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Inflationary α-attractor cosmology: A global dynamical systems perspective

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We study flat Friedmann-Lemaître-Robertson-Walker α-attractor E- and T-models by introducing a dynamical systems framework that yields regularized unconstrained field equations on two-dimensional compact state spaces. This results in both illustrative figures and a complete description of the entire solution spaces of these models, including asymptotics. In particular, it is shown that observational viability, which requires a sufficient number of e-folds, is associated with a particular solution given by a one-dimensional center manifold of a past asymptotic de Sitter state, where the center manifold structure also explains why nearby solutions are attracted to this "inflationary attractor solution." A center manifold expansion yields a description of the inflationary regime with arbitrary analytic accuracy, where the slow-roll approximation asymptotically describes the tangency condition of the center manifold at the asymptotic de Sitter state.

I. INTRODUCTION

Recently, there have been considerable developments as regards large field inflation with plateau-like inflaton potentials, driven by the models' compatibility with observational data [1] and their ties to supergravity and string theory [2–12]. On the theoretical side, it has, for example, been shown that such models naturally arise from phenomenological supergravity. The underlying hyperbolic geometry of the moduli space and the flatness of the Kähler potential in the inflation direction is conveniently described in terms of a field variable ϕ, which gives rise to a kinetic term in the Lagrangian with a pole at the boundary of the moduli space. By making a transformation to a canonical variable φ, the moduli space near its boundary becomes stretched, leading to an inflationary potential $V(\varphi)$ with an asymptotic plateau-like form for $V(\varphi)$. This stretching results in predictions that are quite insensitive to the original form of the potential $V(\phi)$. In particular, this leads to universal properties in $n_s - r$ diagrams, which motivates calling these models α-attractors, where α is a constant parameter describing the exponential asymptotic flattening of the potential $V(\varphi)$ in the Einstein frame. It should be noted that this theoretical perspective unifies and contextualizes results for several previous models such as the Starobinski and the Higgs inflation models. For additional intriguing aspects such as couplings to other fields with couplings which become exponentially small for large fields φ, as well as further background discussions, see [2–12].

We take the Einstein frame formulation as our starting point and restrict the discussion to the flat, spatially homogeneous, isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. We thereby consider the following field equations for a canonically normalized inflaton field φ with a potential $V(\varphi)$[1]:

$$3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = \rho_\varphi, \tag{1a}$$

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2, \tag{1b}$$

$$0 = \ddot{\varphi} + 3H\dot{\varphi} + V_\varphi, \tag{1c}$$

where $V_\varphi = dV/d\varphi$; an overdot signifies the derivative with respect to synchronous proper time, t; $H = \dot{a}/a$ is the Hubble variable, where a is the cosmological scale factor, with evolution equation $\dot{a} = aH$, which decouples from the above equations. Furthermore, our focus will be on E- and T-models defined by the potentials

$$V = V_0\left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi}\right)^{2n}, \tag{2a}$$

$$V = V_0\tanh^{2n}\frac{\varphi}{\sqrt{6\alpha}}, \tag{2b}$$

with $V_0 > 0$, respectively (see Fig. 1).

The modest aim in this paper is to present a dynamical systems formulation that allows us to give a complete global classical description of the solution spaces of the above models, including asymptotic properties, illustrated

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[1]We use reduced Planck units: $c = 1 = 8\pi G = 8\pi M_{\text{Pl}}^{-2}$.

(a) $V = V_0 \left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi}\right)^{2n}$, with $n = \alpha = 1$.

(b) $V = V_0 \tanh^{2n} \frac{\varphi}{\sqrt{6\alpha}}$, with $n = \alpha = 1$
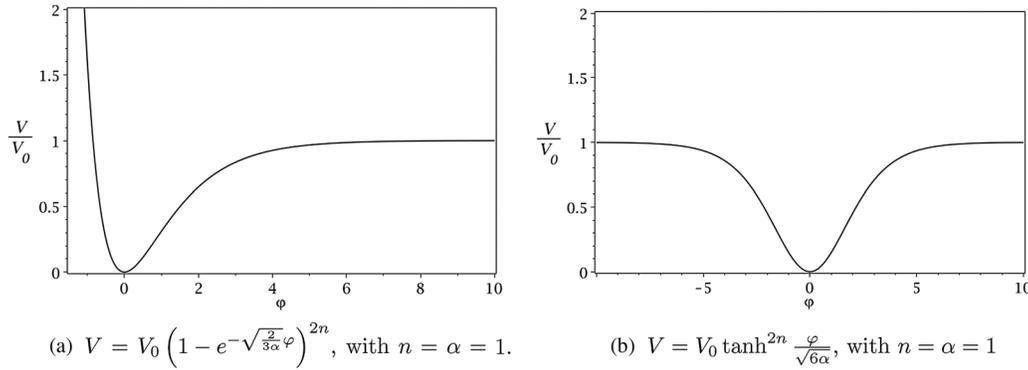
FIG. 1.   The potentials of the E- and T-models.

with pictures describing compactified two-dimensional state spaces. Our motivation for this is threefold:

(i) As will be discussed, the above equations and potentials allow one to rather directly get a feeling for the solution space and its properties, but all aspects are not obvious. It is therefore of value to show how a complete understanding can be obtained in a rigorous manner, especially since this exemplifies how one can address other reminiscent problems.

(ii) By reformulating the problem as a regularized dynamical system on a reduced compactified state space, we provide an example that helps give the dynamical systems community access to inflationary cosmology. This also makes powerful mathematical tools available to an area where such methods have not, in our opinion, been used to their full effect. For example, as will be shown, the universal inflationary properties of the models are intimately connected with certain aspects concerning center manifolds, which also result in approximation methods complementing heuristic methods such as the slow-roll approximation.

(iii) The present dynamical systems formulation can be adapted so that it can be used in more general contexts such as anisotropic, spatially homogeneous cosmology and even for models without any symmetries at all, as shown for perfect fluids in, e.g., Refs. [13] and [14]. Such an extension of the present paper would make it possible to address the issue of initial conditions for inflation in a generic context.

Let us now turn to the system (1). By treating $\dot{\varphi}$ as an independent variable, we obtain a reduced state space (because the equation for the scale factor a decouples) described by the state vector $(H, \dot{\varphi}, \varphi)$ obeying the constraint (1a); i.e., the problem can be regarded as a two-dimensional dynamical system. The E- and T-models are examples of models with a non-negative potential $V(\varphi)$ with a single extremum point, a minimum, conveniently located at $\varphi = 0$, for which $V(0) = 0$. As a consequence, these models admit a Minkowski (fixed point/critical point/equilibrium point) solution at $(H, \dot{\varphi}, \varphi) = (0,0,0)$. Due to (1a), all other

solutions either have a positive or negative H. Since we are interested in cosmology, we consider $H > 0$. It follows that H is monotonically decreasing since Eq. (1b) yields $\dot{H} \leq 0$, where $\dot{H} = 0$ requires $\dot{\varphi} = 0$; moreover, $\dot{\varphi}$ cannot remain zero since $\dot{\varphi} = 0$ implies that $\ddot{\varphi}|_{\dot{\varphi}=0} = -V_\varphi$, where $V_\varphi \neq 0$ since we exclude the only extremum point at $(H, \dot{\varphi}, \varphi) = (0,0,0)$ by assuming that $H > 0$. Thus, the graph of H just passes through an inflection point when $\dot{\varphi} = 0$.

The system (1) can be discussed in terms of the following heuristic picture: Equation (1c) can be interpreted as an equation for a particle of unit mass with a one-dimensional coordinate φ, moving in a potential $V(\varphi)$ with a friction force $-3H\dot{\varphi}$, while $3H^2$ can be viewed as a monotonically decreasing energy in the constraint (1a). Running backwards in time leads to a particle with ever-increasing energy moving in the potential $V(\varphi)$, where $3H^2 \to \infty$. In combination with the shape of a given potential, this yields an intuitive picture of the dynamics. Let us now turn to the specific potentials (2). In the case of the E- and T-models, both potentials behave as $\sim \varphi^{2n}$ for small φ. The potential for the E-models (T-models) has a plateau given by V_0 when $\varphi \to +\infty$ ($\varphi \to \pm\infty$), while V behaves as $V_0 \exp(-2n\sqrt{2/3\alpha}\varphi)$ when $\varphi \to -\infty$. In contrast to the E-models, the potential, and thereby the field equations, of the T-models exhibits a discrete symmetry under the transformation $\varphi \to -\varphi$.

Let us begin our heuristic discussion of the dynamics of the E- and T-models with dynamics toward the future. In both cases, if a solution has obtained an "energy" $3H^2 < V_0$, it follows straightforwardly that the solution will end up at the Minkowski fixed point. But does it do so by just "gliding" down the potential, or does it do so by damped oscillations; i.e., is the motion of φ overdamped or underdamped? This does not follow from the heuristic particle discussion but requires further analysis. Attempting to do so by solving the constraint (1) for H, i.e., by setting $H = \sqrt{\dot{\varphi}^2/2 + V(\varphi)}/\sqrt{3}$ in (1c), leads to an unconstrained two-dimensional dynamical system for $(\dot{\varphi}, \varphi)$. This system, however, has an unbounded state space and differentiability

problems at $\varphi = 0$ for potentials that behave as $\sim \varphi^{2n}$ for small φ. These problems are of course not insurmountable, but they prevent global pictures of the solution spaces that accurately reflect the asymptotic features of the solutions. Nevertheless, it is not difficult to show that the motion is underdamped and that solutions with $3H^2 < V_0$ undergo damped oscillations. But do all solutions end at the Minkowski state, or are there solutions that asymptotically end with an energy V_0 at infinitely large φ? In other words, is the Minkowski state a global future attractor? (As we will see, the answer is "yes" for the present potentials.) One would also perhaps guess that there is a single solution that begins at an infinite value of the scalar field with an initial energy $3H^2 = V_0$, corresponding to an asymptotic de Sitter state, with the solution initially slowly rolling down the potential, which, as we will show, is indeed the case.

What about the past dynamics? From an inflationary point of view, one would perhaps argue that solutions are only physically acceptable with initial data for which $3H^2 \approx V_0$. However, all solutions, except for the single one coming from the de Sitter plateau with an initially infinite scalar field, originate from $3H^2 \to \infty$. One might then take the viewpoint that the previous evolution of solutions before $3H^2 \approx V_0$ is physically irrelevant. However, all this assumes that the inflationary scenario is correct and prevents investigations that either justify this or cast doubt on it. Moreover, even if this is assumed to be correct, one still wants approximations for the solutions before the quasi–de Sitter stage, as done in, e.g., Ref. [7], where a massless state is used for this purpose. But this is intimately connected with the limit $3H^2 \to \infty$, so let us therefore consider this limit from the present heuristic perspective.

Because the potential (2b) is symmetric and bounded, the heuristic description of the dynamics of the T-models is simpler than that for the E-models. We therefore begin with the T-models. Since the potential in this case is bounded, $V(\varphi) \leq V_0$, it follows from (1a) that the limit $3H^2 \to \infty$ implies that $\dot{\varphi}^2/2V(\varphi) \to 0$ and hence that the influence of the potential on the dynamics becomes asymptotically negligible; i.e., the asymptotic state is expected to be that of a massless scalar field. Furthermore, we expect two physically equivalent representations associated with $\varphi \to \pm\infty$ toward the past.

For the E-models the potential is no longer bounded, and this results in a somewhat more complicated situation. Nevertheless, toward the past, irrespective of the value of α, we would expect an open set of models to approach a massless state toward $\varphi \to +\infty$ for similar reasons as for the E-models. Furthermore, we expect that whether or not all solutions end up there depends on the steepness of the potential in the limit $\varphi \to -\infty$. In this limit the potential behaves as an exponential, and problems with an exponential can be viewed as scattering a particle against a potential wall if the potential is steep enough; in such a

situation we expect that the massless state with $\varphi \to +\infty$ describes the past asymptotic behavior of all solutions. If the potential is not steep enough to accomplish this, the situation becomes more complicated. Nevertheless, based on the knowledge of the dynamics of a single exponential, one might guess that there is an open set of solutions that approaches a massless state at $\varphi \to -\infty$ and that there exists a single solution that approaches this limit in a power-law fashion. Moreover, if the steepness of the exponential limit is sufficiently moderate, we expect this solution to describe an early inflationary power-law state.

Although quite helpful, the above heuristic considerations lead to a rather scattered impression of the global dynamics and leave some remaining unanswered questions. In addition, we have not obtained any quantitative results, and, what is even worse, the heuristic picture runs into increasing problems when trying to use it in more general contexts involving additional degrees of freedom. In contrast, the dynamical systems formalism presented below is applicable to more general situations than the present one, which rather acts as a pedagogical example. With this in mind we will now develop a formalism that addresses the above issues and, at a glance, describes the global situation rigorously and also makes techniques available for describing quantitatively correct approximations for solutions in the past and future regimes (indeed, they can be combined to even give global approximations for the solutions to arbitrary accuracy if one is so inclined, see [15]).

To avoid differentiability problems and to obtain asymptotic approximations and a global picture of the solution space, we change variables. We do so by following the treatment of monomial potentials $V \propto \varphi^{2n}$ in [15] and [16], but adapting the formulation to the particular features of the potentials (2a) and (2b).[2] We derive three complementary dynamical systems since the global one is not optimal for quantitative descriptions in all parts of the state space; instead, the global picture should be seen as a collecting ground which gives the overall picture. Since the dependent variables are defined in a similar manner for E- and T-models, while the independent variables differ, we define the three complementary sets of dependent variables in this section, while the independent variables and the dynamical systems and their analysis will be presented in the subsequent two sections.

We begin by defining the Hubble-normalized or, equivalently in the present flat FLRW cases, energy density-normalized dimensionless variables (thereby capturing the

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[2]To treat models with positive potentials, see e.g., [17] and [18]; for examples of other work on scalar fields using dynamical systems methods, see e.g., [19–24]. For a recent rather general discussion on dynamical systems formulations and methods in other cosmological contexts, see [25].

physical essence of the problem), by making the following variable transformation, $(H, \dot{\varphi}, \varphi) \rightarrow (\tilde{T}, \Sigma_\varphi, X)^3$:

$$\tilde{T} = \left(\frac{V_0}{3H^2}\right)^{\frac{1}{2n}} = \left(\frac{V_0}{\rho_\varphi}\right)^{\frac{1}{2n}}, \tag{3a}$$

$$\Sigma_\varphi = \frac{1}{\sqrt{6}}\frac{\dot{\varphi}}{H} = \frac{1}{\sqrt{6}}\frac{d\varphi}{dN}, \tag{3b}$$

$$X = \left(\frac{V(\varphi)}{3H^2}\right)^{\frac{1}{2n}} = \left(\frac{V(\varphi)}{\rho_\varphi}\right)^{\frac{1}{2n}}, \tag{3c}$$

where X takes the following explicit form for the E- and T-models,

$$X = \left(\frac{V_0}{3H^2}\right)^{\frac{1}{2n}}(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi}) = \tilde{T}(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi}), \tag{3d}$$

$$X = \left(\frac{V_0}{3H^2}\right)^{\frac{1}{2n}}\tanh\frac{\varphi}{\sqrt{6\alpha}} = \tilde{T}\tanh\frac{\varphi}{\sqrt{6\alpha}}, \tag{3e}$$

respectively. The quantity $N = \ln(a/a_0)$ in (3b) represents the number of e-folds with respect to some reference time t_0 where $a(t_0) = a_0$. Below, for simplicity, we assume that n is a positive integer.

Since H is monotonically decreasing, it follows that $\tilde{T}$ is monotonically increasing. Furthermore, when $H \rightarrow 0 \Rightarrow \tilde{T} \rightarrow \infty$, $H \rightarrow \infty \Rightarrow \tilde{T} \rightarrow 0$, and $3H^2 \rightarrow V_0 \Rightarrow \tilde{T} \rightarrow 1$. With the above definitions, the Gauss constraint (1a) takes the form

$$1 = \Sigma_\varphi^2 + X^{2n}. \tag{4}$$

The state space thereby has a cylinderlike structure (cylinder structure when $n = 1$). The constraint can be solved globally by introducing an angular variable θ according to

$$\Sigma_\varphi = G(\theta)\sin\theta, \qquad X = \cos\theta, \tag{5a}$$

$$G(\theta) = \sqrt{\frac{1 - \cos^{2n}\theta}{1 - \cos^2\theta}} = \sqrt{\sum_{k=0}^{n-1}\cos^{2k}\theta}, \tag{5b}$$

which leads to an unconstrained dynamical system for the state vector $(\tilde{T}, \theta)$. For future purposes we note that $G \geq 1$ (with $G \equiv 1$ when $n = 1$), and

$$G(0) = \sqrt{n}. \tag{6}$$

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^{3}The notation Σ_φ is used because Σ_φ, in a multidimensional Kaluza-Klein perspective, is analogous to Hubble-normalized shear in anisotropic cosmology, where Σ is standard notation; see, e.g., Ref. [26].

To obtain a bounded (relatively compact) state space with state vector (T, θ), we make the transformation

$$T = \frac{\tilde{T}}{1 + \tilde{T}}, \qquad \tilde{T} = \frac{T}{1 - T}. \tag{7}$$

Thus, T is monotonically increasing where $H \rightarrow 0 \Rightarrow T \rightarrow 1$, $H \rightarrow \infty \Rightarrow T \rightarrow 0$, and $3H^2 \rightarrow V_0 \Rightarrow T \rightarrow \frac{1}{2}$.

The deceleration parameter q is defined and given by

$$q = -\frac{\ddot{a}}{aH^2} = -(1 + H^{-2}\dot{H}) = -1 + 3\Sigma_\varphi^2 = 2 - 3\cos^{2n}\theta. \tag{8}$$

To proceed with the choice of independent variables, we treat the E- and T-models separately in the following two sections, where we also perform a complete local and global dynamical systems analysis of these models. We end the paper with some concluding remarks in Sec. IV, e.g., about the relationship between the center manifold analysis performed in the two E- and T-model sections and the slow-roll approximation.

II. E-MODELS

A. Dynamical systems formulations

Using the dependent variables given in Eq. (3) for the E-models and $N = \ln(a/a_0)$ as the independent variable, where

$$\frac{dN}{dt} = H, \tag{9}$$

results in the following evolution equations for the state vector $(\tilde{T}, \Sigma_\varphi, X)$,

$$\frac{d\tilde{T}}{dN} = \frac{3}{n}\Sigma_\varphi^2\tilde{T}, \tag{10a}$$

$$\frac{d\Sigma_\varphi}{dN} = -3(\Sigma_\varphi X + \bar{\lambda}(\tilde{T} - X))X^{2n-1}, \tag{10b}$$

$$\frac{dX}{dN} = \frac{3}{n}(\Sigma_\varphi X + \bar{\lambda}(\tilde{T} - X))\Sigma_\varphi, \tag{10c}$$

and the constraint

$$1 = \Sigma_\varphi^2 + X^{2n}, \tag{10d}$$

where it has been convenient to define

$$\bar{\lambda} = \frac{2n}{3\sqrt{\alpha}}. \tag{11}$$

The state space is bounded by the conditions that $\tilde{T} > 0$ and that

$$\tilde{T}e^{-\sqrt{\frac{2}{3\alpha}}\varphi} = \tilde{T} - X > 0. \tag{12}$$

Since

$$\frac{d}{dN}(\tilde{T} - X) = \frac{3}{n}(\Sigma_\varphi - \bar{\lambda})\Sigma_\varphi(\tilde{T} - X), \tag{13}$$

it follows that the physical state space is bounded toward the past by the invariant subsets $\tilde{T} = 0$ for $X \leq 0$ and $\tilde{T} - X = 0$ for $X \geq 0$ (recall that $\tilde{T}$ is monotonically increasing toward the future and therefore decreasing toward the past). Their intersection, $\Sigma_\varphi = \pm 1$, $X = 0$, corresponds to two fixed points $M_\pm$. Furthermore, the $\tilde{T} - X = 0$ subset is divided into two disconnected parts by a fixed point dS located at $\tilde{T} = 1 = X$. Due to the regularity of the above equations, we can include the invariant boundary, $(\tilde{T} = 0 \text{ for } X \leq 0) \cup (\tilde{T} - X = 0 \text{ for } X \geq 0)$, which we refer to as the past boundary. Indeed, it is necessary to include this boundary in order to describe past asymptotics since, as will be shown, all solutions originate from the fixed points on this boundary. Note further that the equations on the $\tilde{T} = 0$ subset are identical to those for an exponential potential $V = V_0 e^{-\sqrt{6}\bar{\lambda}\varphi}$ since in this case

$$\frac{d\Sigma_\varphi}{dN} = -3(\Sigma_\varphi - \bar{\lambda})X^{2n} = -3(\Sigma_\varphi - \bar{\lambda})(1 - \Sigma_\varphi^2). \tag{14}$$

Moreover, the equations on the $\tilde{T} = X$ subset are identical to those that correspond to a constant potential, which can be seen by setting $\bar{\lambda} = 0$ in the above equation.

By globally solving the constraint by using Eq. (5), we obtain the following unconstrained dynamical system:

$$\frac{d\tilde{T}}{dN} = \frac{3}{n}(1 - \cos^{2n}\theta)\tilde{T}, \tag{15a}$$

$$\frac{d\theta}{dN} = -\frac{3}{2n}(G\sin 2\theta + 2\bar{\lambda}(\tilde{T} - \cos\theta))G. \tag{15b}$$

Finally, by using T and θ and changing the time variable from N to $\bar{\tau}$ according to

$$\frac{d\bar{\tau}}{dN} = 1 + \tilde{T} = \frac{1}{1 - T}, \tag{16}$$

we obtain the regular dynamical system

$$\frac{dT}{d\bar{\tau}} = \frac{3}{n}T(1 - T)^2(1 - \cos^{2n}\theta), \tag{17a}$$

$$\frac{d\theta}{d\bar{\tau}} = -\frac{3}{2n}[(1 - T)G\sin 2\theta + 2\bar{\lambda}(T - (1 - T)\cos\theta)]G. \tag{17b}$$

Apart from including the past boundary, which in the present variables is given by $T = 0$ when $\cos\theta \leq 0$ and $T - (1 - T)\cos\theta = 0$ for $\cos\theta \geq 0$, we also include the future boundary $T = 1$, which corresponds to $H = 0$ and the final Minkowski state. Thus, the resulting extended state space is given by a finite cylinder with the region $T - (1 - T)\cos\theta < 0$ with $\cos\theta > 0$ removed (see Fig. 2).

B. Dynamical systems analysis

From the definitions, and the fact that H is monotonically decreasing, it follows that $\tilde{T}$ and T are monotonically increasing. This is also seen in Eqs. (10a), (15a), and (17a), although further insight is gained by considering how $\tilde{T}$, and hence T, behaves when $\Sigma_\varphi = 0 \Rightarrow \theta = m\pi$, where $q = -1$:

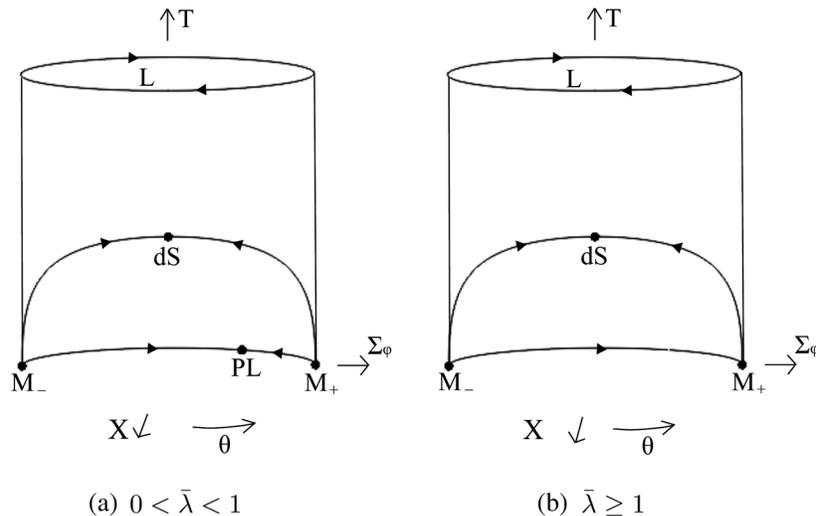(a) $0 < \bar{\lambda} < 1$ (b) $\bar{\lambda} \geq 1$

FIG. 2.   State space and boundary structures for the E-models with $V = V_0(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi})^{2n}$. Recall that $\bar{\lambda} = \frac{2n}{3\sqrt{\alpha}}$.

$$\left.\frac{d\tilde{T}}{dN}\right|_{q=-1} = 0, \qquad \left.\frac{d^2\tilde{T}}{dN^2}\right|_{q=-1} = 0,$$

$$\left.\frac{d^3\tilde{T}}{dN^3}\right|_{q=-1} = \frac{54\bar{\lambda}^2}{n}(\tilde{T} - \cos(m\pi))^2\tilde{T}. \qquad (18)$$

Since $\tilde{T} > 1$ when m is even and $\tilde{T} > 0$ when m is odd in the physical state space, it follows that by viewing the above as the coefficients in a Taylor expansion, $\tilde{T}$, and hence T, is monotonically increasing, although the graphs of $\tilde{T}$ and T go through inflection points when $q = -1$. Furthermore, since

$$\left.\frac{d\theta}{dN}\right|_{q=-1} = -\frac{3\bar{\lambda}}{\sqrt{n}}(\tilde{T} - \cos(m\pi)), \qquad (19)$$

it follows that θ is monotonically decreasing at $q = -1$ and thus that the solution curves in the T, θ state space become horizontal in T at $q = -1$ (see Fig. 3).
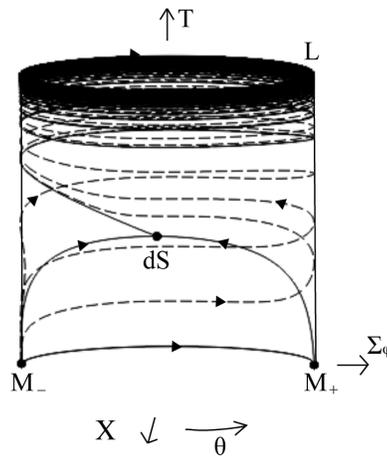
The monotonicity properties of T show that there are no fixed points or recurring orbits in the physical state space (i.e., the extended state space with the future and past invariant boundaries excluded). All orbits originate from the past boundary and end at the future boundary at $T = 1$, where
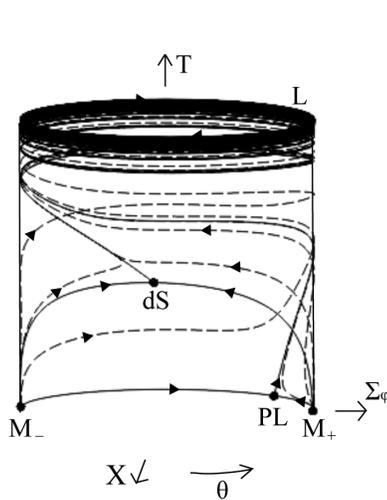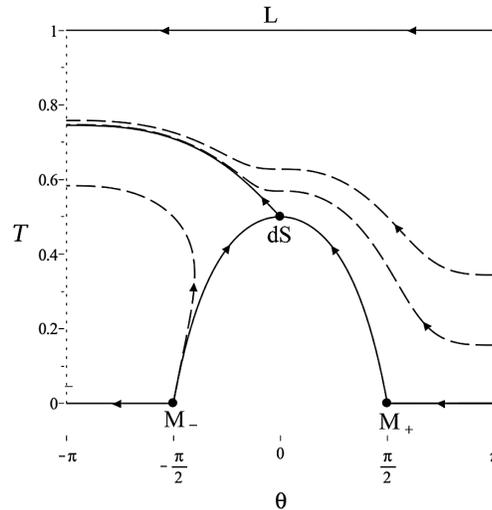

(a) Solution space for $0 < \bar{\lambda} < 1$.
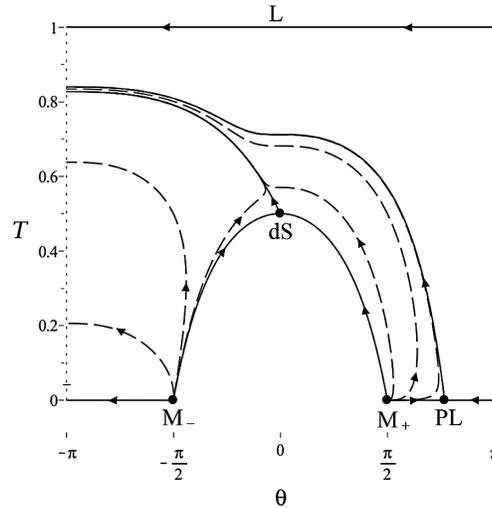
(b) "Unwrapped" solution space for $0 < \bar{\lambda} < 1$.

(c) Solution space for $\bar{\lambda} \geq 1$.

(d) "Unwrapped" solution space for $\bar{\lambda} \geq 1$.

FIG. 3. Representative solutions describing the solution spaces for E-models with $V = V_0(1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi})^{2n}$. In panels (a) and (b), $0 < \bar{\lambda} < 1$, which corresponds to $\alpha > (2n/3)^2$, represented by the values $n = 1$ and $\alpha = 1$. In panels (c) and (d), $\bar{\lambda} \geq 1$, which corresponds to $\alpha \leq (2n/3)^2$, represented by $n = 1$ and $\alpha = 1/4$.

$$\frac{d\theta}{d\bar{\tau}}\bigg|_{T=1} = -\frac{3\bar{\lambda}}{n}G < 0; \qquad (20)$$

i.e., $T = 1$ corresponds to a periodic orbit [i.e., a periodic solution trajectory to the dynamical system (17)], L, with monotonically decreasing θ, and, hence, to a limit cycle for all solutions in the physical state space.

The structure on the past boundary is easily found since it consists of two parts, one corresponding to an exponential potential and one to a constant potential. It consists of fixed points and heteroclinic orbits (orbits that originate and end at distinct fixed points) that join them. Since there are no heteroclinic cycles on the past boundary, it follows that all interior physical orbits originate from the fixed points, which are given by

$$M_{\pm}: \tilde{T} = 0, \qquad T = 0; \qquad \Sigma_{\varphi} = \pm 1;$$
$$X = 0; \qquad \theta = \left(2m \pm \frac{1}{2}\right)\pi, \qquad (21a)$$

$$dS: \tilde{T} = 1, \qquad T = \frac{1}{2}; \qquad \Sigma_{\varphi} = 0;$$
$$X = 1; \qquad \theta = 2m\pi, \qquad (21b)$$

$$PL: \tilde{T} = 0, \qquad T = 0; \qquad \Sigma_{\varphi} = \bar{\lambda};$$
$$X = -(1-\bar{\lambda}^2)^{\frac{1}{2n}}; \qquad \theta = \arccos X, \qquad (21c)$$

where PL only exists on the extended physical state space if $\bar{\lambda} < 1$. This fixed point corresponds to the self-similar solution for an exponential potential, which yields a power-law solution, explaining the nomenclature. If $\bar{\lambda} < 1/\sqrt{3}$ this solution is accelerating since $q = 3\bar{\lambda}^2 - 1$ for PL. The nomenclature $M_{\pm}$ corresponds to a massless scalar field with $q = 2$ (i.e., it corresponds to setting the potential to zero), where $M_{\pm}$ implies $\Delta\varphi = \pm\sqrt{6}N$, due to (3b). Finally, dS stands for de Sitter since $q = -1$ for this fixed point, although note that this de Sitter state corresponds to $\varphi \to \infty$; i.e., it is an asymptotic state and not a physical de Sitter solution with finite constant φ.

A local analysis of the fixed points shows that if $0 < \bar{\lambda} < 1$, then both M_+ and M_- are sources while PL is a saddle with a single solution entering the physical state space. If $\bar{\lambda} \geq 1$ then M_- is a source, while M_+ is a saddle from which no solutions enter the physical state space. Arguably, the most interesting fixed point is dS, which is a center saddle, with a one-dimensional center manifold corresponding to the "attractor solution" or the "inflationary trajectory." To describe this interior state space solution, which originates from dS, we follow [15,16] and perform a center manifold analysis. Since it is more convenient to use $\tilde{T}$ than T, we use the system (15), which results in the following center manifold expansion (without loss of generality, we choose $\theta = 0$ for dS):

$$\theta(\tilde{T}) = -\frac{\bar{\lambda}}{\sqrt{n}}(\tilde{T} - 1)\left(1 - \frac{\bar{\lambda}}{2\sqrt{n}}(\tilde{T} - 1) + \cdots\right). \quad (22)$$

The periodic orbit L corresponds to a blowup of the completely degenerate Minkowski fixed point in the $(\dot{\varphi}, \varphi)$ formulation. As discussed in [15,16], to obtain explicit expressions for future asymptotics, one can use available approximations for late stage behavior when $V \sim \varphi^{2n}$, or one can use the averaging techniques developed in [16]. The overall global solution structure for the E-models is depicted in Fig. 3. Note that L is the true future attractor and that it represents the future asymptotic behavior of *all* physical solutions.

Finally, let us translate the above results to the original scalar field picture. Let us first consider the inflationary solution coming from the dS fixed point. In the vicinity of dS, $\Sigma_{\varphi} = \frac{d\varphi}{dN}/\sqrt{6}$ is negative, which means that toward the past, φ is increasing. It is not difficult to use the approximation (22) to show that φ to leading order can be written in the form $\varphi = A\ln(1 - BN)$, where A and B are positive constants and where $N \to -\infty$ toward the past; i.e., the solution originates from $\varphi \to +\infty$ with a subsequently slowly decreasing φ, as expected. Furthermore, note that initial data with an energy that is close to that of an initial quasi–de Sitter state correspond to $T \approx \frac{1}{2}$, where Fig. 3 conveniently, at a glance, shows how solutions, and their properties, are distributed in terms of such initial data. From this figure it is obvious that there exists an open set with initial data with such initial energies that do not have a quasi–de Sitter phase in their evolution. Moreover, solutions that do have an intermediate de Sitter stage have solution trajectories that shadow the heteroclinic orbits $M_{\pm} \to dS$, which correspond to scalar field models with a constant potential V_0.

From $\Sigma_{\varphi} = \frac{d\varphi}{dN}/\sqrt{6}$ it follows directly that M_- corresponds to a massless initial asymptotic state for which $\varphi \to +\infty$. If $0 < \bar{\lambda} < 1$, i.e., if $\alpha > (2n/3)^2$, then M_+ corresponds to a massless initial asymptotic state with $\varphi \to -\infty$ from which an open set of solutions originates. Similarly, it follows that in this case the single solution that originates from PL corresponds to an initial power-law state for which $\varphi \to -\infty$ toward the past; furthermore, if $\bar{\lambda} < 1/\sqrt{3}$ this is an initial power-law inflation state. Since PL is a saddle, it follows that there is an open set of solutions that undergo an intermediate stage of power-law inflation in this case, but note that (i) this stage is much shorter than the de Sitter stage since PL is a hyperbolic saddle in contrast to the center saddle dS and (ii) this inflationary stage takes place at a much larger energy scale than that of the present quasi–de Sitter inflation. On the other hand, if $\bar{\lambda} \geq 1$, i.e., if $\alpha \leq (2n/3)^2$, then all solutions originate from $\varphi \to +\infty$. Finally, all solutions end at the Minkowski state associated with L.

III. T-MODELS

A. Dynamical systems formulations

Using the dependent variables given in Eq. (3) and a new time variable $\tilde{\tau}$, defined by

$$\frac{d\tilde{\tau}}{dt} = H\tilde{T}^{-1}, \tag{23}$$

results in the following evolution equations for the state vector $(\tilde{T}, \Sigma_\varphi, X)$,

$$\frac{d\tilde{T}}{d\tilde{\tau}} = \frac{3}{n}\Sigma_\varphi^2\tilde{T}^2, \tag{24a}$$

$$\frac{d\Sigma_\varphi}{d\tilde{\tau}} = -3\left(\Sigma_\varphi X\tilde{T} + \frac{1}{2}\bar{\lambda}(\tilde{T}^2 - X^2)\right)X^{2n-1}, \tag{24b}$$

$$\frac{dX}{d\tilde{\tau}} = \frac{3}{n}\left(\Sigma_\varphi X\tilde{T} + \frac{1}{2}\bar{\lambda}(\tilde{T}^2 - X^2)\right)\Sigma_\varphi, \tag{24c}$$

and the constraint

$$1 = \Sigma_\varphi^2 + X^{2n}, \tag{24d}$$

where again

$$\bar{\lambda} = \frac{2n}{3\sqrt{\alpha}}. \tag{25}$$

The constrained dynamical system (24) admits a discrete symmetry $(\Sigma_\varphi, X) \to -(\Sigma_\varphi, X)$, because the potential is invariant under the transformation $\varphi \to -\varphi$.

The state space is bounded by the conditions that $\tilde{T} > 0$ and that

$$\left(\frac{\tilde{T}}{\cosh\frac{\varphi}{\sqrt{6\alpha}}}\right)^2 = \tilde{T}^2 - X^2 > 0. \tag{26}$$

Since

$$\frac{d}{d\tilde{\tau}}(\tilde{T}^2 - X^2) = \frac{6}{n}\Sigma_\varphi\left(\Sigma_\varphi\tilde{T} - \frac{1}{2}\bar{\lambda}X\right)(\tilde{T}^2 - X^2), \tag{27}$$

it follows that the physical state space is bounded toward the past by the invariant subset $\tilde{T}^2 - X^2 = 0$ for $\tilde{T} \geq 0$. Due to the discrete symmetry, this invariant subset consists of two equivalent disconnected parts, one with $X > 0$ and one with $X < 0$, separated by the equivalent (massless state) fixed points $M_\pm$ at $X = 0$, $\Sigma_\varphi = \pm 1$. The equations on the two branches of the boundary subset, defined by $\tilde{T}^2 - X^2 = 0$, are identical to those for a constant potential. Thus, there are also two physically equivalent de Sitter fixed points $dS_\pm$ at $\Sigma_\varphi = 0$, $X = \pm 1$. As for the previous E-models, we use the regularity of the equations to include the above boundary subset in our analysis.

By solving the constraint using Eq. (5), we obtain the following dynamical system:

$$\frac{d\tilde{T}}{d\tilde{\tau}} = \frac{3}{n}(1 - \cos^{2n}\theta)\tilde{T}^2, \tag{28a}$$

$$\frac{d\theta}{d\tilde{\tau}} = -\frac{3}{2n}(G\tilde{T}\sin 2\theta + \bar{\lambda}(\tilde{T}^2 - \cos^2\theta))G. \tag{28b}$$

Changing $\tilde{T}$ to T and the independent variable $\tilde{\tau}$ to $\check{\tau}$ according to

$$\frac{d\check{\tau}}{d\tilde{\tau}} = (1 + \tilde{T})^2 = (1 - T)^{-2} \tag{29}$$

results in

$$\frac{dT}{d\check{\tau}} = \frac{3}{n}T^2(1 - T)^2(1 - \cos^{2n}\theta), \tag{30a}$$

$$\frac{d\theta}{d\check{\tau}} = -\frac{3}{2n}(GT(1 - T)\sin 2\theta + \bar{\lambda}(T^2 - (1 - T)^2\cos^2\theta))G. \tag{30b}$$

Apart from including the past boundary, which in the present variables is given by $T^2 - (1 - T)^2\cos^2\theta = 0$, we also include the future boundary $T = 1$, which corresponds to $H = 0$ and the final Minkowski state. The resulting extended state space is therefore given by a finite cylinder with the region $T^2 - (1 - T)^2\cos^2\theta < 0$ removed (see Fig. 4).

B. Dynamical systems analysis

As for the E-models, since H is monotonically decreasing, it follows that $\tilde{T}$ and T are monotonically increasing. Again, further insight is gained by considering how $\tilde{T}$, and hence T, behave when $\Sigma_\varphi = 0 \to \theta = m\pi$, where $q = -1$:
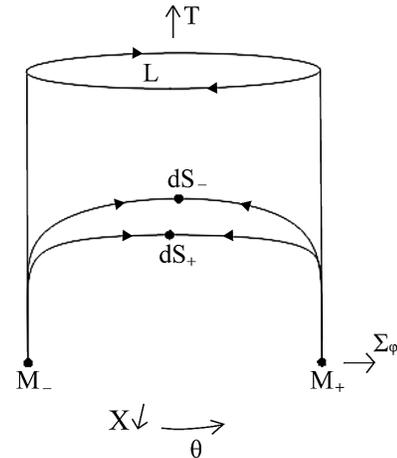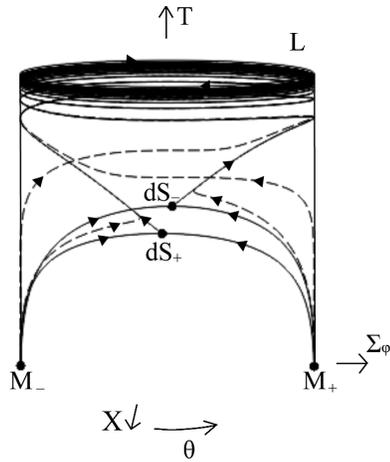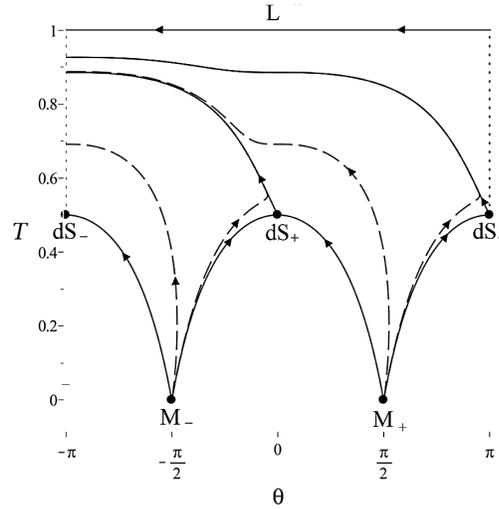

FIG. 4. State space and boundary structures for T-models with $V = V_0\tanh^{2n}\frac{\varphi}{\sqrt{6\alpha}}$.

(a) Solution space for the T-models.

(b) "Unwrapped" solution space for the T-models.

FIG. 5.   Representative solutions describing the solution space for the T-models with the potential $V = V_0 \tanh^{2n} \frac{\varphi}{\sqrt{6\alpha}}$ ($n = 1$, $\alpha = 1$ in the figure).

$$\left.\frac{d\tilde{T}}{d\tilde{\tau}}\right|_{q=-1} = 0, \qquad \left.\frac{d^2\tilde{T}}{d\tilde{\tau}^2}\right|_{q=-1} = 0,$$

$$\left.\frac{d^3\tilde{T}}{d\tilde{\tau}^3}\right|_{q=-1} = \frac{27}{2n}\bar{\lambda}^2(\tilde{T}-1)^2\tilde{T}^2. \tag{31}$$

Since $\tilde{T} > 1$ when $q = -1$, it follows that $\tilde{T}$, and hence T, is monotonically increasing, although the graphs of $\tilde{T}$ and T go through inflection points when $q = -1$. Furthermore, since

$$\left.\frac{d\theta}{d\tilde{\tau}}\right|_{q=-1} = -\frac{3}{2n}\bar{\lambda}(\tilde{T}^2-1)G, \tag{32}$$

it follows that θ is monotonically decreasing at $q = -1$, and thus the solution curves in the T, θ state space, also for the T models, become horizontal in T at $q = -1$ (see Fig. 5).

From the monotonicity of T, and the discrete symmetry that makes the two fixed points M_+ and M_- physically equivalent, it follows that both these fixed points are sources, corresponding to asymptotic massless self-similar states. The two physically equivalent fixed points dS_+ and dS_- are center saddles, each yielding a single (physically equivalent) "inflationary attractor solution" entering the physical state space, which, as for the E-models, corresponds to a center manifold. A center manifold expansion gives the following approximation for the inflationary attractor solution (without loss of generality, we choose dS_+ and $\theta = 0$):

$$\theta(\tilde{T}) = -\frac{\bar{\lambda}}{\sqrt{n}}(\tilde{T}-1)\left(1 - \frac{1}{2}\left(\frac{\bar{\lambda}^2}{n}+1\right)(\tilde{T}-1) + \cdots\right). \tag{33}$$

As in the E-model case, the periodic orbit L corresponds to a blowup of the degenerate Minkowski fixed point in the

$(\dot{\varphi}, \varphi)$ formulation, where L is the future attractor. The overall global solution structure is depicted in Fig. 5. All physical solutions originate from M_+ and M_- (forming two physically equivalent sets of solutions), apart from the two physically equivalent inflationary attractor solutions that originate from $dS_\pm$, and all solutions end at the Minkowski state associated with the future attractor and limit cycle L.

In this case it is quite easy to translate the results to the original scalar field picture. A similar analysis to that for the E-models shows that the open sets of physically equivalent solutions that originate from the massless states M_- and M_+ correspond to initial states for which $\varphi \to +\infty$ and $\varphi \to -\infty$, respectively. The single solutions that come from the de Sitter fixed points dS_+ and dS_- correspond to the limits $\varphi \to +\infty$ and $\varphi \to -\infty$, respectively, from which they slowly evolve. As in the E-model case, all solutions end at the Minkowski state associated with the limit cycle L.

IV. CONCLUDING REMARKS

We begin this final section with some remarks on the relationship between the center manifold and the slow-roll approximation for the inflationary attractor solution. In the slow-roll approximation $H \approx \sqrt{V(\varphi)/3}$ (i.e., $X = 1$) is inserted into $\dot{\varphi} = -2\frac{\partial H}{\partial\varphi}$, which gives

$$\dot{\varphi} \approx -\sqrt{\frac{V}{3}}\left(\frac{V_\varphi}{V}\right). \tag{34}$$

In terms of $(\tilde{T}, \Sigma_\varphi, X)$, this yields the following expressions for E- and T-models:

$$\Sigma_\varphi \approx -\bar{\lambda}(\tilde{T} - X)X^{n-1}, \tag{35a}$$

$$\Sigma_\varphi \approx -\frac{\bar{\lambda}}{2}\left(\tilde{T} - \frac{X^2}{\tilde{T}}\right)X^{n-1}. \tag{35b}$$

In the vicinity of the asymptotic de Sitter state, where $\tilde{T} \approx 1$ and $\theta \approx 0$, and therefore $\Sigma_\varphi \approx \sqrt{n}\theta$ [recall that $G(0) = \sqrt{n}$], $X \approx 1$, these expressions yield

$$\theta(\tilde{T}) \approx -\frac{\bar{\lambda}}{\sqrt{n}}(\tilde{T} - 1), \tag{36}$$

to lowest order. It follows that the slow-roll approximation leads to a curve in the (T, θ) state space that is tangential to the center manifold in the limit toward the de Sitter state from which the center manifold, i.e., the inflationary attractor solution, originates, as is also true for monomial potentials as discussed in [15,16].

The inflationary "attractor" solution, being a one-dimensional center manifold, attracts nearby solutions exponentially rapidly, which then move along the center manifold in a relatively slow power-law manner in the vicinity of the de Sitter fixed point.[4] Thus, the center

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[4]For a similar discussion in the context of quadratic theories of gravity, see the paragraph after Eq. (20) in [27].

manifold structure explains both the attracting nature of the inflationary attractor solution and the fact that nearby solutions obtain a sufficient number of *e*-folds to be physically viable in an inflationary context. Nevertheless, although this holds for an open set of solutions that shadow the past boundary from fixed points that are sources to a de Sitter state on this boundary, it should also be pointed out that there exists an open set of solutions that behave differently, as seen in Figs. 3 and 5. Ruling out these other solutions as physically irrelevant, and explaining the special role of the inflationary attractor solution beyond its center manifold structure, thereby relies on paradigmatic assumptions relating the problem to broader contexts. Examples of such contexts involve various proposed theoretical frameworks as well as, e.g., scale considerations, illustrated in the discussion of, e.g., Ref. [11], and various measures, motivated by, e.g., symplectic structures; for a recent discussion on measures which might be applicable to the present models, see [28].

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