving and collaborative creative reasoning: exploring linear functions using dynamic mathematics software. *Journal of Mathematical Behavior*, 37, 48–62.

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V. & Weigand, H.-G. (2016). Aspects of "Grundvorstellungen" of the concepts of derivative and integral subject matter related didactical perspectives of concept formation. *Journal für Mathematik-Didaktik*, 37 (1), 99–129.

Hansson, Ö. (2006). *Studying the views of preservice teachers on the concept of function* (Doctoral thesis). Lulea University of Technology.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan Publishing.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27–32). Hillsdale: Lawrence Erlbaum.

Jonsson, B. Norqvist, M. Liljekvist, Y. & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *Journal of Mathematical Behavior*, 36, 20–32.

Juter, K. (2006). *Limits of functions: university students' concept development* (Doctoral thesis). Luleå University of Technology.

Juter, K. (2009). Development of students' concept images in analysis. *Nordic Studies in Mathematics Education*, 14 (4), 65–87.

Knuth, E. J. (2000). Understanding connections between equations and graphs. *The Mathematics Teacher*, 93 (1), 48–53.

Kilpatrick, J. (2001). Understanding mathematical literacy: the contribution of research. E*ducational studies in mathematics*, 47 (1), 101–116.

Kontorovich, I. (2018). Why Johnny struggles when familiar concepts are taken to a new mathematical domain: towards a polysemous approach. *Educational Studies in Mathematics*, 97, 5–20.

Magidson, S. (1992). What's in a problem? Exploring slope using computer graphing software. In W. Geeslin & K. Graham (Eds.), *Procedings of the 16th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 64–71). Berkeley: University of California.

Markovits, Z., Eylon, B.-S. & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics*, 6 (2), 18–28.

Moschkovich, J. (1990). Students' interpretations of linear equations and their graphs. In G. Booker, P. Cobb & T. N. Mendicuti (Eds.), *Proceedings of the 14th annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 109–116). Mexico City: PME Program Committee.

Moschkovich, J., Schoenfeld, A. & Arcavi, A. (1993) Aspects of understanding: on multiple perspectives and representations of linear relations, and connections among them. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of function* (pp. 69–100). Hillsdale: Lawrence Erlbaum.

Niss, M. (2006). The structure of mathematics and its influence on the learning process. In J. S. W. Maasz & W. Schlöglmann (Eds.), *New mathematics education research and practise* (pp. 51–62). Rotterdam: Sense Publishers.

Norqvist, M. (2018). The effect of explanations on mathematical reasoning tasks. *International Journal of Mathematical Education in Science and Technology*, 49 (1), 15–30.

Pettersson, A. (2016). *Grafisk och algebraisk representation: gymnasieelevers förståelse av linjära funktioner* (Licentiate thesis). Karlstad University.

Roos, A.-K. (2017). "Grundvorstellungen" and their application to the concept of extreme points. In T. Dooley & G. Gueudet (Eds.), *Proceedings of CERME 10* (pp. 2763–2770). Dublin: Dublin City University and ERME.

Schoenfeld, A. H., Smith, J. P. & Arcavi, A. (1993). Learning: the microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55–175). Hillsdale: Lawrence Erlbaum.

Schwarz, B., Dreyfus, T. & Bruckheimer, M. (1990). A model of the function concept in a three-fold representation. *Computers and Education*, 14 (3), 249–262.

SFS 2010:800. *Skollag* [The education act]. Stockholm: Ministry of Education.

Star, J. R. & Stylianides, G. J. (2013). Procedural and conceptual knowledge: exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics and Technology Education*, 13 (2), 169–181.

Tall, D. (2017). Long-term effects of sense making and anxiety in algebra. In S. Stewart (Ed.), *And the rest is just algebra* (pp. 43–62). Cham: Springer.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12 (2), 151–169.

Viirman, O., Attorps, I. & Tossavainen, T. (2010). Different views – some Swedish mathematics students' concept images of the function concept. *Nordic Studies in Mathematics Education*, 15 (4), 5–24.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14 (3), 293–305.

Vom Hofe, R. & Blum, W. (2016). "Grundvorstellungen" as a category of subject-matter didactics. *Journal für Mathematik-Didaktik*, 27 (1), 225–254.

Watson, A., Ayalon, M. & Lerman, S. (2017). Comparison of students' understanding of functions in classes following English and Israeli national curricula. *Educational Studies in Mathematics*, 97 (3), 255–272.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning*, 7 (1), 42–57.

Zaslavsky, O., Sela, H. & Leron, U. (2002). Being sloppy about slope: the effect of changing the scale. *Educational Studies in Mathematics*, 49 (1), 119–140.

Notes

1   All tasks and more about the study, see the licentiate thesis by Pettersson (2016).

2   "The initial value" (Swedish *startvärde*) is not as well-defined a term in a Swedish school context as in an English-speaking school context.

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