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Corrector estimates for a thermo-diffusion model with weak thermal coupling

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Abstract

The present work deals with the derivation of corrector estimates for the two-scale homogenization of a thermo-diffusion model with weak thermal coupling posed in a heterogeneous medium endowed with periodically arranged high-contrast microstructures. The terminology “weak thermal coupling” refers here to the variable scaling in terms of the small homogenization parameter $\varepsilon$ of the heat conduction-diffusion interaction terms, while the “high-contrast” is thought particularly in terms of the heat conduction properties of the composite material. As main target, we justify the first-order terms of the multi-scale asymptotic expansions in the presence of coupled fluxes, induced by the joint contribution of Sorret and Dufour-like effects. The contrasting heat conduction combined with cross coupling lead to the main mathematical difficulty in the system. Our approach relies on the method of periodic unfolding combined with $\varepsilon$-independent estimates for the thermal and concentration fields and for their coupled fluxes.


Keywords: Homogenization, corrector estimates, periodic unfolding, gradient folding operator, perforated domain, composite media, reaction-diffusion systems, thermo-diffusion.

1 Introduction

This paper deals with the justification of the two-scale asymptotic expansions method applied to a thermo-diffusion problem arising in the context of transport of densities of hot colloids in media made of periodically-distributed microstructures. Following [KAM14], we study a system of two coupled semi-linear parabolic equations, where the diffusivity for the concentration $u_\varepsilon$ is of order $O(1)$ and for the temperature $\theta_\varepsilon$ it is of order $O(\varepsilon^2)$. Here $\varepsilon > 0$ denotes the characteristic length scale of the underlying microstructure. We rigorously justify the expansions $u_\varepsilon(x) \approx u(x) + \varepsilon U(x, x/\varepsilon)$ and $\theta_\varepsilon(x) \approx \Theta(x, x/\varepsilon)$ and prove corrector estimate of the type

$$\| T_\varepsilon^* u_\varepsilon - u \|_{L^\infty(0,T;L^2(\Omega \times Y))} + \| T_\varepsilon^* (\nabla u_\varepsilon) - (\nabla u + \nabla y U) \|_{L^2(0,T;L^2(\Omega \times Y))} + \| T_\varepsilon^* \epsilon \nabla \theta_\varepsilon - \nabla \theta \|_{L^2(0,T;L^2(\Omega \times Y))} \leq \sqrt{\varepsilon} C, \quad (1.1)$$

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where $\Omega \subset \mathbb{R}^d$ denotes the macroscopic domain and $Y \subset [0,1)^d$ is the perforated reference cell. Estimate (1.1) basically gives a quantitative indication of the speed of the (two-scale) convergence between the unknowns of our problem and their limits, which is detailed in the forthcoming sections. This work follows up previous successful attempts of deriving quantitative corrector estimates using periodic unfolding; see e.g. [Gri04, Gri05, OnV07, FMP12, Rei15, Rei16, Rei17]. The unfolding technique allows for deriving homogenization results under minimal regularity assumptions on the data and on the choice of allowed microstructures. The novelty we bring in here is the combination of three aspects: (i) the asymptotic procedure refers to a suitably perforated domain, (ii) presence of a cross coupling in gradient terms, and (iii) lack of compactness for $\theta_\varepsilon$. Our working techniques combine $\varepsilon$-independent a priori estimates for the solutions and periodic unfolding-based estimates such as the periodicity defect in [Gri04] and the folding mismatch in [Rei16]. Estimate (1.1) improves existing convergence rates for semi-linear parabolic equations with possibly non-linear boundary conditions in [FMP12] or small diffusivity in [Rei15] from $\varepsilon^{1/4}$ to $\varepsilon^{1/2}$. This improvement is obtained by studying all equations in the two-scale space $\Omega \times Y$ and by suitably rearranging and controlling occurring error terms $\Delta^\varepsilon_{\text{error}}$.

It is worth noting that the availability of corrector estimates for the thermo-diffusion system allows in principle the construction of rigorously convergent multi-scale numerical methods to capture thermo-diffusion effects in porous media. For instance based on multi-scale finite elements methods, as in e.g. [HoW97, LLL14, MAP14], or heterogeneous multi-scale methods, as in e.g. [EE03]. Interestingly, for the thermo-diffusion system posed in perforated domains such convergent multi-scale numerical methods are yet unavailable.

The paper is structured as follows: In Section 2 we introduce the thermo-diffusion model and prove existence as well as a priori estimates for the solutions of the microscopic problem respective the two-scale limit problem. The periodic unfolding method and auxiliary corrector estimates are presented in Section 3.1 and 3.2, respectively. Finally, the corrector estimates in (1.1) are proved in Section 3.3. In Section 4 different parameter choices are studied. We conclude our paper with a discussion in Section 5.

## 2 A thermo-diffusion model

### 2.1 Model equations. Notation and assumptions

We investigate a system of reaction-diffusion equations which includes mollified cross-diffusion terms and different diffusion length scales. The cross-diffusion terms are motivated by the incorporation of Soret and Dufour effects as outlined in [KAML4]. For more information on phenomenological descriptions of thermo-diffusion, we refer the reader to [deM84]. The concentrations of the transported species through the perforated domain $\Omega_\varepsilon$ are denoted by $u_\varepsilon$, while $\theta_\varepsilon$ is the temperature. The overall interplay between transport and reaction is modeled here by the following system of partial differential equations:

\[
\begin{align*}
\dot{u}_\varepsilon &= \text{div}(d_\varepsilon \nabla u_\varepsilon) + \tau \varepsilon^\alpha \nabla u_\varepsilon \cdot \nabla^\delta \theta_\varepsilon + R(u_\varepsilon) \quad \text{in } \Omega_\varepsilon, \\
\dot{\theta}_\varepsilon &= \text{div}(\varepsilon^2 \kappa_\varepsilon \nabla \theta_\varepsilon) + \mu \varepsilon^\beta \nabla \theta_\varepsilon \cdot \nabla^\delta u_\varepsilon \quad \text{in } \Omega_\varepsilon,
\end{align*}
\]

where $\nabla^\delta$ denotes a smoothed gradient obtained via convolution with a mollifier, see (2.4). This modification of the cross-diffusion model is necessary for proving existence
and boundedness of weak solutions. We supplement equations (2.1) with the Neumann boundary conditions

\[
\begin{align*}
-d_\varepsilon \nabla u_\varepsilon \cdot \nu &= \varepsilon (au_\varepsilon + bv_\varepsilon) & \text{on } \Gamma_\varepsilon, \\
-\varepsilon^2 \kappa_\varepsilon \nabla \theta_\varepsilon \cdot \nu &= \varepsilon g & \text{on } \Gamma_\varepsilon, \\
-d_\varepsilon \nabla u_\varepsilon \cdot \nu &= 0 & \text{on } \partial \Omega_\varepsilon \setminus \Gamma_\varepsilon, \\
-\varepsilon^2 \kappa_\varepsilon \nabla \theta_\varepsilon \cdot \nu &= 0 & \text{on } \partial \Omega_\varepsilon \setminus \Gamma_\varepsilon,
\end{align*}
\]

(2.2)

and the initial conditions

\[
\begin{align*}
u_\varepsilon(0, x) &= u_0^\varepsilon(x) & \text{and } \theta_\varepsilon(0, x) &= \theta_0^\varepsilon(x), x \in \Omega_\varepsilon.
\end{align*}
\]

(2.3)

In this context, \(\nu\) denotes the normal outer unit vector of \(\Omega_\varepsilon\). First of all, it is important to note that the \(\varepsilon\)-scaling for some of the terms in the system is variable in the parameters \(\alpha \geq 0\) and \(\beta \geq 1\). We refer to the suitably scaled heat conduction-diffusion interaction terms \(\varepsilon^\alpha \nabla u_\varepsilon \nabla^2 \theta_\varepsilon\) and \(\varepsilon^\beta \nabla u_\varepsilon \nabla^2 \theta_\varepsilon\) as “weak thermal couplings”, while the “high-contrast” is thought here particularly in terms of the heat conduction properties of the composite material that can be seen in \(\varepsilon^2 \kappa_\varepsilon \nabla \theta_\varepsilon\). The matrix \(d_\varepsilon\) is the diffusivity associated to the concentration of the (diffusive) species \(u_\varepsilon\), \(\kappa_\varepsilon\) is the heat conductivity, while \(\tau_\varepsilon := \tau \varepsilon^\alpha\) and \(\mu_\varepsilon := \mu \varepsilon^\beta\) are the Soret and Dufour coefficients. The \(\varepsilon^2\)-scaling of \(\kappa_\varepsilon\) is associated with “slow diffusion” because it enables large temperature gradients \(\nabla \theta_\varepsilon\) of order \(O(1/\varepsilon)\).

The limiting temperature \(\Theta\) only undergoes diffusion on the microscopic scale, cf. (2.8). In contrast, the diffusivity \(d_\varepsilon\) and the concentration gradient \(\nabla u_\varepsilon\) are of order \(O(1)\) and then we speak of “classical diffusion”.

The critical parameter choices are \(\alpha \in \{0, 1\}\) and \(\beta \in \{1, 2\}\). The case \(\beta=1\) corresponds to the slow diffusion scaling of the heat conductivity and it is the only case which leads to a microscopic cross-diffusion term in the limit, see (2.8). For \(\beta \geq 2\) the \(\varepsilon^\beta\)-cross-diffusion term vanishes as \(\varepsilon\) tends to zero, see (4.6). The \(\varepsilon^\alpha\)-cross-diffusion term vanishes for all \(\alpha \geq 1\) in (2.8) and (4.6), whereas an averaged macroscopic cross-diffusion term appears for \(\alpha=0\) in the limit, see (4.1).

Note that \(d_\varepsilon, \kappa_\varepsilon, \tau,\) and \(\mu\) are either positive definite matrices, or they are positive real numbers. Furthermore, the reaction term \(R(\cdot)\) models the Smoluchovski interaction production. In the original model from [KAM14], the function \(v_\varepsilon\) is an additional unknown modeling the mass of deposited species on the pore surface \(\Gamma_\varepsilon\), and it is shown to possess the regularity \(v_\varepsilon \in H^1(0, T; L^2(\Gamma_\varepsilon)) \cap L^\infty((0, T) \times \Gamma_\varepsilon)\). Here we assume \(v_\varepsilon\) as given data. We point out that the linear boundary terms are relevant for the regularity of solutions, but that they are not required to prove the convergence rate of order of \(\sqrt{\varepsilon}\) in (1.1).
To deal with perforated domains we employ the method of periodic unfolding as presented in \cite{CDZ06}. Let $Y = [0,1]^d$ denote the standard reference cell. We fix here and for all the following assumptions on the domain and the microstructure.

**Assumptions 2.1.** Our geometry is designed as follows:

(i) The domain $\Omega = \prod_{i=1}^d [0,l_i)$ is a $d$-polytope with length $l_i > 0$ for all $1 \leq i \leq d$.

(ii) The reference hole $Y_{\text{hole}} \subseteq Y$ is an open set (possibly disconnected) with Lipschitz boundary $\Gamma = \partial Y_{\text{hole}}$ and the perforated cell $Y_\ast := Y \setminus Y_{\text{hole}}$ satisfies $Y_\ast \neq \emptyset$. Moreover $Y_\ast$ is a connected Lipschitz domain and $\partial Y_\ast \cap \partial Y$ is identical on all faces of $Y$.

The set of all nodal points is given via $N_\varepsilon := \{ \lambda \in \mathbb{Z}^d | \varepsilon(\lambda + Y) \subseteq \Omega \}$. With this we define the perforated domain $\Omega_\varepsilon$, which is connected, the pore part $Y_{\text{hole},\varepsilon}$, and the pore boundaries $\Gamma_\varepsilon$ via

$$
\Omega_\varepsilon := \bigcup_{\lambda \in N_\varepsilon} \varepsilon(\lambda + Y^\circ), \quad Y_{\text{hole},\varepsilon} := \bigcup_{\lambda \in N_\varepsilon} \varepsilon(\lambda + Y_{\text{hole}}), \quad \Gamma_\varepsilon := \bigcup_{\lambda \in N_\varepsilon} \varepsilon(\lambda + \Gamma),
$$

where $A^\circ$ denotes the interior of the set $A$. Both sets are open and form together the original domain $\overline{\Omega} = \overline{Y_{\text{hole},\varepsilon}} \cup \overline{\Omega_\varepsilon}$. Indeed we have $\Gamma_\varepsilon = \partial Y_{\text{hole},\varepsilon} \subseteq \partial \Omega_\varepsilon$. The assumptions on the domain guarantee the existence of suitable extensions from $\Omega_\varepsilon$ to $\Omega$ (cf. Theorem A.2). Also traces exist and are well-defined on the boundaries $\partial \Omega_\varepsilon$ and $\Gamma$. With this, perforated domains with isolated holes as well as the prominent “pipe-model” for porous media are included in our considerations, see Figure 2.2. The boundary of the perforated domain $\Omega_\varepsilon$ is given by $\partial \Omega_\varepsilon = (\partial \Omega \cup \Gamma_\varepsilon) \setminus (\partial \Omega_\ast \cap \Gamma_\varepsilon)$. Indeed intersected pore structures at the boundary $\partial \Omega \cap \Gamma_\varepsilon \neq \emptyset$ as in Figure 2.2(ii) are not excluded. We also point out that microscopic inclusions need not be connected, cf. Figure 2.2(i).

**Remark 2.2.** In the following we denote by $\varepsilon$ a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of numbers satisfying $1/\varepsilon_n \in \mathbb{N}$. This implies that all microscopic cells $\varepsilon(\lambda + Y_\ast)$, for $\lambda \in N_\varepsilon$, are contained in $\Omega_\varepsilon$ and no intersected cells occur on the boundary $\partial \Omega$.

If $\varepsilon$ satisfied $1/\varepsilon \in \mathbb{R} \setminus \mathbb{N}$, there could arise intersected cells at the boundary such that $\Omega_\varepsilon$ does not have a Lipschitz boundary anymore, cf. Figure 2.2(iii).

**Assumptions 2.3.** We impose the following restrictions on the data:

(i) The diffusion matrices $d_\varepsilon$ and $\kappa_\varepsilon$ are given via

$$
d_\varepsilon(x) := \mathbb{D}(\frac{x}{\varepsilon}) \quad \text{and} \quad \kappa_\varepsilon(x) := \mathbb{K}(\frac{x}{\varepsilon}),
$$

where $\mathbb{D}, \mathbb{K} \in L^\infty(Y_\ast; \mathbb{R}^{d \times d})$ are symmetric and uniformly elliptic, i.e.

$$
\exists C_{\text{elip}} > 0, \forall (\xi, y) \in \mathbb{R}^d \times Y_\ast : \quad C_{\text{elip}} |\xi|^2 \leq \mathbb{D}(y)\xi \cdot \xi \leq C_{\text{elip}}^{-1} |\xi|^2.
$$
(ii) The constants $\tau, \mu, a, b, g$ are non-negative.

(iii) The reaction term $R : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, i.e.

$$\exists L > 0, \forall s_1, s_2 \in \mathbb{R} : \quad |R(s_1) - R(s_2)| \leq L|s_1 - s_2|.$$  

Moreover, it is $R(s) = 0$ for all $s < 0$.

(iv) The sink/source term $v_\varepsilon$ is given via $v_\varepsilon(t, x) := \mathbb{V}(t, x, x/\varepsilon)$ for any data $\mathbb{V} \in C([0, T]; W^{1, \infty}(\Omega; L^2(\Gamma))).$

Here we denote with $a \cdot b$ the scalar product of vectors in $\mathbb{R}^d$ and set

$$L^\infty_+(\Omega) := \{ \varphi \in L^\infty(\Omega) \mid \varphi \geq 0 \text{ a.e. in } \Omega \}.$$  

For technical reasons, we introduce the mollified gradient $\nabla^\delta$ which is given as follows: for $\delta > 0$, we introduce the mollifier

$$J_\delta(x) := \begin{cases} C \exp(1/(|x|^2 - \delta^2)), & |x| < \delta, \\ 0, & |x| \geq \delta, \end{cases}$$

where the constant $C > 0$ is selected such that $\int_{\mathbb{R}^d} J_\delta \, dx = 1$. Using $J_\delta$ we define for $u_\varepsilon \in L^1(\Omega_\varepsilon)$ the mollified gradient

$$\nabla^\delta u_\varepsilon := \nabla \left[ \int_{B_\delta(x)} J_\delta(x-z) u_\varepsilon^{ex}(z) \, dz \right] \quad \text{with} \quad u_\varepsilon^{ex}(x) := \begin{cases} u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ 0, & x \in \mathbb{R}^d \setminus \Omega_\varepsilon. \end{cases}$$  

(2.4)

where $B_\delta(x)$ denotes the ball centered at $x \in \mathbb{R}^d$ with radius $\delta$. Notice that the intersection $\Omega_\varepsilon \cap B_\delta(x)$ does not have a Lipschitz boundary in general, however, the intersection of two Lebesgue measurable sets is Lebesgue measurable. With this, $u_\varepsilon^{ex}$ is indeed well-defined in $L^1(B_\delta(x))$ for $x \in \Omega_\varepsilon$. According to [Evans, Sec. C.4] there holds $\nabla^\delta u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and

$$\exists C_\delta > 0, \forall u_\varepsilon \in L^2(\Omega_\varepsilon) : \quad \|\nabla^\delta u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C_\delta \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$  

(2.5)

We assume throughout this text that $\varepsilon$ and $\delta$ are chosen such that $\delta > 2\varepsilon \text{diam}(Y)$ holds. This assumption arises in Lemma 3.4.

### 2.2 Existence of solutions and a priori estimates

This subsection is devoted to the existence of weak solutions to our original microscopic problem with $\alpha \geq 0$ and $\beta \geq 1$.

**Theorem 2.4.** Let the Assumptions 2.1 and 2.3 hold and let the initial condition $(u_\varepsilon^0, \theta_\varepsilon^0)$ satisfy

$$\exists C_0, M_0 > 0 : \quad \|u_\varepsilon^0\|_{H^1(\Omega_\varepsilon)} + \|\theta_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} + \varepsilon \|\nabla \theta_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \leq C_0,$n

$$0 \leq u_\varepsilon^0(x), \theta_\varepsilon^0(x) \leq M_0 \quad \text{for a.e. } x \in \Omega_\varepsilon.$$  

Then there exists for every $\varepsilon > 0$ a unique solution $(u_\varepsilon, \theta_\varepsilon)$ of (2.1)–(2.3) with $u_\varepsilon, \theta_\varepsilon \in H^1(0, T; L^2(\Omega_\varepsilon)) \cap L^\infty(0, T; H^1(\Omega_\varepsilon)) \cap L^\infty_+(0, T) \times \Omega_\varepsilon).$

Moreover the solution is non-negative, i.e. $0 \leq u_\varepsilon, \theta_\varepsilon \leq M$ almost everywhere in $[0, T] \times \Omega_\varepsilon$, and uniformly bounded

$$\|u_\varepsilon\|_{H^1(0, T; L^2(\Omega_\varepsilon))} + \|\nabla u_\varepsilon\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} + \|\theta_\varepsilon\|_{H^1(0, T; L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \theta_\varepsilon\|_{L^\infty(0, T; L^2(\Omega_\varepsilon))} \leq C,$$  

(2.6)

where the constants $M, C > 0$ are independent of $\varepsilon$.  

5
Proof. The existence of solutions, non-negativity, and uniform boundedness follow from the Lemmata 3.2–3.6 and Theorem 3.8 in [KAM14] by replacing \( \kappa_\varepsilon \) and \( \tau_\varepsilon \) with \( \varepsilon^2 \kappa_\varepsilon \) and \( \varepsilon \tau_\varepsilon \), respectively. Note that the proof can be generalized from diffusion coefficients \( d_\varepsilon, \kappa_\varepsilon \in \mathbb{R} \) to symmetric matrices as in Assumption 2.3(i). In equation (35) respective (57) in [KAM14] it holds for \( A \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{d \times d}) \) and \( u \in H^1(\Omega) \):

\[
\frac{d}{dt} \int_{\Omega} A \nabla u \cdot \nabla u \, dx = \int_{\Omega} A \nabla \dot{u} \cdot \nabla u \, dx + \int_{\Omega} A \nabla u \cdot \nabla \dot{u} \, dx = 2 \int_{\Omega} A \nabla u \cdot \nabla \dot{u} \, dx. \tag{2.7}
\]

This argumentation also requires linear boundary terms. Otherwise one has to argue as in [Tem88, Thm. 3.2] or [MRT14, Prop. 1] and differentiate the whole equation with respect to time and then use a second Grönewall argument.

\[\square\]

Remark 2.5. Since our solutions are uniformly bounded in \( L^\infty((0, T) \times \Omega_\varepsilon) \), we may consider reaction terms with arbitrary growth as in [KAM14].

2.3 The two-scale limit system

If not stated otherwise, we set \( \alpha = \beta = 1 \) for the rest of the paper. In the limit \( \varepsilon \to 0 \), we obtain the following two-scale system

\[
\dot{u} = \text{div}(d_{\text{eff}} \nabla u) + R(u) - \frac{1}{V_\varepsilon} (au + bv_0) \quad \text{in } \Omega, \tag{2.8}
\]

\[
\dot{\Theta} = \text{div}_y (K \nabla_y \Theta) + \mu \nabla_y \Theta \cdot \nabla^\delta u \quad \text{in } \Omega \times Y_\varepsilon,
\]

where the mollified gradient \( \nabla^\delta u \) is defined for \( u \in L^1(\Omega) \) via

\[
\nabla^\delta u := \nabla \left[ \int_{B_\delta(x)} J_\delta(x-z) u_{\text{ex}}(z) \, dz \right] \quad \text{with } u_{\text{ex}}(x) := \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^d \setminus \Omega. \end{cases} \tag{2.9}
\]

System (2.8) is supplemented with the boundary conditions

\[
-d_{\text{eff}} \nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega, \\
-K \nabla_y \Theta \cdot \nu_{Y_\varepsilon} = g \Theta \quad \text{on } \Omega \times \Gamma, \\
\Theta \text{ is periodic} \quad \text{on } \Omega \times \partial Y \setminus \Gamma \tag{2.10}
\]

and the initial conditions

\[
u(0, x) = u^0(x) \quad \text{and } \Theta(0, x, y) = \Theta^0(x, y). \tag{2.11}
\]

Here \( \nu \) and \( \nu_{Y_\varepsilon} \) denote the normal outer unit vector of \( \Omega \) and \( Y_\varepsilon \), respectively. A similar, but purely macroscopic, limit system as in (2.8)–(2.11) has been derived in [KAM14]. For limit systems that reveal the same two-scale structure, we refer to e.g. [Pet07, Mei08, Rei15]. To capture the oscillations in the limit we define the space of \( Y_\varepsilon \)-periodic functions \( H^1_{\text{per}}(Y_\varepsilon) \subseteq H^1_{\text{per}}(Y) \) via

\[
H^1_{\text{per}}(Y_\varepsilon) := \{ \Phi \in H^1(Y_\varepsilon) | \Phi|_{\Gamma_i} = \Phi|_{\Gamma_{-i}} \},
\]

where \( \Gamma_i \) and \( \Gamma_{-i} \) are opposite faces of the unit cube \( Y \) with \( \partial Y = \bigcup_{i=1}^d (\Gamma_i \cup \Gamma_{-i}) \). With this, the effective coefficients are given via the standard unit cell problem (see e.g. [All92])

\[
\forall \xi \in \mathbb{R}^d : \quad d_{\text{eff}} \xi \cdot \xi = \min_{\Phi \in H^1_{\text{per}}(Y_\varepsilon)} \int_{Y_\varepsilon} D[\nabla_y \Phi + \xi] : [\nabla_y \Phi + \xi] \, dy. \tag{2.12}
\]
Theorem 2.6. Let the Assumptions 2.1 and 2.3 hold and let the initial values satisfy
\[ u \text{ boundary data} \]
Note that the integral is taken over \( Y \) and not the average \( \bar{f}_{Y_x} \). In full, formula (2.12) reads \( 1/Y' |_{Y_x} \) with \( |Y'| = 1 \) here. The corrector \( U \in L^2(\Omega; H^1_{pe}(Y)) \) minimizes the quadratic functional in (2.12) for \( \xi = \nabla u(x) \). And the corrector \( U \) is the unique minimizer when demanding in addition that the microscopic average satisfies \( \int_{Y_x} U(x, y) \, dy = 0 \). For the boundary data \( v_0 \), we obtain in the limit \( \varepsilon \to 0 \) the usual average
\[ \forall (t, x) \in [0, T] \times \Omega : \quad v_0(t, x) = \int_{\Gamma} \mathcal{V}(t, x, y) \, dy. \]

Finally, we state the existence and uniqueness of solutions for the limit system.

**Theorem 2.6.** Let the Assumptions 2.1 and 2.3 hold and let the initial value \( (u^0, \Theta^0) \) satisfy \( u^0 \in H^1(\Omega) \cap L^\infty(\Omega) \) and \( \Theta^0 \in L^2(\Omega; H^1_{pe}(Y)) \cap L^\infty(\Omega \times Y) \). There exists a unique solution \((u, \Theta)\) of (2.8)–(2.11) with
\[ u \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap L^\infty(0, T \times \Omega), \]
\[ \Theta \in H^1(0, T; H^1(\Omega; L^2(Y))) \cap L^\infty(0, T; H^1(\Omega; H^1_{pe}(Y))) \cap L^\infty((0, T) \times \Omega \times Y). \]

**Proof.** The existence and boundedness of unique solutions \((u, \Theta)\) follow by Galerkin approximation as in e.g. [Emm04, Sect. 8.4]. The standard Galerkin approximation has been adapted to the two-scale setting in [MuN10]. In [MuN10, Thm. 4], the initial values satisfy \( u^0 \in H^2(\Omega) \) and \( \Theta^0 \in L^2(\Omega; H^1_{pe}(Y)) \), however, this assumption can be relaxed to \( u^0 \in H^1(\Omega) \) and \( \Theta^0 \in L^2(\Omega; H^1_{pe}(Y)) \) in our setting as follows: To prove the \( L^\infty(0, T) \)-estimates for the gradients and the \( L^2(0, T) \)-estimates for the time derivative we can simplify the proof of [MuN10, Thm. 4] after equations (40)–(42) by exploiting that the boundary terms are linear and arguing as in [KAM14, Eq. (35)].

Moreover, the higher \( x \)-regularity of \( \Theta \) follows by differentiating the \( \Theta \)-equation w.r.t. \( x \in \Omega \) and using the smoothness of \( \nabla^3 u \), see e.g. [MuN10, Thm. 5] or [Rei15, Prop. 2.3.17].

Notice that the space \( H^1(\Omega; H^1_{pe}(Y)) \) is a subspace of \( H^1(\Omega; H^1(Y)) \), which is endowed with the norm
\[ \|U\|_{H^1(\Omega; H^1(Y))} := \left( \int_{\Omega} \|U(x, \cdot)\|_{H^1(Y)}^2 + \|\nabla_x U(x, \cdot)\|_{H^1(Y)}^2 \, dx \right)^{1/2}. \]

**Remark 2.7.** We can identify the spaces \( H^1(\Omega; H^1(Y)) \) and \( H^1(Y; H^1(\Omega)) \) and there holds
\[ \|U\|_{H^1(\Omega; H^1(Y))}^2 = \int_{\Omega \times Y} |U(x, y)|^2 + |\nabla_x U(x, y)|^2 + |\nabla_y U(x, y)|^2 + |\nabla_{xy} U(x, y)|^2 \, dx \, dy. \]
In particular the spaces \( L^p(\Omega \times Y) \), \( L^p(\Omega; L^p(Y)) \), and \( L^p(Y; L^p(\Omega)) \) can be identified for all \( 1 \leq p < \infty \) since they are separable.

# 3 Corrector Estimates

## 3.1 Periodic unfolding and folding of two-scale functions

To rigorously justify the structure of the two-scale expansions \( u_\varepsilon(x) \approx u(x) + \varepsilon U(x, x/\varepsilon) \) and \( \theta_\varepsilon(x) \approx \Theta(x, x/\varepsilon) \), we employ the periodic unfolding method [CDG02]. The idea of the method is to map functions from the periodic domain \( \Omega_\varepsilon \) to the fixed two-scale domain
\( \Omega \times Y \), and then allow for corrector estimates under general regularity assumptions on the data.

The usual two-scale decomposition is given via the mappings \([\cdot]: \mathbb{R}^d \rightarrow \mathbb{Z}^d\) and \(\{\cdot\}: \mathbb{R}^d \rightarrow Y\). For \(x \in \mathbb{R}^d\), \([x]\) denotes the component-wise application of the standard Gauss bracket and \(\{x\} := x - [x]\) is the remainder. Following [CD+12] we introduce the periodic unfolding operator on domains with holes for \(1 \leq p \leq \infty\) via

\[
\mathcal{T}_\varepsilon^* : L^p(\Omega_\varepsilon) \rightarrow L^p(\Omega \times Y_\varepsilon); \quad (\mathcal{T}_\varepsilon^* u)(x,y) := u \left(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y\right).
\]

Note that we do not need to extend \(u\) by 0 outside \(\Omega_\varepsilon\), since there occur no intersected cells at the boundary \(\partial \Omega\), cf. also Remark 2.2. We have indeed \(\varepsilon([\frac{x}{\varepsilon}] + y) \in \Omega_\varepsilon\) for all \((x,y) \in \Omega \times Y_\varepsilon\) such that \(\mathcal{T}_\varepsilon^*\) is well-defined. In the same manner, the boundary unfolding operator is given by

\[
\mathcal{T}_\varepsilon^b : L^p(\Gamma_\varepsilon) \rightarrow L^p(\Omega \times \Gamma); \quad (\mathcal{T}_\varepsilon^bu)(x,y) := u \left(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y\right).
\]

Both operators satisfy the integral identity

\[
\int_{\Omega_\varepsilon} u \, dx = \int_{\Omega \times Y_\varepsilon} \mathcal{T}_\varepsilon^* u \, dx \, dy \quad \text{and} \quad \varepsilon \int_{\Gamma_\varepsilon} u \, d\sigma(x) = \int_{\Omega \times \Gamma} \mathcal{T}_\varepsilon^b u \, dx \, d\sigma(y) \quad (3.1)
\]

for all \(u \in L^1(\Omega_\varepsilon)\) and \(u \in L^1(\Gamma_\varepsilon)\), respectively. For \(p = 2\) we also introduce the dual operator of \(\mathcal{T}_\varepsilon^*\), namely, the folding (averaging) operator

\[
\mathcal{F}_\varepsilon^* : L^2(\Omega \times Y_\varepsilon) \rightarrow L^2(\Omega_\varepsilon); \quad (\mathcal{F}_\varepsilon^* U)(x) := \int_{\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + Y} U(z, \{ \frac{z}{\varepsilon} \}) \, dz,
\]

where \(\int_A u \, dz = |A|^{-1} \int_A u \, dz\) denotes the usual average. We point out that the proof of the corrector estimates relies heavily on the integral identities in (3.1) and the duality of \(\mathcal{T}_\varepsilon^*\) and \(\mathcal{F}_\varepsilon^*\) is not used. Moreover, we will use the original periodic unfolding and folding operators in the whole domain \(\Omega\) respective \(Y\) as in [CDG02]

\[
\mathcal{T}_\varepsilon : L^2(\Omega) \rightarrow L^2(\Omega \times Y); \quad (\mathcal{T}_\varepsilon u)(x,y) := u \left(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y\right),
\]

\[
\mathcal{F}_\varepsilon : L^2(\Omega \times Y) \rightarrow L^2(\Omega); \quad (\mathcal{F}_\varepsilon U)(x) := \int_{\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + Y} U(z, \{ \frac{z}{\varepsilon} \}) \, dz.
\]

To derive quantitative estimates for the differences \(\varepsilon - \Theta\) and \(\theta - \Theta\), we need to test the weak formulation of the original system with \(H^1(\Omega_\varepsilon)\)-functions which are one-scale pendants of the limiting solution \((u, \Theta)\). There are two options to naively fold a two-scale function \(U(x,y)\), namely

\[
u_\varepsilon(x) = U(x, \frac{x}{\varepsilon})|_{\Omega_\varepsilon} \quad \text{and} \quad u_\varepsilon^*(x) = (\mathcal{F}_\varepsilon^* U)(x).
\]

The function \(u_\varepsilon\) is only well-defined in \(H^1(\Omega_\varepsilon)\), if at least \(x \mapsto U(x,y)\) belongs to \(C^1(\Omega)\), see [LNW02] for regularity of two-scale functions. However our limit \((u, \Theta)\) (respective the corrector \(U\) for \(\nabla u\)) does not satisfy strong differentiability in general. The second option \(u_\varepsilon^*\) is neither a suitable test function, since it is not \(H^1(\Omega_\varepsilon)\)-regular. To overcome this regularity issue, we define the gradient folding operator following [MT07, Han11, MRT14, Rei16]:

\[
\mathcal{G}_\varepsilon : L^2(\Omega; H^1_{\text{per}}(Y)) \rightarrow H^1(\Omega); \quad \mathcal{G}_\varepsilon U := \widehat{u}_\varepsilon,
\]
where for every \( U \in L^2(\Omega; H^1_{\text{per}}(Y)) \), the function \( \hat{u}_\varepsilon \in H^1(\Omega) \) is the solution of the elliptic problem
\[
\int_\Omega (\hat{u}_\varepsilon - F_\varepsilon U) \varphi + (\varepsilon \nabla \hat{u}_\varepsilon - F_\varepsilon (\nabla U)) \cdot \varepsilon \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1(\Omega).
\]

Note that \( \hat{u}_\varepsilon \) is uniquely determined by the Lax–Milgram Lemma implying the well-definedness of \( G_\varepsilon \). Thus \( (G_\varepsilon U)|_{|\Omega} \) yields an admissible test function in \( H^1(\Omega_\varepsilon) \). Both folding operators, \( F_\varepsilon \) and \( G_\varepsilon \), are linear and bounded operators satisfying
\[
\|F_\varepsilon U\|_{L^2(\Omega)} \leq \|U\|_{L^2(\Omega \times Y)} \quad \text{and} \quad \|G_\varepsilon U\|_\varepsilon \leq 2\|U\|_{L^2(\Omega; H^1(\varepsilon Y))},
\]
(3.2)
where the first estimate is due to Jensen’s inequality, while the second one is due to Hölder’s inequality. We point out that for \( \varphi \in H^1(\Omega) \) there holds
\[
\|T_\varepsilon \varphi\|_{L^2(\Omega; H^1(\varepsilon Y))} = \|\varphi\|_{L^2(\Omega)} + \varepsilon \|\nabla \varphi\|_{L^2(\Omega)},
\]
(3.3)
which follows from the properties (A3) and (A4) in Lemma A.1.

### 3.2 Auxiliary error estimates

We are now collecting several results which are essential ingredients in the proof of our corrector estimates \([1,1]\). We point out that \( F_\varepsilon^* T_\varepsilon^* u = u \) for all \( u \in L^2(\Omega_\varepsilon) \), since \( x \mapsto (T_\varepsilon^* u)(x, y) \) is constant on each cell \( \varepsilon (\lambda + Y_\varepsilon) \). However the reverse application of periodic unfolding and folding yields the identity only approximately, namely, \( T_\varepsilon^* F_\varepsilon^* U = U + O(\varepsilon) \) in Lemma 3.1 below.

Note that \( u \in H^1(\Omega) \) also belongs to the space \( H^1(\Omega_\varepsilon) \) since \( \Omega_\varepsilon \subset \Omega \) and we can apply the unfolding operator via \( T_\varepsilon^* u := T_\varepsilon^* (\chi_\varepsilon u) \), where \( \chi_\varepsilon \) denotes the characteristic function of the set \( \Omega_\varepsilon \). For the sake of brevity \( \chi_\varepsilon \) is omitted in the following.

**Lemma 3.1.** For all \( U \in H^1(\Omega; L^2(Y_\varepsilon)) \) and \( u \in H^1(\Omega) \) we have
\[
\|T_\varepsilon^* F_\varepsilon^* U - U\|_{L^2(\Omega \times Y_\varepsilon)} \leq \varepsilon C \|\nabla U\|_{L^2(\Omega \times Y_\varepsilon)} \quad \text{and} \quad \|T_\varepsilon^* u - u\|_{L^2(\Omega \times Y_\varepsilon)} \leq \varepsilon C \|u\|_{H^1(\Omega)},
\]
respectively, where \( C > 0 \) only depends on the domains \( \Omega \) and \( Y_\varepsilon \).

**Proof.** The proof for the first estimate is based on the application of the Poincaré–Wirtinger inequality on each cell \( \varepsilon (\lambda + Y_\varepsilon) \) with \( \text{diam}(\varepsilon (\lambda + Y_\varepsilon)) \leq \varepsilon \text{diam}(Y) \)
\[
\int_{\varepsilon (\lambda + Y_\varepsilon)} \left| \int_{\varepsilon (\lambda + Y_\varepsilon)} U(z, y) \, dz - U(x, y) \right|^2 \, dx \, dy \leq (\varepsilon \text{diam}(Y))^2 \|\nabla x U\|^2_{L^2(\lambda + Y_\varepsilon)},
\]
(cf. also \[\text{[Gri04, Rei16]}\]). The second estimate follows from the first one with
\[
\|T_\varepsilon^* u - u\|_{L^2(\Omega \times Y_\varepsilon)} \leq \|T_\varepsilon^* F_\varepsilon^* u - u\|_{L^2(\Omega \times Y_\varepsilon)} + \|F_\varepsilon^* u - u\|_{H^1(\Omega_\varepsilon)}.
\]
Note that \( F_\varepsilon^* u \) is indeed well-defined for one-scale functions. \( \square \)

We emphasize that the same arguments as in Lemma 3.1 also yield the analogous estimate on the whole unit cell \( Y \), i.e. for \( U \in H^1(\Omega; L^2(Y)) \) it holds
\[
\|T_\varepsilon F_\varepsilon U - U\|_{L^2(\Omega \times Y)} \leq \varepsilon \|\nabla U\|_{L^2(\Omega \times Y)}.
\]
(3.4)

Having defined two folding operators, \( F_\varepsilon \) being dual to \( T_\varepsilon \) and \( G_\varepsilon \) assuring \( H^1 \)-regularity, we call their difference folding mismatch and control it as follows.
Theorem 3.2 ([Rei16 Thm. 3.4, Eq. (3.8)]). For $U \in H^1(\Omega; H^1_{\text{per}}(Y))$ it holds

$$
\|G_\varepsilon U - F_\varepsilon U\|_{L^2(\Omega)} + \|\varepsilon \nabla G_\varepsilon U - F_\varepsilon (\nabla_y U)\|_{L^2(\Omega)} \leq \varepsilon C\|U\|_{H^1(\Omega; H^1(Y))},
$$

where $C > 0$ only depends on $\Omega$ and $Y$.

In particular, the application of gradient folding and periodic unfolding is close to the identity, namely, $T_\varepsilon G_\varepsilon U = U + O(\varepsilon)$.

Corollary 3.3. For $U \in H^1(\Omega; H^1_{\text{per}}(Y))$ it holds

$$
\|T_\varepsilon G_\varepsilon U - U\|_{L^2(\Omega; H^1(Y))} \leq \varepsilon C\|U\|_{H^1(\Omega; H^1(Y))},
$$

where $C > 0$ only depends on $\Omega$ and $Y$.

Proof. Inserting $\pm T_\varepsilon F_\varepsilon U$ respective $\pm T_\varepsilon F_\varepsilon (\nabla_y U)$, applying the triangle inequality, and using the norm preservation of $T_\varepsilon$ as in (A.2) gives

$$
\|T_\varepsilon G_\varepsilon U - U\|_{L^2(\Omega; H^1(Y))} \leq \|G_\varepsilon U - F_\varepsilon U\|_{L^2(\Omega)} + \|\varepsilon \nabla (G_\varepsilon U) - F_\varepsilon (\nabla_y U)\|_{L^2(\Omega)}
+ \|T_\varepsilon F_\varepsilon U - U\|_{L^2(\Omega \times Y)} + \|T_\varepsilon F_\varepsilon (\nabla_y U) - \nabla_y U\|_{L^2(\Omega \times Y)}
\leq \varepsilon C\|U\|_{H^1(\Omega; H^1(Y))},
$$

where the last inequality follows from Theorem 3.2 and estimate (3.4).

To control the mollified gradient we prove:

Lemma 3.4. For $\delta > 2\varepsilon \text{diam}(Y)$ and all $u \in L^2(\Omega)$ and $(x,y) \in \Omega \times Y_*$ we have

$$
|T_\varepsilon^* (\nabla^\delta u) - \nabla^\delta u|(x,y) \leq \sqrt{\varepsilon} C_\delta \|u\|_{L^2(\Omega)},
$$

where $C_\delta > 0$ depends on the mollifier $J_\delta$.

Proof. According to [Eva98 Thm. 6] we obtain for every $(x,y) \in \Omega \times Y_*$

$$
[T_\varepsilon^* (\nabla^\delta u)](x,y) = T_\varepsilon^* \left( \int_{B_\delta(x)} \nabla J_\delta(x-z) u^\text{ex}(z) \, dz \right) (x,y)
= \int_{B_\delta(s[\delta] + \varepsilon y)} \nabla J_\delta(s[\delta] + \varepsilon y - z) u^\text{ex}(z) \, dz.
$$

For $\delta > 2\varepsilon \text{diam}(Y)$, we define the following $d$-dimensional annulus

$$
B_{\text{diff}} := B(x, \delta + \varepsilon \text{diam}(Y)) \setminus B(x, \delta - \varepsilon \text{diam}(Y))
$$

of thickness $\varepsilon 2\text{diam}(Y)$ and with volume $|B_{\text{diff}}| \leq \varepsilon \text{Const}(\delta, Y)$. For brevity, we set

$$
B_1 := B_\delta(x), \quad B_2 := B_\delta(s[\delta] + \varepsilon y), \quad B_3 := B_1 \cap B_2
$$

such that we have the following relations, cf. Figure 3.1

$$(B_1 \setminus B_3) \cup (B_2 \setminus B_3) \subseteq B_{\text{diff}} \quad \text{and} \quad B_1 \cup B_2 \subseteq B_3 \cup B_{\text{diff}}.$$
With $|\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y - x| \leq \varepsilon \text{diam}(Y)$ for all $(x, y) \in \Omega \times Y$, we arrive at
\[
\left| \mathcal{T}_\varepsilon^* (\nabla^e u) - \nabla^e u \right|(x, y) \\
\leq \left| \int_{B_\delta(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y)} \nabla J_\delta(\varepsilon[\frac{x}{\varepsilon}] + \varepsilon y - z) u^e(x) \nabla z - \int_{B_\delta(x)} \nabla J_\delta(x - z) u^e(z) \nabla z \right| \\
\leq C \| u \|_{L^2(\Omega)},
\]
where $C > 0$ only depends on the domains $\Omega$ and $Y$. This proves the assertion. \hfill \Box

Since unfolded Sobolev functions $T_\varepsilon u \in L^2(\Omega; H^1(Y)) \supseteq L^2(\Omega; H^1_{\text{per}}(Y))$ are in general not $Y$-periodic, we need to control the so-called periodicity defect, cf. [Gri04, Gri05]. Therefore, we will use the following two theorems for the case of slow diffusion and classical diffusion, respectively.

**Theorem 3.5 (Gri05 Thm. 2.2)**. For every $\varphi \in H^1(\Omega)$, there exists a $Y$-periodic function $\Phi_\varepsilon \in L^2(\Omega; H^1_{\text{per}}(Y))$ such that
\[
\| \Phi_\varepsilon \|_{H^1(Y; L^2(\Omega))} \leq C \left( \| \varphi \|_{L^2(\Omega)} + \varepsilon \| \nabla \varphi \|_{L^2(\Omega)} \right),
\]
\[
\| T_\varepsilon \varphi - \Phi_\varepsilon \|_{H^1(Y; H^1(\Omega)^*)} \leq \sqrt{\varepsilon} C \left( \| \varphi \|_{L^2(\Omega)} + \varepsilon \| \nabla \varphi \|_{L^2(\Omega)} \right),
\]
where the constant $C > 0$ only depends in the domains $\Omega$ and $Y$.

**Theorem 3.6 (Gri05 Thm. 2.3)**. For every $\varphi \in H^1(\Omega)$, there exists a $Y$-periodic function $\Phi_\varepsilon \in L^2(\Omega; H^1_{\text{per}}(Y))$ such that
\[
\| \Phi_\varepsilon \|_{H^1(Y; L^2(\Omega))} \leq C \| \varphi \|_{H^1(\Omega)},
\]
\[
\| \nabla \varphi + \nabla y \Phi_\varepsilon - T_\varepsilon(\nabla \varphi) \|_{L^2(Y; H^1(\Omega)^*)} \leq \sqrt{\varepsilon} C \| \varphi \|_{H^1(\Omega)},
\]
where the constant $C > 0$ only depends in the domains $\Omega$ and $Y$.

### 3.3 Main Theorem and its proof

Having collected all preliminaries, we can now state and prove the corrector estimates for our thermo-diffusion model with $\alpha = \beta = 1$. 

---

**Figure 3.1**: Depiction of the sets $B_1$, $B_2$, $B_3$, and $B_{\text{diff}}$. 

\[
\text{Figure 3.1: Depiction of the sets } B_1, B_2, B_3, \text{ and } B_{\text{diff}}.
\]
Theorem 3.7. Let \((u_\varepsilon, \theta_\varepsilon)\) and \((u, \Theta)\) denote the unique solution of system (2.1)–(2.3) and (2.8)–(2.11), respectively, according to Theorem 2.4 and Theorem 2.6. If the initial values satisfy

\[ \exists C_0 > 0 : \| \mathcal{T}_\varepsilon u_0^\varepsilon - u_0 \|_{L^2(\Omega \times Y_\varepsilon)} + \| \mathcal{T}_\varepsilon^* \theta_0^\varepsilon - \Theta_0 \|_{L^2(\Omega \times Y_\varepsilon)} \leq \sqrt{\varepsilon} C_0, \]  

then we have

\[
\begin{align*}
\| \mathcal{T}_\varepsilon u_\varepsilon - u \|_{L^\infty(0,T;L^2(\Omega \times Y_\varepsilon))} + \| \mathcal{T}_\varepsilon^* (\nabla u_\varepsilon) - (\nabla u + \nabla_y U) \|_{L^2(0,T;L^2(\Omega \times Y_\varepsilon))} \\
+ \| \mathcal{T}_\varepsilon^* \theta_\varepsilon - \Theta \|_{L^\infty(0,T;L^2(\Omega \times Y_\varepsilon))} + \| \mathcal{T}_\varepsilon^* (\varepsilon \nabla \theta_\varepsilon) - \nabla_y \Theta \|_{L^2(0,T;L^2(\Omega \times Y_\varepsilon))} \leq \sqrt{\varepsilon} C,
\end{align*}
\]

where the constant \(C > 0\) depends on the given data and the norms in (2.6) and (2.13).

Proof. Note that the domain \(\Omega\) is convex, bounded, and has a Lipschitz boundary. Since \(\dot{u}\) and \(v_0\) belong to the space \(L^2((0,T) \times \Omega)\), we can apply [Gris65 Thm. 3.2.1.3] and obtain that the limit \(u(t, \cdot)\) belongs to the better space \(H^2(\Omega)\).

If not stated otherwise, the following notion of weak formulation is to be understood pointwise in \([0,T]\). Moreover, the dependence of the solutions \((u_\varepsilon, \theta_\varepsilon)\) and \((u, \Theta)\) on \(t \in [0,T]\) is only written explicitly when needed.

Part A: Slow diffusion. In order to apply the auxiliary error estimates in the Theorems 3.2, 3.5, and 3.6 which are formulated for the whole domain \(\Omega\) and the unit cell \(Y\), we extend the solutions \(\theta_\varepsilon\) and \(\Theta\) suitably. According to Theorem A.2 there exist extension operators \(E_1\) and \(E_2\) such that

\[
\begin{align*}
\| E_1 \Theta \|_{H^1(\Omega;H^1_{per}(Y))} &\leq C_{ext} \| \Theta \|_{H^1(\Omega;H^1_{per}(Y))}, \\
\| E_2 \theta_\varepsilon \|_{L^2(\Omega)} &\leq C_{ext} \| \theta_\varepsilon \|_{L^2(\Omega)}, \quad \| \nabla (E_2 \theta_\varepsilon) \|_{L^2(\Omega)} \leq C_{ext} \| \nabla \theta_\varepsilon \|_{L^2(\Omega)},
\end{align*}
\]

where \(C_{ext}\) does not depend on \(\varepsilon\). For brevity, we write \(\tilde{\theta}_\varepsilon\) and \(\tilde{\Theta}\) for \(E_2 \theta_\varepsilon\) and \(E_1 \Theta\), respectively.

Note that for \(u_\varepsilon \in H^1(\Omega_\varepsilon)\) and \(u \in H^1(\Omega)\) the following two norms are equivalent up to an error of order \(O(\varepsilon)\), i.e.

\[ \| \mathcal{T}_\varepsilon u_\varepsilon - u \|_{L^2(\Omega \times Y_\varepsilon)} - \| u_\varepsilon - u \|_{L^2(\Omega_\varepsilon)} \leq \varepsilon C \| u \|_{H^1(\Omega)}, \]  

which is due to \(\| \mathcal{T}_\varepsilon u_\varepsilon - u \|_{L^2(\Omega \times Y_\varepsilon)} \leq \varepsilon C \| u \|_{H^1(\Omega)}\) by Lemma 3.1. Recall \(\beta = 1\).

Step 1: Reformulation of \(\theta_\varepsilon\)-equation. The weak formulation of the \(\theta_\varepsilon\)-equation reads

\[
\int_{\Omega_\varepsilon} \tilde{\theta}_\varepsilon \psi \, dx = \int_{\Omega_\varepsilon} -\kappa_\varepsilon \varepsilon \nabla \theta_\varepsilon \cdot \varepsilon \nabla \psi + \mu \varepsilon \nabla \theta_\varepsilon \cdot \nabla^\varepsilon u_\varepsilon \psi \, dx - \int_{\Gamma_\varepsilon} \varepsilon \mu \varepsilon \theta_\varepsilon \psi \, d\sigma
\]

for all admissible test functions \(\psi \in H^1(\Omega_\varepsilon)\). Applying the periodic unfolding operators \(\mathcal{T}_\varepsilon\) and \(\mathcal{T}_\varepsilon^*\), with \((\mathcal{T}_\varepsilon^* \kappa_\varepsilon)(x,y) = \mathcal{K}(y)\), and exploiting the integral identities in (3.1) as well as properties (A2) and (A4) in Lemma A.1 gives

\[
\begin{align*}
\int_{\Omega \times Y_\varepsilon} \mathcal{T}_\varepsilon^* \tilde{\theta}_\varepsilon \mathcal{T}_\varepsilon^* \psi \, dx \, dy &= \int_{\Omega \times Y_\varepsilon} -\mathcal{K} \nabla_y (\mathcal{T}_\varepsilon^* \tilde{\theta}_\varepsilon) \cdot \nabla_y (\mathcal{T}_\varepsilon^* \psi) + \mu \nabla_y (\mathcal{T}_\varepsilon^* \tilde{\theta}_\varepsilon) \cdot \mathcal{T}_\varepsilon^* (\nabla^\varepsilon u_\varepsilon) \mathcal{T}_\varepsilon^* \psi \, dx - \int_{\Omega \times Y_\varepsilon} g \mathcal{T}_\varepsilon^b \tilde{\theta}_\varepsilon \mathcal{T}_\varepsilon^b \psi \, dx \, d\sigma(y).
\end{align*}
\]
We choose \( \psi := \theta_{\varepsilon} - G_{\varepsilon} \bar{\Theta} \) in (3.8) such that \((G_{\varepsilon} \Theta)|_{\Omega_{\varepsilon}}\) is by construction of the gradient folding operator an admissible test function in \( H^{1}(\Omega_{\varepsilon}) \). Thus (3.8) equals

\[
\begin{align*}
\int_{\Omega \times Y_{e}} T_{\varepsilon}^* \dot{\theta}_{\varepsilon} T_{\varepsilon}^* (\theta_{\varepsilon} - G_{\varepsilon} \bar{\Theta})
&= \int_{\Omega \times Y_{e}} -\mathcal{K} \nabla_{g} (T_{\varepsilon}^* \dot{\theta}_{\varepsilon}) \cdot \nabla_{g} [T_{\varepsilon}^* (\theta_{\varepsilon} - G_{\varepsilon} \bar{\Theta})] + \mu \nabla_{g} (T_{\varepsilon}^* \theta_{\varepsilon}) \cdot T_{\varepsilon}^* (\nabla^{\delta} u_{e}) T_{\varepsilon}^* (\theta_{\varepsilon} - G_{\varepsilon} \bar{\Theta})
\end{align*}
\]

Adding \( \pm \Theta \) respective \( \pm \nabla_{g} \theta \) gives

\[
\begin{align*}
\int_{\Omega \times Y_{e}} T_{\varepsilon}^* \dot{\theta}_{\varepsilon} (T_{\varepsilon}^* \theta_{\varepsilon} - \Theta)
&= \int_{\Omega \times Y_{e}} -K \nabla_{g} (T_{\varepsilon}^* \dot{\theta}_{\varepsilon}) \cdot \nabla_{g} (T_{\varepsilon}^* \theta_{\varepsilon} - \Theta) + \mu \nabla_{g} (T_{\varepsilon}^* \theta_{\varepsilon}) \cdot T_{\varepsilon}^* (\nabla^{\delta} u_{e}) (T_{\varepsilon}^* \theta_{\varepsilon} - \Theta)
\end{align*}
\]

Exploiting (\( \Omega \times T \))
\[\epsilon \geq 0 \] and the higher \( \varepsilon \)-regularity \( \Theta \in L^2(0, T; H^1(\Omega_{\varepsilon}, Y_{e})) \) as well as applying Theorem 3.2 and Corollary 3.3 yields

\[
\begin{align*}
\int_{0}^{T} |\Delta_{\varepsilon}^{g_{\varepsilon} \bar{\Theta}}| dt &\leq C \left\| T_{\varepsilon}^* G_{\varepsilon} \bar{\Theta} - \Theta \right\|_{L^2(0, T; H^1(\Omega_{\varepsilon}, Y_{e})))},
\end{align*}
\]

where the constant \( C \) does not depend on \( \varepsilon \).

**Step 3: Reformulation of \( \Theta \)-equation.** The weak formulation of the \( \Theta \)-equation reads

\[
\begin{align*}
\int_{\Omega \times Y_{e}} \dot{\Theta} \psi dx dy &= \int_{\Omega \times Y_{e}} -K \nabla_{g} \Theta \cdot \nabla_{g} \psi + \mu \nabla_{g} \Theta \cdot \nabla^{\delta} u \psi dx dy
\end{align*}
\]
for all admissible test functions $\Psi \in L^2(\Omega; H^1_{per}(Y_*))$. We choose $\Psi_\varepsilon \in L^2(\Omega; H^1_{per}(Y))$ according to Theorem 3.3 such that we can control the periodicity defect of $T_\varepsilon \psi_2$ for arbitrary functions $\psi_2 \in H^1(\Omega)$. With $T_\varepsilon \psi_2 = T_\varepsilon^* \psi_2$ almost everywhere in $\Omega \times Y_*$ and $\Psi_\varepsilon |_{\Omega \times Y_*} \in L^2(\Omega; H^1_{per}(Y_*))$, we obtain

$$
\int_{\Omega \times Y_*} \dot{\Theta} T_\varepsilon^* \psi_2 \, dx \, dy = \int_{\Omega \times Y_*} -\mathcal{K} \nabla_y \Theta \cdot \nabla_y T_\varepsilon^* \psi_2 + \mu \nabla_y \Theta \cdot \nabla^\delta u T_\varepsilon^* \psi_2 \, dx \, dy \\
- \int_{\Omega \times \Gamma} g \dot{\Theta} T_\varepsilon^* \psi_2 \, dx \, d\sigma(y) + \Delta_{per}^\Theta,
$$

(3.12)

where the periodicity defect $\Delta_{per}^\Theta$ is given via

$$
\Delta_{per}^\Theta := \int_{\Omega \times Y_*} \dot{\Theta} (T_\varepsilon^* \psi_2 - \Psi_\varepsilon) - \mathcal{K} \nabla_y \Theta \cdot \nabla_y (\Psi - T_\varepsilon^* \psi_2) + \mu \nabla_y \Theta \cdot \nabla^\delta u (\Psi_\varepsilon - T_\varepsilon^* \psi_2) \, dx \, dy \\
- \int_{\Omega \times \Gamma} g \dot{\Theta} (\Psi_\varepsilon - T_\varepsilon^* \psi_2) \, dx \, d\sigma(y).
$$

Applying Hölder’s inequality with respect to the dual spaces $H^1(\Omega)$ and $H^1(\Omega)^*$ as well as using embedding (3.10) yields

$$
|\Delta_{per}^\Theta| \leq (1 + C_{emb}) \left\{ \|\dot{\Theta}\|_{L^2(\Omega; H^1(\Omega))} + C_{clips}^{-1} \|\nabla_y \Theta\|_{L^2(\Omega; H^1(\Omega))} + \|\nabla_y \Theta \cdot \nabla^\delta u\|_{L^2(\Omega; H^1(\Omega))} \right\}
$$

$$
\quad + g \left\{ \|\Theta\|_{L^2(\Omega; H^1(\Omega))} \right\} \|T_\varepsilon^* \psi_2 - \Psi_\varepsilon\|_{H^1(\Omega; H^1(\Omega)^*)}.
$$

According to the higher $x$-regularity of $\Theta$ in (2.13) we have $\nabla_y \Theta \cdot \nabla^\delta u \in L^2(\Omega; H^1(\Omega))$ with $\nabla^\delta u \in W^{1,\infty}(\Omega)$ such that we can apply Theorem 3.5 and obtain

$$
|\Delta_{per}^\Theta| \leq C \|T_\varepsilon^* \psi_2 - \Psi_\varepsilon\|_{H^1(\Omega; H^1(\Omega)^*)} \leq C \|T_\varepsilon^* \psi_2 - \Psi_\varepsilon\|_{H^1(\Omega; H^1(\Omega)^*)} \\
\quad \leq \sqrt{\varepsilon} C \left( \|\psi_2\|_{L^2(\Omega)} + \varepsilon \|\nabla \psi_2\|_{L^2(\Omega)} \right).
$$

(3.13)

We choose the test function $\psi_2 = \tilde{\theta}_\varepsilon - G_\varepsilon \tilde{\Theta}$ with $\psi_2 = \psi$ almost everywhere in $\Omega_\varepsilon$ and $\psi = \theta_\varepsilon - G_\varepsilon \tilde{\Theta}$ from Step 1. Exploiting the norm identity in (3.3), the triangle inequality, and Corollary 3.3 gives

$$
\|\psi_2\|_{L^2(\Omega)} + \varepsilon \|\nabla \psi_2\|_{L^2(\Omega)} = \|T_\varepsilon (\tilde{\theta}_\varepsilon - G_\varepsilon \tilde{\Theta})\|_{L^2(\Omega; H^1(\Omega))} \\
\quad \leq \|T_\varepsilon \tilde{\theta}_\varepsilon - \tilde{\Theta}\|_{L^2(\Omega; H^1(\Omega))} + \|T_\varepsilon G_\varepsilon \tilde{\Theta} - \tilde{\Theta}\|_{L^2(\Omega; H^1(\Omega))} \\
\quad \leq C_{ext} \|T_\varepsilon^* \theta_\varepsilon - \Theta\|_{L^2(\Omega; H^1(\Omega))} + \varepsilon C.
$$

Notice that $T_\varepsilon \tilde{\theta}_\varepsilon \in L^2(\Omega; H^1(\Omega))$ is one possible extension of $T_\varepsilon^* \theta_\varepsilon \in L^2(\Omega; H^1(\Omega))$ satisfying the properties in Theorem A.2. Furthermore applying Young’s inequality with $\eta_1 > 0$ in (3.13), we arrive at

$$
\int_0^T |\Delta_{per}^\Theta| \, dt \leq \eta_1 \|T_\varepsilon^* \theta_\varepsilon - \Theta\|_{L^2((0,T) \times \Omega; H^1(\Omega))}^2 + \varepsilon C_{\eta_1}.
$$

Inserting $\psi := \tilde{\theta}_\varepsilon - G_\varepsilon \tilde{\Theta}$ into (3.12) yields

$$
\int_{\Omega \times Y_*} \dot{\Theta} T_\varepsilon^* (\theta_\varepsilon - G_\varepsilon \tilde{\Theta}) \, dx \, dy = \int_{\Omega \times Y_*} \{-\mathcal{K} \nabla_y \Theta \cdot \nabla_y T_\varepsilon^* (\theta_\varepsilon - G_\varepsilon \tilde{\Theta}) \\
\quad + \mu \nabla_y \Theta \cdot \nabla^\delta u T_\varepsilon^* (\theta_\varepsilon - G_\varepsilon \tilde{\Theta})\} \, dx \, dy \\
- \int_{\Omega \times \Gamma} g \dot{\Theta} T_\varepsilon^* (\theta_\varepsilon - G_\varepsilon \tilde{\Theta}) \, dx \, d\sigma(y) + \Delta_{per}^\Theta.
$$
Adding $\pm \Theta$, and respectively $\pm \nabla_y \Theta$, as in Step 1 gives
\[
\int_{\Omega \times Y_*} \hat{\Theta}(T^*_\varepsilon \theta_\varepsilon - \Theta) \, dx \, dy = \int_{\Omega \times Y_*} -K \nabla_y \Theta \cdot \nabla_y (T^*_\varepsilon \theta_\varepsilon - \Theta) + \mu \nabla_y \Theta \cdot \nabla^\delta u(T^*_\varepsilon \theta_\varepsilon - \Theta) \, dx \, dy \\
- \int_{\Omega \times \Gamma} g \Theta(T^b_\varepsilon \theta_\varepsilon - \Theta) \, dx \, d\sigma(y) + \Delta^\Theta_{\text{per}} + \Delta^\Theta_{\text{fold}},
\] (3.14)
where the folding mismatch $\Delta^\Theta_{\text{fold}}$ is determined by
\[
\Delta^\Theta_{\text{fold}} := \int_{\Omega \times Y_*} \hat{\Theta}(T^*_\varepsilon \theta_\varepsilon \sim \Theta) - K \nabla_y \Theta \cdot \nabla_y (\Theta - T^*_\varepsilon \theta_\varepsilon \sim \Theta) + \mu \nabla_y \Theta \cdot \nabla^\delta u(\Theta - T^*_\varepsilon \theta_\varepsilon \sim \Theta) \, dx \, dy \\
- \int_{\Omega \times \Gamma} g \Theta(\Theta - T^b_\varepsilon \theta_\varepsilon \sim \Theta) \, dx \, d\sigma(y).
\]

The estimation of $\Delta^\Theta_{\text{fold}}$ follows along the lines of $\Delta^\Theta_{\text{fold}}$ in Step 1 using the boundedness of the limit $(u, \Theta)$, in particular, the boundedness of $||\hat{\Theta}||_{L^2((0,T) \times \Omega \times Y_*)}$. Finally, the estimation of all error terms in Steps 1–3 yields
\[
\int_0^T |\Delta^\Theta_{\text{fold}}| + |\Delta^\Theta_{\text{per}}| + |\Delta^\Theta_{\text{fold}}| \, dt \leq \varepsilon C_{\text{en}} + \eta \|T^*_\varepsilon \theta_\varepsilon - \Theta\|_{L^2((0,T) \times \Omega; H^1(Y_*))}^2.
\] (3.15)

**Step 4: Derivation of Grönwall-type estimates.** Subtracting equation (3.14) from (3.9) and using $\frac{1}{2} \frac{d}{dt} ||\Psi||_{L^2(\Omega \times Y_*)}^2 = \int_{\Omega \times Y_*} \hat{\Psi} \hat{\Psi} \, dx \, dy$ gives
\[
\frac{1}{2} \frac{d}{dt} ||T^*_\varepsilon \theta_\varepsilon - \Theta||_{L^2(\Omega \times Y_*)}^2 = \int_{\Omega \times Y_*} \left\{ -K[\nabla_y (T^*_\varepsilon \theta_\varepsilon - \Theta)] \cdot [\nabla_y (T^*_\varepsilon \theta_\varepsilon - \Theta)] \\
+ \mu[\nabla_y (T^*_\varepsilon \theta_\varepsilon) \cdot T^*_\varepsilon (\nabla^\delta u_\varepsilon) - \nabla_y \Theta \cdot \nabla^\delta u](T^*_\varepsilon \theta_\varepsilon - \Theta) \right\} \, dx \, dy \\
- \int_{\Omega \times \Gamma} g |T^b_\varepsilon \theta_\varepsilon - \Theta|^2 \, dx \, d\sigma(y) \\
+ \Delta^\Theta_{\text{fold}} - \Delta^\Theta_{\text{per}} - \Delta^\Theta_{\text{fold}}.
\] (3.16)

We continue by estimating each term on the right-hand side in (3.16) separately. Exploiting the interpolation inequality (cf. e.g. [LM72])
\[
\exists C_{\text{int}} > 0, \forall \Psi \in L^2(\Omega; H^1(Y_*)) : \|\Psi\|_{L^2(\Omega \times \Gamma)}^2 \leq C_{\text{int}} \|\Psi\|_{L^2(\Omega \times Y_*)} \|\Psi\|_{L^2(\Omega; H^1(Y_*))}
\]
and then Young’s inequality with $\eta_2 > 0$ lead to
\[
||T^b_\varepsilon \theta_\varepsilon - \Theta||_{L^2(\Omega \times \Gamma)}^2 \leq C_{\eta_2} ||T^*_\varepsilon \theta_\varepsilon - \Theta||_{L^2(\Omega \times Y_*)}^2 + \eta_2 ||T^*_\varepsilon \theta_\varepsilon - \Theta||_{L^2(\Omega; H^1(Y_*))}^2.
\] (3.17)

Reformulating the $\mu$-term gives
\[
\int_{\Omega \times Y_*} \mu[\nabla_y (T^*_\varepsilon \theta_\varepsilon) \cdot T^*_\varepsilon (\nabla^\delta u_\varepsilon) - \nabla_y \Theta \cdot \nabla^\delta u](T^*_\varepsilon \theta_\varepsilon - \Theta) \, dx \, dy \\
= \int_{\Omega \times Y_*} \mu[\nabla_y (T^*_\varepsilon \theta_\varepsilon) \cdot |T^*_\varepsilon (\nabla^\delta u_\varepsilon) - \nabla^\delta u|(T^*_\varepsilon \theta_\varepsilon - \Theta) \, dx \, dy \\
+ \int_{\Omega \times Y_*} \mu \nabla_y (T^*_\varepsilon \theta_\varepsilon - \Theta) \cdot \nabla^\delta u(T^*_\varepsilon \theta_\varepsilon - \Theta) \, dx \, dy.
\] (3.18)
Using in (3.19) that \( \| \nabla^\delta u \|_{L^\infty(\Omega \times Y)} = \| \nabla^\delta u \|_{L^\infty(\Omega)} \leq C_\delta \| u \|_{L^2(\Omega)} \) is bounded as well as Hölder’s and Young’s inequality with \( \eta_3 > 0 \) gives
\[
\left\| \int_{\Omega \times Y} \mu \nabla_y (T^\delta \Theta - \Theta) \cdot \nabla^\delta u (T^\delta \Theta - \Theta) \, dx \, dy \right\|
\leq \mu C_\delta \| u \|_{L^2(\Omega)} \| \nabla_y (T^\delta \Theta - \Theta) \|_{L^2(\Omega \times Y)} \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)}
\leq C_\eta \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)}^2 + \eta_3 \| \nabla_y (T^\delta \Theta - \Theta) \|_{L^2(\Omega \times Y)}^2.
\]
(3.20)

In a similar manner, we obtain for (3.18) by adding \( \pm T^\delta (\nabla^\delta u) \) and using Lemma 3.4
\[
\left\| \int_{\Omega \times Y} \mu \nabla_y (T^\delta \Theta) \cdot [T^\delta (\nabla^\delta u_e) - \nabla^\delta u (T^\delta \Theta - \Theta)] \, dx \, dy \right\|
\leq \mu \| \nabla_y (T^\delta \Theta) \|_{L^2(\Omega \times Y)} \| T^\delta (\nabla^\delta u_e) - \nabla^\delta u (T^\delta \Theta - \Theta) \|_{L^2(\Omega \times Y)}
\leq C \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)} \left\{ \| T^\delta (\nabla^\delta u_e) - T^\delta (\nabla^\delta u) \|_{L^\infty(\Omega \times Y)} + \| T^\delta (\nabla^\delta u) - \nabla^\delta u \|_{L^\infty(\Omega \times Y)} \right\}
\leq C_\delta \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)} \left\{ \| u_e - u \|_{L^2(\Omega)} + \sqrt{\varepsilon} \| u \|_{L^2(\Omega)} \right\}
\leq C \left\{ \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)}^2 + \| u_e - u \|_{L^2(\Omega)}^2 + \varepsilon \| u \|_{L^2(\Omega)}^2 \right\}.
\]
(3.21)

Overall we can estimate equation (3.16) using the uniform ellipticity of \( K \) and \( g \geq 0 \) as well as (3.17), (3.20), and (3.21) such that
\[
\frac{1}{2} \frac{d}{dt} \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)}^2 \leq (\eta_1 + \eta_2 + \eta_3 - C_{\text{elip}}) \| \nabla_y (T^\delta \Theta - \Theta) \|_{L^2(\Omega \times Y)}^2
+ C \left\{ \| T^\delta \Theta - \Theta \|_{L^2(\Omega \times Y)}^2 + \| u_e - u \|_{L^2(\Omega)}^2 \right\}
+ |\Delta^\delta \Theta| + |\Delta^\theta_{\text{per}}| + |\Delta^\theta_{\text{fold}}| + \varepsilon \| u \|_{L^2(\Omega)}^2.
\]

Choosing \( \eta_i = C_{\text{elip}} / 6 \), integrating over \([0, t]\) with \( 0 < t \leq T \), as well as recalling (3.7) and (3.15) yields
\[
\| T^\delta \Theta(t) - \Theta(t) \|_{L^2(\Omega \times Y)}^2 + C_{\text{elip}} \| \nabla_y (T^\delta \Theta - \Theta) \|_{L^2(\Omega \times Y)}^2
\leq C \left\{ \| T^\delta \Theta(t) - \Theta(t) \|_{L^2((0, t) \times \Omega \times Y)}^2 + \| T^\delta u_e - u \|_{L^2((0, t) \times \Omega \times Y)} \right\}
+ \| T^\delta \Theta^0 - \Theta^0 \|_{L^2(\Omega \times Y)}^2.
\]
(3.22)

Part B: Classical diffusion. We point out that the higher regularity of the limit \( u \in H^2(\Omega) \) implies the higher \( x \)-regularity of the corrector \( U \in H^1(\Omega; H^1_{\text{per}}(Y)) \) which is the unique minimizer of the unit cell problem \((2.12)\) with \( \xi = \nabla u(x) \). As in Part A, we denote by \( \tilde{U} \in H^1(\Omega; H^1_{\text{per}}(Y)) \) the extension of \( U \) by \( \mathcal{E}_1 \) according to Theorem A.2.

Step 1: Reformulation of \( u_e \)-equation. The weak formulation of the \( u_e \)-equation is given via \( (\alpha = 1) \)
\[
\int_{\Omega_e} \dot{u}_e \varphi \, dx = \int_{\Omega_e} -d_e \nabla u_e \cdot \nabla \varphi + \varepsilon \tau \nabla u_e \cdot \nabla \delta \varphi + R(u_e) \varphi \, dx - \int_{\Gamma_e} \varepsilon (a u_e + b v_e) \varphi \, d\sigma
\]
for all test functions \( \varphi \in H^1(\Omega_e) \). First of all note that the cross-diffusion term
\[
\Delta^\delta u_{\text{cross}} := \int_{\Omega_e} \varepsilon \tau \nabla u_e \cdot \nabla \delta \varphi \, dx
\]
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is of order $O(\varepsilon)$ thanks to Hölder’s inequality and the boundedness in (2.6) and (2.5)

$$
|\Delta_{\text{cross}}^u| \leq \varepsilon \tau C_\theta \||\nabla u_x||_{L^2(\Omega_x)}||\theta_x||_{L^2(\Omega_x)}\|\phi\|_{L^2(\Omega_x)}.
$$

Applying the unfolding operators $T^*_\varepsilon$ and $T^b_\varepsilon$, in particular, rewriting $(T^*_\varepsilon d_x(x,y) = D(y)$ and $(T^b_\varepsilon v_x)(x,y) = V(\varepsilon(x/\varepsilon)+y), y)$, and using the integral identities in (3.1) gives

$$
\int_{\Omega \times Y_x} T^*_\varepsilon \hat{u}_x T^*_\varepsilon \varphi \, dx \, dy = \int_{\Omega \times Y_x} -\mathcal{D} T^*_\varepsilon (\nabla u_x) T^*_\varepsilon (\nabla \varphi) + R (T^*_\varepsilon u_x) T^*_\varepsilon \varphi \, dx \, dy
\nonumber
- \int_{\Omega \times \Gamma} (a T^b_\varepsilon u_x + bV) T^b_\varepsilon \varphi \, dx \, ds(y) + \Delta_{\text{cross}}^u + \Delta_{\text{app}}^u,
$$

(3.23)

wherein we replaced the boundary term $T^b_\varepsilon v_x$ with $V$ and created the approximation error

$$
\Delta_{\text{app}}^u := \int_{\Omega \times \Gamma} b (V - T^b_\varepsilon v_x) T^b_\varepsilon \varphi \, dx \, ds(y).
$$

Using that $|x - \varepsilon ((x/\varepsilon)+y)| \leq \varepsilon \text{diam}(Y)$ holds for all $(x,y) \in \Omega \times \Gamma$, we obtain the pointwise estimate $|(T^b_\varepsilon v_x)(x,y) - V(x,y)| \leq \varepsilon C \|\nabla_x V\|_{L^\infty(\Omega)}$ thanks to the Lipschitz continuity of $x \mapsto V(x,y)$. Together with embedding (3.10), the approximation error is bounded by

$$
|\Delta_{\text{app}}^u| \leq \varepsilon C \|T^*_\varepsilon \varphi\|_{L^2(\Omega;H^1(\Gamma))}.
$$

We choose the test function $\varphi := u_x - (u + \varepsilon G_\varepsilon \tilde{U})|_{\Omega_x} \in H^1(\Omega_x)$ in (3.23) such that

$$
\int_{\Omega \times Y_x} T^*_\varepsilon \hat{u}_x (T^*_\varepsilon u_x - u) \, dx \, dy
\nonumber
= \int_{\Omega \times Y_x} -\mathcal{D} T^*_\varepsilon (\nabla u_x) [T^*_\varepsilon (\nabla u_x) - (\nabla u + \nabla_y U)] + R (T^*_\varepsilon u_x) (T^*_\varepsilon u_x - u) \, dx \, dy
\nonumber
- \int_{\Omega \times \Gamma} (a T^b_\varepsilon u_x + bV) (T^b_\varepsilon u_x - u) \, dx \, ds(y) + \Delta_{\text{cross}}^u + \Delta_{\text{app}}^u + \Delta_{\text{fold}}^u,
$$

(3.24)

where we added $\pm u$ and $\pm [\nabla u + \nabla_y U]$, and created the folding mismatch

$$
\Delta_{\text{fold}}^u := \int_{\Omega \times Y_x} \{(R(T^*_\varepsilon u_x) - T^*_\varepsilon \hat{u}_x)[u - T^*_\varepsilon (u + \varepsilon G_\varepsilon \tilde{U})]
\nonumber
-\mathcal{D} T^*_\varepsilon (\nabla u_x) [\nabla u + \nabla_y U - T^*_\varepsilon (\nabla (u + \varepsilon G_\varepsilon \tilde{U}))] \} \, dx \, dy
\nonumber
- \int_{\Omega \times \Gamma} (a T^b_\varepsilon u_x + bV) [u - T^b_\varepsilon (u + \varepsilon G_\varepsilon \tilde{U})] \, dx \, ds(y).
$$

(3.25)

The boundary term in (3.25) is controlled via

$$
\|T^*_\varepsilon u - u\|_{L^2(\Omega \times \Gamma)} \leq C_{\text{emb}} \|T^*_\varepsilon u - u\|_{L^2(\Omega;H^1(\Gamma))}
\nonumber
= C_{\text{emb}} \left( \|T^*_\varepsilon u - u\|^2_{L^2(\Omega \times \Gamma)} + \|\varepsilon T^*_\varepsilon (\nabla u)\|^2_{L^2(\Omega \times Y_x)} \right)^{1/2} \leq \varepsilon C \|u\|_{H^1(\Omega)},
$$

(3.26)

while noting that $\nabla_y u = 0$. Exploiting the higher regularity $u \in H^2(\Omega)$ we obtain with Lemma 3.1 and Theorem 3.2

$$
\|T^*_\varepsilon u - u\|_{L^2(\Omega \times \Gamma)} + \|T^*_\varepsilon (\nabla u) - \nabla u\|_{L^2(\Omega \times Y_x)} \leq \varepsilon C \|u\|_{H^2(\Omega)},
$$

$$
\|T^*_\varepsilon [\varepsilon \nabla (G_\varepsilon \tilde{U})] - \nabla_y U\|_{L^2(\Omega \times Y_x)} \leq \|T^*_\varepsilon [\varepsilon \nabla (G_\varepsilon \tilde{U})] - \nabla_y U\|_{L^2(\Omega \times Y)} \leq \varepsilon C_{\text{ext}} \|U\|_{H^1(\Omega;H^1(\Gamma))}.
$$
With this, we can estimate the gradient term of $\Delta_{\text{fold}}^u$ in (3.25). The remaining term $\varepsilon T_e^* G_e \tilde{U}$ is of order $O(\varepsilon)$ since $\|G_e U\|_{L^2(\Omega \times Y)}$ is bounded according to (3.2). Applying Hölder’s inequality and the boundedness of $\varepsilon$ we can estimate the gradient term of $\Delta u$ as in Step 1 and $\tilde{\phi}$ to the space $H^1_\text{per}$ for all test functions $\phi$, that we can control the periodicity defect of $u$ in (2.6), we obtain for the folding mismatch $\varepsilon$

$$\int_0^T |\Delta_{\text{fold}}^u| \, dt \leq \varepsilon C,$$  

(3.27)

where the constant $C$ does not depend on $\varepsilon$.

**Step 2: Reformulation of $u$-equation.** The weak formulation reads

$$\int_{\Omega \times Y_e} \tilde{u} \varphi_2 \, d\sigma(y) = \int_{\Omega \times Y_e} -D[\nabla u + \nabla \tilde{y}] : [\nabla \varphi_2 + \nabla \tilde{\phi}] + R(u) \varphi_2 \, d\sigma(y)$$

for all $\varphi_2 \in H^1(\Omega)$ which is equivalent to, see e.g. [CDZ06],

$$\int_{\Omega \times Y_e} \tilde{u} \varphi_2 \, d\sigma(y) = \int_{\Omega \times Y_e} -D[\nabla u + \nabla \tilde{y}] : [\nabla \varphi_2 + \nabla \tilde{\phi}] + R(u) \varphi_2 \, d\sigma(y)$$

The periodicity defect is given via

$$\Delta_{\text{per}}^u := \int_{\Omega \times Y_e} [R(u) - \tilde{u}] (\varphi_2 - T_e^* \varphi_2) - D[\nabla u + \nabla \tilde{y}] : [\nabla \varphi_2 + \nabla \tilde{\phi} - T_e^* (\nabla \varphi_2)] \, d\sigma(y)$$

and it is controlled by applying Lemma 3.1 Theorem 3.6 with $D[\nabla \tilde{u} + \nabla \tilde{y}] \in H^1(\Omega; L^2(Y_e))$, arguing as in (3.26) for the boundary term, and using the boundedness of $u$ in (2.13) via

$$|\Delta_{\text{per}}^u| \leq \sqrt{\varepsilon} C \|\varphi_2\|_{H^1(\Omega)}.$$  

(3.29)

We choose $\varphi_2 := \bar{\varphi} \in H^1(\Omega)$, where $\varphi = u - (u + \varepsilon G_e \tilde{U})|_{\Omega_e}$ is the same test function as in Step 1 and $\bar{\varphi} = E_2 \varphi$ as in Theorem A.2. The $H^1$-norm in (3.29) is controlled via $\|\varphi_2\|_{H^1(\Omega)} \leq C_{\text{ext}} \|\bar{\varphi}\|_{H^1(\Omega)}$ and

$$\|\varphi\|_{H^1(\Omega_e)}^2 = \|T_e^* u - T_e^* (u + \varepsilon G_e \tilde{U})\|_{L^2(\Omega \times Y_e)}^2 + \|T_e^* (\nabla u) - T_e^* (\nabla u + \varepsilon G_e \tilde{U})\|_{L^2(\Omega \times Y_e)}^2$$

$$\leq \|T_e^* u - u\|_{L^2(\Omega \times Y_e)}^2 + \|T_e^* (\nabla u) - (\nabla u + \nabla \tilde{y})\|_{L^2(\Omega \times Y_e)}^2 + \varepsilon^2 C$$

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by using (3.2) for \( \varepsilon \| T_\varepsilon G_\varepsilon \tilde{U} \|_{L^2(\Omega \times Y)} \), Lemma 3.1 for \( \| T_\varepsilon^* (\nabla u - \nabla \tilde{u}) \|_{L^2(\Omega \times Y)} \), Theorem 3.2 for \( \| T_\varepsilon (\varepsilon \nabla G_\varepsilon U) - \nabla \tilde{U} \|_{L^2(\Omega \times Y)} \), as well as the \( H^2(\Omega) \)-regularity of \( u \) respective the \( H^1(\Omega) \)-regularity of \( U \). Applying Young’s inequality with \( \eta_1 > 0 \) in (3.29) yields

\[
|\Delta_{\text{per}}^u| \leq \varepsilon C_{\eta_1} + \eta_1 \left( \| T_\varepsilon^* u_{\varepsilon} - u \|_{L^2(\Omega \times Y)}^2 + \| T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U) \|_{L^2(\Omega \times Y)}^2 \right). \tag{3.30}
\]

Inserting \( \varphi_2 = \tilde{\varphi} \) into (3.28) with \( T_\varepsilon^* \varphi_2 = T_\varepsilon^* \varphi \) almost everywhere in \( \Omega \times Y \), we obtain

\[
\int_{\Omega \times Y} \hat{u} (T_\varepsilon^* u_{\varepsilon} - u) \, dx \, dy = \int_{\Omega \times Y} \{ -D [\nabla u + \nabla \tilde{y} U] : [T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U)] + R(u)(T_\varepsilon^* u_{\varepsilon} - u) \} \, dx \, dy
\]

\[
- \int_{\Omega \times \Gamma} (au + \nabla \tilde{y} (T_\varepsilon^* u_{\varepsilon} - u)) \, dx \, d\sigma(y) + \Delta_{\text{per}}^u + \Delta_{\text{fold}}^u, \tag{3.31}
\]

and another folding mismatch

\[
\Delta_{\text{fold}}^u := \int_{\Omega \times Y} \{ (R(u) - \hat{u}) [u - T_\varepsilon^* (u + \varepsilon G_\varepsilon \tilde{U})]
\]

\[
- D [\nabla u + \nabla \tilde{y} U] : [\nabla u + \nabla \tilde{y} U - T_\varepsilon^* (\nabla u_{\varepsilon} + \varepsilon \nabla G_\varepsilon \tilde{U})] \} \, dx \, dy
\]

\[
- \int_{\Omega \times \Gamma} (au + \nabla \tilde{y} [u - T_\varepsilon^* (u + \varepsilon G_\varepsilon \tilde{U})]) \, dx \, d\sigma(y). \tag{3.32}
\]

The folding mismatch \( \Delta_{\text{fold}}^u \) has the same form as \( \Delta_{\text{fold}}^u \) in (3.25) when replacing \( u \) with \( T_\varepsilon^* u_{\varepsilon} \). Hence, we have as in (3.27)

\[
\int_0^T |\Delta_{\text{fold}}^u| \, dt \leq \varepsilon C.
\]

**Step 3: Derivation of Grönwall-type estimates.** Subtracting equation (3.31) from (3.24) yields with \( \Delta_{\varepsilon} := \Delta_{\text{cross}}^u + \Delta_{\text{app}}^u + \Delta_{\text{fold}}^u - \Delta_{\text{fold}}^u \)

\[
\frac{1}{2} \frac{d}{dt} \| T_\varepsilon^* u_{\varepsilon} - u \|_{L^2(\Omega \times Y)}^2 = \int_{\Omega \times Y} \{ -D [T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U)] : [T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U)] + R(T_\varepsilon^* u_{\varepsilon} - R(u))(T_\varepsilon^* u_{\varepsilon} - u) \} \, dx \, dy
\]

\[
- \int_{\Omega \times \Gamma} a |T_\varepsilon^b u_{\varepsilon} - u|^2 \, dx \, d\sigma(y) + \Delta_{\varepsilon} - \Delta_{\text{per}}^u.
\]

Using the uniform ellipticity of \( D_\varepsilon \), \( a \geq 0 \), the Lipschitz continuity of \( R \), and the estimations of the periodic defect \( \Delta_{\text{per}}^u \) in (3.30) gives

\[
\frac{1}{2} \frac{d}{dt} \| T_\varepsilon^* u_{\varepsilon} - u \|_{L^2(\Omega \times Y)}^2 \leq -C_{\text{clip}} \| T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U) \|_{L^2(\Omega \times Y)}^2 + L \| T_\varepsilon^* u_{\varepsilon} - u \|_{L^2(\Omega \times Y)}^2
\]

\[
+ \eta_1 \left( \| T_\varepsilon^* u_{\varepsilon} - u \|_{L^2(\Omega \times Y)}^2 + \| T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U) \|_{L^2(\Omega \times Y)}^2 \right)
\]

\[
+ \varepsilon C_{\eta_1} + |\Delta_{\varepsilon}|. \tag{3.33}
\]

Choosing \( \eta_1 = C_{\text{clip}}/2 \) and integrating over \((0, t)\) with \( 0 < t \leq T \), we arrive at

\[
\| T_\varepsilon^* u_{\varepsilon}(t) - u(t) \|_{L^2(\Omega \times Y)}^2 + C_{\text{clip}} \| T_\varepsilon^* (\nabla u_{\varepsilon}) - (\nabla u + \nabla \tilde{y} U) \|_{L^2(\Omega \times Y)}^2
\]

\[
\leq C \left\{ \| T_\varepsilon^* u_{\varepsilon} - u \|_{L^2((0, t) \times \Omega \times Y)}^2 + \| T_\varepsilon^* \theta_{\varepsilon} - \Theta \|_{L^2((0, t) \times \Omega \times Y)}^2 + \varepsilon \right\}
\]

\[
+ \| T_\varepsilon^* u_{\varepsilon}^0 - u^0 \|_{L^2(\Omega \times Y)}^2. \tag{3.34}
\]
Final Step. We add (3.22) and (3.34) and finally obtain for all $t \in [0, T]$

$$
\|T_\epsilon^* u_\epsilon(t) - u(t)\|_{L^2(\Omega \times Y)}^2 + \|T_\epsilon^* \theta_\epsilon(t) - \Theta(t)\|_{L^2(\Omega \times Y)}^2
+ C_{\text{elip}} \left\{ \|T_\epsilon^* (\nabla u_\epsilon) - (\nabla u + \nabla y U)\|_{L^2((0,T) \times \Omega \times Y)}^2 + \|T_\epsilon^* (\epsilon \nabla \theta_\epsilon) - \nabla y \Theta\|_{L^2((0,T) \times \Omega \times Y)}^2 \right\}
\leq C \left\{\|T_\epsilon^* u_\epsilon - u\|_{L^2((0,T) \times \Omega \times Y)}^2 + \|T_\epsilon^* \theta_\epsilon - \Theta\|_{L^2((0,T) \times \Omega \times Y)}^2 + \epsilon \right\}
+ \|T_\epsilon^* u_\epsilon^0 - u^0\|_{L^2(\Omega \times Y)}^2 + \|T_\epsilon^* \theta_\epsilon^0 - \Theta^0\|_{L^2(\Omega \times Y)}^2.
$$

The application of Grönwall’s Lemma and the convergence of the initial values in (3.5) complete the proof of (3.6). \hfill \square

4 Different parameter choices for $(\alpha, \beta)$

The existence and uniform boundedness of the solutions $(u_\epsilon, \theta_\epsilon)$ for the original system (2.1)–(2.3) as in Theorem 2.4 is independent of the parameters $(\alpha, \beta)$. However the choice of parameters determines the coupling via cross-diffusion in the limit $\epsilon \to 0$. Besides the critical parameter choice $\alpha = \beta = 1$, we identify two more critical regimes, namely, $(\alpha, \beta) = (0, 1)$ and $(\alpha, \beta) = (1, 2)$, which will be discussed hereafter.

4.1 The case $(\alpha, \beta) = (0, 1)$

In this case the two-scale limit system includes an additional macroscopic cross-diffusion term and is of the form

$$
\begin{aligned}
\dot{u} &= \text{div}(d_{\text{eff}} \nabla u) + \tau_{\text{eff}} \nabla u \cdot \nabla \delta \theta_0 + R(u) - \frac{\partial \Theta}{\partial \tau}(au + bv_0) \quad \text{in } \Omega, \\
\dot{\Theta} &= \text{div}_y (K \nabla_y \Theta) + \mu \nabla_y \Theta \cdot \nabla \delta u \quad \text{in } \Omega \times Y,
\end{aligned}
$$

(4.1)

where the effective Soret coefficient $\tau_{\text{eff}}$ is given via $\tau_{\text{eff}} = \tau(|Y_1|/|Y|) = |Y_2| \tau$. The mollified gradient $\nabla \delta \theta_0$ is given via formula (2.9) and $\theta_0 \in L^2(\Omega)$ denotes the microscopic average

$$
\theta_0(x) := \int_{Y_\epsilon} \Theta(x, y) \, dy.
$$

(4.2)

Equations (4.1) are supplemented with the boundary conditions in (2.10) and the initial conditions in (2.11). We recover our main corrector estimates (1.1) for $(\alpha, \beta) = (0, 1)$.

Theorem 4.1. Let $(u_\epsilon, \theta_\epsilon)$ and $(u, \Theta)$ denote the unique solution of system (2.1)–(2.3) and (4.1), (2.10)–(2.11), respectively. If the initial values satisfy condition (3.5), then the corrector estimates (3.6) hold true.

Proof. The proof follows the lines of Theorem 3.7 and we only specify the modifications here. Part A (slow diffusion) of the proof remains as it is. In Part B (classical diffusion) the weak formulation of the $u_\epsilon$-equations reads for $\alpha = 0$

$$
\int_{\Omega_\epsilon} \dot{u}_\epsilon \varphi \, dx = \int_{\Omega_\epsilon} -d_\epsilon \nabla u_\epsilon \cdot \nabla \varphi + \tau \nabla u_\epsilon \cdot \nabla \delta \varphi + R(u_\epsilon) \varphi \, dx - \int_{\Gamma_\epsilon} \epsilon (au_\epsilon + bv_\epsilon) \varphi \, d\sigma
$$

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with the particular test function \( \varphi = u - (u + \varepsilon G_\varepsilon \tilde{U}) \) \( \mid \Omega \) \( \in H^1(\Omega) \). In the weak formulation of the \( u \)-equation arises the additional cross-diffusion term, namely,

\[
\int_{\Omega \times Y_\varepsilon} \dot{\varphi} \, dxdy = \int_{\Omega \times Y_\varepsilon} \left\{ -\nabla \cdot \left[ \nabla u + \nabla_\varepsilon \right] \cdot [\nabla \varphi + \nabla_\varepsilon \Phi] + \tau \left[ \nabla u + \nabla_\varepsilon U \right] \cdot \nabla \theta_0 \varphi_2 \\
+ R(u) \varphi_2 \right\} \, dxdy - \int_{\Omega \times \Gamma} (au + bV) \varphi_2 \, d\sigma(y).
\]

Notice that \( \int Y_\varepsilon \nabla_\varepsilon U(x, y) \, dy = 0 \) due to periodicity, and hence, we have for all \( \varphi_2 \in H^1(\Omega) \) the equivalence

\[
\int_{\Omega \times Y_\varepsilon} \tau \left[ \nabla u + \nabla_\varepsilon U \right] \cdot \nabla \theta_0 \varphi_2 \, dxdy = \int_{\Omega} \tau_{\text{eff}} \nabla u \cdot \nabla \theta_0 \varphi_2 \, dx.
\]

Now we can reformulate both weak formulations as in the steps 1–3 in Part B by choosing \( \varphi_2 = \mathcal{E}_2 \varphi \) and noting that \( \Delta_{\text{cross}} = O(1) \). It remains to control the difference of the cross-diffusion terms

\[
\int_{\Omega \times Y_\varepsilon} \tau \mathcal{T}_\varepsilon^* \left( \nabla u_\varepsilon \right) \cdot \mathcal{T}_\varepsilon^* \left( \nabla \theta_\varepsilon \right) \mathcal{T}_\varepsilon^* \left( \varphi \right) - \tau \left[ \nabla u + \nabla_\varepsilon U \right] \cdot \nabla \theta_0 \varphi_2 \, dxdy. \tag{4.3}
\]

Since the gradient of the test functions does not occur in integral (4.3), we can neglect the corrector term \( \varepsilon G_\varepsilon \tilde{U} \), which is of order \( O(\varepsilon) \), in what follows. Moreover, we can replace \( \mathcal{T}_\varepsilon^* \varphi_2 \) and \( \mathcal{T}_\varepsilon^* (u_\varepsilon - u) \) with \( \mathcal{T}_\varepsilon^* u_\varepsilon - u \), respectively, by creating another error of order \( O(\varepsilon) \) according to Lemma 3.1 and \( \mathcal{E}_2 u_\varepsilon, u \in H^1(\Omega) \) uniformly bounded. With this, the integral in (4.3) is equivalent to

\[
\int_{\Omega \times Y_\varepsilon} \tau \mathcal{T}_\varepsilon^* \left( \nabla u_\varepsilon \right) - \left( \nabla u + \nabla_\varepsilon U \right) \cdot \mathcal{T}_\varepsilon^* \left( \nabla \theta_\varepsilon \right) \mathcal{T}_\varepsilon^* \left( u_\varepsilon - u \right) \\
+ \tau \left[ \nabla u + \nabla_\varepsilon U \right] \cdot \left[ \mathcal{T}_\varepsilon^* \left( \nabla \theta_\varepsilon \right) - \nabla \theta_0 \right] \left( \mathcal{T}_\varepsilon^* u_\varepsilon - u \right) \, dxdy + O(\varepsilon).
\]

By using Hölder’s and Young’s inequality with \( \eta_2 > 0 \) as well as (4.4), we control the latter expression as in (3.21) via

\[
\eta_2 \left\| \mathcal{T}_\varepsilon^* \left( \nabla u_\varepsilon \right) - \left( \nabla u + \nabla_\varepsilon U \right) \right\|_{L^2(\Omega \times Y_\varepsilon)} + C_2 \left\| \mathcal{T}_\varepsilon^* u_\varepsilon - u \right\|_{L^2(\Omega \times Y_\varepsilon)} + C_\delta \left\| \mathcal{T}_\varepsilon^* \theta_\varepsilon - \Theta \right\|_{L^2(\Omega \times Y_\varepsilon)} + \varepsilon C_\delta.
\]

Adding this term to the right-hand side in (3.33), we choose \( \eta_1 = \eta_2 = C_\text{clip}/4 \) and obtain estimate (3.34). The corrector estimates (3.6) follow as in the Final Step.

**Lemma 4.2.** Let \( \theta_\varepsilon \) and \( \theta_0 \) be as in Theorem 4.1 and equation (4.2). Then it holds

\[
\left\| \mathcal{T}_\varepsilon^* \left( \nabla \theta_\varepsilon \right) - \nabla \theta_0 \right\|_{L^\infty(\Omega \times Y_\varepsilon)} \leq C_\delta \left( \left\| \mathcal{T}_\varepsilon^* \theta_\varepsilon - \Theta \right\|_{L^2(\Omega \times Y_\varepsilon)} + \varepsilon C \right), \tag{4.4}
\]

where \( C_\delta, C > 0 \) do not depend on \( \varepsilon \).

**Proof.** Fix \( \delta > 0 \) and let \( x \in \Omega \) be such that \( B_\delta (x) \subseteq \Omega \), otherwise we extend all functions with zero outside of \( \Omega \) as in (2.9). According to Lemma 3.4, we have for \( \theta_\varepsilon \in L^2(\Omega) \)

\[
\left\| \mathcal{T}_\varepsilon^* \left( \nabla \theta_\varepsilon \right) - \nabla \theta_\varepsilon \right\|_{L^\infty(\Omega \times Y_\varepsilon)} \leq \sqrt{\varepsilon} C_\delta \left\| \theta_\varepsilon \right\|_{L^2(\Omega)}.
\]


We can replace \( T \). Therefore the difference in (4.5) can be controlled via
\[
\frac{\partial B}{\partial \theta}(x) \subseteq \Omega. \quad \text{We denote by} \quad \tilde{B}_\delta(x) \quad \text{the subset of all microscopic cells} \quad \varepsilon(\lambda+Y) \quad \text{that fit completely into} \quad B_\delta(x), \quad \text{i.e.}
\[
\tilde{B}_\delta(x) := \{ x \in \mathbb{R}^d \mid \varepsilon([\overline{\varepsilon}] + Y^o) \subseteq B_\delta(x) \}.
\]
The remaining cells intersected by the boundary \( \partial B_\delta(x) \) are denoted \( \Lambda_\varepsilon := B_\delta(x) \setminus \tilde{B}_\delta(x) \), see Figure [4.1]. Thus we have
\[
|\nabla^\delta \theta_\varepsilon - \nabla^\delta \theta_0|(x) = \int_{\tilde{B}_\delta(x)} \nabla J_\delta(x-z)\theta^e_\varepsilon(z) - \nabla J_\delta(x-z)\theta_0(z) \, dz + \int_{\Lambda_\varepsilon} \nabla J_\delta(x-z)[\theta^e_\varepsilon(z) - \theta_0(z)] \, dz.
\]
Notice that the volume of \( \Lambda_\varepsilon \) is of order \( O(\varepsilon) \), namely, \( |\Lambda_\varepsilon| \leq \varepsilon \text{diam}(Y) \text{meas}(\partial B_\delta(x)) \) with \( \text{meas}(\cdot) \) denoting the surface measure. With this and Hölder’s inequality we obtain
\[
\left| \int_{\Lambda_\varepsilon} \nabla J_\delta(x-z)[\theta^e_\varepsilon(z) - \theta_0(z)] \, dz \right| \leq \sqrt{\varepsilon} C_\delta \left( \| \theta_e \|_{L^2(\Omega_\varepsilon)} + \| \Theta \|_{L^2(\Omega \times \mathbb{R})} \right).
\]
The integral identity as well as \( (T^*_\varepsilon \theta^e_\varepsilon)(z,y) = 0 \) for all \( (z,y) \in B_\delta(x) \times Y_\text{hole} \) give
\[
\int_{\tilde{B}_\delta(x)} \nabla J_\delta(x-z)\theta^e_\varepsilon(z) \, dz = \int_{\tilde{B}_\delta(x) \times Y} (T^*_\varepsilon)[\nabla J_\delta(x-\cdot)](z,y)) \, dz \, dy = \int_{\tilde{B}_\delta(x) \times Y_\varepsilon} (T^*_\varepsilon \theta^e_\varepsilon)(z,y) \, dz \, dy.
\]
We can replace \( T^*_\varepsilon(\nabla J_\delta) \) with \( \nabla J_\delta \) by exploiting the Lipschitz continuity of the mollifier
\[
|\nabla J_\delta(x-\varepsilon[\overline{\varepsilon}] - \varepsilon y) - \nabla J_\delta(x-z)| \leq \varepsilon \| J_\delta \|_{C^\infty(\mathbb{R}^d)} \text{diam}(Y).
\]
Therefore the difference in (4.5) can be controlled via
\[
\left| \int_{\tilde{B}_\delta(x) \times Y_\varepsilon} \nabla J_\delta(x-z)[(T^*_\varepsilon \theta_\varepsilon)(z,y) - \Theta(z,y)] \, dz \, dy \right| + \varepsilon C_\delta \leq C_\delta \left( \| T^*_\varepsilon \theta_\varepsilon - \Theta \|_{L^2(\Omega \times \mathbb{R})} + \varepsilon C \right),
\]
which proves estimate (4.4).
4.2 The case $\alpha \ge 1$ and $\beta \ge 2$

In this case the two-scale limit system does not include any cross-diffusion term anymore and is of the form

$$\dot{u} = \text{div}(d_{\text{eff}} \nabla u) + R(u) - \frac{[\partial T]}{[\nabla]}(au + bv_0) \quad \text{in } \Omega, \quad \dot{\Theta} = \text{div}_y(K\nabla_y \Theta) \quad \text{in } \Omega \times Y_*.$$  \hspace{1cm} (4.6)

In particular the concentration $u$ and the temperature $\Theta$ decouple in the limit $\varepsilon \to 0$ for this choice of parameters $(\alpha, \beta)$. Equations (4.6) are supplemented with the boundary conditions in (2.10) and the initial conditions in (2.11). Again we recover the corrector estimates (1.1) for all $\alpha \ge 1$ and $\beta \ge 2$.

**Theorem 4.3.** Let $(u_\varepsilon, \theta_\varepsilon)$ and $(u, \Theta)$ denote the unique solution of system (2.1)–(2.3) and (4.6), (2.10), (2.11), respectively. If the initial values satisfy condition (3.5), then the corrector estimates (3.6) hold true.

**Proof.** In Step 1 of Part A of the proof for Theorem 3.7 the cross-diffusion term reads

$$\varepsilon^{\beta-1} \int_{\Omega_\varepsilon} \mu \varepsilon \nabla \theta_\varepsilon \cdot \nabla \delta u_\varepsilon \psi \, dx$$

and is bounded by $\varepsilon^{\beta-1} \mu C_\delta \|\varepsilon \nabla \theta_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\varepsilon u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\psi\|_{L^2(\Omega_\varepsilon)} = O(\varepsilon^{\beta-1})$ with $\beta-1 \ge 1$. Hence the estimation of the cross-diffusion term in (3.18)–(3.19) is omitted. In the same manner the cross-diffusion term in Step 1 of Part B satisfies $\Delta_{\text{cross}} u_\varepsilon = O(\varepsilon^\alpha)$ with $\alpha \ge 1$. The remaining steps of the proof follow analogously.

**Remark 4.4.**
1. For the intermediate parameter regime $\alpha \in (0, 1)$ and $\beta \in (1, 2)$, we obtain in the limit $\varepsilon \to 0$ the same two-scale system (4.6). In this case we also obtain corrector estimates, however, the convergence rate $\sqrt{\varepsilon}$ is not valid, but it is replaced by the lower rate $\min\{\varepsilon^{\alpha/2}, \varepsilon^{(\beta-1)/2}\}$.

2. The parameters $\alpha \ge 0$ and $\beta \ge 1$ can be chosen independently. For instance the choice $(\alpha, \beta) = (0, 2)$ yields the two-scale system

$$\dot{u} = \text{div}(d_{\text{eff}} \nabla u) + \tau_{\text{eff}} \nabla u \cdot \nabla \delta \theta_0 + R(u) - \frac{[\partial T]}{[\nabla]}(au + bv_0) \quad \text{in } \Omega, \quad \dot{\Theta} = \text{div}_y(K\nabla_y \Theta) \quad \text{in } \Omega \times Y_*,$$

with the boundary conditions in (2.10) and the initial conditions in (2.11). Again the arguments of the Theorems 4.1 and 4.3 give the corrector estimates (3.6).

5 Discussion

Our corrector estimates generalize the qualitative homogenization result obtained in [KAM14] in two ways: on the one hand we prove quantitative estimates. On the other hand, we consider slow thermal diffusion as well as the variable scaling $\varepsilon^\alpha$ and $\varepsilon^\beta$ in front of the cross-diffusion terms. Under slightly more general assumptions on the data with respect to the $x$-dependence, our estimates imply in particular the rigorous qualitative homogenization limit for this system.
Possible generalizations concerning the data. Our analysis allows for not-exactly periodic coefficients such as $d_\varepsilon(x) := \mathbb{D}(x, x/\varepsilon)$ with $\mathbb{D} \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{L}^\infty(Y_\varepsilon))$ as in [Rei16]. The coefficients $\tau_\varepsilon$ and $\mu_\varepsilon$ as well as the reaction term $R_\varepsilon$ may also be not-exactly periodic in the same manner. For instance the choice $\mu_\varepsilon(x) := M(x, x/\varepsilon)$ with $M \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{L}^\infty(Y_\varepsilon))$ yields for $\beta=1$ in the limit $\varepsilon \to 0$ the microscopic cross-diffusion term $M(x, y) \nabla_x \Theta \cdot \nabla^\delta u$. Similarly the choice $\tau_\varepsilon(x) := T(x, x/\varepsilon)$ with $T \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{L}^\infty(Y_\varepsilon))$ yields for $\alpha=0$ the limiting macroscopic cross-diffusion term $\tau_{\text{eff}}(x) \nabla u \cdot \nabla^\delta \theta_0$, where $\tau_{\text{eff}}(x)$ is given via a unit cell problem as for $d_{\text{eff}}$, cf. [KAM14]. Moreover all coefficients may additionally depend Lipschitz continuously on time.

The sink/source term $v_\varepsilon$ may be less regular by choosing $v_\varepsilon(t, x) := [\mathcal{F}_0\mathcal{V}(x, \cdot, \cdot)](t)$ to capture, if needed, possible spatial discontinuities in $\mathcal{V} \in C([0, T]; H^1(\Omega; L^2(\Gamma)))$.

On the boundary $\Gamma_\varepsilon$ we may consider globally Lipschitz continuous reaction terms $g : \mathbb{R} \to \mathbb{R}$. In this case, the boundary term in (3.16) is controlled by $L \| T^0 \theta_0 - \Theta \|_{L^2(\Omega \times \Gamma)}$, where $L > 0$ denotes the global Lipschitz constant. Non-linear boundary terms may require better initial values to derive the $L^2$-regularity of the time derivatives as in [FMP12, Rei15], however the error estimates hold as they are.

On the choice of the initial values. For given $u^0 \in H^1(\Omega) \cap L^\infty(\Omega)$ the obvious choice is $u_\varepsilon^0 = u^0|_{\Omega_\varepsilon}$ such that the assumption $\| T_\varepsilon^* u_\varepsilon^0 - u_\varepsilon^0 \|_{L^2(\Omega_\varepsilon \times Y_\varepsilon)} \leq \sqrt{\varepsilon} C_0$ is satisfied. Perturbations of the form $u_\varepsilon^0 = u^0 + \varepsilon V(x, x/\varepsilon)$, which preserve non-negativity, are possible as well.

In the case of slow diffusion such a direct choice is not possible mainly because $\theta_\varepsilon^0$ and $\Theta^0$ belong to spaces of dimension $d$ and $2d$, respectively. Let $\Theta^0 \in H^1(\Omega; H^1_{\text{per}}(Y_\varepsilon)) \cap L^\infty(\Omega \times Y_\varepsilon)$ be given. One possible choice is $\theta_\varepsilon^0 = \mathcal{G}_\varepsilon \theta^0$, however we are not able to prove $\theta_\varepsilon^0 \in L^\infty(\Omega_\varepsilon)$ in this case, since $L^\infty(\Omega_\varepsilon)$ is not a Hilbert space. Hence, we assume strong differentiability, such as $\Theta^0 \in C^1(\overline{\Omega}; H^1_{\text{per}}(Y_\varepsilon))$ or $\Theta^0 \in H^1(\Omega; C^1_{\text{per}}(Y_\varepsilon))$, so that $\theta_\varepsilon^0 = \Theta^0(x, x/\varepsilon)$ is well-defined in $H^1(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$.

## A Properties of periodic unfolding

We recall elementary properties for the periodic unfolding operator $T_\varepsilon^*$ on domains with holes and the boundary unfolding operator $T_\varepsilon^b$ as well as extensions operators. The analogous properties also hold for the periodic unfolding operator $T_\varepsilon$ in the whole domain.

**Lemma A.1.** Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$.

(A1) The operators $T_\varepsilon^*$ and $T_\varepsilon^b$ are linear and bounded.

(A2) The product rule

$$T_\varepsilon^*(uv) = (T_\varepsilon^* u)(T_\varepsilon^* v) \quad \text{and} \quad T_\varepsilon^b(uv) = (T_\varepsilon^b u)(T_\varepsilon^b v)$$

holds for all $u \in L^p(\Omega_\varepsilon)$, $v \in L^q(\Omega_\varepsilon)$ and $u \in L^p(\Gamma_\varepsilon)$, $v \in L^q(\Gamma_\varepsilon)$, respectively.

(A3) The norms are preserved via

$$\| T_\varepsilon^* u \|_{L^p(\Omega_\varepsilon \times Y_\varepsilon)} = \| u \|_{L^p(\Omega_\varepsilon)} \quad \text{and} \quad \| T_\varepsilon^b u \|_{L^p(\Omega_\varepsilon \times \Gamma)} = \sqrt{\varepsilon} \| u \|_{L^p(\Gamma_\varepsilon)}$$

for all $u \in L^p(\Omega_\varepsilon)$ and $u \in L^p(\Gamma_\varepsilon)$, respectively.

(A4) If $u \in H^1(\Omega_\varepsilon)$, then it is $T_\varepsilon^* u \in L^p(\Omega; H^1(Y_\varepsilon))$ with $T_\varepsilon^*(\varepsilon \nabla u) = \nabla u(T_\varepsilon^* u)$.

(A5) For all $u \in L^p(\Omega_\varepsilon)$ it holds $T_\varepsilon^*[R(u)] = R(T_\varepsilon^* u)$ where $R : \mathbb{R} \to \mathbb{R}$ is an arbitrary function.
Proof. Assertions (A1)-(A4) follow from \[ CD^\ast_12 \] and (A5) from \[ [T^\ast_\varepsilon R(u)](x, y) = R(u(\varepsilon/[x/\varepsilon]+\varepsilon y)] = [T^\ast_\varepsilon(T^\ast_\varepsilon u)](x, y) \] for all \((x, y) \in \Omega \times Y^\ast_\varepsilon\).

\[ \square \]

Theorem A.2 (\[ ACP^\ast 92, H\ddot{o}B14 \]). Let Assumption 2.1 hold. Then there exists a constant \( C_{\text{ext}} > 0 \) depending on the fixed domains \( \Omega \) and \( Y^\ast_\varepsilon \) such that the following is true.

1. There exists a linear operator \( \mathcal{E}_1 : \mathcal{H}^1_{\text{per}}(Y^\ast_\varepsilon) \to \mathcal{H}^1_{\text{per}}(Y) \) such that for every \( u \in \mathcal{H}^1_{\text{per}}(Y^\ast_\varepsilon) \) it holds
   \[ (\mathcal{E}_1 u)|_{Y^\ast_\varepsilon} = u, \quad \|\mathcal{E}_1 u\|_{L^2(Y)} \leq C_{\text{ext}} \|u\|_{L^2(Y^\ast_\varepsilon)}, \quad \|\nabla_y(\mathcal{E}_1 u)\|_{L^2(Y)} \leq C_{\text{ext}} \|\nabla_y u\|_{L^2(Y^\ast_\varepsilon)}. \]

2. There exists a family of linear operators \( \mathcal{E}_2 : \mathcal{H}^1(\Omega^\ast_\varepsilon) \to \mathcal{H}^1(\Omega) \) such that for every \( u \in \mathcal{H}^1(\Omega^\ast_\varepsilon) \) it holds
   \[ (\mathcal{E}_2 u)|_{\Omega^\ast_\varepsilon} = u, \quad \|\mathcal{E}_2 u\|_{L^2(\Omega)} \leq C_{\text{ext}} \|u\|_{L^2(\Omega^\ast_\varepsilon)}, \quad \|\nabla(\mathcal{E}_2 u)\|_{L^2(\Omega)} \leq C_{\text{ext}} \|\nabla u\|_{L^2(\Omega^\ast_\varepsilon)}. \]

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