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The logarithmic Cardy case: Boundary states and annuli

Jürgen Fuchs a,*, Terry Gannon b, Gregor Schaumann c, Christoph Schweigert d

a Teoretisk fysik, Karlstads Universitet, Universitetsgatan 21, S-65188, Karlstad, Sweden
b Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G2G1, Canada
c Fakultät für Mathematik, Universität Wien, Austria
d Fachbereich Mathematik, Universität Hamburg, Bereich Algebra und Zahlentheorie, Bundesstraße 55, D-20146 Hamburg, Germany

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Abstract

We present a model-independent study of boundary states in the Cardy case that covers all conformal field theories for which the representation category of the chiral algebra is a — not necessarily semisimple — modular tensor category. This class, which we call finite CFTs, includes all rational theories, but goes much beyond these, and in particular comprises many logarithmic conformal field theories.

We show that the following two postulates for a Cardy case are compatible beyond rational CFT and lead to a universal description of boundary states that realizes a standard mathematical setup: First, for bulk fields, the pairing of left and right movers is given by (a coend involving) charge conjugation; and second, the boundary conditions are given by the objects of the category of chiral data. For rational theories our proposal reproduces the familiar result for the boundary states of the Cardy case. Further, with the help of sewing we compute annulus amplitudes. Our results show in particular that these possess an interpretation as partition functions, a constraint that for generic finite CFTs is much more restrictive than for rational ones.

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1. Introduction

Two-dimensional conformal field theory, or CFT for short, is of fundamental importance in many areas, including the theory of two-dimensional critical systems in statistical mechanics, string theory, and quasi one-dimensional condensed matter systems. For understanding issues like percolation probabilities, open string perturbation theory in D-brane backgrounds, or defects in condensed matter physics, one must study CFT on surfaces with boundary. Of particular interest in applications is one of the simplest surfaces of this type, namely a disk with one bulk field insertion. From a more theoretical perspective, these correlators offer the most direct way to gain insight into boundary conditions, and have been frequently used to this end.

Basic symmetries of a conformal field theory are encoded in a chiral symmetry algebra, which can be realized as a vertex operator algebra. Here we consider the situation that the representation category \( \mathcal{C} \) of the chiral algebra has the structure of a ribbon category; this structure encodes in particular information about conformal weights and about the braiding and fusing matrices in a basis independent form. We will be interested in theories for which the category \( \mathcal{C} \) exhibits suitable finiteness properties and has dualities and a non-degenerate braiding (for details see Definition 2.5). We refer to ribbon categories with the relevant properties as modular tensor categories. Modular categories in this sense are not required to be semisimple, and indeed there are many interesting systems, such as critical dense polymers [16], for which \( \mathcal{C} \) is non-semisimple. For brevity, we will refer to conformal field theories whose chiral data are described by such a category as finite conformal field theories. The class of finite CFTs includes, besides all rational CFTs, in particular all rigid finite logarithmic CFTs. In terms of vertex operator algebras, the relevant notion of finiteness is, basically, \( C_2 \)-cofiniteness [45,57]; see [13] for precise statements and examples.

In the present paper we are concerned with specific correlators for finite CFTs: with boundary states and with annulus partition functions. Boundary states and boundary conditions are a feature of full local conformal field theory, in which left- and right-movers are adequately combined. In the special case of rational CFTs, for which \( \mathcal{C} \) is a semisimple modular tensor category, the structure of a full conformal field theory is fully understood [24,27] and can be implemented in the framework of vertex operator algebras [46]. This includes in particular the proper description as well as classification of boundary conditions. The simplest possibility – known as the Cardy case – is that the boundary conditions are just the objects of the tensor category \( \mathcal{C} \), while in the general case they are the objects of a module category over \( \mathcal{C} \).

Beyond semisimplicity, much less is known, but there has been substantial recent progress. Specifically, structural properties of the space of bulk fields and their role for fulfilling the modular invariance and sewing constraints have been understood [31], and systematic model-independent results for correlators of finite CFTs on closed world sheets have been obtained [31, 34,35]. In contrast, no model-independent results are available for correlators of non-semisimple finite CFTs on world sheets with boundary.

The present paper takes the first steps towards filling this gap. Concerning boundary conditions and boundary states, our starting point consists of the following two statements which can be expected to be valid under very general circumstances, even beyond the realm of finite CFTs:

(BC) First, the boundary conditions for a given local conformal field theory should be the objects of some category \( \mathcal{M} \). This category may be realized in various guises, e.g. as the (homotopy) category of matrix factorizations in a Landau–Ginzburg formulation, or as a category of modules over a Frobenius algebra in the TFT approach [27] to rational CFT.
For a finite CFT based on a modular tensor category $\mathcal{C}$, the category $\mathcal{C}$ itself is a natural candidate for the category of boundary conditions. If this is a valid choice and thus determines a consistent local CFT, then it is appropriate to refer to that full local CFT, following the parlance for rational theories, as the \textit{Cardy case}.

(BS) Second, an essential feature of a \textit{boundary state} is that it associates to a given boundary condition an element of some vector space. In a Landau–Ginzburg formulation, this space is a center (or its derived version, a Hochschild complex). In a more abstract approach to conformal field theory, the appropriate notion is the center of the category $\mathcal{C}$, i.e. the space $\text{End}(\text{Id}_\mathcal{C})$ of natural endo-transformations of the identity functor of $\mathcal{C}$. (This generalizes the fact that for the category $A$-mod of modules over an associative algebra $A$, $\text{End}(\text{Id}_{A\text{-mod}})$ can be identified with the center of $A$ as an algebra.) With the help of standard categorical manipulations this vector space can be expressed as

$$\text{End}(\text{Id}_\mathcal{C}) = \int_{c \in \mathcal{C}} \text{Hom}_\mathcal{C}(c, c) \cong \int_{c \in \mathcal{C}} \text{Hom}_\mathcal{C}(c^* \otimes c, 1) \cong \text{Hom}_\mathcal{C}(L, 1)$$

with $1$ the tensor unit of $\mathcal{C}$ and with the object $L$ of $\mathcal{C}$ given by $L = \int_{c \in \mathcal{C}} c^* \otimes c$. (The \textit{end} $\int_{c}$ and \textit{coend} $\int^{c}$ appearing here are categorical limit and colimit constructions, respectively; for the functors in question they exist in any finite tensor category.)

Now the map from boundary conditions to the vector space $\text{Hom}_\mathcal{C}(L, 1)$ is a decategorification. It is thus natural to expect that it factorizes over the Grothendieck ring $K_0(\mathcal{C})$, which is the decategorification of the category $\mathcal{C}$. Such a factorization over the Grothendieck ring is generally afforded by \textit{characters}. We should therefore expect that boundary states are characters for representations of some suitable algebraic structure; as we will see, the latter is precisely the object $L$, endowed with a natural Hopf algebra structure. Let us note that a similar description is known from two-dimensional topological field theories, as studied in [9, Sect. 7]. In that case the role of the center is played by the zeroth Hochschild homology of smooth projective schemes (or of more general spaces), and the homomorphism from the Grothendieck ring to the center is given by the Chern character [9, Prop. 13].

\subsection{1.1. Boundary states in rational CFT}

As we will now explain, the paradigm outlined above is indeed realized in the semisimple case. In that case, boundary states can be regarded as the characters of specific $L$-modules which are given by the objects of $\mathcal{C}$ together with a canonical $L$-action on them. We will now give a detailed account of this interpretation of the structure of boundary states of a rational CFT with semisimple modular tensor category $\mathcal{C}$ in the Cardy case. In the Cardy case of a rational CFT, one first selects a finite set $(x_i)_{i \in I}$ of representatives for the isomorphism classes of simple objects of $\mathcal{C}$. Boundary states are then conventionally written as linear combinations of so-called \textit{Ishibashi states}; for each $i \in I$ there is one Ishibashi state $|i\rangle$. An Ishibashi state is in fact nothing but a canonical vector spanning a space of two-point conformal blocks on the sphere, namely the one with the two chiral insertions given by $x_i$ and $x_\tau$. Boundary conditions are thus labeled by objects $x$ of $\mathcal{C}$, and elementary boundary conditions by isomorphism classes of simple objects $x_a$ of $\mathcal{C}$ with $a \in I$. The boundary state $|x_a\rangle$ associated with the elementary boundary condition $x_a$ is expanded in Ishibashi states as
\[ \langle x_a \rangle = \sum_{i \in I} \frac{S_{ia}}{\sqrt{S_{ii}}} |i\rangle, \tag{1.2} \]

where \( S_{ij} \) are the entries of the modular S-matrix – the non-degenerate matrix which represents the transformation \( \tau \mapsto -1/\tau \) on the (vertex algebra) characters of the theory – and \( 0 \in I \) is the label for the identity field, i.e. for the tensor unit \( 1 \) of \( \mathcal{C} \).

The formula (1.2) can be conveniently understood via the relation [22] to three-dimensional topological field theory. Namely, the boundary state \( \langle x_a \rangle \) can be constructed as the topological invariant that the TFT functor \( \text{tft} \) associates to a certain ribbon link in the three-ball:

\[ \langle x_a \rangle = \sum_{i \in I} \text{tft}(x_i) \tag{1.3} \]

By construction this is a vector in the space of two-point conformal blocks on the sphere; expanding it in the basis

\[ |i\rangle = \text{tft}(x_i) \tag{1.4} \]

of Ishibashi states yields the expression (1.2).\(^1\)

Now when evaluating the invariant (1.3), the ribbon link appearing in the picture is interpreted as a morphism in the category \( \mathcal{C} \). Moreover, with the help of the duality structure on \( \mathcal{C} \) we can bend down the \( i \)-line in the so obtained morphism according to

\[ x_i \quad \mapsto \quad x_i^\gamma \tag{1.5} \]

Upon summation over \( i \in I \), the morphism on the right hand side of (1.5) is indeed precisely the character \( \chi_{x_a}^L \) of a simple \( L \)-module \( (x_a, \rho_a) \) with \( \rho_a \) a canonical action of \( L \) on the object \( x_a \in \mathcal{C} \).

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\(^1\) The precise normalization depends in fact on the conventions for the two-point functions of bulk fields on the sphere; see e.g. [22, Sect. 4.3].
1.2. Boundary states in finite CFT

A crucial observation is now that by making use of the coend structure of \( L \) the result just described for rational CFT actually generalizes directly to non-semisimple finite CFTs. A detailed justification of this statement will be given in Section 2.3. Let us point out that the characters of \( L \)-modules appearing here are not to be confused with characters in the sense of vertex operator algebras. However, as will be explicated in Remark 2.18, they indeed directly correspond to chiral genus-1 one-point functions for vertex algebra representations.

Thus the TFT construction of correlators of rational CFTs precisely yields a standard mathematical structure that is still present for arbitrary finite conformal field theories – a lattice \( K_0(\mathcal{C}) \hookrightarrow \text{End}(\text{Id}_\mathcal{C}) \cong \text{Hom}_\mathcal{C}(L, 1) \). Moreover, as a generic feature of decategorification, this lattice comes with the additional structure of a distinguished basis – in our case, the characters \( \chi^L_{x_a} \) of the simple \( L \)-modules \( \langle x_a, \rho_a \rangle \). For non-semisimple \( \mathcal{C} \) the lattice is not of maximal rank, i.e. the characters \( \chi^L_{x_a} \) do not span the whole space \( \text{Hom}_\mathcal{C}(L, 1) \).

To allow for an interpretation of this result in CFT terms, we need to identify the vector space \( \text{Hom}_\mathcal{C}(L, 1) \) with a space of conformal blocks. The space of conformal blocks in question is not the one of zero-point blocks on the torus (which is also isomorphic to \( \text{Hom}_\mathcal{C}(L, 1) \) [55]), but the one for a disk with one bulk field insertion. As will be explained in Section 3.1, this follows by combining recent developments [29] concerning conformal blocks for surfaces with boundary with an appropriate expression for the space of bulk fields in the Cardy case. More specifically, we need to describe the latter as an object, and in fact even as a commutative Frobenius algebra, in the category \( \overline{\mathcal{C}} \boxtimes \mathcal{C} \), i.e. in the Deligne product of \( \mathcal{C} \) with its reverse \( \overline{\mathcal{C}} \). (\( \overline{\mathcal{C}} \) is the same category as \( \mathcal{C} \), but with reversed braiding and twist, which accounts for the opposite chirality of left- and right-movers.) For rational CFTs, the space of bulk fields of the Cardy case is realized by the object

\[
\bigoplus_{i \in I} x_i^\vee \boxtimes x_i \in \overline{\mathcal{C}} \boxtimes \mathcal{C} ,
\] (1.6)

which in particular gives rise to the charge-conjugate (sometimes also called diagonal) torus partition function.

We need to generalize this expression to arbitrary finite CFTs. We do so by the following further natural hypothesis about the Cardy case:

(F) We assume that for any finite CFT the bulk object in the Cardy case is the coend

\[
\hat{F} := \int_{c \in \mathcal{C}} c^\vee \boxtimes c \in \overline{\mathcal{C}} \boxtimes \mathcal{C} .
\] (1.7)

The object \( \hat{F} \) combines left- and right-movers in the same way as in rational CFT: it pairs each object with its charge-conjugate, modulo dividing out all morphisms between objects. When \( \mathcal{C} \) is semisimple, this leaves one representative out of each isomorphism class of simple objects, so that \( \hat{F} \) reduces to the Cardy bulk algebra (1.6).
The assumption (F) about the bulk object is logically independent from the assumptions (BC) and (BS) about boundary conditions and boundary states made above. It is remarkable that

1. by the proper notion of modularity of braided finite tensor categories, \( \hat{F} \in \mathcal{C} \boxtimes \mathcal{C} \) gives an object \( F \) in the Drinfeld center \( \mathcal{Z} \mathcal{C} \) of \( \mathcal{C} \) that is a commutative symmetric Frobenius algebra in \( \mathcal{Z} \mathcal{C} \), whereby also \( \hat{F} \) naturally is such an algebra (see Section 2.2);
2. with this Frobenius algebra in \( \mathcal{Z} \mathcal{C} \), the conformal blocks for the correlator of one bulk field on the disk can be shown to be canonically isomorphic to the center \( \text{Hom}_{\mathcal{C}}(L, 1) \) (see Section 3.1).

Boundary states must satisfy a number of consistency requirements. Most notably, upon sewing they must lead to annulus amplitudes that in the open-string channel can be expanded in terms of characters. In the non-semisimple case this is a non-trivial requirement, as it excludes contributions from so-called pseudo-characters. Moreover, the coefficients in such an expansion must be non-negative integers, as befits a partition function of open string states or boundary fields. It is then a further remarkable observation of our paper that the setup laid out above furnishes consistent annulus partition functions, with coefficients taking values in the positive integer cone.

The format of the paper is as follows: We start in Section 2 by presenting pertinent results about (not necessarily semisimple) modular tensor categories and about algebraic structures internal to them, in particular the Frobenius algebra structure on the coend bulk object (1.7) and characters and cocharacters of \( L \)-modules. Important input needed for this description has become available only recently [28,61–63] and has not been adapted to the CFT setting before. Taking \( \mathcal{C} \), as a module category over itself, as the category of boundary conditions, in Section 3 we then obtain the spaces of conformal blocks for incoming and outgoing boundary states and present, in Postulates 3.2 and 3.3, our precise proposal for the boundary states. Afterwards, in Section 4, these boundary states are used to obtain, via sewing, annulus amplitudes. We show that the open-string channel annulus amplitudes can be expressed as non-negative integral linear combinations of characters, so that they can be consistently be interpreted as partition functions. As a further consistency check, we show that the annulus amplitudes are compatible with the natural proposal that the boundary fields can be described as internal Hom objects for \( \mathcal{C} \) as a module category over itself. We therefore conjecture that the boundary operator products can be expressed through the structure maps of these internal Homs. The Appendix provides additional information about some of the mathematical tools that are used in the main text.

As an illustration, for three specific classes of models – the rational case, the logarithmic \((p, 1)\) triplet models [20,39,49] whose boundary states have been studied in [40,41,43], and the case that \( \mathcal{C} \) is the representation category of a finite-dimensional factorizable ribbon Hopf algebra – we present further details about the Cardy case bulk object (Example 2.7), the spaces of conformal blocks for boundary states (Example 3.1) and their subspaces spanned by (co)characters (Example 3.6), and finally the annulus amplitudes obtained from them by sewing (Example 4.3).

2. Structures in modular tensor categories

2.1. Modular tensor categories beyond semisimplicity

In this section we present the class of categories relevant to us. We also survey pertinent structure in these categories, to be used freely in the rest of the paper. For applications to con-
formal field theory, the categories in question should be thought of as (being ribbon equivalent to) representation categories of appropriate $C_2$-cofinite vertex operator algebras. All categories considered in this paper will be $\mathbb{C}$-linear.\(^2\)

**Definition 2.1 (Finite category).** A $\mathbb{C}$-linear category $C$ is called finite iff

1. $C$ has finite-dimensional spaces of morphisms;
2. every object of $C$ has finite length;
3. $C$ has enough projective objects;
4. there are finitely many isomorphism classes of simple objects.

**Remark 2.2.** A $\mathbb{C}$-linear category is finite if and only if it is equivalent to the category $A$-mod of finite-dimensional modules over a finite-dimensional $\mathbb{C}$-algebra $A$.

**Definition 2.3 (Finite tensor category).** A finite tensor category is a rigid monoidal finite $\mathbb{C}$-linear category with simple tensor unit.

The tensor product $\otimes$ in a finite tensor category is automatically exact in each argument. As a consequence, the Grothendieck group $K_0(C)$ of $C$ inherits a ring structure; it is thus referred to as the Grothendieck ring, or fusion ring, of $C$. A semisimple finite tensor category is also called a fusion category. Without loss of generality we take the monoidal structure to be strict, i.e. assume that the tensor product is strictly associative and that the monoidal unit $1$ obeys $c \otimes 1 = c = 1 \otimes c$ for all objects $c$.

The categories of our interest are not only monoidal, i.e. endowed with a tensor product, and rigid, i.e. endowed with left and right dualities, but have further structure: they are also braided and have a twist, or balancing, satisfying compatibility relations which correspond to properties of ribbons embedded into three-space. (For more details see e.g. Chapter XIV of [48] and Section 2.1 of [27].)

**Definition 2.4 (Ribbon category).** A ribbon category (or tortile category) is a balanced braided rigid monoidal category.

A ribbon category is in particular endowed with a canonical pivotal structure, i.e. a choice of monoidal natural isomorphism between the (left or right) double dual functor and the identity functor or, equivalently, with a sovereign structure, i.e. a choice of monoidal natural isomorphism between the left and the right dual functors. The pivotal structure can be expressed through the twist together with the dualities and braiding or, conversely, the twist through the pivotal structure together with the dualities and braiding. A ribbon category is also spherical, i.e. the left and right trace of any endomorphism are equal.

We denote the right and left dual of an object $c$ of a rigid category by $c^\vee$ and $\check{c}$, respectively, and the corresponding evaluation and coevaluation morphisms by $ev_c : c^\vee \otimes c \to 1$ and $coev_c : 1 \to c \otimes c^\vee$, and by $\check{ev}_c : c \otimes \check{c} \to 1$ and $\check{coev}_c : 1 \to \check{c} \otimes c$, respectively. For the braiding between objects $c$ and $d$ of a braided category we write $\beta_{c,d} : c \otimes d \xrightarrow{\cong} d \otimes c$, and for the twist on an object $c$ of a ribbon category we write $\theta_c : c \xrightarrow{\cong} c$. Henceforth we will, for the sake of

\(^2\) The statements below remain true if $\mathbb{C}$ is replaced by any algebraically closed field $k$. In the CFT application, $k = \mathbb{C}$.
brevity, often tacitly identify each object of a ribbon category with its double dual, i.e. suppress the pivotal structure, since it can be restored unambiguously.

A coend is a specific colimit that, morally, amounts to summing over all objects of a category while at the same time dividing out all relations among them that are implied by morphisms in the category. It vastly generalizes the direct sum $\bigoplus_{i \in I}$, as appearing e.g. in the expression (1.6) for the bulk state space in the semisimple case, to which it reduces if the category is finitely semisimple (for more details see e.g. [32]). Coends are defined through a universal property; thus if a coend exists, it is unique up to unique isomorphism. In any finite tensor category $C$ and for any object $c \in C$ the exactness of the tensor product and of the duality functor guarantee that the specific coend

$$Z(c) := \int x^\vee \otimes c \otimes x$$

exists as an object in $C$, see Theorem 3.43 of [61]. If $C$ is semisimple, then $Z(c)$ is a finite direct sum $\bigoplus_{i \in I} x_i^\vee \otimes c \otimes x_i$. The prescription (2.1) defines an endofunctor $Z$ of $C$ that admits an algebra structure, i.e. is a monad, even a Hopf monad, on $C$ (for details see Appendix A.3). Of particular interest is the object

$$Z(1) = \int x^\vee \otimes x \in C,$$  

which has a natural structure of algebra in $C$. To appreciate these statements, note that the notions of an algebra and coalgebra, as well as Frobenius algebra, can be defined in any monoidal category $C$ (e.g., in any category of endofunctors) in full analogy with the category of vector spaces. When $C$ is in addition braided, the same holds for the notion of a Hopf algebra.

If the finite category $C$ is braided, then the object $Z(1)$ in fact has a canonical structure of a Hopf algebra in $C$, and the Hopf monad $Z$ can be obtained by tensoring with $Z(1)$. When regarding $Z(1)$ as a Hopf algebra we write

$$Z(1) := L.$$  

We denote the multiplication, unit, comultiplication, counit and antipode of $L$ by $\mu \equiv \mu_L$, $\eta \equiv \eta_L$, $\Delta \equiv \Delta_L$, $\varepsilon \equiv \varepsilon_L$ and $s \equiv s_L$, respectively. The Hopf algebra $L$ also comes with a natural pairing

$$\omega : L \otimes L \to 1,$$  

which has the structure of a Hopf pairing, i.e. satisfies the compatibility relations

$$\omega \circ (\mu \otimes id_L) = \omega \circ [id_L \otimes ((\omega \otimes id_L) \circ (id_L \otimes \Delta))], \quad \omega \circ (\eta \otimes id_L) = \varepsilon,$$

$$\omega \circ (id_L \otimes \mu) = \omega \circ [(id_L \otimes \omega) \circ (\Delta \otimes id_L)] \otimes id_L], \quad \omega \circ (id_L \otimes \eta) = \varepsilon$$

with the structural morphisms of $L$ as a bialgebra.

As a coend, $L = Z(1)$ comes with a morphism $i^Z_{x}(1) : x^\vee \otimes x \to L$ for each $x \in C$, forming a dinatural family. This implies that a morphism $f : L \to c$ is uniquely determined by the dinatu-
A braided category is characterized by a natural family $f : x \otimes x \to c$ of morphisms, and likewise for morphisms with source object $L \otimes L$ etc. For instance, the following formula determines the Hopf pairing $\omega$ uniquely:

$$\omega \circ (I^L_x \otimes I^L_y) = (ev_x \otimes ev_y) \circ (id_{x^\vee} \otimes (\beta_{y,x} \circ \beta_{x,y}^\vee) \otimes id_y).$$

(2.6)

Analogously, the defining formulas for the structural morphisms of the Hopf algebra $L$ read

$$\mu \circ (I^L_x \otimes I^L_y) := \eta \circ (id_{x^\vee} \otimes \beta_{y,x} \otimes id_y),$$

$$\Delta \circ I^L_x := (I^L_x \otimes I^L_x) \circ (id_{x^\vee} \otimes \coev_x \otimes id_x),$$

$$\varepsilon \circ I^L_x := ev_x,$$

(2.7)

Here in the formula for $\mu$ the trivial identifications of $id_{x^\vee} \otimes id_{y^\vee}$ with $id_{(x^\vee, y^\vee)}$ and of $id_y \otimes id_x$ with $id_{y \otimes x}$ are implicit. A graphical interpretation of the expressions for the coproduct $\Delta$ and the counit $\varepsilon$ looks as follows:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram1} \\
= \\
\includegraphics[width=0.3\textwidth]{diagram2}
\end{array}
\]

(2.8)

The corresponding description of the product $\mu$ will be provided in formula (2.20) below. (The picture for the antipode will not be needed; it can e.g. be found in [30, Eq. (4.19)].)

To any monoidal category $\mathcal{A}$ there is canonically associated a braided monoidal category $\mathcal{Z}(\mathcal{A})$, called the monoidal center, or Drinfeld center, of $\mathcal{A}$. The objects of $\mathcal{Z}(\mathcal{A})$ are pairs $(a, \gamma)$ consisting of an object $a \in \mathcal{A}$ and a natural family $\gamma = (\gamma_b)_{b \in \mathcal{A}}$ of isomorphisms $\gamma_b : a \otimes b \cong b \otimes a$ satisfying one half of the properties of a braiding and accordingly called a half-braiding.\(^4\) The associativity constraint of $\mathcal{Z}(\mathcal{A})$ is the same as the one of $\mathcal{C}$ (and thus in the present context, by strictness, taken to be trivial), while the braiding of $\mathcal{Z}(\mathcal{C})$ is (see e.g. [17, Prop. 8.5.1])

$$\left(\beta_{\mathcal{Z}(\mathcal{A})}\right)_{(a, \gamma), (a', \gamma')} = \gamma_{a'}.$$

(2.9)

Forgetting the half-braiding furnishes an exact monoidal functor

$$U : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}.$$

(2.10)

If $\mathcal{C}$ has a (right, say) duality, then so has its Drinfeld center $\mathcal{Z}(\mathcal{C})$, with the same evaluation and coevaluation morphisms $ev_a$ and $\coev_a$ as in $\mathcal{C}$ and with dual objects

$$(a, \gamma)^\vee = (a^\vee, \gamma^\vee),$$

(2.11)

where

\[^4\] We follow the convention used e.g. in [14]; in [17] the half-braiding is defined in the opposite manner.
\[ (\gamma^\cdot)^b := (\text{ev}_a \otimes \text{id}_b \otimes \text{id}_{a^\vee}) \circ (\text{id}_{a^\vee} \otimes \gamma^{\cdot b}_{a^\vee} \otimes \text{id}_{a^\vee}) \circ (\text{id}_{a^\vee} \otimes \text{id}_b \otimes \text{coev}_a) \] (2.12)

is the partial dualization of the inverse half-braiding of \( a \). In particular, if, as in the case of our interest, \( C \) is ribbon, then \( \mathcal{Z}(C) \) naturally comes with a ribbon structure.\(^5\) Henceforth we usually reserve the symbol \( C \) for braided categories; given a braided category \( C \), we write \( \beta \) for the braiding in \( C \) and \( \beta^\cdot \mathcal{Z} \) for the braiding in its center \( \mathcal{Z}(C) \).

We denote by \( \overline{C} \) the reverse of a finite ribbon category \( C \), i.e. the same monoidal category, but with inverse braiding and twist. For any finite ribbon category \( C \) there is a canonical braided functor

\[ \Xi_C : \overline{C} \otimes C \to \mathcal{Z}(C) \] (2.13)

from the enveloping category of \( C \), i.e. the Deligne product of \( \overline{C} \) with \( C \), to the Drinfeld center of \( C \). As a functor, \( \Xi_C \) maps the object \( u \otimes v \in \overline{C} \otimes C \) to the tensor product \( u \otimes v \in C \) endowed with the half-braiding \( \gamma_{u \otimes v} \) that has components

\[ \gamma_{u \otimes v; c} = (\beta^{-1}_{c,u} \otimes \text{id}_v) \circ (\text{id}_u \otimes \beta_{v,c}) \] (2.14)

for \( c \in C \). We will freely use the graphical calculus for morphisms in the braided monoidal category \( C \). We then have the following graphical description of the half-braiding (2.14):

\[ \gamma_{u \otimes v; c} = \]

(2.15)

with the individual braiding in the picture being braiding in \( C \). The braided monoidal structure on the functor \( \Xi_C \) is given by the coherent family \( \text{id}_u \otimes \beta_{v,x} \otimes \text{id}_y \) of isomorphisms from \( u \otimes v \otimes x \otimes y \) to \( u \otimes x \otimes v \otimes y \) (for details see Appendix A.1).

In the theories relevant to us the braiding obeys a non-degeneracy condition. This condition can be formulated in several equivalent ways:

**Definition 2.5 (Modular tensor category).** A modular tensor category is a finite ribbon category \( C \) which satisfies one of the following equivalent \([63]\) conditions:

- The canonical functor \( \Xi_C \) (2.13) is a braided equivalence.
- The Hopf pairing \( \omega \) (2.4) on the coend \( L \in \mathcal{C} \) is non-degenerate.
- The linear map \( \text{Hom}_C(1, L) \to \text{Hom}_C(L, 1) \) that is induced by the Hopf pairing \( \omega \) is an isomorphism of vector spaces.
- The category \( C \) has no non-trivial transparent objects, i.e. any object having trivial monodromy with every object is a finite direct sum of copies of the tensor unit \( \textbf{1} \).

Modularity of \( C \) is crucial for constructing a modular functor in the sense of \([55]\).

---

\(^5\) Recall that we suppress the pivotal structure of \( C \). Likewise we suppress the pivotal structure for \( \mathcal{Z}(C) \). This is consistent because \([17, \text{Exc. 7.13.6}]\) a pivotal structure of \( C \) induces one for \( \mathcal{Z}(C) \).
Example 2.6. (i) Let us stress that for $\mathcal{C}$ being modular it is not required that it is semisimple. If $\mathcal{C}$ is in addition semisimple, corresponding to the case of rational CFTs, then the conditions in Definition 2.5 are equivalent [6,59] to the familiar requirement that the modular $S$-matrix is non-degenerate.

In the semisimple case, each indecomposable object is simple, so that in particular up to isomorphism there are finitely many indecomposable objects. For instance, in the case of the category $\mathcal{C}_{g,\ell}$ that is relevant for the WZW model based on a semisimple Lie algebra $g$ and positive integer $\ell$, the isomorphism classes of indecomposable objects are labeled by the finitely many integrable highest weights of the untwisted affine Lie algebra $g^{(1)}$ at level $\ell$.

(ii) Classes of logarithmic conformal field theories for which the representation category of the chiral algebra is known to be a (non-semisimple) modular tensor category are the symplectic fermion models [1,15,19] and the $(p,1)$ triplet models [2,65]. While, like in the semisimple case, the corresponding categories have finitely many simple objects up to isomorphism, already for the arguably simplest logarithmic CFT, the $(2,1)$ triplet model, there are uncountably many isomorphism classes of indecomposable objects [21].

(iii) The category $H$-mod of finite-dimensional modules over any finite-dimensional factorizable ribbon Hopf (or, more generally, weak quasi-Hopf) algebra $H$ is a modular tensor category. It is semisimple iff $H$ is semisimple as an algebra.

2.2. The Cardy bulk algebra

The property of a braided finite tensor category of being modular has important consequences. Two of these are of particular interest to us: First, recall that in a full local conformal field theory the chiral degrees of freedom of left and right movers taken together are described in terms of the enveloping category $\mathcal{C} \boxtimes \mathcal{C}$. In the case of a modular tensor category we can replace $\mathcal{C} \boxtimes \mathcal{C}$ by the Drinfeld center $\mathcal{Z}(\mathcal{C})$. Working with $\mathcal{Z}(\mathcal{C})$ allows us in the semisimple case to use the conformal blocks of the topological field theory of Turaev–Viro type that is associated with $\mathcal{C}$ (see e.g. [5]). For Turaev–Viro type theories the obstruction in the Witt group to the existence of boundary conditions [37] vanishes, so that such conformal blocks are also available for surfaces with boundary. (The boundary Wilson lines are then labeled by $\mathcal{C}$ itself, as befits a Cardy case.)

Second, a modular tensor category is in particular unimodular [18, Prop. 4.5]. It follows (see Theorems 4.10 and 5.6 of [61]) that the forgetful functor $U$ from $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$ in (2.10) has a two-sided adjoint $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$, i.e. is a Frobenius functor. (There are then corresponding Frobenius algebras in $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$, and associated with them [47, Thm. 8.2] Frobenius monads on $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$.) As a result the object

$$F := I(1) \in \mathcal{Z}(\mathcal{C})$$

(2.16)

has a natural structure of a commutative Frobenius algebra in the braided tensor category $\mathcal{Z}(\mathcal{C})$. When transported to the enveloping category $\mathcal{C} \boxtimes \mathcal{C}$, the object (2.16) is nothing but the coend $\hat{F}$ introduced in (1.7), i.e. we have $\mathcal{Z}_{\mathcal{C}}(\hat{F}) = F$. As explained there, this object is expected to furnish the bulk state space for the Cardy case; its algebra structure provides the bulk field operator products, while the non-degenerate Frobenius form encodes the two-point correlation function of bulk fields on the sphere. To constitute the bulk object of a consistent full CFT, $F$
must not only be a commutative symmetric Frobenius algebra, but in addition also modular [31];
this has so far only been fully established for the cases that the category $\mathcal{C}$ is semisimple (see [32,
Thm. 3.4] and [31]) or that it is the representation category of a Hopf algebra [33, Cor. 5.11].

Example 2.7. (i) In the finitely semisimple case the Cardy case bulk algebra as an object of $\overline{\mathcal{C}} \boxtimes \mathcal{C}$
is the direct sum $\hat{F}_{\text{fss}} = \bigoplus_{i \in I} x_i \otimes x_i$ as given in (1.6), while the Frobenius algebra $F \in \mathcal{Z}(\mathcal{C})$
can be written as the object

$$F_{\text{fss}} = \bigoplus_{i \in I} x_i \otimes x_i \quad (2.17)$$

in $\mathcal{C}$ together with the half-braiding described explicitly e.g. in [5, Thm. 2.3]. The Frobenius
algebra structure (on $\hat{F}_{\text{fss}}$) is given in [26, Lemma 6.19].

(ii) It follows from Corollary 5.1.8 of [51] that the coend $\hat{F} = \int^{c \in \mathcal{C}} c \otimes c \in \overline{\mathcal{C}} \boxtimes \mathcal{C}$ can be
written as

$$\hat{F} \cong (P \boxtimes P)/N , \quad (2.18)$$

where $P = \bigoplus_{i \in I} P_i$, the direct sum of (representatives for the isomorphism classes of) all
indecomposable projective objects of $\mathcal{C}$, is a projective generator, and $N$ is the subspace obtained
by acting with $f \boxtimes id \rho - id \rho \boxtimes f$ for all $f \in \text{End}_{\mathcal{C}}(P)$. A representation-theoretic description of
$N$ has been given in [41, Sect. 3.3] for a class of models that includes in particular the logarithmic
$p, 1$ triplet models, as the kernel of a pairing defined in terms of three-point conformal blocks with one insertion from the vertex algebra itself. In the particular case of the $(2, 1)$ triplet model (of Virasoro central charge $-2$) which describes symplectic fermions, this kernel can be
expressed in terms of the zero mode of the chiral fermion field [40, Eq. (2.11)].

It is also known that for the $(p, 1)$ models the class $[F]$ in the Grothendieck ring is given by
$\sum_{i \in I}[x_i \otimes P_i]$, with $x_i \cong x_i^{\vee}$ the simple objects and $P_i$ the indecomposable projectives (which
are the projective covers of the $x_i$) [41, Sect. 4.4] or, what is the same (see e.g. [34]), by
$\sum_{i, j \in I}C_{ij}[x_i \otimes x_j]$ with $(C_{ij})$ the Cartan matrix of the category. For comparison, the Cartan
matrix of a semisimple category is the identity matrix, so that $[F_{\text{fss}}] = \sum_{i \in I}[x_i^{\vee} \otimes x_i]$, in agreement with (2.17).

(iii) For $\mathcal{C} = H$-mod the category of finite-dimensional left modules over a finite-dimen-
sional factorizable ribbon Hopf algebra $H$, the enveloping category $\overline{\mathcal{C}} \boxtimes \mathcal{C}$ is braided equivalent to
the category of finite-dimensional $H$-bimodules [33, App. A.2]. The coend $\hat{F}$ is then the dual
vector space $H^*$ endowed with the co-regular left and right $H$-actions [33, App. A.1], while
$Z(1) \in H$-mod is $H^*$ with the co-adjoint left $H$-action [66, Sect. 4.5].

It should, however, be appreciated that a decomposition into a direct sum of factorized objects,
as for $\hat{F}_{\text{fss}}$, no longer occurs when $\mathcal{C}$ is non-semisimple, not even in the Cardy case. For a general,
not necessarily semisimple, modular tensor category, the Cardy case bulk algebra in $\overline{\mathcal{C}} \boxtimes \mathcal{C}$ is the
coend $\hat{F}$ as given in (1.7), while in $\mathcal{Z}(\mathcal{C})$ it is the object $F$ consisting of the object in $\mathcal{C}$ given by
the coend
\[ U(F) = Z(1) = \int c \in C c^\vee \otimes c, \quad (2.19) \]

together with a half-braiding. In particular, the object \( Z(1) \) of \( C \), besides having a structure of Hopf algebra in \( C \), also naturally comes with a half-braiding \( \gamma \) such that \( (Z(1), \gamma) = F \in \mathcal{Z}(C) \) has a structure of a symmetric commutative Frobenius algebra in \( \mathcal{Z}(C) \), and this Frobenius structure is unique up to a scalar (see [14, Lemma 3.5] and [61, Thm. 5.6]). This half-braiding can be obtained explicitly by realizing that \( Z(Z(1)) \cong Z(1) \otimes Z(1) \) and using the dinatural morphisms for \( Z(Z(1)) \) as a coend \((2.1)\) together with the product \( \mu_F \). It is, however, best described with the help of the monad structure on the endofunctor \( c \mapsto Z(c) \) and realizing that modules over the monad \( Z \) are the same as objects with a half-braiding. The object \( Z(c) \) has a canonical structure of a \( Z \)-module.

We will need to know the Frobenius algebra structure on the object \( F \) of \( \mathcal{Z}(C) \) explicitly. Let us first describe the algebra structure on \( F \). The relevant commutative associative multiplication on \( \tilde{F} \in \mathcal{C} \otimes \mathcal{C} \) has been described in [36, Prop. 2.3]. We need to transport this product along the functor \( \mathcal{E}_C \). When doing so we must account for the monoidal structure on \( \mathcal{E}_C \): For \( H : \mathcal{D} \to \mathcal{D}' \) a tensor functor with monoidal structure \( \varphi \) and \( A \) an algebra in \( \mathcal{D} \) with product \( m' \), the corresponding product \( m \) on the algebra \( H(A) \) is the composition \( H(m) \circ \varphi_{A,A} : H(A) \otimes H(A) \to H(A \otimes A) \to H(A) \). As shown in Lemma A.1, the monoidal structure on \( \mathcal{E}_C \) is given by a braiding; we then find the following description of the multiplication morphism \( \mu_F = \mathcal{E}_C(\mu_{\tilde{F}}) \circ \varphi_{\tilde{F},\tilde{F}} \) in \( \mathcal{Z}(C) \):

\[ \mu_F \circ (\iota_x Z(1) \otimes \iota_y Z(1)) = \]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (Z1) at (0,0) {$Z(1)$};
\node (iota_x) at (-1,-1) {$\iota_x Z(1)$};
\node (iota_y) at (1,-1) {$\iota_y Z(1)$};
\node (x) at (-2,-2) {$x^\vee$};
\node (y) at (2,-2) {$y^\vee$};
\node (x_bar) at (-1,-3) {$x$};
\node (y_bar) at (1,-3) {$y$};
\draw[->] (Z1) -- (iota_x);
\draw[->] (Z1) -- (iota_y);
\draw[->] (iota_x) -- (x);
\draw[->] (iota_y) -- (y);
\end{tikzpicture}
\end{array}
\]

Here the two un-labeled coupons stand for the (trivial) identifications \( \text{id}_{x^\vee} \otimes \text{id}_{y^\vee} = \text{id}_{(y^\vee \otimes x^\vee)} \) and \( \text{id}_y \otimes \text{id}_x = \text{id}_{y^\vee \otimes x^\vee} \), respectively.

**Lemma 2.8.** (i) The morphism \( \mu_F \) in \( \mathcal{Z}(C) \) defined by \((2.20)\) is a commutative associative multiplication for \( F \).

(ii) The morphism \( \eta_F : = \iota_Z 1 \) is a unit for the product \( \mu_F \).

**Proof.** (i) Associativity is guaranteed by the fact that \( \mu_F \) is obtained from an associative product for \( \tilde{F} \). It can also be verified directly through an exercise in braid gymnastics, which pictorially looks as follows:
Commutativity is seen as follows. With the help of the dinatural family $t^Z_{x,y}$, the braiding $\beta^Z_{x,y}$ can be expressed in terms of the braiding $\beta^Z_{x,y}$ such that

$$\mu_F \circ (t^Z_{x,y} \otimes t^Z_{y,z}) = \mu_F \circ (t^Z_{y,z} \otimes t^Z_{x,y}).$$

Moreover, dinaturality of $t^Z_{x,y}$ implies the identity

$$t^Z_{x,y} \circ (id_{x \otimes y} \otimes \beta_{x,y}) = t^Z_{x,y} \circ (\beta_{x,y} \otimes id_{x \otimes y}).$$

Combining these equalities with the definition (2.20) of $\mu_F$, commutativity boils down to the relation

$$\beta^Z_{y,x} \circ (id_{y \otimes x} \otimes \beta_{y,x}) = (id_{y \otimes x} \otimes \beta_{y,x}) \circ (id_{x \otimes y} \otimes \beta_{x,y}) \circ (id_{x \otimes y} \otimes \beta_{x,y} \otimes id_y).$$

After inserting the expression (2.15) for $\beta^Z$, this reduces to the identity

among braids and is thus indeed satisfied.

(ii) By definition of the product one has $\mu_F \circ (t^Z_{1,1} \otimes t^Z_{x,1}) = t^Z_{x,1} = \mu_F \circ (t^Z_{x,1} \otimes t^Z_{1,1})$ for all $x \in C$, i.e. $t^Z_{1,1}$ satisfies the properties of a unit for $\mu_F$. □
It is worth noting that the product and unit of \( F \in \mathcal{Z}(\mathcal{C}) \) are the same, when regarded as morphisms in \( \mathcal{C} \), as those of the Hopf algebra \( L = \mathcal{Z}(\mathbf{1}) = U(F) \in \mathcal{C} \),

\[
\eta_F = \eta \quad \text{and} \quad \mu_F = \mu .
\]

This should in fact not come as a surprise. Indeed, the antipode of the Hopf algebra \( L \) is invertible, and \( L \) admits a left integral \( \Lambda \in \text{Hom}_\mathcal{C}(\mathbf{1}, L) \) and a right cointegral \( \lambda \in \text{Hom}_\mathcal{C}(L, \mathbf{1}) \) such that \( \lambda \circ \Lambda \in \text{End}_\mathcal{C}(\mathbf{1}) \) is invertible. By definition, a left integral \( \Lambda \) of a Hopf algebra \( H \) is a morphism in \( \text{Hom}_\mathcal{C}(\mathbf{1}, H) \) that intertwines the trivial and regular left \( H \)-actions, i.e. satisfies

\[
\Lambda \circ \varepsilon = \mu \circ (\text{id}_H \otimes \Lambda) .
\]

Similarly, a right cointegral \( \lambda \) obeys by definition

\[
\eta \circ \lambda = (\lambda \otimes \text{id}_H) \circ \Delta .
\]

Since the category \( \mathcal{C} \) is unimodular, so is the Hopf algebra \( L \), i.e. the left integral is also a right integral, i.e. satisfies \( \Lambda \circ \varepsilon = \mu \circ (\lambda \otimes \text{id}_L) \), implying in particular that the integral is invariant under the antipode, \( s \circ \Lambda = \Lambda \). Now it is known [30, App. A.2] that in a linear ribbon category any Hopf algebra with invertible antipode and (co)integrals for which \( \lambda \circ \Lambda \) is invertible, also carries a natural Frobenius algebra structure, with the same product and unit, but with different coalgebra structure.

Moreover, the Hopf algebra structure on the object \( L = U(F) \) in \( \mathcal{C} \) together with the integral and cointegral of \( L \) also induces a coalgebra structure on \( U(F) \) that is part of its Frobenius algebra structure in \( \mathcal{C} \) [30, App. A.2]. In view of (2.26) one would expect that also the coalgebra structure on \( F \) is such that upon forgetting the half-braiding on \( F \) it reproduces this Frobenius coalgebra structure on \( U(F) \in \mathcal{C} \). This indeed turns out [62, Sect. 5] to be the case. In particular, the Frobenius counit is given by the cointegral of \( L \). This conclusion also coincides with the expectations from CFT. Indeed, the counit \( \varepsilon_F \) is provided by the one-point correlator of bulk fields on the sphere [31], and by comparison with the one-point correlators for semisimple \( \mathcal{C} \) one expects that, as a morphism in \( \mathcal{C} \), it is a (non-zero) cointegral of the Hopf algebra \( L \),

\[
\varepsilon_F = \lambda .
\]

Actually, this is already implied by the fact [62, Sect. 5.1] that the relevant morphism space is one-dimensional:

\[
\text{Hom}_{\mathcal{Z}(\mathcal{C})}(F, \mathbf{1}_{\mathcal{Z}(\mathcal{C})}) = \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}(\mathbf{1}), \mathbf{1}_{\mathcal{Z}(\mathcal{C})}) \cong \text{Hom}_\mathcal{C}(\mathbf{1}, U(\mathbf{1}_{\mathcal{Z}(\mathcal{C})})) = \text{Hom}_\mathcal{C}(\mathbf{1}, \mathbf{1}) \cong \mathbb{C} .
\]

Given the counit, the Frobenius form \( \kappa : F \otimes F \to \mathbf{1} \) is given by

\[
\kappa = \varepsilon_F \circ \mu_F = \lambda \circ \mu .
\]

This is indeed non-degenerate, as required for a Frobenius form; a side inverse \( \kappa^- \), satisfying \((\kappa \otimes \text{id}_F) \circ (\text{id}_F \otimes \kappa^-) = \text{id}_F = (\text{id}_F \otimes \kappa) \circ (\kappa^- \otimes \text{id}_F) \) is given by (see e.g. [30, Eq. (3.33)])

\[
\kappa^- = (\text{id}_L \otimes s) \circ \Delta \circ \Lambda
\]

with \( s \) and \( \Delta \) the antipode and coproduct of \( L \), and \( \Lambda \) an integral of \( L \) satisfying

\[
\lambda \circ \Lambda = 1 .
\]
As for any Frobenius algebra [38], the Frobenius coproduct can be obtained from the product
\( \mu_F = \mu \) by appropriately composing with \( \kappa \) and its side inverse. Concretely, with the help of the isomorphism

\[
\Phi := (\kappa \otimes \text{id}_{F^\vee}) \circ (\text{id}_F \otimes \text{coev}_F) \in \text{Hom}_C(F, F^\vee)
\] (2.34)

we can write (compare [62, Eq. (5.5)])

\[
\Delta_F = (\text{id}_F \otimes \mu) \circ (\text{id}_F \otimes \Phi^{-1} \otimes \text{id}_F) \circ (\text{coev}_F \otimes \text{id}_F) = (\mu \otimes \text{id}_F) \circ (\text{id}_F \otimes \Phi^{-1} \otimes \text{coev}_F).
\] (2.35)

By using the explicit expression

\[
\Phi^{-1} = (\text{ev}_F \otimes \text{id}_F) \circ (\text{id}_{F^\vee} \otimes \kappa^-)
\] (2.36)

for the inverse of (2.34), this can be rewritten as

\[
\Delta_F = (\text{id}_F \otimes \mu) \circ (\text{id}_F \otimes \text{coev}_F \otimes \text{id}_F) \circ ((\Delta \circ \Lambda) \otimes \text{id}_F) = (\mu \otimes \text{id}_F) \circ (\text{id}_F \otimes \kappa^-),
\] (2.37)

thereby reproducing formula (A.10) of [30]. A mirror version of this formula is valid as well; graphically, the two formulas look like

\[
\Delta_F = \begin{array}{c}
\mu \\
\kappa^- \\
\end{array} = \begin{array}{c}
\mu \\
\kappa^- \\
\end{array}
\] (2.38)

It follows e.g. that the integral satisfies (compare [62, Prop. 5.5]) \( \Lambda = (\text{id}_F \otimes (\varepsilon_L \circ \Phi^{-1})) \circ \text{coev}_F \).

Let us also show that \( F \in Z(\mathcal{C}) \) has trivial twist. By definition, the twist on the ribbon category \( \mathcal{C} \boxtimes \mathcal{C} \) is given by \( \theta^{-1} \boxtimes \theta \) with \( \theta \) the twist on \( \mathcal{C} \). Accordingly the following statement is not surprising:

**Lemma 2.9.** For \( u, v \in \mathcal{C} \) the twist of the object \( \Xi_C(u \boxtimes v) \in Z(\mathcal{C}) \) can be expressed as

\[
\theta_{\Xi_C(u \boxtimes v)} = \theta^{-1}_u \otimes \theta_v
\] (2.39)

in terms of the twist \( \theta \) in \( \mathcal{C} \).

**Proof.** Expressing the twist of the ribbon category \( Z(\mathcal{C}) \) through the braiding and dualities and using the explicit form (2.14) for the braiding of \( \Xi_C(u \boxtimes v) \) in \( Z(\mathcal{C}) \), we have, pictorially:
Lemma 2.10. (i) $F$ has trivial twist in $Z(C)$.
(ii) $F$ is a symmetric Frobenius algebra in $Z(C)$.

Proof. (i) By the naturality of the twist we have $\theta_F^Z(C) \circ i_x^{Z(1)} = i_x^{Z(1)} \circ \theta_x^Z(C)$, where on the right hand side $x^\vee \otimes x$ is, by construction, the object $\Xi C(x^\vee \otimes x)$ of $Z(C)$. Lemma 2.9 thus implies $\theta_F^Z(C) \circ i_x^{Z(1)} = i_x^{Z(1)} \circ (\theta_x^{-1} \otimes \theta_x)$. By dinaturality of $i_x^{Z(1)}$ this becomes

$$\theta_F^Z(C) \circ i_x^{Z(1)} = i_x^{Z(1)} \circ (id_x^\vee \otimes (\theta_x^{-1} \circ \theta_x)) = i_x^{Z(1)} = id_F \circ i_x^{Z(1)}.$$  

(2.41)

This holds for every $x \in C$; the claim thus follows by dinaturality.

(ii) That $F$ is Frobenius is a direct consequence of the construction of the coalgebra structure from the Hopf algebra structure and (co-)invariant of $L$. That the Frobenius form $(2.31)$ is symmetric follows by combining the facts that the product $\mu_F$ is commutative and that $F$ has trivial twist. □

Remark 2.11. In the case of semisimple $C$, in which $F = \bigoplus_{i \in I} x_i^\vee \otimes x_i$, Lemma 2.10(i) follows immediately by combining (2.39) with $\theta_{x^\vee} = \theta_x$. Lemma 2.10(ii) follows in this case from the fact that, by Lemma 6.19(ii) of [26], $\tilde{F} \in \overline{C \otimes C}$ is symmetric Frobenius. (By the same lemma, for semisimple $C$ the Frobenius algebra $F$ is also special; this ceases to be true for non-semisimple $C$.)

By construction, the morphisms $\mu$, $\eta$, $\Delta_F$ and $\varepsilon_F$ also endow, when regarded as morphisms in $C$, the object $U(F) = Z(1) \in C$ with the structure of a Frobenius algebra. However, since the braiding and twist in $C$ are different from those in $Z(C)$, unlike $F = \bigoplus_{i \in I} 1 \in Z(C)$ this Frobenius algebra is not commutative, and $U(F)$ does not have trivial twist. (The existence of a two-sided adjunction between $C$ and $Z(C)$ does not require $C$ to be braided, so the lack of commutativity is not so surprising.) Moreover, even though $U(F)$ is a symmetric algebra, the Frobenius form $\kappa$ is not symmetric on the nose when regarded as a morphism in $C$, but there is a minor deviation from being symmetric. Indeed, combining the expression (2.31) for $\kappa$ with general properties of the cointegral, and with the fact that the square of the antipode $s$ of $L$ is an inner automorphism, one concludes (compare [60, Thm. 3]) that

$$\kappa \circ \beta_{L,L} = \kappa \circ (id_L \otimes s^{-2}).$$  

(2.42)

In other words, $s^2$ is an inner Nakayama automorphism for the Frobenius algebra $U(F)$. Going from $C$ to $Z(C)$ changes the braiding and requires a trivial Nakayama automorphism. (Note that
this means that $S^4_L$ is the identity in $\mathcal{Z}(C)$. This might be expected to be the case for a consistent full CFT even beyond finite CFTs, because the anomaly in the action of the modular group that is generically present [55] on the chiral level should cancel out between left and right movers.)

2.3. Characters and cocharacters

As outlined in the Introduction, the boundary states for the Cardy case furnish the standard mathematical structure of a ring homomorphism from the fusion ring $K_0(C)$ to the endomorphisms of the identity functor of $C$. We now construct such a homomorphism concretely with the help of the Hopf algebra $L$. As ingredients we need the notions of algebras and their modules in rigid monoidal categories.

The representation theory of (associative, unital) algebras in a rigid monoidal category can be studied in full analogy with the classical case of algebras over $\mathbb{C}$. In particular, characters are defined as partial traces in the same way as (see e.g. [53, Sect. 1.5]) for a $\mathbb{C}$-algebra. (Below we will only be interested in the case that $C$ is ribbon and thus spherical, so that we do not have to distinguish between left and right traces.) In more detail, for $A \in C$ an algebra, an $A$-module $(m, \rho_m)$ is an object $m \in C$ together with a representation morphism $\rho_m : A \otimes m \to m$ that satisfies the usual compatibility requirements with the product and unit of $A$.

**Definition 2.12 (Character).** The character of an $A$-module $(m, \rho_m)$ in $C$ is the partial trace

$$\chi^A_m := \tilde{\text{ev}}_m \circ (\rho_m \otimes id_{m'}) \circ (id_A \otimes \text{coev}_m) \in \text{Hom}_C(A, 1)$$

(2.43)

of $\rho_m$.

Here $\tilde{\text{ev}}_x : x \otimes x^\vee \to 1$ is the (left) evaluation morphism, and $\text{coev}_x : 1 \to x \otimes x^\vee$ is the (right) coevaluation. Pictorially,

$$\chi^A_m = \xymatrix{ & m \ar[dl] \ar[dr] & \\
A & & }

(2.44)$$

Characters split under extensions, i.e. for any short exact sequence

$$0 \to m' \to m \to m'' \to 0$$

(2.45)

of $A$-modules we have

$$\chi^A_m = \chi^A_{m'} + \chi^A_{m''}.$$  

(2.46)

As a consequence, the character of any module is a sum of irreducible characters, i.e. of characters of simple modules. Isomorphic modules have equal characters.

From now on we assume that $C$ is braided. If $A$ is even a bialgebra, then one can form tensor products of representations: The tensor product of two $A$-modules $(m, \rho^A_m)$ and $(n, \rho^A_n)$ is the object $m \otimes n$ together with the representation morphism

$$\rho^A_{m\otimes n} := (\rho_m \otimes \rho_n) \circ (id_A \otimes \beta_{A,m} \otimes id_n) \circ (\Delta_A \otimes id_m \otimes id_n),$$

(2.47)
with $\Delta_A$ the coproduct of $A$. Characters satisfy

$$\chi^A_{m \otimes n} = (\chi^A_m \otimes \chi^A_n) \circ \Delta_A ,$$

(2.48)
i.e. they are multiplicative with respect to the convolution product on $\text{Hom}_C(A, \mathbf{1})$ that is induced by the coalgebra structure of $A$ (and the algebra structure of $\mathbf{1}$).

In the case of the Hopf algebra $L = Z(\mathbf{1})$, every object $m$ in $C$ turns out to have a canonical structure of an $L$-module $(m, \rho^L_m)$. The action $\rho^L_m : L \otimes m \rightarrow m$ of $L$ on $m$ is defined via a partial monodromy. More explicitly, $\rho^L_m$ can be expressed with the help of the dinatural family of morphisms $\iota^L_x : x^\vee \otimes x \rightarrow L$ that comes with the coend structure of $L$, and of the braiding $\beta$ and the right evaluation $\text{ev}$ of $C$, as [35, Sect. 2.2]

$$\rho^L_m \circ (\iota^L_x (1) \otimes \text{id}_m) := (\text{ev}_x \otimes \text{id}_m) \circ [\text{id}_{x^\vee} \otimes (\beta_{m,x} \circ \beta_{x,m})]$$

(2.49)
for all $x \in C$. Pictorially, this action and the resulting character are given by

$$\rho^L_m \circ (\iota^L_x (1) \otimes \text{id}_m) = \chi^L_m \circ \iota^L_x (1) =$$

(2.50)

Notice in particular that for semisimple $C$ the morphism $\chi^L_m \circ \iota^L_x (1)$ coincides with the morphism on the right hand side of (1.5) and that in this case the monodromy appearing in (2.50) gives rise to the $S$-matrix factor in the formula (1.2) for the boundary state. Moreover, any family of characters of pairwise non-isomorphic simple $L$-modules $(m, \rho^L_m)$ is linearly independent in the vector space $\text{Hom}_C(L, \mathbf{1})$ [62, Thm. 4.1].

By (2.47) the $L$-modules $(m, \rho^L_m)$ form a full monoidal subcategory of the category $C_L$ of all $L$-modules in $C$. The latter category $C_L$ is in fact braided equivalent to the Drinfeld center $Z(C)$, which is best understood when formulated [7, Thm. 8.13] in terms of the central monad on $C$ (compare Appendix A.3).

Through the structure of $L$ as a coend, every object $m$ in $C$ is also canonically an $L$-comodule $(m, \delta^L_m)$, with the coaction given by

$$\delta^L_m := (\iota^L_m (1) \otimes \text{id}_m) \circ (\text{id}_m \otimes \text{coev}_m) \in \text{Hom}_C(m, L \otimes m) ,$$

(2.51)
and with corresponding cocharacter

$$\check{\chi}^L_m := (\text{id}_L \otimes \text{coev}_m) \circ (\delta^L_m \otimes \text{id}_m) \circ \text{coev}_m = \iota^L_m \circ \text{coev}_m \in \text{Hom}_C(\mathbf{1}, L) .$$

(2.52)

Pictorially,
The character and cocharacter of the trivial $L$-module are given by the counit and unit of $L$, respectively: $\chi^L_1 = \varepsilon$ and $\hat{\chi}^L_1 = \eta$. Like characters, also cocharacters split under extensions, and they are multiplicative under the tensor product of comodules, so that [30, Sect. 4.5]

$$
\hat{\chi}^L_{m \otimes n} = \mu \circ (\hat{\chi}^L_m \otimes \hat{\chi}^L_n).
$$

(2.54)

The action $\rho^L_m$ and coaction $\delta^L_m$ are connected by the Hopf pairing $\omega$ according to $(\omega \otimes \text{id}_m) \circ (\text{id}_L \otimes \delta^L_m) = \rho^L_m$. Recall from Definition 2.5 that bijectivity of the linear map

$$
\Omega : \text{Hom}_C(\mathbf{1}, L) \rightarrow \text{Hom}_C(L, \mathbf{1})
$$

$$
\hat{\alpha} \mapsto \omega \circ (\text{id}_L \otimes \hat{\alpha})
$$

(2.55)

is one of the equivalent definitions of modularity. Thus for modular $C$ the relation between the action and coaction can be inverted. In particular, for the characters and cocharacters we have

$$
\chi^L_m = \Omega(\hat{\chi}^L_m) \quad \text{and} \quad \hat{\chi}^L_m = \Omega^{-1}(\chi^L_m).
$$

(2.56)

The algebra structure of $L$ and the (trivial) coalgebra structure of $\mathbf{1}$ supply a convolution product on $\text{Hom}_C(\mathbf{1}, L)$:

$$
\hat{\alpha} \ast \hat{\beta} := \mu_L \circ (\hat{\alpha} \otimes \hat{\beta})
$$

(2.57)

for $\hat{\alpha}, \hat{\beta} \in \text{Hom}_C(\mathbf{1}, L)$. Analogously, the coalgebra structures of $L$ and $U(\mathbb{F})$ supply two convolution products on $\text{Hom}_C(L, \mathbf{1})$:

$$
\alpha \ast_F \beta := (\alpha \otimes \beta) \circ \Delta \quad \text{and} \quad \alpha \ast_F \beta := (\alpha \otimes \beta) \circ \Delta_F
$$

(2.58)

for $\alpha, \beta \in \text{Hom}_C(L, \mathbf{1})$. The multiplicativity relations (2.48) and (2.54) for the characters and cocharacters tell us that the (co-)character maps respect these products,

$$
\chi^L_{m \otimes n} = \chi^L_m \ast_L \chi^L_n \quad \text{and} \quad \hat{\chi}^L_{m \otimes n} = \hat{\chi}^L_m \ast \hat{\chi}^L_n.
$$

(2.59)

Since they also split under extensions, we have

**Lemma 2.13.** The characters $\chi^L_m$ and $\hat{\chi}^L_m$ define ring homomorphisms

$$
\chi^L : K_0(C) \rightarrow \text{Hom}_C(L, \mathbf{1}) \quad \text{and} \quad \hat{\chi}^L : K_0(C) \rightarrow \text{Hom}_C(\mathbf{1}, L),
$$

(2.60)

respectively.

**Remark 2.14.** There are two distinguished pairs of isomorphisms between the morphism spaces $\text{Hom}_C(L, \mathbf{1})$ and $\text{Hom}_C(\mathbf{1}, L)$: composition with the Hopf pairing $\omega$ as in (2.55) and its side inverse $\omega^-$ on the one hand, and composition with the Frobenius form (2.31) and its side inverse $\kappa^-$ on the other hand. The defining property of the Hopf pairing is equivalent to the first of these pairs of isomorphisms being intertwiners between the convolution product on $\text{Hom}_C(\mathbf{1}, L)$ and the convolution product $\ast_L$ on $\text{Hom}_C(L, \mathbf{1})$. 

$$
\delta^L_m = \iota^{Z, L}_m,
$$

$$
\hat{\chi}^L_m = \eta^L_m.
$$

(2.53)
Remark 2.15. Composing a cocharacter with a character gives a Hopf link morphism. Indeed, by direct calculation we have
\[ \chi^L_y \circ \delta^L_x (2.56) = \omega \circ (\delta^L_x \otimes \delta^L_y) = S^{\otimes}_{x,y}, \]
with the Hopf link morphisms \( S^{\otimes} \) defined as
\[ S^{\otimes}_{x,y} := (e_x \otimes e_y) \circ (id_{x^\vee} \otimes ((\beta^C)_{y,x} \circ (\beta^C)_{x,y}) \otimes id_{y^\vee}) \circ (c_{\circ} e_x \otimes c_{\circ} e_y). \]

Using the naturality of the braiding so as to deform the strands in the picture (2.62) suitably one shows that \( S^{\otimes}_{x,y} = \tilde{S}^{\otimes}_{y^\vee x} = S^{\otimes}_{y^\vee,x^\vee} \), where \( \tilde{S}^{\otimes} \) is defined analogously as \( S^{\otimes} \), but with the monodromy \( (\beta^C)_{y,x} \circ (\beta^C)_{x,y} \) replaced by its inverse.

If \( C \) is semisimple, then the matrix \( (S^{\otimes}_{i,j})_{i,j \in I} \) is non-degenerate – it is then just the (unnormalized) modular S-matrix. In contrast, for non-semisimple \( C \) the pairing between the spaces of characters and cocharacters furnished by \( S^{\otimes} \) is degenerate. Indeed, for any projective object \( p \in \mathcal{C} \) and any \( x \in \mathcal{C} \) the morphism \( S^{\otimes}_{p,x} \in \text{End}_C(1) = \mathbb{I} \) factorizes through the projective object \( p^\vee \otimes p \otimes x \otimes x^\vee \) and can therefore only be non-zero if the tensor unit \( 1 \) is projective which, in turn, is equivalent to \( C \) being semisimple.

2.4. The modular S-transformation

If \( C \) is a semisimple modular tensor category, then the modular S-transformation of vertex algebra characters is described, up to normalization, by the modular S-matrix \( (S^{\otimes}_{i,j})_{i,j \in I} \) (see e.g. [44]). In contrast, in the non-semisimple case this is no longer the case, as is indicated by the degeneracy just mentioned. However, it is known from [55] that morphism spaces in \( \mathcal{C} \) of the form \( \text{Hom}_C(x, L) \) or \( \text{Hom}_C(L, y) \) with \( x, y \in \mathcal{C} \) admit a natural projective action of the modular group \( \text{SL}(2, \mathbb{Z}) \) by post- or precomposition, respectively, with endomorphisms of the object \( L \).

This generalizes to projective actions of mapping class groups at any genus \( g \) for morphisms involving the object \( L^{\otimes g} \). This provides us in particular with an \( \text{SL}(2, \mathbb{Z}) \)-action on the spaces \( \text{Hom}_C(1, L) \) or \( \text{Hom}_C(L, 1) \) which contain the cocharacters and characters of \( L \). Specifically, for the modular S-transformation the endomorphism of \( L \) in question is [55]

\[ S_L := (e \otimes id_L) \circ Q_L \circ (id_L \otimes \Lambda) \]
with \( Q_L \), the endomorphism of \( L \otimes L \) defined by
\[ Q_L \circ (\theta^Z_x \otimes \theta^Z_y) := (\theta^Z_x \otimes \theta^Z_y) \circ (id_{x^\vee} \otimes [\beta_{y,x} \circ \beta_{x,y^\vee}] \otimes id_y). \]

Moreover, the integral \( \Lambda \) can be normalized such that this morphism squares to the inverse antipode [55, Thm. 2.1.9],

\[ (S_L)^2 = s^{-1}. \]
In the sequel we assume that this normalization has been chosen (this determines $\Lambda$ up to a sign).

In the case that $C = H$-mod is the representation category of a factorizable ribbon Hopf algebra, $S_L$ is the composition of the Drinfeld and Frobenius maps of $H$ (see e.g. [64]). This generalizes as follows:

**Lemma 2.16.** The morphism $S_L$ can be expressed as

$$S_L = (\kappa \otimes id_L) \circ (id_L \otimes \omega^-) \quad (2.66)$$

in terms of the Hopf pairing $\omega$ (2.4) and of the Frobenius form $\kappa$ (2.31).

**Proof.** By direct calculation one sees that

$$\begin{align*}
(\omega \otimes id_L) \circ (id_L \otimes \Delta) \circ (t^Z_1(y) \otimes id_L) &= (\text{coev}_y \otimes id_L) \circ (id_L \otimes \beta_{x,y}) \circ (id_L \otimes \beta_{x,y}) \circ (id_L \otimes \omega^-) \\
&= (\epsilon \otimes id_L) \circ Q_L \circ (id_L \otimes \omega^-).
\end{align*} \quad (2.67)$$

Here the first equality follows by combining the definitions of $\Delta$ and $\omega$ and the second by combining those of $\epsilon$ and $Q_L$. By dinaturality, (2.67) implies that $(\omega \otimes id_L) \circ (id_L \otimes \Delta) = (\epsilon \otimes id_L) \circ Q_L$. Pre-composing this equality with $id_L \otimes \omega^- \otimes id_L$ and post-composing with the antipode gives, by recalling the expression for $\kappa^-$,

$$\begin{align*}
(\omega \otimes id_L) \circ (id_L \otimes \kappa^-) &= (\epsilon_L \otimes s) \circ Q_L \circ (id_L \otimes \Lambda) = s \circ S_L \overset{(2.65)}{=} S^{-1}_L. \quad (2.68)
\end{align*}$$

Taking inverses, we finally arrive at the claimed formula (2.66).  \[\square\]

We also have (see e.g. [51, Lemma 5.2.4])

$$\omega \circ \beta_{L,L} = \omega \circ (s^{-1} \otimes s^{-1}) \quad (2.69)$$

or, equivalently,

$$\beta_{L,L} \circ \omega^- = (s_L \otimes s_L) \circ \omega^- \quad (2.70)$$

When combined with (2.42) and (2.66) it then follows by direct calculation that the formulas relating the S-automorphism to the Hopf pairing and Frobenius form are left–right-symmetric:

**Lemma 2.17.** The S-endomorphism of $L$ and its inverse obey

$$\begin{align*}
(id_L \otimes \kappa) \circ (\omega^- \otimes id_L) &= S_L = (\kappa \otimes id_L) \circ (id_L \otimes \omega^-) \quad (2.71)
\end{align*}$$

and

$$\begin{align*}
(id_L \otimes \omega) \circ (\kappa^- \otimes id_L) &= S^{-1}_L = (\omega \otimes id_L) \circ (id_L \otimes \kappa^-) \quad (2.72)
\end{align*}$$
Pictorially,

\[
\kappa \omega^- = S_L = \kappa \\
\omega^- = \omega \kappa^- = S_L^{-1} = \omega \\
\kappa \\
\text{(2.73)}
\]

and

\[
\omega = (id_L \otimes \omega \otimes id_L) \circ (id_L \otimes s^{-1} \otimes id_L \otimes id_L) \circ (\kappa^\perp \otimes \kappa^\perp) \\
= (id_L \otimes id_L \otimes \omega) \circ (id_L \otimes [\Delta \circ \Lambda] \otimes s) \circ \Delta \circ \Lambda \\
= (id_L \otimes s \otimes \omega) \circ (id_L \otimes [\Delta \circ \Lambda] \otimes id_L) \circ \Delta \circ \Lambda .
\]

Combining this result with the relation (2.65) between \( S_L \) and the antipode, one can derive the following formulas for the inverse of the Hopf pairing:

\[
\omega^- = (id_L \otimes \omega \otimes id_L) \circ (id_L \otimes s^{-1} \otimes id_L \otimes id_L) \circ (\kappa^\perp \otimes \kappa^\perp) \\
= (id_L \otimes id_L \otimes \omega) \circ (id_L \otimes [\Delta \circ \Lambda] \otimes s) \circ \Delta \circ \Lambda \\
= (id_L \otimes s \otimes \omega) \circ (id_L \otimes [\Delta \circ \Lambda] \otimes id_L) \circ \Delta \circ \Lambda.
\]

Let us also mention that it follows directly from the defining properties of a Hopf pairing and the non-degeneracy of \( \omega \) that if \( \Lambda \) is a right integral of \( L \), then \( \Omega(\Lambda) \) is a left cointegral, and vice versa, and analogously for left integrals and right cointegrals (see e.g. [50, Thm. 5]). We can then fix the relative normalization of \( \Lambda \) and \( \lambda \) in such a way that

\[
\lambda = \Omega(\Lambda) \circ s^{-1}
\]

or, equivalently,

\[
\lambda = \tilde{\Omega}(\Lambda)
\]

with the linear map \( \tilde{\Omega}: \text{Hom}_C(1, L) \to \text{Hom}_C(L, 1) \) being defined, similarly as \( \Omega \) in (2.55), by \( \tilde{\Omega}(\hat{a}) := \omega \circ (\hat{a} \otimes id_L) \).

Finally note that

\[
\chi_Y \circ S_L \circ \hat{\chi}_x = \kappa \circ (\hat{\chi}_x \otimes \hat{\chi}_y),
\]

which may be compared to (2.61).

**Remark 2.18.** It is tempting to interpret the (co)characters \( \chi \) and \( \hat{\chi} \) in the context of \( C_2 \)-cofinite vertex operator algebras. To do so, note that the appropriate notion of character for a module \( M \) over a vertex operator algebra \( \mathcal{V} \) is as a chiral genus-1 one-point function.
\[ \chi_M^\omega(v, \tau) = \text{Tr}_M o_M(v) q^{L_0-c/24}, \]  
\[
(2.79)
\]

with \( v \in \mathcal{F} \) and \( o_M(v) \) the grade-preserving endomorphism of \( M \) induced by \( v \) (rather than as a Virasoro-specialized character \( \chi_M^\omega(\tau) \), which is obtained when taking \( v \) to be the vacuum vector; those are typically not linearly independent). In conformal field theory terms, these characters span a subspace of the space of one-point blocks on the torus with insertion the tensor unit; if the representation category \( \mathcal{F} \)-Rep is non-semisimple, then this is a proper subspace, with a (non-canonical) complement given by \([57]\) certain pseudo-characters. (While the characters are power series in \( q \), the pseudo-characters involve extra factors of \( \tau = \ln q/2\pi i \); see e.g. [43, Eq. A.2] for explicit expressions for the logarithmic \( (p, 1) \) triplet models.) In our setting, this space is the morphism space \( \text{Hom}_\mathcal{F}\text{-Rep}(1, L) \) (or the isomorphic space \( \text{Hom}_\mathcal{F}\text{-Rep}(L, 1) \), but as we will see below, the former description is more appropriate). Now as an object of the category \( \mathcal{C} = \mathcal{F} \)-Rep, \( M \) has a natural structure of \( L \)-module and \( L \)-comodule, so that we can consider its character \( \chi_M^L \) and cocharacter \( \hat{\chi}_M^L \) as in \((2.50)\) and \((2.53)\). Thinking of these as categorical variants of the characters \((2.79)\) fits perfectly with the fact that the morphisms \( \chi_M^L \) and \( \hat{\chi}_M^L \), for \( M \) the simple \( \mathcal{F} \)-modules, are linearly independent and span subspaces of \( \text{Hom}_\mathcal{C}(L, 1) \) and \( \text{Hom}_\mathcal{C}(1, L) \), respectively. Likewise it fits with the observation that, at least for \( \mathcal{C} = H \)-mod, the torus partition function can be expanded as a bilinear form in the characters \( \chi_m^L \) (with coefficients that are given by the Cartan matrix of \( C \)) \([34]\).

Of course, once we have switched to the categorical setting, the dependence on the parameter \( \tau \) (the modulus of the torus) is no longer visible. Now it follows directly from the definition of \( T_L \) that the cocharacter \( \hat{\chi}_M^L \) satisfies
\[
T_L \circ \hat{\chi}_M^L = [\chi_M^L] \circ (\theta_M \otimes id_{M^\vee}) \circ \text{coev}_{M^\vee} \tag{2.80}
\]
with \( \theta_M \) the twist isomorphism of \( M \), so that acting with \( T_L \) on a simple cocharacter just multiplies it with its twist eigenvalue. This indicates that it is \( \hat{\chi}_M^L \) that more directly corresponds to \( \chi_M^\omega(v, \tau) \), while the character \( \chi_M^L \) rather corresponds to \( \chi_M^\omega(v, -1/\tau) \). This interpretation is corroborated by the fact that \( L \) can be regarded as the dual of a more fundamental algebraic structure (compare Example 2.7(iii)), as well as by the following observation: Combining the relation \((2.56)\) between characters and cocharacters with the formula \((2.71)\) for \( S_L \), we can write \( \hat{\chi}_m^L = (id_L \otimes (\chi_m^L \circ S_L)) \circ \kappa^- \). Now the morphisms \( \kappa \) and \( \kappa^- \) constitute a possible choice for the (right, say) evaluation and coevaluation morphisms \( ev_L \) and \( \text{coev}_L \). With this choice, the previous equality can be rewritten as
\[
\chi_m^L \circ S_L = (\hat{\chi}_m^L)^\vee. \tag{2.81}
\]
Thus in short, up to a duality the characters and cocharacters of our interest are related by a modular S-transformation.

3. Boundary states

We henceforth assume that the category of boundary conditions is given by the category \( \mathcal{C} \) of chiral sectors. Besides being a natural choice, another attractive feature of \( \mathcal{C} \) as a module category over itself is that it is an exact module category, whereby it comes with a lot of additional structure, such as relative Serre functors \([28]\).

A boundary state assigns to a boundary condition an element of a vector space of conformal blocks; the relevant space of conformal blocks will be described in Section 3.1. As explained in
Section 1, this map constitutes a decategorification and therefore should factorize over the fusion ring, i.e. the Grothendieck ring of $C$. As also explained there, by comparison with rational CFT this is achieved by realizing boundary states as characters of the $L$-modules with the canonical $L$-action (2.49). In this section we discuss this issue in more detail, accounting in particular also for the need to distinguish between incoming and outgoing bulk field insertions. Later, in Section 4, we will show that it also gives rise to sensible annulus partition functions.

3.1. Blocks for boundary states

We first establish that the conformal blocks for boundary states are indeed (canonically isomorphic to) the vector space $\mathcal{End}_C(Id_C)$. This requires some explanation, because conformal blocks for non-semisimple conformal field theories on world sheets with boundary have so far not been discussed systematically.

Pertinent calculations in the semisimple case have been performed in [22,27]. They made use of the fact that the complex double of a disk is a two-sphere. Accordingly, to describe a disk with one bulk field insertion, without loss of generality it was possible to work with two-point blocks on the sphere with chiral insertions given by simple objects which, in turn, can be viewed as conformal blocks for a three-dimensional topological field theory of Reshetikhin–Turaev type. In contrast, for non-semisimple $C$ the Cardy bulk algebra $F$ does not decompose into a direct sum of products of left and right movers any longer, so that we should keep $F$ as a single insertion. We therefore consider a different geometry: we directly work with a disk with a single puncture in its interior, and this puncture is labeled by the entire bulk algebra $F \in \mathcal{Z}(C)$. When doing so, then unlike in [22,27] the relevant conformal blocks are those of a topological field theory of Turaev–Viro type; our formulation is made possible by recent insight which allows one to consider such theories also for surfaces with boundary, and in particular for a disk.

The vector space of conformal blocks for boundary states can thus be obtained as follows. First, by the results of [37] a boundary condition for the three-dimensional topological field theory associated with $C$ can be characterized by a central functor $\mathcal{Z}(C) \to \mathcal{W}$ from the category $\mathcal{Z}(C)$ of bulk Wilson lines to a fusion category $\mathcal{W}$ of boundary Wilson lines. In the Cardy case, $\mathcal{W}$ coincides with $C$ and this functor is nothing but the forgetful functor $U$ from (2.10). Second, by adiabatically moving the bulk insertion labeled by $F$ into the boundary, we obtain a boundary insertion, and this boundary insertion is labeled by the object $U(F) = Z(1) \in C$ (recall from (2.19) that this has the structure of a coend). Finally there aren’t any other insertions involved. We conclude that the vector space in which the boundary states take their value is

$$B_{\text{b.s.}}^{\text{out}} = \text{Hom}_C(1, Z(1))$$  \hspace{1cm} (3.1)

for the case of outgoing boundary states, respectively

$$B_{\text{b.s.}}^{\text{in}} = \text{Hom}_C(Z(1), 1)$$  \hspace{1cm} (3.2)

for incoming boundary states.

This conclusion holds in the first place only as long as $C$ is semisimple, which is assumed in the setting of [37]. It is natural to expect that, analogously as in the case of bulk correlators [31], expressions for boundary conformal blocks remain correct in the non-semisimple case once they are written, as we did, in a form that makes sense without assuming semisimplicity. And indeed a detailed study of a non-semisimple generalization of the Turaev–Viro state sum construction [29] confirms this expectation.
Now recall from the Introduction that the boundary states are expected to furnish a ring homomorphism from the fusion ring $K_0(C)$ of $C$ to the center of $C$, i.e. to the linear natural endo-transformations $\mathcal{E}nd C(Id C)$ of the identity functor (not to be confused with the monoidal (Drinfeld) center $Z(C)$ of $C$). For compatibility, the center must be isomorphic to the spaces $(3.1)$ and $(3.2)$ of conformal blocks obtained above. This is indeed the case: As already pointed out in formula $(1.1)$ in the Introduction, for any finite tensor category $C$ the space $\mathcal{E}nd C(Id C)$ is isomorphic as a $C$-algebra to the morphism space $\text{Hom}_C(L, 1)$ with $L \equiv Z(1)$ (see also [50, Lemma 4] and [51, Prop. 5.2.5]). An isomorphism from $\text{Hom}_C(L, 1)$ to $\mathcal{E}nd C(Id C)$ (as vector spaces, and even as algebras) is furnished by composition with the natural coaction of $L$, i.e.

$$\text{Hom}_C(L, 1) \ni \alpha \mapsto \left( (id_x \otimes (\alpha \circ i_x^{Z(1)})) \circ (\text{coev}_x \otimes id_x) \right)_{x \in C},$$

and an inverse to this linear map is given by composition with the counit of $L$, i.e. a natural transformation $g = (g_x) \in \mathcal{E}nd C(Id C)$ is mapped to the element $\alpha_g \in \text{Hom}_C(L, 1)$ that is determined by

$$\alpha_g \circ i_x^{Z(1)} = \varepsilon \circ i_x^{Z(1)} \circ (id_x \otimes g_x) = \text{ev}_x \circ (id_x \otimes g_x).$$

Since $L$ is self-dual, the space $\text{Hom}_C(L, 1)$ is, in turn, isomorphic to $\text{Hom}_C(1, L)$. Indeed, these two spaces are even isomorphic as algebras (with convolution product). One way to see this is via isomorphisms to the space $\text{Hom}_{Z(C)}(F, F)$ of endomorphisms of the bulk object $F$ in the center of $C$; these will be presented in Section 4.1 below.

**Example 3.1.** (i) If $C$ is semisimple, we have

$$\text{Hom}_C(L, 1) = \bigoplus_{i \in I} \text{Hom}_C(x_i^* \otimes x_i, 1) \cong \bigoplus_{i \in I} \text{End}_C(x_i),$$

so that a natural basis is given by the evaluation morphisms $\text{ev}_{x_i} \in \text{Hom}_C(x_i^* \otimes x_i, 1)$, respectively by the identity morphisms $id_{x_i} \in \text{End}_C(x_i)$. The latter are precisely the Ishibashi states depicted in $(1.4)$.

(ii) For the logarithmic $(p, 1)$ triplet models, the space $\text{Hom}_C(L, 1)$ is $(3p-1)$-dimensional. A basis of this space has been obtained in [40, Sect. 3.1] and [41, Sect. 5.1] (compare also [43, Sect. 5.2]). There, these conformal blocks were determined by interpreting them as bilinear forms on $\mathcal{H}^{\text{bulk}} \times \mathcal{Y}$ that are compatible with $\mathcal{Y}$-actions and solving the constraints, or chiral Ward identities, which implement this compatibility requirement.

(iii) For $C = H$-$\text{mod}$ the category of finite-dimensional left modules over a finite-dimensional factorizable ribbon Hopf algebra $H$, $\text{Hom}_C(L, 1) \cong \text{Hom}_H(k, H)$ is the center of $H$ and $\text{Hom}_C(1, L) \cong \text{Hom}_H(H, k)$ is the space of class functions, see e.g. [12].

**3.2. Boundary states as (co)characters**

Recall that a boundary state captures information about one-point functions of bulk fields on a disk with specified boundary condition $m$. As explained in Section 1.1, in rational CFT it can be expressed as a finite sum $(1.3)$ over simple objects $x_i$. In the non-semisimple situation the restriction to simple objects is, however, not natural. Category theory gives us a clear instruction what to do: consider all objects (and also account for all morphisms). Hence we let $x$ be an arbitrary object of $C$ and use an evaluation morphism to bend down the $x$-line in the way indicated in picture $(1.5)$ – which we repeat here for convenience (with appropriate change of labels):
Further, conceptually the summing over simple objects that is performed in the semisimple case is a way of implementing the functoriality of conformal blocks. Its natural non-semisimple generalization is to take the coend of the relevant functor over all objects \( x \). Now we know from (2.19) that the coend \( \int^{x \in C} x^\vee \otimes x \) is nothing but \( Z(1) \), and so we indeed arrive at a vector in the space (3.2) of conformal blocks for incoming boundary states.

Likewise, by using the coevaluation instead of the evaluation we can bend the \( x \)-line upwards instead of downwards, according to

\[
\begin{array}{c}
\text{} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{6}{6} \text{ For defining the dinatural family of the end one could alternatively use the Frobenius form instead of the Hopf pairing. By the universality of the end the two families must be isomorphic, and indeed they differ just by composition with } S_L. \]
and with the formula (2.56) for the relation between characters and cocharacters, it follows that, similarly as in (3.8), we have

\[ B_{\text{b.s.}}^{\text{out}} \ni m_\ast^{\text{out}} = j_x Z^{(1)} L x. \]  

(3.10)

We can again express this result as a

**Postulate 3.3. Outgoing boundary states are \( L \)-cocharacters.**

That, as seen in Lemma 2.13, the characters and cocharacters neatly provide us with ring homomorphisms from the fusion ring \( K_0(C) \) to the center of \( C \), convincingly supports these conclusions.

**Remark 3.4.** If \( C \) is non-semisimple, then the characters and cocharacters, and thus the outgoing and incoming boundary states, only span proper subspaces of the respective spaces \( B_{\text{b.s.}}^{\text{out}} \) and \( B_{\text{b.s.}}^{\text{in}} \) of conformal blocks. For the torus partition function the same pattern is known [34, Rem. 4(ii)] to be present if \( C \) is the representation category of a non-semisimple Hopf algebra, and is expected to arise analogously if \( C \) is any non-semisimple finite CFT.

**Remark 3.5.** For a non-semisimple algebra, characters do not fully specify a representation, since they split under extensions (see formula (2.46)). Hence in the non-semisimple case, boundary conditions cannot be fully distinguished by their boundary states, and thus cannot be classified completely with the help of boundary states. (Likewise, the torus partition function does not fully specify the bulk state space.)

**Example 3.6.** (i) If \( C \) is semisimple, then the characters \( \chi_m^L \) span all of \( \text{Hom}_C(L, 1) \). Also note that the action (2.49) of \( L \) involves a double braiding. For semisimple \( C \), this gives rise to the appearance of entries of the (Hopf link) S-matrix \( S^\circ \) and thereby reproduces the familiar formula (1.2) for the Cardy case boundary states of rational CFTs.

(ii) For the logarithmic \( (p, 1) \) triplet models, there are \( 2p \) simple objects and thus \( 2p \) Cardy boundary states. This agrees with the findings in [40, Sect. 3.3] and [41, Sect. 5.2]. Note, however, that the approach of [40,41] for determining boundary states is quite different from ours. Namely, it consists of making an ansatz for the boundary states as linear combinations of Ishibashi states which form a basis of the relevant space of conformal blocks (similar to the states \(| i \rangle \) that are given by (1.4) for the rational case) and then imposing the so-called Cardy condition for the resulting annulus amplitudes. The solution to this constraint, which is found with the help of explicit expressions for the characters of the triplet vertex operator algebra and their modular transformations, is not unique. In [43, Sect. 5.3] additional boundary states were suggested, yielding a total of \( 3p - 1 \) boundary states; including these leads to annulus coefficients which are integral but, unlike those for the \( 2p \) boundary states obtained in [41] (see Example 4.3 below) can be negative.

(iii) For \( C = \text{H-mod} \) the category of finite-dimensional left modules over a finite-dimensional factorizable ribbon Hopf algebra \( H \), the subspace of \( \text{Hom}_C(L, 1) \), i.e. of the center of \( H \), that is spanned by the characters \( \chi_m^L \) is the Reynolds ideal of \( H \), i.e. the intersection of the socle of \( H \) with the center, while the subspace of \( \text{Hom}_C(1, L) \), i.e. of the class functions, is the character algebra of \( H \); see e.g. [12,54]. Further, the \( L \)-characters \( \chi_m^L \) are related to the \( H \)-characters by the Drinfeld map [34, Lemma 6], while with the help of the explicit form of the dinatural morphisms \( t_m^{Z(1)} \) (see [33, Prop. A.6]) one shows that the \( L \)-cocharacters
just coincide with the $H$-characters as morphisms in $H$-mod. Also, the fact that the Hopf algebra $H$ is a symmetric algebra allows one to understand aspects of the complement of the character algebra in the space of class functions in terms of linear functions on the endomorphism rings of projective $H$-modules [3].

4. Annulus amplitudes

In this final section we compute the annulus amplitudes that result from our proposal for the boundary states for the Cardy case of a finite CFT. Concretely, we obtain the correlator for an annulus with boundary conditions $m$ and $n$ and without field insertions by sewing the one-point correlators for bulk fields on two disks having boundary conditions $m$ and $n$, respectively, i.e. of the boundary states for $m$ and $n$. As in Section 3 we start by describing the relevant spaces of conformal blocks.

4.1. Blocks for annulus amplitudes

As pointed out above, for non-semisimple conformal field theories, conformal blocks for world sheets with boundary have so far not been constructed systematically. But for partition functions, i.e. correlators without field insertions, it is quite irrelevant whether the bulk object is of factorized form, as in the semisimple case, or not. We can therefore transfer the insight [22,27] that the conformal blocks for a correlator are those for the complex double of the relevant conformal surface from the semisimple to the general case. Now the complex double of an annulus is a torus. Moreover, the space of zero-point conformal blocks on a torus is known [31,56] to be isomorphic to the morphism spaces $\text{Hom}_C(1, L)$ and $\text{Hom}_C(L, 1)$, i.e. to the spaces (3.1) and (3.2) for outgoing and incoming boundary states which, in turn, are (see (1.1)) isomorphic as algebras to the natural endo-transformations of the identity functor:

$$\text{Bl}_{\text{ann}} \cong \text{B}^\text{out}_{b.s.} \cong \text{B}^\text{in}_{b.s.} \cong \text{End}(\text{Id}_C).$$

(4.1)

To be precise, when specifying an annulus partition function we must also tell in which ‘channel’ it is considered – closed-string channel or open-string channel – or, in the more precise terminology of the Lego–Teichmüller game [4,31], with which auxiliary marking the annulus is endowed. In accordance with the observation about the connection with vertex algebra characters in Remark 2.18, the open-string channel should correspond to the space $\text{Hom}_C(1, L)$. It is this space that we mean in the sequel when referring to annulus blocks, i.e. we set

$$\text{Bl}_{\text{ann}} = \text{Hom}_C(1, L) = \text{B}^\text{out}_{b.s.}.$$  

(4.2)

For the discussion of sewing, yet another description of this space will be convenient, namely as a morphism space in the Drinfeld center:

$$\text{Bl}_{\text{ann}} \cong \text{End}_{Z(C)}(F).$$  

(4.3)

To arrive at this description we make use of the fact (see Appendix A.3) that for a modular tensor category $C$ the Drinfeld center $Z(C)$ is equivalent to the category of $L$-modules in $C$. Recall from Section 2.2 that $L = Z(1) = U(F)$ has a structure of a Frobenius algebra in $C$, and that the induction functor $I$ from $C$ to $Z(C)$ and forgetful functor $U : Z(C) \to C$ form a bi-adjoint pair. The adjunction morphisms are given by the elementary representation-theoretic formulas (A.10) and (A.11), respectively, specialized to the case at hand, in which $I(1) = F$. For completeness, let us write down the linear isomorphisms
\[
\hat{\phi} : \text{Hom}_C(1, L) = \text{Hom}_C(1, U(F)) \leftrightarrow \text{Hom}_{\mathcal{Z}(C)}(I(1), F) = \text{Hom}_{\mathcal{Z}(C)}(F, F) : \hat{\psi}
\]

(4.4)

and

\[
\varphi : \text{Hom}_C(L, 1) = \text{Hom}_C(U(F), 1) \leftrightarrow \text{Hom}_{\mathcal{Z}(C)}(F, I(1)) = \text{Hom}_{\mathcal{Z}(C)}(F, F) : \psi
\]

(4.5)

which express that \( I \) is, respectively, a left and right adjoint of \( U \), explicitly. They are given by

\[
\hat{\phi}(\hat{\alpha}) = \mu \circ (id_F \otimes \hat{\alpha}) \quad \text{and} \quad \hat{\psi}(g) = g \circ \eta,
\]

(4.6)

and by

\[
\varphi(\alpha) = (id_F \otimes (\alpha \circ \mu)) \circ ((\Delta_F \circ \eta) \otimes id_F) = (id_F \otimes \alpha) \circ \Delta_F
\]

(4.7)

and

\[
\psi(g) := \varepsilon_F \circ g,
\]

(4.8)

respectively. (The second expression in (4.7) follows from the first by using the Frobenius and unit axioms.)

Note that in the formulas above the endomorphisms of \( F \) are described as morphisms in \( \mathcal{C} \). That they are indeed even morphisms in \( \mathcal{Z}(\mathcal{C}) \) may be verified by direct calculation, but can be seen more conceptually by noting that they can also be expressed in terms of the central monad on \( \mathcal{C} \) (compare the formulas (A.18) and (A.19)). Further, the linear maps (4.4) are in fact algebra isomorphisms, with the multiplication on \( \text{Hom}_{\mathcal{Z}(\mathcal{C})}(F, F) \) being composition and the one on \( \text{Hom}_C(1, L) \) given by the convolution product (2.57) [62, Thm. 3.9].

**Remark 4.1.** The compatibility between bulk and boundary theories is of widespread interest in various settings of quantum field theory. The isomorphisms (4.1) and (4.3) can be viewed as one aspect of such a compatibility: \( \text{Hom}_C(L, 1) \) and \( \text{Hom}_C(1, L) \) have a natural interpretation as spaces for boundary states, while \( \text{End}_{\mathcal{Z}(\mathcal{C})}(F) \) appears naturally when studying bulk fields.

### 4.2. Sewing of boundary blocks

For understanding the relation between boundary states and annulus amplitudes we need to know how factorization relates conformal blocks for disks to annulus blocks, i.e. to find four sewing maps

\[
\text{Bl}^{\text{in/out}}_{\text{b.s.}} \otimes_k \text{Bl}^{\text{in/out}}_{\text{b.s.}} \longrightarrow \text{Bl}_{\text{ann}}
\]

(4.9)

from the tensor product of the conformal blocks for incoming and/or outgoing boundary states to the space of annulus blocks. Clearly, these maps are not bijections; this reflects the fact that no summation over the bulk insertions on the boundary disks is implied.

The following two ingredients are crucial for being able to understand these sewing maps: First, with the help of the results of Section 4.1 we can rewrite the maps (4.9) as the composition of known isomorphisms with a sewing map \( s \) that only involves morphism spaces in \( \mathcal{Z}(\mathcal{C}) \) or, in other words, conformal blocks for the bulk theory. Specifically,
Thus question direct channel)

\[ \text{Bl}_{\text{b.s.}}^{\text{out}} \otimes_k \text{Bl}_{\text{b.s.}}^{\text{in}} = \text{Hom}_C(1, L) \otimes_k \text{Hom}_C(L, 1) \xrightarrow{\cong \phi \otimes \varphi} \text{End}_\mathcal{C}(F) \otimes_k \text{End}_\mathcal{C}(F) \]

\[ \xrightarrow{\sim} \text{End}_\mathcal{C}(F) \xrightarrow{\cong \psi} \text{Bl}_{\text{ann}} \]

(4.10)

for the case of a sewing of incoming and outgoing boundary blocks, and analogously for other combinations.

Second, the sewing of morphism spaces of \( \mathcal{Z}(C) \simeq \mathcal{C} \otimes \mathcal{C} \) is fully understood \([31, 56]\); it is realized by the structural maps of appropriate coends in \( \mathcal{Z}(C) \). Moreover, for the morphism spaces in question these structural maps just amount to composition of linear maps (see e.g. \([32, \text{Cor. 2.3}]\)). Thus the sewing map \( s \) in (4.10) takes the simple form

\[ s : \text{End}_\mathcal{C}(F) \otimes_k \text{End}_\mathcal{C}(F) \ni f_1 \otimes f_2 \mapsto f_2 \circ f_1 \in \text{End}_\mathcal{C}(F). \]

(4.11)

Let us write down the resulting sewing maps (4.9) explicitly. We start with the sewing of outgoing to outgoing boundary blocks, which will confirm the identification (4.2) of the (open-string channel) annulus amplitude. We have

\[ \hat{\alpha} \otimes \hat{\beta} \xrightarrow{\hat{\phi} \otimes \hat{\psi}} (\mu \circ (id_F \otimes \hat{\alpha})) \otimes (\mu \circ (id_F \otimes \hat{\beta})) \]

\[ \xrightarrow{s} \mu \circ (\mu \otimes \hat{\beta}) \circ (id_F \otimes \hat{\alpha}) \]

(4.12)

\[ = \mu \circ (id_F \otimes [\mu \circ (\hat{\alpha} \otimes \hat{\beta})]) \equiv \hat{\phi}(\hat{\alpha} \ast \hat{\beta}) \xrightarrow{\hat{\psi}} \hat{\alpha} \ast \hat{\beta} \]

for any \( \hat{\alpha}, \hat{\beta} \in \text{Hom}_C(1, L) \), where the equality holds by associativity. In short, \( \text{Bl}_{\text{b.s.}}^{\text{out}} - \text{Bl}_{\text{b.s.}}^{\text{out}} \)-sewing amounts to convolution in \( \text{Hom}_C(1, L) \).

The sewing of outgoing to incoming boundary blocks (or vice versa) amounts to a convolution as well provided, however, that we also include a modular S-transformation, as might be anticipated from the observation (2.81). Indeed, directly we have

\[ \hat{\alpha} \otimes \beta \xrightarrow{\hat{\phi} \otimes \varphi} (\mu \circ (id_F \otimes \hat{\alpha})) \otimes ((id_F \otimes \beta) \circ \Delta_F) \]

\[ \xrightarrow{s} (id_F \otimes \beta) \circ \Delta_F \circ \mu \circ (id_F \otimes \hat{\alpha}) = [id_F \otimes (\beta \circ \mu)] \circ (\Delta_F \otimes \hat{\alpha}) \]

(4.13)

for any \( \hat{\alpha} \in \text{Hom}_C(1, L) \) and \( \beta \in \text{Hom}_C(L, 1) \), where the equality uses the Frobenius axiom. By direct rewriting, using the first of the expressions in (2.37) for the Frobenius coproduct \( \Delta_F \) and formula (2.32) for \( \kappa^- \), we then have

\[ s \circ (\hat{\phi} \otimes \varphi)(\hat{\alpha} \otimes (\beta \circ S_L)) = (id_L \otimes (\beta \circ S_L \circ \mu)) \circ (\Delta_F \otimes \hat{\alpha}) \]

\[ = (id_L \otimes (\beta \circ S_L \circ \mu)) \circ (id_L \otimes \mu \otimes \hat{\alpha}) \circ (\kappa^- \otimes id_L). \]

(4.14)

Inserting the explicit form (2.66) of the endomorphism \( S_L \) and using associativity of \( \mu \) together with the invariance of the Frobenius form this, in turn, implies that

\[ s \circ (\hat{\phi} \otimes \varphi)(\hat{\alpha} \otimes (\beta \circ S_L)) = (\mu \otimes \beta) \circ (\mu \otimes \omega^-) \circ (id_L \otimes \hat{\alpha}) \]

\[ = \mu \circ (id_L \otimes [\mu \circ (\hat{\alpha} \otimes \hat{\beta})]) \equiv \hat{\phi}(\hat{\alpha} \ast \hat{\beta}), \]

(4.15)

where \( \hat{\beta} \in \text{Hom}_C(1, L) \) is defined analogously as in the relation (2.56) between characters and cocharacters, i.e.
\[ \hat{\beta} := (id_L \otimes \beta) \circ \omega^- . \]

Thus also the sewing of an outgoing and an incoming boundary block can be described as a convolution in the space Hom_C(1, L), provided that before sewing the incoming boundary block is precomposed with an S-transformation. For performing the convolution, the incoming boundary block is transformed to an outgoing one via the side inverse of the Hopf pairing.

Finally, the sewing of two incoming boundary blocks amounts to flipped $\Delta_F$-convolution: we have

\[
\begin{align*}
\alpha \otimes \beta \xmapsto{\psi \otimes \psi} & (id_F \otimes \alpha) \circ \Delta_F \otimes (id_F \otimes \beta) \circ \Delta_F \\
& \xmapsto{s} \ (id_F \otimes \beta) \circ (\Delta_F \otimes \alpha) \circ \Delta_F = \left( id_F \otimes \left( (\beta \otimes \alpha) \circ \Delta_F \right) \right) \circ \Delta_F \equiv \varphi (\beta \ast_F \alpha) \\
& \xmapsto{\psi} \beta \ast_F \alpha
\end{align*}
\]

for any $\alpha, \beta \in \text{Hom}_C(L, 1)$, where the equality holds by coassociativity. In view of the result (4.15) above, it is natural to consider alternatively the sewing after precomposing both of the incoming boundary blocks with $S_L$,

\[ (\alpha \circ S_L) \otimes (\beta \circ S_L) \xmapsto{\beta \ast F (\alpha \circ S_L)} . \]

A straightforward calculation shows that the sewn expression in (4.18) can be rewritten as

\[ (\beta \circ S_L) \ast_F (\alpha \circ S_L) = \kappa \circ \left( id_L \otimes (\hat{\alpha} \ast \hat{\beta}) \right) \equiv \omega \circ \left( id_L \otimes \left( S_L \circ (\hat{\alpha} \ast \hat{\beta}) \right) \right) \]

with $\hat{\beta}$ as in (4.16) and analogously for $\hat{\alpha}$. Thus also in this case we get an expression that is a convolution product, up to suitable insertions of $S_L$-automorphisms and using $\omega$ and its side inverse to relate incoming and outgoing boundary blocks.

To summarize, the sewing of two boundary blocks amounts to a suitable convolution.

**Remark 4.2.** The occurrence of an additional S-transformation when switching between outgoing and incoming boundary blocks should not come as a surprise once one remembers that according to formula (2.81), characters and cocharacters are related to each other by a duality morphism combined with an S-transformation. More significantly, this behavior can be appreciated through a comparison with the description of sewing that is natural in the context of the TFT construction of the correlators of rational CFTs. In that context, sewing combines an outgoing and an incoming block, and it automatically involves a modular S-transformation; see e.g. Sections 5.2 and 5.3 of [23] or Section 2.2 of [25] for details. For the same type of sewing, a shadow of that S-transformation appears in the present approach, namely as the morphism $S_L$ in (4.15).

### 4.3. Annulus amplitudes via sewing of boundary states

Knowing the sewing maps on the respective spaces of boundary blocks, we can proceed to the sewing of the boundary states proposed above as specific elements of those spaces.

Consider first the sewing of two outgoing boundary states corresponding to boundary conditions $m$ and $n$. According to (3.10) these are given by the cocharacters $\hat{x}^L_m$ and $\hat{x}^L_n$. The sewing relation (4.12), combined with (2.59) tells us that the (open-string channel) annulus amplitude with boundary conditions $m, n \in \mathcal{C}$ is

\[ B_{\text{ann}}^{\cdot m} \ni A_{mn} = \hat{x}^L_m \ast \hat{x}^L_n = \hat{x}^L_{m \otimes n} . \]
Now the mapping $[m] \mapsto \hat{\chi}^L_m$ is an injective homomorphism of monoids from the Grothendieck ring $K_0(C)$ to $\text{Hom}_C(\mathbf{1}, L)$ \cite[Cor. 4.2]{62}, so that the annulus amplitude can be written as

$$A_{mn} = \hat{\chi}^L_{m \otimes n} = \sum_{l \in I} N_{mn}^{-1} \hat{\chi}^L_l$$

(4.21)

with $N_{mn}^{-1} \in \mathbb{Z}_{\geq 0}$ the structure constants of $K_0(C)$. Thus $A_{mn}$ can be expanded as a linear combination of cocharacters, with coefficients given by the structure constants of the Grothendieck ring. In particular, $A_{mn}$ lies completely in the subspace of the space $\text{Hom}_C(\mathbf{1}, L)$ of conformal blocks that is spanned by the cocharacters – recall that this is a proper subspace of non-zero codimension unless $C$ is semisimple. In short, the annulus coefficients are fusion coefficients.

Since according to Remark 2.18 the cocharacters correspond to the genus-1 one-point functions of vertex algebra representations, and since fusion coefficients are by definition non-negative integers, we see in particular that the annulus amplitude $A_{mn}$ naturally admits an interpretation as a partition function that counts (open-string) states, precisely like in the semisimple case. This constitutes a rather non-trivial check of the interpretation of boundary states as (co)characters.

**Example 4.3.** (i) For semisimple $C$, the formula (4.21) reproduces the familiar result \cite{10} for the Cardy case annulus amplitude in the open-string channel. Note that the semisimple case is too degenerate – in that case the cocharacters span all of $\text{Hom}_C(\mathbf{1}, L)$ – to illustrate the non-trivial feature of (4.21) that the ‘pseudo-cocharacters’ do not contribute to the amplitude.

(ii) Annulus amplitudes for the logarithmic $(p, 1)$ triplet models have been computed in \cite[Sect. 3.2]{40} and \cite[Sect. 5.2]{41}. Specifically, once cocharacters are identified with vertex algebra one-point functions, (4.21) reproduces formula (5.10) of \cite{41}, which in the context considered there arises as an assumption that precedes the determination of the boundary states.

The other types of sewings considered in the previous subsection amount to the following statements for annulus amplitudes. For the sewing of an incoming and an outgoing boundary state, the formula (4.15) yields

$$\hat{\chi}^L_m \otimes \chi^L_n \xrightarrow{\text{sew}} \hat{\chi}^L_m \ast \Omega^{-1}(\chi^L_n \circ S_L^{-1})$$

(4.22)

while according to (4.19), sewing two incoming boundary states is given by

$$\chi^L_m \otimes \chi^L_n \xrightarrow{\text{sew}} S_L \circ \left(\Omega^{-1}(\chi^L_m \circ S_L^{-1}) \ast \Omega^{-1}(\chi^L_n \circ S_L^{-1})\right)$$

(4.23)

Here $\Omega : \text{Hom}_C(\mathbf{1}, L) \rightarrow \text{Hom}_C(L, \mathbf{1})$ is the composition with the Hopf pairing, as defined in (2.55), which exchanges characters and cocharacters.

The presence of the modular $S$-transformation $S_L \in \text{End}_C(L)$ in (4.22) has a natural interpretation in terms of the TFT construction of the correlators, see Remark 4.2 above. Another way to understand it is obtained by realizing that the annulus amplitude described by (4.22) is in the closed-string channel – the formula thus tells us that the annulus amplitudes in the open- and closed-string channels are related by an $S$-transformation.

**4.4. Boundary fields**

Finally we note that the interpretation of the annulus amplitudes as partition functions implies constraints on the “objects of boundary fields”. More specifically, the boundary fields that
change a boundary condition $m$ to the boundary condition $n$ must be compatible with the annulus amplitude $A_{m,n}$ as given by (4.21). There is a priori no guarantee that those constraints can be satisfied at all. But inspection shows that this is indeed possible, namely by taking these fields to be given by the corresponding internal Hom for $C$ as a (right) $C$-module category, i.e. by

$$B_{mn} := \text{Hom}(m, n) \cong m^\vee \otimes n \in C.$$  

(4.24)

This natural prescription generalizes the finding in the semisimple case; that it is compatible with the annulus amplitudes gives additional support to our ansatz for the boundary states.

The collection of objects $B_{mn}$ is not only naturally associated with the structure of $C$ as a module category over itself, but also comes with further structure: It can be endowed with natural algebroid and coalgebroid structures that fit together as a Frobenius algebroid in $C$. The non-zero components of the product of the algebroid and of the coproduct of the coalgebroid are

$$\mu_{l,m,n} := \text{id}_l^\vee \otimes \bar{\epsilon}_m \otimes \text{id}_n : \quad B_{lm} \otimes B_{mn} \to B_{ln} \quad \text{and}$$

$$\Delta_{l,m,n} := \text{id}_l^\vee \otimes \text{coev}_m \otimes \text{id}_n : \quad B_{ln} \to B_{lm} \otimes B_{mn}.$$  

(4.25)

while the components of the unit and counit are $\eta_m = \text{coev}_m \circ \text{id}_m : 1 \to B_{mm}$ and $\varepsilon_m = \text{ev}_m : B_{mm} \to 1$, respectively. The Frobenius compatibility condition satisfied by the morphisms (4.25) reads

$$(\text{id}_m^\vee \otimes k \otimes \mu_{k,l,n}) \circ (\Delta_{m,k,l} \otimes \text{id}_l^\vee \otimes n) = \Delta_{m,k,n} \circ \mu_{m,k,n}$$

(4.26)

$$= (\mu_{m,k,l} \otimes \text{id}_l^\vee \otimes n) \circ (\text{id}_m^\vee \otimes k \otimes \Delta_{k,l,n}).$$

Associativity of the algebroid is in fact a consequence of the realization by internal Homs, even beyond the setting of finite tensor categories [42, Sect. 3.3].

In particular, the objects $B_{mm}$ which describe boundary fields that do not change the boundary condition admit a structure of symmetric Frobenius algebra, with product $\mu_{m,m,m}$, unit $\eta_m$, coproduct $\Delta_{m,m,m}$ and counit $\varepsilon_m$, which is in addition special iff $\dim C(m) \neq 0$.

The algebroid structure of the objects $B_{mn}$ should realize the operator product of boundary fields, in much the same way as the multiplication $\mu$ on the bulk object $F$ realizes the operator product expansion of bulk fields (and makes the associativity property of that operator product precise). We expect that the description of boundary fields as internal Homs will generalize to the non-Cardy case, in which the boundary conditions should still be the objects of some suitable module category $\mathcal{M}$ over $\mathcal{C}$.

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Appendix A

A.1. A canonical functor from the enveloping category to the Drinfeld center

Here we provide further details about the canonical functor $\Xi_C: \overline{C} \boxtimes C \to Z(C)$ that is introduced in (2.13) for braided finite tensor categories $C$. Specifying a monoidal structure on $\Xi_C$ amounts to giving coherent isomorphisms

$$\varphi_{u,v,x,y} : u \otimes v \otimes x \otimes y = \Xi_C(u \boxtimes v) \otimes Z(C) \Xi_C(x \boxtimes y) \longrightarrow \Xi_C((u \boxtimes v) \otimes \Xi_C(x \boxtimes y)) = u \otimes x \otimes v \otimes y \quad (A.1)$$

that obey the relevant hexagon and triangle identities. It is easily checked that any odd power of the braiding (in $C$) of the second and third tensor factor, i.e.

$$\varphi_{u,v,x,y}^{(m)} := \text{id}_u \otimes (\beta^{2m+1})_{v,x} \otimes \text{id}_y \quad (A.2)$$

for any $m \in \mathbb{Z}$ satisfies these requirements.

In order that the functor is even braided monoidal, the morphisms $\varphi$ must in addition make the diagrams

$$u \otimes v \otimes x \otimes y \xrightarrow{\varphi_{u,v,x,y}} u \otimes x \otimes v \otimes y \quad (A.3)$$

commute. Here $\beta^{\overline{C} \boxtimes C}$ is the braiding on the enveloping category $\overline{C} \boxtimes C$ that is induced from the one of $C$, i.e.

$$(\beta^{\overline{C} \boxtimes C})_{u \otimes x, v \otimes y} = \beta_{u,x}^{-1} \otimes \beta_{v,y}, \quad (A.4)$$

while $\beta^Z$ is the braiding in $Z(C)$ which just coincides with the half-braiding, i.e. is given by (2.14) with $c = x \otimes y$ ($\beta^Z_{u \otimes v, x \otimes y}$ must not be confused with the braiding of $u \otimes v$ with $x \otimes y$ in $C$). Pictorially,

$$\begin{align*}
(\beta^{\overline{C} \boxtimes C})_{u \otimes x, v \otimes y} = & \quad \begin{array}{c}
\includegraphics[width=0.3\textwidth]{braiding_diagram1} \\
\includegraphics[width=0.3\textwidth]{braiding_diagram2}
\end{array} \\
\beta^Z_{u \otimes v, x \otimes y} = & \quad \begin{array}{c}
\includegraphics[width=0.3\textwidth]{braiding_diagram3} \\
\includegraphics[width=0.3\textwidth]{braiding_diagram4}
\end{array}
\end{align*} \quad (A.5)$$

One finds that this requirement is solved uniquely by setting $m = 0$ in (A.2), i.e. we have

**Lemma A.1.** The braided monoidal structure (A.1) on the functor $\Xi_C$ is given by

$$\varphi_{u,v,x,y} = \text{id}_u \otimes \beta_{v,x} \otimes \text{id}_y \quad (A.6)$$

for $u, v, x, y \in C$. 

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*Note: The diagrams and equations are placeholders for actual content.*
Proof. That the diagram (A.3) commutes if \( m = 0 \) is seen diagrammatically as follows:

\[
\varphi_{u,v,x,y} = \varphi_{u,v,x,y}
\]

This picture also makes it clear that the requirement cannot be fulfilled with any other odd power of the braiding. \( \square \)

A.2. Adjoints of the forgetful functor

Given an (associative, unital) algebra in a monoidal category \( C \), we denote by \( \mathcal{M} \equiv A\text{-mod} \) the category of left \( A \)-modules in \( C \). Further, write

\[
U: \quad \mathcal{M} \rightarrow C
\]

\[
m \mapsto \hat{m}
\]

(A.8)

for the forgetful functor and

\[
I: \quad C \rightarrow \mathcal{M}
\]

\[
x \mapsto A \otimes x
\]

(A.9)

for the induction functor. In case a functor \( F \) has a right or left adjoint, we denote it by \( F^{r.a.} \) and \( F^{l.a.} \), respectively.

As is well known, and easy to check, the mappings \( \hat{\psi} : \text{Hom}_{\mathcal{M}}(I(x), m) \rightarrow \text{Hom}_{C}(x, U(m)) \) and \( \hat{\phi} : \text{Hom}_{C}(x, U(m)) \rightarrow \text{Hom}_{\mathcal{M}}(I(x), m) \) defined by

\[
\hat{\psi}(f) := f \circ (\eta \otimes id_{x}) \quad \text{and} \quad \hat{\phi}(g) := \rho_{m} \circ (id_{A} \otimes g),
\]

(A.10)

for \( x \in C \) and \( m \equiv (\hat{m}, \rho_{m}) \in \mathcal{M} \), are each other’s inverse, so that we have

Lemma A.2. \( I \) is left adjoint to \( U \), so that we can write \( U^{l.a.} = I \).

Remark A.3. This statement is analogous to the classical result that for an embedding \( R \subset S \) of rings the induction functor \( S \otimes_R - \) is left adjoint to the forgetful functor \( U: S\text{-mod} \rightarrow R\text{-mod.} \) In this setting, a right adjoint of \( U \) is given by the coinduction functor \( \text{Hom}_{R}(S, -) \).

Lemma A.4. If \( A = (A, \mu, \eta, \Delta, \varepsilon) \) is a Frobenius algebra, then \( I \) is right adjoint to \( U \), so that we can write \( U^{r.a.} = I \).

Proof. Consider the mappings \( \psi : \text{Hom}_{\mathcal{M}}(m, I(x)) \rightarrow \text{Hom}_{C}(U(m), x) \) and \( \varphi : \text{Hom}_{C}(U(m), x) \rightarrow \text{Hom}_{\mathcal{M}}(m, I(x)) \) that, for \( x \in C \) and \( m \equiv (\hat{m}, \rho_{m}) \in \mathcal{M} \), are defined by
\(\psi(f) := (\varepsilon \otimes \text{id}_c) \circ f\) and \(\varphi(g) := (\text{id}_A \otimes (g \circ \rho_m)) \circ ((\Delta \circ \eta) \otimes \text{id}_m).\) (A.11)

(That \(\varphi(g)\) is a morphism in \(\mathcal{M}\) follows by a twofold use of the Frobenius relation for the product and coproduct together with unitality \(A\).) These mappings are each other’s inverse: we have \(\psi \circ \varphi = \text{id}_{\text{Hom}_{\mathcal{C}}(U(m), x)}\) by first using the defining property of the counit \(\varepsilon\) and then the compatibility of \(\rho_m\) with the unit, as well as \(\varphi \circ \psi = \text{id}_{\text{Hom}_{\mathcal{A}}(m, I(x))}\) by first using the module morphism property, then the Frobenius relation and then the defining properties of unit and counit. \(\square\)

Thus for a Frobenius algebra the functors \(I\) and \(U\) are two-sided adjoints to each other; in other words, induction and coinduction coincide. They thus form what is called a strongly adjoint pair of functors \([58]\), or a pair of Frobenius functors, or Frobenius pair \([11]\). Such functors are particularly well-behaved, e.g. they are exact, preserve limits and colimits, and preserve injective and projective objects (see e.g. \([8]\)).

A.3. The central monad

Consider the mapping

\[
c \mapsto Z(c) := \int x \in \mathcal{C} x^\vee \otimes c \otimes x, \tag{A.12}\]

sending an object \(c\) of \(\mathcal{C}\) to a coend as in formula (2.1). This furnishes an endofunctor of \(\mathcal{C}\); moreover, this endofunctor carries a natural algebra structure and is thus a monad on \(\mathcal{C}\).

**Definition A.5 (Monad).** A monad \(\mathbb{T} = (T, m, e)\) on a category \(\mathcal{C}\) is an algebra in the monoidal category of endofunctors of \(\mathcal{C}\), i.e. an endofunctor \(T : \mathcal{C} \to \mathcal{C}\) together with a natural transformation \(m = (m_c)_{c \in \mathcal{C}} : T \circ T \Rightarrow T\) and a natural transformation \(e = (e_c)_{c \in \mathcal{C}} : \text{id}_\mathcal{C} \Rightarrow T\) that obey the associativity and unit properties

\[
m_c \circ T(m_c) = m_c \circ m_{T(c)} \quad \text{and} \quad m_c \circ T(e_c) = \text{id}_{T(c)} = m_c \circ e_{T(c)} \tag{A.13}\]

for all \(c \in \mathcal{C}\).

Note that naturality of \(m\) and \(e\) mean that

\[
m_d \circ T^2(f) = T(f) \circ m_c \quad \text{and} \quad e_d \circ f = T(f) \circ e_c \tag{A.14}\]

for any morphism \(f : c \to d\) in \(\mathcal{C}\). A virtue of the monad concept is that it does not require \(\mathcal{C}\) itself to be monoidal. In case \(\mathcal{C}\) is monoidal, then for any algebra \(A \in \mathcal{C}\) the endofunctors \(- \otimes A\) and \(A \otimes -\) admit natural structures of monads on \(\mathcal{C}\).

Modules over monads are defined analogously as modules over algebras: A (left) module over a monad \(\mathbb{T} = (T, m, e)\) on \(\mathcal{C}\) (also called a \(\mathbb{T}\)-algebra) consists of an object \(m\) of \(\mathcal{C}\) and a morphism \(\rho_m : T(m) \to m\) such that \(\rho_m \circ m_m = \rho_m \circ T(\rho_M) : T^2(m) \to m\) and \(\rho_m \circ e_m = \text{id}_m\). A morphism \(f : m \to n\) of \(\mathbb{T}\)-modules satisfies by definition

\[
f \circ \rho_m = \rho_n \circ T(f) \tag{A.15}\]

with \(\rho_m : T(m) \to m\) and \(\rho_n : T(n) \to n\) the respective representation morphisms.

For any finite tensor category \(\mathcal{C}\) the prescription (A.12) provides a distinguished monad on \(\mathcal{C}\), the central monad \(Z\). The product and unit of \(Z\) are directly defined in terms of the dinatural families \(\int Z(c)\) of the coends \(Z(c)\):
\[ \epsilon_c := \iota^{Z(c)}_1 \quad \text{and} \quad m_c \circ \iota^{Z(c)}_y \circ (\text{id}_{Z'} \otimes \iota^{Z(c)}_x \otimes \text{id}_y) := \iota^{Z(c)}_{x \otimes y} \quad (A.16) \]

(in the formula for m, use of the Fubini theorem for iterated coends is implicit). It can be shown [7, Thm. 8.13] that the Drinfeld center \( Z(C) \) is equivalent to the category \( Z\text{-mod} \) of \( Z \)-modules in \( C \) as a braided monoidal category. In particular, a morphism \( f : m \to n \) of \( Z \)-modules is a morphism in \( Z(C) \).

The central monad \( Z \) is even a bimonad; it has a comonoidal structure given by morphisms \( \epsilon^Z : Z(1) \to 1 \) and \( \Delta^Z_{c \otimes c'} : Z(c \otimes c') \to Z(c) \otimes Z(c') \) for all \( c, c' \in C \) that are defined by

\[ \epsilon^Z := \iota^{Z(1)}_1 \]

\[ \Delta^Z_{c \otimes c'} := (\iota^{Z(c)}_x \otimes \iota^{Z(c')}_x) \circ (\text{id}_{Z'} \otimes \text{id}_c \otimes \text{coev}_x \otimes \text{id}_{c'} \otimes \text{id}_x), \quad (A.17) \]

respectively, for all \( x \in C \).

Also note that \( Z(1) = L \). Accordingly, structural insight about statements involving the object \( L \in C \) can favorably be obtained by formulating them in terms of the more canonical monad \( Z \).

Here are a few examples: First, the algebra structure on \( L \) is a specialization of the algebra structure on \( Z \), namely \( \mu = m_1 \) and \( \eta = e_1 \). Second, the convolution product \((2.57)\) on \( \text{Hom}_C(1, L) \), which in the monad setting reads \( \hat{\alpha} \ast \hat{\beta} = m_1 \circ Z(\hat{\beta}) \circ \hat{\alpha} \), is a special case of a product that exists on the morphism space \( \text{Hom}_C(1, T(1)) \) for any monad \( T \) on a monoidal category. Third, the adjunction isomorphisms \((4.6)\) can be expressed as

\[ \hat{\phi} : \text{Hom}_C(1, L) \ni \hat{\alpha} \longmapsto m_1 \circ Z(\hat{\alpha}) \in \text{Hom}_{Z(C)}(F, F) \quad (A.18) \]

and

\[ \hat{\psi} : \text{Hom}_{Z(C)}(F, F) \ni g \longmapsto g \circ e_1 \in \text{Hom}_C(1, L), \quad (A.19) \]

respectively, showing in particular that \( \hat{\psi}(\hat{\alpha}) \) is indeed an element of \( \text{Hom}_{Z(C)}(F, F) \) rather than only of \( \text{Hom}_C(F, F) \supset \text{Hom}_{Z(C)}(F, F) \). And fourth, that \( L \) has a natural Hopf algebra structure if \( C \) is braided can be seen, without spelling out the structural morphisms explicitly, as a consequence of

**Lemma A.6.** For \( C \) a braided finite tensor category, the endofunctors \( Z \) and \( L \otimes - \) are isomorphic as bimonads. A pair of mutually inverse isomorphisms \( \zeta : L \otimes - \Rightarrow Z \) and \( \xi : Z \Rightarrow L \otimes - \) is given by

\[ \zeta_c \circ (\iota^L_x \otimes \text{id}_c) := \iota^{Z(c)}_x \circ (\text{id}_{Z'} \otimes \beta_{x,c}) \quad \text{and} \quad \xi_c \circ \iota^{Z(c)}_x := (\iota^L_x \otimes \text{id}_c) \circ (\text{id}_{Z'} \otimes \beta_{x,c}^{-1}). \quad (A.20) \]

**Proof.** That \( \xi_c \circ \zeta_c = \text{id}_{L \otimes c} \) and \( \zeta_c \circ \xi_c = \text{id}_{Z(c)} \) is seen by direct computation. That these natural transformations furnish an isomorphism of bimonads follows from the fact that, via the braiding, the functors \( x \otimes - \otimes y \setminus \setminus \) and \( x \otimes y \setminus \setminus \) are isomorphic. Indeed, that the units and counits are intertwined is immediate, and that the products and coproducts are intertwined is verified by the calculations
\[\xi_c \circ m_c \circ \zeta_{Z(c)} \circ (t^1_y \otimes \zeta_c) \circ (id_{y^\vee} \otimes id_y \otimes t^1_x \otimes id_c)\]
\[= \xi_c \circ m_c \circ t^1_y Z_{Z(c)} \circ (id_{y^\vee} \otimes \beta_{y,Z(c)}) \circ (id_{y^\vee} \otimes id_y \otimes [t^1_x Z(c) \circ (id_{y^\vee} \otimes \beta_{x,c})])\]
\[= \xi_c \circ m_c \circ t^1_y Z_{Z(c)} \circ (id_{y^\vee} \otimes t^1_x Z(c) \otimes id_y) \circ (id_{y^\vee} \otimes id_{y^\vee} \otimes \beta_{x,c} \otimes id_y)\]
\[\circ (id_{y^\vee} \otimes \beta_{y,x^\vee \otimes x} \otimes id_c)\]
\[\xi_c \circ t^1_y Z_{Z(c)} \circ (id_{y^\vee} \otimes \beta_{y,x^\vee \otimes x} \otimes id_x)\]
\[= (t^1_y \otimes id_c) \circ (id_{y^\vee} \otimes id_{y^\vee} \otimes \beta_{x^\vee,y,c}^{-1}) \circ (id_{y^\vee} \otimes id_{x^\vee} \otimes \beta_{x,y,c})\]
\[\circ (id_{y^\vee} \otimes \beta_{y,x^\vee \otimes x} \otimes id_c)\]
\[= (t^1_y \otimes id_c) \circ (id_{y^\vee} \otimes \beta_{y,x^\vee \otimes x} \otimes id_{y,c}) \otimes id_c = (\mu_L \circ (t^1_y \otimes t^1_x)) \otimes id_c\]
and
\[\xi_c \circ \xi_{c'} \circ \Delta^Z_{Z(c)} \circ \zeta_{c \otimes c'} \circ (t^1_x \otimes id_c \otimes id_{c'})\]
\[= (t^1_y \otimes id_c \otimes t^1_x \otimes id_{c'}) \circ (id_{y^\vee} \otimes id_{y^\vee} \otimes \beta_{x^\vee,y,c}^{-1} \otimes id_{x^\vee} \otimes \beta_{x^\vee,c}^{-1})\]
\[\circ (id_{y^\vee} \otimes id_c \otimes \coev_{y} \otimes id_{c'} \otimes id_x) \circ (id_{y^\vee} \otimes \beta_{x,c} \otimes id_{c'})\]
\[= (t^1_y \otimes id_c \otimes t^1_x \otimes id_{c'}) \circ (id_{x^\vee} \otimes id_{x^\vee} \otimes \beta_{x^\vee,c} \otimes id_{x^\vee} \otimes id_{c'})\]
\[\circ (id_{y^\vee} \otimes \beta_{x,c} \otimes id_{c'})\]
\[= (id_{x^\vee} \otimes \beta_{x,c} \otimes (\Delta_L \otimes id_c) \circ (t^1_x \otimes id_c) \otimes id_{c'})\]
respectively (the right hand side of (A.22) defines the comonoidal structure of \(L \otimes -\)). \(\square\)

Actually we could choose to replace the braiding and inverse braiding in (A.20) by an odd power of them. This would not have any effect on the resulting structural morphisms for the Hopf algebra \(L\), but only change the braiding in the definition of the comonoidal structure of the bimonad \(L \otimes -\).

References


