Symmetry methods and some nonlinear differential equations

Background and illustrative examples

Symmetrimetoder och några icke-linjära differentialekvationer
Bakgrund och illustrativa exempel

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Abstract

Differential equations, in particular the nonlinear ones, are commonly used in formulating most of the fundamental laws of nature as well as many technological problems, among others. This makes the need for methods in finding closed form solutions to such equations all-important. In this thesis we study Lie symmetry methods for some nonlinear ordinary differential equations (ODE). The study focuses on identifying and using the underlying symmetries of the given first order nonlinear ordinary differential equation. An extension of the method to higher order ODE is also discussed. Several illustrative examples are presented.

Sammanfattning

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Chapter 1

Introduction

New mathematical models of fundamental laws of nature and of technological problems are constantly formulated in the form of nonlinear differential equations [7, 12]. The Norwegian mathematician Marius Sophus Lie (1842 – 1899) dedicated most of his life to the theory of continuous groups (today known as Lie groups), and their impact on differential equations [5]. Lie discovered that the standard solution methods uses groups of symmetries of the equations to obtain the solutions. Consequently, exact solutions can be found through a systematic use of symmetries [9, 11, 12].

Today, Lie group analysis is a fundamental tool in many diverse areas, such as analysis, differential geometry, number theory, differential equations, atomic structure and high energy physics [19]. One application of symmetry methods is in the study of mathematical models in epidemiology [2, 15]. For example, Lie group analysis has been applied to models which describes human immunodeficiency virus (HIV) transmission in male homosexual/bisexual groups [16].

In this thesis we study Lie symmetries with focus on nonlinear ordinary differential equations. The symmetry methods are especially important when finding solutions for such types of equations, since most of the standard solution methods become insufficient in these cases [5, 12]. The idea of symmetry methods is basically to find a new coordinate system, that makes the resulting differential equation easier to solve [22]. The purpose of this thesis is to identify the Lie symmetries of a given first order ordinary differential equation, and then illustrate how they can be used to solve the given equation.

This thesis is organized as follows. In Chapter 2, we outline some of the main mathematical basics of Lie symmetries. We focus on how first order ordinary differential equations can be solved using symmetries, and extend this to higher order ordinary differential equations. In Chapter 3, we present illustrative examples. Most of the
examples are adapted from various sources, reexamined from symmetry perspective. Moreover, some of the examples are exercises from the references, which the author has solved using the solution method that is studied in the thesis. Several of the examples are also graphically illustrated. The figures are made by the author using the program Wolfram Mathematica, and with some guidance from the supervisor. In Chapter 4, we summarize the thesis and mention some highlights on recent developments in symmetry methods.
Chapter 2

Lie symmetries of ODE

2.1 Symmetries and Lie groups

To clarify the concept of symmetries of ordinary differential equations, it is convenient to first study symmetries in other contexts. For example, symmetries of geometrical objects are transformations that leaves the objects exactly the same. Besides from mapping the object to itself, the transformations must also preserve the structural properties of the original item to be symmetries. These mappings include rotations, translations and reflections.

Consider an equilateral triangle. After rotations of $2\pi/3$, $4\pi/3$ and $2\pi$ about its centre, the triangle is apparently unchanged. The same goes for flips about its three axes. Thus, these operations preserve the geometrical orientation of the triangle. This means that an equilateral triangle has six symmetries and is said to be invariant with respect to these operations. These symmetries are discrete, since they do not depend upon some continuous parameter. Moreover, every geometrical object has a trivial symmetry, which is the transformation when every point of the object is mapped to itself. To rotate an equilateral triangle $2\pi$ about its centre is a trivial symmetry [5, 11, 21].

We will now consider symmetries of algebraic equations. The graph of $f(x) = x^2$ is symmetric due to reflections across the $y$-axis, since $f(-x) = (-x)^2 = x^2 = f(x)$. Moreover, the graph of $f(x) = \sin(x)$ is symmetric due to horizontal translations by $2\pi$. This holds since $f(x + 2\pi) = \sin(x + 2\pi) = \sin(x) = f(x)$. Thus, these transformations are symmetries of $f$, since the graph of $f$ is mapped to itself [22]. However, these are also examples of symmetries that are discrete. In this thesis, we are interested in symmetries that depend upon some continuous parameter. For example, this is the case when the unit circle is rotated by any amount about its centre [11].
We will now develop this concept to symmetries of ordinary differential equations. For further references, we consider $x$ to be an independent variable and $y$ to be a dependent variable. We have that $y = y(x)$, which will apply throughout the thesis.

When dealing with ordinary differential equations and their symmetries, we consider point transformations depending on (at least) one arbitrary parameter $\epsilon \in \mathbb{R}$ [8, 11, 12]. In [12], this transformation is explained as

$$
\Gamma \epsilon : \begin{align*}
\hat{x} &= \varphi(x, y; \epsilon), \\
\hat{y} &= \psi(x, y; \epsilon),
\end{align*}
$$

for functions $\varphi$ and $\psi$ such that

$$
\Gamma_0 : \begin{align*}
\hat{x} &= \varphi(x, y; 0) = x, \\
\hat{y} &= \psi(x, y; 0) = y,
\end{align*}
$$

is the identity transformation. Under the transformation (2.1), we have that an arbitrary point $P = (x, y)$ in the plane is mapped to the point $P' = (\hat{x}, \hat{y})$. We write this as $P' = \Gamma(\epsilon)P$. Consequently, the inverse transformation is given by

$$
\Gamma^{-1} \epsilon : \begin{align*}
x &= \varphi^{-1}(\hat{x}, \hat{y}; \epsilon), \\
y &= \psi^{-1}(\hat{x}, \hat{y}; \epsilon),
\end{align*}
$$

i.e. $\Gamma^{-1} \epsilon (P') = P$. Thus, the identity transformation (2.2) may be written as $\Gamma_0(P) = P$.

The next definition is interpreted from [11].

**Definition 2.1.** A smooth transformation (2.1) is invertible if its Jacobian determinant is nonzero, that is

$$
\hat{x}_x \hat{y}_y - \hat{x}_y \hat{y}_x \neq 0,
$$

where $\hat{x}_x = \partial \hat{x} / \partial x$, etc.

In [11, 12, 20], they give presentations of Lie groups. In the following definition, we outline the interpretation we have made.

**Definition 2.2.** A set of smooth invertible point transformations (2.1) that satisfies (2.2) is called a one-parameter (continuous) group, if it contains the inverse (2.3) and the composition $\Gamma_{\epsilon_1} \Gamma_{\epsilon_2} = \Gamma_{\epsilon_1 + \epsilon_2}$, for every $\Gamma_{\epsilon_1}, \Gamma_{\epsilon_2}$ in the set. Such a set is also known as a Lie group of transformations.

A one-parameter group of transformations constitutes a symmetry group of an ordinary differential equation, if the transformations map one solution curve to another. The resulting solution curve also have to satisfy the original equation. Thus, the transformations must leave the form of the differential equation invariant. This is called the
symmetry condition for an ordinary differential equation. Symmetries that satisfy this condition are called Lie symmetries, or Lie point symmetries [3, 11].

The symmetry condition for a first order ordinary differential equation is presented by [11] as described below.

Consider a first order ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

where $f$ is an arbitrary function of $x$ and $y$. Then the symmetry condition becomes

$$\frac{d\hat{y}}{d\hat{x}} = f(\hat{x}, \hat{y}) \quad \text{when} \quad \frac{dy}{dx} = f(x, y).$$

Since,

$$\frac{d\hat{y}}{d\hat{x}} = \frac{d\hat{y}(x, y)}{d\hat{x}(x, y)} = \frac{\hat{y}_x dx + \hat{y}_y dy}{\hat{x}_x dx + \hat{x}_y dy},$$

Lie symmetries of (2.4) satisfies the constraint

$$\frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y} = f(\hat{x}, \hat{y}),$$

where $y' = dy/dx$.

The Lie symmetries that we are dealing with are sets under a local group, which means that the group action is not necessarily defined over the entire plane. The conditions only need to apply in some neighbourhood of $\epsilon = 0$ [11, 20].

The following example is based on contents from [11, 20, 22], to illustrate examples of Lie symmetries of the simplest ordinary differential equation.

**Example 2.1.** Consider the ordinary differential equation

$$\frac{dy}{dx} = 0.$$  

The solution to (2.6) is $y = c$, where $c \in \mathbb{R}$. This yields that the graph of solutions of (2.6) are horizontal lines in the plane. Thus, for a parameter $\epsilon \in \mathbb{R}$, one symmetry of (2.6) is translations in the $y$-direction, i.e. $(\hat{x}, \hat{y}) = (x, y + \epsilon)$. This is true since the transformation maps the solution $y = c$ to the solution $y = c + \epsilon$, which also satisfies (2.6). Another symmetry that (2.6) possesses is the scaling $(\hat{x}, \hat{y}) = (e^\epsilon x, e^\epsilon y)$, since it maps horizontal lines to other horizontal lines. For $\epsilon \neq 0$, these transformations will stretch or shrink the lines, but horizontal lines will be preserved as sets. Moreover, for
the symmetry of translations in the $x$-direction, i.e. $(\hat{x}, \hat{y}) = (x + \epsilon, y)$, every solution curve is mapped to itself. This is a trivial symmetry. In each case mentioned above, $\epsilon = 0$ corresponds to the identity transformation.

From now on we will refer to Lie symmetries only as symmetries.

### 2.2 Symmetries in solving first order ODE

In this section we give a brief description on the method of symmetries as applied to first order ordinary differential equations. At the end of the section, we present a method on how the theory can be applied to solve such differential equations.

#### One-parameter group

The material in the following paragraph is based on contents from [9, 11, 20].

Consider a first order ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y). \tag{2.7}$$

We express the derivative $dy/dx$ as a coordinate $p$ and rewrite (2.7) as

$$F(x, y, p) = f(x, y) - p = 0,$$

which will be referred to as the surface equation, corresponding to (2.7). We want to find a one-parameter group which leaves the surface equation invariant. That is, the transformations

$$\hat{x} = \varphi(x, y; \epsilon), \quad \hat{y} = \psi(x, y; \epsilon), \quad \hat{p} = \vartheta(x, y, p; \epsilon), \tag{2.8}$$

for a parameter $\epsilon \in \mathbb{R}$. For each of the expressions in (2.8), we make a Taylor series expansion in the parameter $\epsilon$ near $\epsilon = 0$,

$$\hat{x} = \varphi(x, y; \epsilon) = x + \epsilon \xi(x, y) + \mathcal{O}(\epsilon^2), \quad \varphi(x, y; 0) = x,$$
$$\hat{y} = \psi(x, y; \epsilon) = y + \epsilon \eta(x, y) + \mathcal{O}(\epsilon^2), \quad \psi(x, y; 0) = y,$$
$$\hat{p} = \vartheta(x, y, p; \epsilon) = p + \epsilon \zeta(x, y, p) + \mathcal{O}(\epsilon^2), \quad \vartheta(x, y, p; 0) = p. \tag{2.9}$$
Because of (2.5), the Taylor series of $\dot{p}$ in (2.9) can also be expressed as

$$
\dot{p} = \frac{dy}{dx} = \frac{d\hat{y}/dx}{d\hat{x}/dx} = \frac{p + \epsilon(\eta_x + \eta_y)}{1 + \epsilon(\xi_x + \xi_y)} = p + \epsilon[\eta_x + (\eta_y - \xi_x)p - \xi_y p^2].
$$

This yields that

$$
\zeta(x, y, p) = \eta_x + (\eta_y - \xi_x)p - \xi_y p^2. \tag{2.10}
$$

Geometrically, $(\xi, \eta)$ is the tangent vector (at the point $(x, y)$) to the curve described by the transformed points $(\hat{x}, \hat{y})$, where

$$
(\xi(x, y), \eta(x, y)) = \left( \frac{d\hat{x}}{de} \bigg|_{\epsilon=0}, \frac{d\hat{y}}{de} \bigg|_{\epsilon=0} \right). \tag{2.11}
$$

Thus, $(\xi, \eta)$ is called the tangent vector field of the one-parameter group. In general, the one-parameter group as expressed in (2.9) is easier to find than (2.8). However, if the tangent vector field is known, the symmetries $(\hat{x}, \hat{y})$ can be constructed using (2.11).

In the solution method we study in this section, we have that $\xi$ is represented by

$$
\xi(x, y) = \sum_{i,j} \xi_{ij} x^i y^j, \quad 0 \leq i, j, \quad i + j \leq d_\xi. \tag{2.12}
$$

The same applies for $\eta$, where $d_\xi$ and $d_\eta$ are of finite degrees. Most of the results are independent of this, but for simplicity we restrict our attention to such types of functions.

**The determining equation**

In the following paragraph, we present material based on contents from [9, 12, 19].

**Definition 2.3.** A function $F(x, y, p)$ is said to be an invariant function under the one-parameter group (2.9) if $F(\hat{x}, \hat{y}, \hat{p}) = F(x, y, p)$, that is

$$
F(\varphi(x, y; \epsilon), \psi(x, y; \epsilon), \vartheta(x, y, p; \epsilon)) = F(x, y, p),
$$

is identical in the variables $x, y, p$ and the parameter $\epsilon$.

**Definition 2.4.** The infinitesimal generator is defined as

$$
X \equiv X(x, y, p) = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \zeta(x, y, p) \frac{\partial}{\partial p}. \tag{2.13}
$$

Sometimes, it is expressed only as $X(x, y) = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$. 

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The symmetries of a first order ordinary differential equation can be described using the Taylor series expansions (2.9) as well as the infinitesimal generator (2.13). For a given infinitesimal generator (2.13), the one-parameter group (2.8) can also be expressed by the following exponential transformations,

\[
\begin{align*}
\hat{x} &= e^{\epsilon X} x, \\
\hat{y} &= e^{\epsilon X} y, \\
\hat{p} &= e^{\epsilon X} p,
\end{align*}
\] (2.14)

where

\[
e^{\epsilon X} = 1 + \frac{\epsilon}{1!} X + \frac{\epsilon^2}{2!} X^2 + \ldots + \frac{\epsilon^a}{a!} X^a + \ldots
\]

**Theorem 2.1.** A function \(F(x, y, p)\) is invariant under the one-parameter group (2.9) if and only if \(F\) solves \(XF = 0\), that is

\[
\xi(x, y) \frac{\partial F}{\partial x} + \eta(x, y) \frac{\partial F}{\partial y} + \zeta(x, y, p) \frac{\partial F}{\partial p} = 0.
\] (2.15)

**Proof.** Let \(F(x, y, p)\) be an invariant function. We can expand the exponential transformations (2.14) into a function \(F(x, y, p)\) according to the following,

\[
F(\hat{x}, \hat{y}, \hat{p}) = e^{\epsilon X} F(x, y, p) = \left(1 + \frac{\epsilon}{1!} X + \frac{\epsilon^2}{2!} X^2 + \ldots + \frac{\epsilon^a}{a!} X^a + \ldots\right) F(x, y, p).
\] (2.16)

Then

\[
F(\hat{x}, \hat{y}, \hat{p}) = F(x, y, p) \iff e^{\epsilon X} F(x, y, p) = F(x, y, p).
\]

This yields that

\[
\frac{d}{d\epsilon} \left[e^{\epsilon X} F(x, y, p)\right]_{\epsilon=0} = \frac{d}{d\epsilon} F(x, y, p)_{\epsilon=0} \iff XF(x, y, p) = 0,
\]

i.e. \(F\) solves (2.15).

Conversely, let \(F(x, y, p)\) be a solution to (2.15). Since \(X(x, y, p)F(x, y, p) = 0\), we also have that \(X^2 F = X(XF) = 0, \ldots, X^a F = 0\) in (2.16). The conclusion is that \(F(\hat{x}, \hat{y}, \hat{p}) = F(x, y, p)\), i.e. \(F(x, y, p)\) is an invariant function, which proves the theorem. \(\square\)

Thus, if (2.15) is fulfilled, the surface equation \(F(x, y, p) = 0\) is left invariant under the transformations (2.9). We will refer to the partial differential equation (2.15) as the determining equation.

Based on linear algebra, the functions \(\xi(x, y)\) and \(\eta(x, y)\) can be determined from (2.15). Since we assume that \(\xi\) and \(\eta\) are polynomials of finite degree, we can use this
for the determining equation. Then it reduces to an equation of monomials in \( x \) and \( y \), constant coefficients and the unknowns \( \xi_{ij} \) and \( \eta_{ij} \) (see (2.12)).

New system of coordinates

The material in the remaining part of this section is based on contents from [9, 11, 17, 22].

Having the functions \( \xi(x, y) \) and \( \eta(x, y) \), the function \( \zeta(x, y, p) \) can be determined from (2.10), and then the infinitesimal generator \( X \) can be constructed from (2.13). Then the symmetry of the differential equation (2.7) is known.

**Theorem 2.2.** A first order ordinary differential equation with a translational symmetry in the dependent variable is separable.

**Proof.** For \( \epsilon \in \mathbb{R} \), let \( (\hat{x}, \hat{y}) = (x, y + \epsilon) \) be a symmetry of the differential equation

\[
\frac{dy}{dx} = f(x, y).
\]

According to (2.5),

\[
f(x, y + \epsilon) = f(\hat{x}, \hat{y}) = \frac{d\hat{y}}{d\hat{x}} = \frac{\hat{y}_x + y\hat{y}_y}{\hat{x}_x + y\hat{x}_y} = \frac{dy}{dx} = f(x, y).
\]

This shows that \( f \) is independent of \( y \). Therefore we have that \( \frac{dy}{dx} = f(x) \), which is separable. \( \square \)

Theorem 2.2 yields that if we have a way to convert any general symmetry of (2.7) into a translational symmetry in the dependent variable, then it is easy to obtain the solution to the differential equation. To make this happen, we want to find a new system of coordinates \( r(x, y) \), \( s(x, y) \) and \( t(x, y, p) \). Here, \( r \) is the independent variable, \( s \) is the dependent variable and \( t \) is the new constraint between \( r \) and \( s \).

**Theorem 2.3.** A one-parameter group \( \hat{x} = \varphi(x, y; \epsilon) \) and \( \hat{y} = \psi(x, y; \epsilon) \), with the infinitesimal generator

\[
X(x, y) = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y},
\]

(2.17)

can be reduced by a change of variables

\[
r = r(x, y), \quad s = s(x, y),
\]

(2.18)

to the translation group

\[
\hat{r} = r, \quad \hat{s} = s + \epsilon,
\]
using the infinitesimal generator

\[ X(r, s) = \frac{\partial}{\partial s}. \]  

(2.19)

These sets of new coordinates are called canonical coordinates.

Proof. The change of variables (2.18) transforms (2.17) according to the following,

\[ \text{XF}(r, s) = \text{XF}(r(x, y), s(x, y)) = \xi \left( \frac{\partial F}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F}{\partial s} \frac{\partial s}{\partial x} \right) + \eta \left( \frac{\partial F}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial F}{\partial s} \frac{\partial s}{\partial y} \right) \]

\[ = \left( \xi \frac{\partial r}{\partial x} + \eta \frac{\partial r}{\partial y} \right) \frac{\partial F}{\partial r} + \left( \xi \frac{\partial s}{\partial x} + \eta \frac{\partial s}{\partial y} \right) \frac{\partial F}{\partial s}. \]

However, \( F \) is an arbitrary function, so the infinitesimal generator in the new set of coordinates is

\[ X(r, s) = (Xr) \frac{\partial}{\partial r} + (Xs) \frac{\partial}{\partial s}. \]  

(2.20)

We have that (2.20) yields (2.19) if we define \( r \) and \( s \) by solving the partial differential equations

\[ Xr \equiv \xi(x, y) \frac{\partial r}{\partial x} + \eta(x, y) \frac{\partial r}{\partial y} = 0, \]

\[ Xs \equiv \xi(x, y) \frac{\partial s}{\partial x} + \eta(x, y) \frac{\partial s}{\partial y} = 1. \]  

(2.21)

Thus, Theorem 2.3 leads to an ordinary differential equation in the new coordinates that is separable and may be integrated directly. Since these sets of new coordinates are canonical coordinates, they also have to satisfy the condition

\[ r_x s_y - r_y s_x \neq 0. \]

Then the transformation will be invertible. However, the canonical coordinates cannot be defined if \( \xi(x, y) = \eta(x, y) = 0 \), because then the equation for \( s \) in (2.21) has no solutions.

In addition to (2.21), we add the differential equation

\[ Xt = \left( \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \zeta(x, y, p) \frac{\partial}{\partial p} \right) t(x, y, p) = 0, \]

to find the last coordinate \( t \). These three differential equations are the determining equa-
tions for the new coordinates. By the method of characteristics, we have the relations

\[ \frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} = \frac{dp}{\zeta(x, y, p)}, \] (2.22)

where \( r \) is constructed from the first differential equation in (2.22), by solving for the constant of integration. If \( \xi(x, y) = 0 \) and \( \eta(x, y) \neq 0 \), we use \( r = x \).

The expressions for the coordinates \( s \) and \( t \) are not unique. We do not seek for all solutions of the differential equations either, merely some. If we construct \( s \) from (2.21) by searching for solutions depending only on the single variable \( x \), that is

\[ \xi(x, y) \frac{ds}{dx} = 1 \leftrightarrow s(x, y) = \int \frac{dx}{\xi(x, y)}, \]

then we should use the second differential equation in (2.22) to construct \( t \).

For the new coordinate system, the constraint equation is

\[ \frac{ds}{dr} = \frac{s_x + ps_y}{r_x + pr_y}, \] (2.23)

according to (2.5). Since an ordinary differential equation is invariant under the group of translations in the \( s \)-direction for canonical coordinates, both the surface equation and the constraint equation have to be independent of \( s \) in the new coordinate system.

Below we present a summary on how symmetries can be identified and used to solve first order ordinary differential equations.

1. Find the surface equation, \( F(x, y, p) = 0 \). If \( \frac{\partial}{\partial y} F(x, y, p) = 0 \), then the ordinary differential equation is of the form \( \frac{dy}{dx} = f(x) \), which is separable. On the other hand, if \( \frac{\partial}{\partial y} F(x, y, p) \neq 0 \), then we continue with the method described below.

2. Construct the determining equation \( XF(x, y, p) = 0 \). Solve \( F(x, y, p) = 0 \) for \( p \) and substitute \( p = p(x, y) \) into the determining equation. Assume that \( \xi(x, y) \) and \( \eta(x, y) \) are expressions of zeroth degree, i.e. constants.

3. Substitute the expressions of \( \xi \) and \( \eta \) into the determining equation.

4. Compare the terms with the same monomials. This gives a set of linear equations for the wanted parameters, which has to be equal to zero. This will often lead to more equations than unknowns. If the rank of the system is equal to the number of unknowns, then there will only be a trivial solution, since the equations are
homogeneous. In that case, increase the degree of \( \xi \) and \( \eta \) by one, and repeat from 3. Otherwise, continue to 5.

5. If there exist a nontrivial solution to the system of equations, determine \( \zeta(x, y, p) \) from (2.10) and then the infinitesimal generator from (2.13).

6. Construct \( s(x, y), r(x, y) \) and \( t(x, y, p) \) and write down the transformations between the coordinates.

7. After some calculations, the constraint equation (2.23) will include the coordinate \( t \). Therefore, use the new surface equation, \( F(r, -t) = 0 \), to solve for \( t \) as a function of \( r \). Then substitute \( t(r) \) into the constraint equation. Now we only have to integrate this, to find the relation between \( s \) and \( r \).

8. Finally, use the inverse relation \( x = x(r, s) \) and \( y = y(r, s) \) to find the solution of the original problem.

In Section 3.2, we present several examples to illustrate the method presented above.

2.3 Extension to higher order ODE

In this section we extend the method described in Section 2.2 and present the use of symmetries when solving ordinary differential equations of second order (and higher).

The prolongation formula

The next two definitions are interpreted from [11].

**Definition 2.5.** The total derivative with respect to \( x \) is defined as

\[
D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \ldots + y^{(b+1)} \partial_{y^{(b+1)}} + \ldots,
\]

where \( y^{(k)} = d^k y/dx^k \) and \( \partial_x = \partial/\partial x \), etc.

**Definition 2.6.** For an ordinary differential equation of order \( n \), the \( k^{th} \) prolongation formula is generated by

\[
\eta^{(k)}(x, y, y', \ldots, y^{(k)}) = D_x \eta^{(k-1)} - y^{(k)} D_x \xi,
\]

for \( k = 1, \ldots, n \) and \( \eta^{(0)} = \eta \).
We have that \( \eta^{(k)} \) (for \( k = 1, \ldots, n \)) arises in the infinitesimal generator when the ordinary differential equation is of order \( n \). In the notation \( \eta^{(k)} \), \( (k) \) stands for an index, not a derivative.

Using [5, 9, 11], we will present what Definition 2.6 yields in the case of a second order ordinary differential equation. That is, for the differential equation

\[
F(x, y, y', y'') = f(x, y, y') - y'' = 0,
\]

the first prolongation formula is

\[
\eta^{(1)}(x, y, y') = D_x \eta - y'D_x \xi = \eta_x + (\eta_y - \xi_x)y' - \xi_y(y')^2,
\]

and the second prolongation formula is

\[
\eta^{(2)}(x, y, y', y'') = D_x \eta^{(1)} - y''D_x \xi \\
= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 - \xi_{yy}(y')^3 + (\eta_y - 2\xi_x - 3\xi_y y')y''.
\]

Then the infinitesimal generator becomes

\[
X(x, y, y', y'') = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta^{(1)}(x, y, y') \frac{\partial}{\partial y'} + \eta^{(2)}(x, y, y', y'') \frac{\partial}{\partial y''}.
\]

This leads to the determining equation

\[
X(x, y, y', y'')F(x, y, y', y'') = 0.
\]

Recalling the method described in Section 2.2, the symmetries can now be found in a similar way. However, we will not handle the corresponding details in this thesis. Even for higher order ordinary differential equations, the infinitesimal generator is sometimes expressed only as \( X(x, y) = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \).

**Reduction of order**

What we want to achieve when we apply the method of symmetries to an ordinary differential equation of higher order, is to reduce the order of the differential equation. Before we consider this concept, we will present the ideas of Lie algebras.

Consider an ordinary differential equation of order \( n \geq 2 \). Then there exist a vector space \( \mathcal{L}_h \), that is the set of all infinitesimal generators of the differential equation. Every
$X \in \mathcal{L}_h$ can be written as

$$X = \sum_{i=1}^{h} c_i X_i, \quad c_i \in \mathbb{R},$$

where \( \{X_1, \ldots, X_h\} \) form a basis for \( \mathcal{L}_h \) of dimension \( h \) [11, 12].

The following two definitions are interpreted from [12].

**Definition 2.7.** Consider two infinitesimal generators

$$X_1 = \xi_1(x, y) \frac{\partial}{\partial x} + \eta_1(x, y) \frac{\partial}{\partial y}, \quad X_2 = \xi_2(x, y) \frac{\partial}{\partial x} + \eta_2(x, y) \frac{\partial}{\partial y}.$$  

Then the Lie bracket is defined as

$$[X_1, X_2] = X_1 X_2 - X_2 X_1,$$

or equivalent,

$$[X_1, X_2] = (X_1 \xi_2 - X_2 \xi_1) \frac{\partial}{\partial x} + (X_1 \eta_2 - X_2 \eta_1) \frac{\partial}{\partial y}.$$  

**Definition 2.8.** The vector space \( \mathcal{L}_h \) is called a Lie algebra if the Lie bracket \( [X, Y] \in \mathcal{L}_h \) when \( X, Y \in \mathcal{L}_h \).

The order of the ordinary differential equation generate restrictions upon \( h \). A differential equation of second order has \( h \in \{0, 1, 2, 3, 8\} \). For order \( n \geq 3 \), then \( h \leq n + 4 \). Moreover, a linear ordinary differential equation of order \( n \geq 3 \) has \( h \in \{n + 1, n + 2, n + 4\} \). All \( X \in \mathcal{L}_h \) generates a set of symmetries, which constitutes an \( h \)-parameter (local) Lie group [11, 13].

In the following example, we illustrate how the infinitesimal generator can look like for an ordinary differential equation of second order. The example is adapted from [11].

**Example 2.2.** The second order ordinary differential equation \( y'' = 0 \) has a tangent vector field \( (\xi(x, y), \eta(x, y)) \), where

$$\xi(x, y) = c_1 + c_3 x + c_5 y + c_7 x^2 + c_8 xy,$$

$$\eta(x, y) = c_2 + c_4 y + c_6 x + c_7 xy + c_8 y^2,$$

and \( c_1, \ldots, c_8 \) are arbitrary constants. Thus, the infinitesimal generator takes the form

$$X = \sum_{i=1}^{8} c_i X_i,$$
where

\[ X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x, \quad X_4 = y\partial_y, \quad X_5 = y\partial_x, \]
\[ X_6 = x\partial_y, \quad X_7 = x^2\partial_x + xy\partial_y, \quad X_8 = xy\partial_x + y^2\partial_y. \]

Hence, \( h = 8 \), which is possible for an ordinary differential equation of second order.

In [12], they give a presentation of subalgebras. In the following definition, we outline the interpretation we have made.

**Definition 2.9.** Suppose \( \mathcal{L}_h \) is a Lie algebra. Then a subspace \( \mathcal{L}_m \) of dimension \( m \leq h \) is said to be a subalgebra to \( \mathcal{L}_h \) if \([X_i, X_j] \in \mathcal{L}_m\) for \( i, j = 1, \ldots, m \). Further, \( \mathcal{L}_m \) is said to be an ideal of \( \mathcal{L}_h \) if \([X_i, X_j] \in \mathcal{L}_m\) for \( i = 1, \ldots, m; j = 1, \ldots, h \).

The next definition is interpreted from [11, 17, 18].

**Definition 2.10.** A Lie algebra \( \mathcal{L}_h \) is said to be solvable if there exists a chain of subalgebras

\[ \{0\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots \subset \mathcal{L}_h, \]

such that \( \dim(\mathcal{L}_m) = m \) and \( \mathcal{L}_{m+1} \) is an ideal of \( \mathcal{L}_m \) for each \( m \).

We will now present how the order of an ordinary differential equation can be reduced by using symmetries.

Consider an ordinary differential equation of order \( n \geq 2 \), which has a known one-parameter group. Then the order of the differential equation can be reduced to \( n - 1 \). Thus, if several symmetries can be found, the order can be gradually reduced. Moreover, if an ordinary differential equation of order \( n \) has an \( h \)-parameter group (where \( h \leq n \)), the order of the differential equation can be reduced to \( n - h \) under the condition that the Lie algebra is solvable. Thus, we can use this to solve ordinary differential equations of higher order [5, 9, 11, 17].

In the remaining part of this section we present material from [11], to illustrate how we can reduce the order of an ordinary differential equation if a one-parameter group is known.

Consider an ordinary differential equation of order \( n \) of the form

\[ y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad n \geq 2. \quad (2.24) \]

If \( X(x, y) \) is the infinitesimal generator of a one-parameter group of (2.24), where \( r(x, y) \)
and $s(x, y)$ are the canonical coordinates, we have that

$$X(r, s) = \frac{\partial}{\partial s}.$$ 

Writing (2.24) in terms of canonical coordinates yields that

$$s^{(n)} = \Omega(r, s, s', \ldots, s^{(n-1)}), \quad s^{(k)} = \frac{d^k s}{dr^k}, \quad (2.25)$$

for some function $\Omega$. Since (2.25) is invariant under the group of translations in the $s$-direction, we have that

$$\Omega_s = 0.$$ 

Therefore, (2.25) takes the form

$$s^{(n)} = \Omega(r, s', \ldots, s^{(n-1)}).$$ 

Thus, for $v = ds/dr$, writing (2.24) in terms of canonical coordinates reduces the order of (2.24) to the following ordinary differential equation of order $n - 1$,

$$v^{(n-1)} = \Omega(r, v, \ldots, v^{(n-2)}), \quad v^{(k)} = \frac{d^{k+1} s}{dr^{k+1}}, \quad (2.26)$$

As a result of this, we can solve (2.24) by first solving the lower order ordinary differential equation (2.26).

In Chapter 3, we present some examples to illustrate the method of reduction of order for second order ordinary differential equations.

2.4 Lie’s integrating factor

Integrating factors used as standard solution method can be generalized using the method of symmetries. This is illustrated in this section. The result is presented in the following theorem, which has been interpreted from [8, 12, 18]. The proof has been adapted from [18].

Theorem 2.4. Consider a first order ordinary differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0. \quad (2.27)$$

If (2.27) possesses the infinitesimal generator $X(x, y) = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$ and if
\( \xi M + \eta N \neq 0 \), then the integrating factor of (2.27) is
\[
\mu(x, y) = \frac{1}{\xi(x, y)M(x, y) + \eta(x, y)N(x, y)},
\]
which is called Lie’s integrating factor.

Proof. We rewrite (2.27) as
\[
\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}.
\]
We let \( p = dy/dx \), which leads to the surface equation
\[
F(x, y, p) = \frac{M(x, y)}{N(x, y)} + p = 0.
\]
We can write the infinitesimal generator as (2.13). Then the determining equation \( XF(x, y, p) = 0 \) becomes
\[
\xi(x, y) \frac{\partial F}{\partial x} + \eta(x, y) \frac{\partial F}{\partial y} + \zeta(x, y, p) \frac{\partial F}{\partial p} = 0.
\]
After some calculations, where we have used (2.10) and substituted \( p(x, y) = -\frac{M(x, y)}{N(x, y)} \) into the determining equation, we get that
\[
\left( \xi \frac{\partial M}{\partial x} + \eta \frac{\partial M}{\partial y} \right) N - \left( \xi \frac{\partial N}{\partial x} + \eta \frac{\partial N}{\partial y} \right) M + \frac{\partial \eta}{\partial x} M^2 - \frac{\partial \xi}{\partial x} N^2 - \left( \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x} \right) MN = 0.
\]
(2.29)

For \( \mu(x, y) \) to be an integrating factor of (2.27), we must have that the differential equation
\[
\mu Mdx + \mu Ndy = 0,
\]
is exact. This means that the following derivatives must be satisfied,
\[
\frac{\partial}{\partial y} (\mu M) = \frac{\partial}{\partial x} (\mu N).
\]
(2.30)

Substituting the formula for \( \mu \) in (2.28), the differential equation (2.30) becomes
\[
\mu^2 \left\{ \eta \left( N \frac{\partial M}{\partial y} - M \frac{\partial N}{\partial y} \right) - \frac{\partial \xi}{\partial y} M^2 - \frac{\partial \eta}{\partial y} MN \right\} = \mu^2 \left\{ \xi \left( M \frac{\partial N}{\partial x} - N \frac{\partial M}{\partial x} \right) - \frac{\partial \xi}{\partial x} MN - \frac{\partial \eta}{\partial x} N^2 \right\}.
\]
(2.31)
Comparing (2.31) with (2.29) proves the theorem.

We present an example in Section 3.2 where we use Lie’s integrating factor.

2.5 Invariant solutions

In this section we give a brief description of invariant solutions of ordinary differential equations and its significants in understanding symmetry methods. The material is based on contents from [4, 11, 12].

Consider an ordinary differential equation of order \( n \) of the form

\[
F(x, y, y', \ldots, y^{(n)}) = 0.
\]

(2.32)

From Section 2.2, we have the one-parameter group

\[
\begin{align*}
\hat{x} &= \varphi(x, y; \epsilon) = x + \epsilon \xi(x, y) + O(\epsilon^2), \\
\hat{y} &= \psi(x, y; \epsilon) = y + \epsilon \eta(x, y) + O(\epsilon^2).
\end{align*}
\]

(2.33)

Definition 2.11. A curve defined implicitly by \( \phi(x, y) = 0 \) is said to be an invariant curve under the transformations (2.33) if \( \phi(\hat{x}, \hat{y}) = 0 \) whenever \( \phi(x, y) = 0 \).

In other words, \( \phi(x, y) = 0 \) is an invariant curve under the transformations (2.33) if the tangent to the curve \( \phi(x, y) = 0 \) is parallel to the tangent vector \( (\xi(x, y), \eta(x, y)) \) at each point \( (x, y) \).

Definition 2.12. The curve \( \phi(x, y) = 0 \) is said to be an invariant solution of (2.32) under the transformations (2.33), if the differential equation (2.32) has a one-parameter group (2.33) and

1. \( \phi(x, y) = 0 \) is an invariant curve under (2.33),
2. \( \phi(x, y) = 0 \) solves (2.32).

For a first order ordinary differential equation \( F(x, y, y') = 0 \) with the infinitesimal generator \( X(x, y) = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \), the characteristic is defined as

\[
Q(x, y) = F\left(x, y, \frac{\eta(x, y)}{\xi(x, y)}\right),
\]

(2.34)

if we assume that \( \xi \neq 0 \). For the algebraic equation \( Q(x, y) = 0 \), three cases arise:
1. \( Q(x, y) = 0 \) does not define any curve \( \phi(x, y) = 0 \) in the plane, i.e. solving \( Q(x, y) = 0 \) for \( y \) does not admit any real solutions.

2. \( Q(x, y) \equiv 0 \).

3. \( Q(x, y) \not\equiv 0 \), but \( Q(x, y) = 0 \) define curves \( \phi(x, y) = 0 \) in the plane.

In the first case mentioned above, the ordinary differential equation \( F(x, y, y') = 0 \) has no invariant solutions. In the second case, each solution of \( F(x, y, y') = 0 \) is an invariant solution. The symmetries are said to be trivial when this happens. In the third case, any curve \( \phi(x, y) = 0 \) is an invariant solution of \( F(x, y, y') = 0 \). This means that the solution of \( F(x, y, y') = 0 \) expressed implicitly by \( \phi(x, y) = 0 \), is mapped to itself under the symmetry (2.33).

The reason why invariant solutions are so important in the symmetry methods, is that they can be used to construct general solutions to the differential equations. For first order ordinary differential equations, this is possible if two infinitesimal generators are known. We provide an example of this in Section 3.2, together with several other examples of invariant solutions. The examples are graphically illustrated.
Chapter 3

Illustrative examples

3.1 Standard methods and symmetry

In this section we provide examples to illustrate how standard methods used to solve ordinary differential equations can be seen from the light of symmetries. The material is based on contents from [11, 22]. In these examples, we do not focus on how we can determine the symmetries, merely how they can be used. Therefore, the reader is not supposed to see these symmetries only by looking at the differential equations.

Example 3.1 (Homogeneous equations). Let $G$ be an arbitrary function that depends on the ratio of $y$ to $x$. Then a homogeneous first order differential equation is of the form

$$
\frac{dy}{dx} = G\left(\frac{y}{x}\right), \quad (x \neq 0).
$$

(3.1)

One symmetry of (3.1) is $(\hat{x}, \hat{y}) = (e^x, e^y)$. According to (2.11), the tangent vector field is $(\xi(x, y), \eta(x, y)) = (x, y)$. Now we can find the set of new coordinates similarly as in Section 2.2. This yields the following canonical coordinates (we will not include $t$),

$$
r(x, y) = yx^{-1}, \quad s(x, y) = \ln|x|.
$$

(3.2)

Thus, the constraint equation (2.23) becomes

$$
\frac{ds}{dr} = \frac{s_x + G\left(\frac{y}{x}\right) s_y}{r_x + G\left(\frac{y}{x}\right) r_y} = \frac{1}{r} \frac{1}{G(r)} = \frac{1}{G(r) - r},
$$

(3.3)

which is a separable equation. Now the solution to (3.1) can be found by integrating (3.3) and then change back to the original coordinates using (3.2).
**Example 3.2** (Integrating factor). Let $F$ and $G$ be arbitrary functions of $x$. Then an inhomogeneous first order linear differential equation is of the form

$$y' + F(x)y = G(x). \tag{3.4}$$

Multiplying (3.4) with $e^{\int_0^x F \, d\tau}$ and integrating both sides, we get that the solution of (3.4) is

$$y = e^{-\int_0^x F \, d\tau} \int e^{\int_0^x F \, d\tau} G(x) \, dx.$$

We will now illustrate how we can derive the solution of (3.4), by using symmetries instead of a standard method.

To begin with, we have that $y_h(x) = e^{-\int_0^x F \, d\tau}$ is a solution to the homogeneous differential equation

$$y' + F(x)y = 0.$$

This yields that $(\hat{x}, \hat{y}) = (x, y + e y_h(x))$ is a symmetry of (3.4). According to (2.11), the tangent vector field is $(\xi(x, y), \eta(x, y)) = (0, y_h(x))$. In the case when $\xi$ is identically zero, we have that $r(x, y) = x$. As in Section 2.2, we construct $s(x, y)$ by solving $X s = 1$. Then we have that

$$\eta(x, y) \frac{\partial s}{\partial y} = 1 \iff s_y = y_h(x)^{-1}.$$

Thus, $s(x, y) = y y_h(x)^{-1}$. This yields that the constraint equation (2.23) becomes

$$\frac{ds}{dr} = \frac{s_x + s_y \frac{dy}{dx}}{r_x + r_y \frac{dy}{dx}} = -y y_h \left( \frac{dy}{dx} \right) + \frac{1}{y_h} \left( -F(x)y + \frac{1}{y_h} [G(x) - F(x)y] = \frac{G(r)}{y_h(r)} \right),$$

which has the solution

$$s(r) = \int \frac{G(r)}{y_h(r)} \, dr = \int e^{\int_0^r F \, d\tau} G(r) \, dr.$$

Changing back to the original coordinates yields that

$$y = y_h(x)s(x, y) = e^{-\int_0^x F \, d\tau} \int e^{\int_0^x F \, d\tau} G(x) \, dx,$$

which is the solution to (3.4).

**Example 3.3** (Reduction of order). Consider the homogeneous second order linear differential equation

$$y'' + h(x)y' + q(x)y = 0, \tag{3.5}$$

...
where \( h \) and \( q \) are arbitrary functions of \( x \). We will use (3.5) to illustrate how the order of an ordinary differential equation can be reduced from two to one by using symmetry.

One symmetry of (3.5) is \((\hat{x}, \hat{y}) = (x, e^y)\). According to (2.11), this yields that the tangent vector field is \((\xi(x, y), \eta(x, y)) = (0, y)\). Since \( \xi \) is identically zero, \( r(x, y) = x \).

Similarly as described in Section 2.2, \( s(x, y) = \int y^{-1} dy = \ln |y| \).

Choosing \( v \) to be \( ds/dr \), we have that

\[
v = \frac{ds}{dr} = \frac{y'}{y}.
\]

In the new coordinates, this yields that (3.5) becomes

\[
\frac{dv}{dr} = \frac{y''}{y} - \frac{y'^2}{y^2} = -[h(r)v + q(r) + v^2],
\]

which is a first order ordinary differential equation. Thus, we have reduced the order of the differential equation from two to one. Now we can solve (3.6) and then change back to the original coordinates, to find the solution to (3.5).

### 3.2 Nonlinear ODE and symmetry

In this section we illustrate the symmetry methods presented in the previous chapter. Examples 3.4–3.7 are adapted from various sources. In these examples, we focus on how symmetries can be used and not how they can be determined. In Example 3.8–3.10, the author has solved given ordinary differential equations by using the method described in Section 2.2. In these examples we also examine if there are any invariant solutions to the given differential equations.

**Example 3.4** (Lie’s integrating factor). We use an example from [12] to illustrate how Lie’s integrating factor can be used to solve a first order ordinary differential equation.

Consider the Riccati-type of equation

\[
y' = \frac{2}{x^2} - y^2, \quad (x \neq 0). \tag{3.7}
\]

We rewrite (3.7) as

\[
dy + \left(y^2 - \frac{2}{x^2}\right) \, dx = 0, \tag{3.8}
\]

which is of the form of (2.27). The differential equation (3.7) has a one-parameter group
whose infinitesimal generator is

\[ X(x, y) = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \]

Substitution of

\[ \xi(x, y) = x, \quad \eta(x, y) = -y, \quad M(x, y) = y^2 - \frac{2}{x^2}, \quad N(x, y) = 1, \]

into (2.28) yields the integrating factor

\[ \mu(x, y) = \frac{x}{x^2y^2 - xy - 2}. \]

We multiply (3.8) with the integrating factor, which leads to the differential equation

\[ \frac{x}{x^2y^2 - xy - 2} dy + \frac{1}{x^2y^2 - xy - 2} \frac{x^2y^2 - 2}{x} dx = 0. \quad (3.9) \]

We notice that (3.9) is exact, and can therefore be solved using a standard procedure.

In this special case, we can also use the following:

Since,

\[ \frac{x^2y^2 - 2}{x} = y + \frac{x^2y^2 - xy - 2}{x}, \]

we can rewrite (3.9) as

\[ \frac{d(xy)}{x^2y^2 - xy - 2} + \frac{dx}{x} = 0. \quad (3.10) \]

Using \( z = xy \) we have that

\[ \frac{d(xy)}{x^2y^2 - xy - 2} = \frac{dz}{z^2 - z - 2} = \frac{1}{3} \left( \frac{1}{z - 2} - \frac{1}{z + 1} \right) dz, \]

and, hence,

\[ \int \frac{1}{z^2 - z - 2} dz = \frac{1}{3} \ln \left( \frac{z - 2}{z + 1} \right). \]

Changing back to the standard variables, (3.10) takes the form

\[ d \left( \frac{1}{3} \ln \left( \frac{xy - 2}{xy + 1} \right) + \ln x \right) = 0. \]

Integration gives that

\[ y = \frac{2x^3 + C}{x(x^3 - C)}, \]

23
is the solution to (3.7), where $C$ is an arbitrary constant. The calculations above requires that $xy - 2 \neq 0$ and $xy + 1 \neq 0$. Since $y = 2/x$ and $y = -1/x$ satisfies (3.7), it is necessary to add these solutions to the differential equation.

The tangent vector field of (3.7) is $(\xi(x, y), \eta(x, y)) = (x, -y)$. According to (2.34), the characteristic is

$$Q(x, y) = \frac{2}{x^2} - y^2 + \frac{y}{x}.$$

Then, $Q(x, y) = 0$ has the solutions $y = 2/x$ and $y = -1/x$. Thus, these solutions are invariant solutions of (3.7).

The basic features of the solutions to (3.7) are illustrated in the form of stream plots, or flow lines, in Figure 3.1. The invariant solution $y = -1/x$ is marked with a blue curve and $y = 2/x$ is marked with a red curve.

Example 3.5 (Reduction of order). We use an example from [11] to illustrate how the order of an ordinary differential equation can be reduced by using symmetry. See also Example 3.3.

Consider the nonlinear second order ordinary differential equation

$$y'' = \frac{y'^2}{y} + \left(y - \frac{1}{y}\right)y'.$$

The differential equation (3.11) possesses a one-parameter group whose infinitesimal
generator is

\[ X(x, y) = \frac{\partial}{\partial x}. \]

Thus, the tangent vector field is \((\xi(x, y), \eta(x, y)) = (1, 0)\). Using this, we can find the set of new coordinates as in Section 2.2. This yields the canonical coordinates

\[ r(x, y) = y, \quad s(x, y) = x. \tag{3.12} \]

Choosing \( v \) to be \( \dot{s} = ds/dr = (y')^{-1} \), the differential equation (3.11) becomes

\[ \frac{dv}{dr} = -\frac{y''}{y^3} = -\frac{v}{r} + \left(\frac{1}{r} - r\right)v^2, \]

which is a Bernoulli-type of equation and can be solved if we rewrite it as a linear differential equation for \( v^{-1} \). If we instead choose \( v = y' \), i.e. \( v = (\dot{s})^{-1} \), then (3.11) reduces to

\[ \frac{dv}{dr} = \frac{y''}{y'} = v + r - \frac{1}{r}, \tag{3.13} \]

which directly leads to a linear differential equation. The general solution of (3.13) is

\[ v(r) = r^2 - 2c_1r + 1, \quad c_1 \in \mathbb{R}. \]

Then we have that

\[ s(r) = \int \frac{1}{r^2 - 2c_1r + 1} dr. \tag{3.14} \]

Solving (3.14) and changing back to the original coordinates using (3.12), leads to the following general solution of (3.11),

\[ y = \begin{cases} 
  c_1 - \sqrt{c_1^2 - 1} \tanh \left( \sqrt{c_1^2 - 1}(x + c_2) \right) & , c_1^2 > 1, \\
  c_1 - (x + c_2)^{-1} & , c_1^2 = 1, \\
  c_1 + \sqrt{1 - c_1^2} \tan \left( \sqrt{1 - c_1^2}(x + c_2) \right) & , c_1^2 < 1,
\end{cases} \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Example 3.6 (Invariant solutions).** We use an example from [4] to illustrate an example of invariant solutions. Here, the geometrical outcome is the one that is of interest.

Consider the first order ordinary differential equation

\[ y' = \frac{x\sqrt{x^2 + y^2} + y(x^2 + y^2 - 1)}{x(x^2 + y^2 - 1) - y\sqrt{x^2 + y^2}}, \tag{3.15} \]
The differential equation (3.15) possesses a one-parameter group of rotations, whose infinitesimal generator is

\[ X(x, y) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \]

Therefore, we know that the tangent vector field is \((\xi(x, y), \eta(x, y)) = (y, -x)\). According to (2.34), the characteristic is

\[ Q(x, y) = - \left( \frac{x}{y} + \frac{x \sqrt{x^2 + y^2} + y(x^2 + y^2 - 1)}{x(x^2 + y^2 - 1) - y \sqrt{x^2 + y^2}} \right). \]

Thus, \(Q(x, y) = 0\) becomes

\[ \frac{(x^2 + y^2)(x^2 + y^2 - 1)}{y[x(x^2 + y^2 - 1) - y \sqrt{x^2 + y^2}]} = 0. \]

This yields that the circle \(x^2 + y^2 = 1\) is an invariant solution of (3.15).

A family of solution curves of (3.15) is illustrated in Figure 3.2. The invariant solution is marked with a red curve.

\[ \text{Figure 3.2: Some solution curves of (3.15), including the invariant solution.} \]

**Example 3.7** (General solutions obtained from invariant solutions). We use an example from [12] to illustrate how the general solution of a first order ordinary differential equation with two known infinitesimal generators can be obtained using invariant solutions.
Consider the ordinary differential equation
\[ y' = \frac{y}{x} + \frac{y^2}{x^3}, \]  
(3.16)
for which the surface equation becomes
\[ F(x, y, y') = \frac{y}{x} + \frac{y^2}{x^3} - y' = 0. \]

Two known infinitesimal generators of (3.16) are
\[ X_1(x, y) = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \]
\[ X_2(x, y) = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}. \]

Therefore, we know that the tangent vector field under \( X_2 \) is \( (\xi_2(x, y), \eta_2(x, y)) = (x, 2y) \).

Then the characteristic is
\[ Q(x, y) = F\left(x, y, \frac{\eta_2}{\xi_2}\right) = \frac{y^2}{x^3} - \frac{y}{x}. \]

Thus, \( Q(x, y) = 0 \) yields the invariant solutions \( y = 0 \) and \( y = x^2 \).

The differential equation (3.16) possesses the symmetry
\[ \hat{x} = \frac{x}{1 - \epsilon x}, \quad \hat{y} = \frac{y}{1 - \epsilon x}, \]  
(3.17)
under \( X_1 \). According to Definition 2.12, we should have that
\[ \hat{y} = \hat{x}^2, \]  
(3.18)
if \( y = x^2 \) is an invariant solution of (3.16). Substitution of (3.17) into (3.18) yields that
\[ y = \frac{x^2}{1 - \epsilon x}. \]  
(3.19)

We denote the parameter \( \epsilon \) in (3.19) with an arbitrary constant \( C \). Then we have that
\[ y = \frac{x^2}{1 - C x}, \]
is the solution to (3.16), as it can easily be verified.
A family of solution curves of (3.16) is illustrated in Figure 3.3. The invariant solution \( y = 0 \) is marked with a red curve and \( y = x^2 \) is marked with a blue curve.

![Figure 3.3: Some solution curves of (3.16), including the two invariant solutions.](image)

**Example 3.8.** In [11], they present the following Riccati-type of equation,

\[
y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, \quad (x \neq 0).
\]  

(3.20)

We will use (3.20) as an example to illustrate how the method described in Section 2.2 works. We rewrite (3.20) by letting \( p = y' \). Then we have the surface equation

\[
F(x, y, p) = xy^2 - \frac{2y}{x} - \frac{1}{x^3} - p = 0.
\]  

(3.21)

Therefore, the determining equation \( XF(x, y, p) = 0 \) is

\[
\xi(x, y) \frac{\partial F}{\partial x} + \eta(x, y) \frac{\partial F}{\partial y} + \zeta(x, y, p) \frac{\partial F}{\partial p} = 0 \Leftrightarrow
\xi \left( y^2 + \frac{2y}{x^2} + \frac{3}{x^4} \right) + \eta \left( 2xy - \frac{2}{x} \right) + \left[ \eta_x + (\eta_y - \xi_x)p - \xi_y p^2 \right](-1) = 0,
\]

where we have used (2.10). Since \( p = y' \), we substitute \( p(x, y) = xy^2 - 2y/x - 1/x^3 \) into
the determining equation. This yields that

\[
\xi \left( y^2 + \frac{2y}{x^2} + \frac{3}{x^4} \right) + \eta \left( 2xy - \frac{2}{x} \right) - \eta x - \\
(\eta_y - \xi_x) \left( xy^2 - \frac{2y}{x} - \frac{1}{x^3} \right) + \xi_y \left( xy^2 - \frac{2y}{x} - \frac{1}{x^3} \right)^2 = 0. 
\]  

(3.22)

To begin with, we assume that \( \xi \) and \( \eta \) are expressions of zeroth degree, i.e. \( \xi(x, y) = \xi_{00} \) and \( \eta(x, y) = \eta_{00} \). Then (3.22) becomes

\[
\xi_{00} \left( y^2 + \frac{2y}{x^2} + \frac{3}{x^4} \right) + \eta_{00} \left( 2xy - \frac{2}{x} \right) = 0. 
\]

This leads to five equations for the two unknowns, \( \xi_{00} \) and \( \eta_{00} \). How the coefficients of the monomials depend on the unknown parameters can be described as

\[
\begin{align*}
y^2 & : \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \eta_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
x y & : \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \eta_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
y/x^2 & : \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \eta_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
1/x & : \begin{bmatrix} 0 & -2 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \eta_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
1/x^4 & : \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \eta_{00} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\end{align*}
\]

where the expressions on the left are the monomials. The above set of equations has rank two, so we only have the trivial solution \( \xi_{00} = \eta_{00} = 0 \). Therefore, we increase the degree of \( \xi \) and \( \eta \) by one, i.e. \( \xi(x, y) = \xi_{00} + \xi_{10}x + \xi_{01}y \) and \( \eta(x, y) = \eta_{00} + \eta_{10}x + \eta_{01}y \). Then (3.22) becomes

\[
\begin{align*}
(\xi_{00} + \xi_{10}x + \xi_{01}y) \left( y^2 + \frac{2y}{x^2} + \frac{3}{x^4} \right) & + (\eta_{00} + \eta_{10}x + \eta_{01}y) \left( 2xy - \frac{2}{x} \right) - \eta_{10} - \\
(\eta_{01} - \xi_{10}) \left( xy^2 - \frac{2y}{x} - \frac{1}{x^3} \right) & + \xi_{01} \left( xy^2 - \frac{2y}{x} - \frac{1}{x^3} \right)^2 = 0.
\end{align*}
\]
This leads to 15 equations for the six unknown parameters, which can be described as

\[
\begin{align*}
y^3 & : \begin{bmatrix} 0 & 0 & -3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \end{bmatrix} = 0 \\
y^2 & : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{10} \\ \xi_{01} \\ \eta_{01} \end{bmatrix} = 0 \\
x^2y^4 & : \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
xy^2 & : \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{10} \\ \eta_{00} \end{bmatrix} = 0 \\
x^2y & : \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
xy & : \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{10} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
1 & : \begin{bmatrix} 0 & 0 & 0 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
y/x & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
y/x^2 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
y/x^4 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
y^2/x^2 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
1/x & : \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
1/x^2 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
1/x^3 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
1/x^4 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
1/x^6 & : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{00} \\ \xi_{01} \\ \eta_{00} \\ \eta_{01} \end{bmatrix} = 0 \\
\end{align*}
\]

We solve the above system of equations and discover that there exist a nontrivial solution, that is

\[
2\xi_{10} = -\eta_{01},
\]

\[
\xi_{00} = \xi_{01} = \eta_{00} = \eta_{10} = 0.
\]

Up to some overall scaling factor, we can choose \( \xi_{10} = 1 \) and \( \eta_{01} = -2 \). Then we have that

\[
\xi(x, y) = x,
\]

\[
\eta(x, y) = -2y. \tag{3.23}
\]

The respective derivatives of (3.23) inserted in (2.10) yields that

\[
\zeta(x, y, p) = -3p.
\]

Therefore, the infinitesimal generator is

\[
X = x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - 3p \frac{\partial}{\partial p}. \tag{3.24}
\]

Now we want to use (3.24) to determine the new coordinates. First, we construct the
dependent coordinate $s$,

$$s(x, y) = \int \frac{dx}{\xi} = \int \frac{1}{x} dx = \ln x.$$  

Then we construct the independent coordinate $r$,

$$\frac{dx}{\xi} = \frac{dy}{\eta} \Leftrightarrow$$

$$\frac{dx}{x} = \frac{dy}{-2y} \Leftrightarrow$$

$$\int \frac{1}{y} dy = -2 \int \frac{1}{x} dx \Leftrightarrow$$

$$y = c_1 x^{-2}, \quad c_1 \in \mathbb{R}.$$  

Thus, $r(x, y) = c_1 = x^2 y$. Finally, we construct $t$,

$$\frac{dp}{\zeta} = \frac{dy}{\eta} \Leftrightarrow$$

$$\frac{dp}{-3p} = \frac{dy}{-2y} \Leftrightarrow$$

$$2 \int \frac{1}{p} dp = 3 \int \frac{1}{y} dy \Leftrightarrow$$

$$p^2 = c_2 y^3, \quad c_2 \in \mathbb{R}.$$  

Thus, $t(x, y, p) = c_2 = p^2 y^{-3}$. Therefore, we have the following transformations between the coordinate systems,

$$s = \ln x \Leftrightarrow x = e^s,$$

$$r = x^2 y \Leftrightarrow y = re^{-2s}, \quad (3.25)$$

$$t = p^2 y^{-3} \Leftrightarrow p = \pm t^{1/2} r^{3/2} e^{-3s}.$$  

The surface equation (3.21) transforms to

$$e^{-3s} [r^2 - 2r - 1 - t^{1/2} r^{3/2}] = 0.$$  

So the new surface equation (independent of $s$) is $r^2 - 2r - 1 - t^{1/2} r^{3/2} = 0$. Solving for $t$ yields that $t(r) = (r^2 - 2r - 1)^2 r^{-3}$. Then the constraint equation, with the new
coordinates and the expression for $t$, becomes

$$\frac{ds}{dr} = \frac{s_x + p s_y}{r_x + p r_y} = \frac{1}{2x^2 y + px^3} = \frac{1}{r^2 - 1}.$$

This yields that

$$s(r) = \int \frac{1}{r^2 - 1} \, dr = \frac{1}{2} \ln \left( \frac{1 - r}{1 + r} \right) + c_3, \quad c_3 \in \mathbb{R}.$$

Changing back to the original coordinates using (3.25), we obtain that

$$y = \frac{c - x^2}{x^4 + cx^2}, \quad c \in \mathbb{R},$$

is the solution to (3.20). The result is unchanged even if we use the negative $p$-value in (3.25). Our result is in agreement with the result in [11], obtained by a different method.

According to (3.23), the tangent vector field of (3.20) is

$$(\xi(x, y), \eta(x, y)) = (x, -2y).$$

Then the scale symmetry $(\hat{x}, \hat{y}) = (e^\xi x, e^{-2\xi} y)$ satisfies (2.11). We claim that this symmetry also satisfies (2.5).

**Proposition 3.1.** For the ordinary differential equation (3.20), the scale symmetry $(\hat{x}, \hat{y}) = (e^\xi x, e^{-2\xi} y)$ satisfies (2.5).

**Proof.** We have that

$$\hat{y}_x = 0, \quad \hat{y}_y = e^{-2\xi}, \quad \hat{x}_x = e^\xi, \quad \hat{x}_y = 0.$$

Evaluation of the left side of (2.5) yields that

$$\frac{\hat{y}_x + \left( xy^2 - \frac{2y}{x^3} - \frac{1}{x^3} \right) \hat{y}_y}{\hat{x}_x + \left( xy^2 - \frac{2y}{x^3} - \frac{1}{x^3} \right) \hat{x}_y} = e^{-3\xi} \left( xy^2 - \frac{2y}{x} - \frac{1}{x^3} \right),$$

and the right side of (2.5) becomes

$$f(\hat{x}, \hat{y}) = e^\xi x(e^{-2\xi} y)^2 - \frac{2e^{-2\xi} y}{e^\xi x} - \frac{1}{(e^\xi x)^3} = e^{-3\xi} \left( xy^2 - \frac{2y}{x} - \frac{1}{x^3} \right).$$

Thus, $(\hat{x}, \hat{y}) = (e^\xi x, e^{-2\xi} y)$ satisfies (2.5) and is indeed a symmetry of (3.20). \qed
According to (2.34), the characteristic is
\[ Q(x, y) = xy^2 - \frac{1}{x^3}. \]
Thus, \( Q(x, y) = 0 \) yields that
\[ y = \pm x^{-2}. \]
Consequently, the symmetry acts nontrivially on all the solution curves of (3.20), except for the invariant solutions \( y = \pm x^{-2} \).

A family of solution curves of (3.20) is illustrated in Figure 3.4. The invariant solution \( y = x^{-2} \) is marked with a red curve and \( y = -x^{-2} \) is marked with a blue curve.

![Figure 3.4: Some solution curves of (3.20), including the two invariant solutions.](image)

**Example 3.9.** Consider the following differential equation, which comes from an exercise in [11],
\[ y' = e^{-x}y^2 + y + e^x. \]  
(3.26)

We use (3.26) as a second example to illustrate the method given in Section 2.2. The surface equation is
\[ F(x, y, p) = e^{-x}y^2 + y + e^x - p = 0. \]  
(3.27)

Thus, the determining equation is
\[ \xi(-e^{-x}y^2 + e^x) + \eta(2e^{-x}y + 1) - \eta_x - (\eta_y - \xi_x)(e^{-x}y^2 + y + e^x) + \xi_y(e^{-x}y^2 + y + e^x)^2 = 0, \]  
(3.28)
where we have used (2.10) and \( p(x, y) = e^{-x}y^2 + y + e^x \). Assuming that \( \xi(x, y) = \xi_{00} \) and \( \eta(x, y) = \eta_{00} \) lead to the trivial solution \( \xi_{00} = \eta_{00} = 0 \). Therefore, we increase the degree by one and let \( \xi(x, y) = \xi_{00} + \xi_{10}x + \xi_{01}y \) and \( \eta(x, y) = \eta_{00} + \eta_{10}x + \eta_{01}y \). Then (3.28) becomes

\[
(\xi_{00} + \xi_{10}x + \xi_{01}y)(-e^{-x}y^2 + e^x) + (\eta_{00} + \eta_{10}x + \eta_{01}y)(2e^{-x}y + 1) - \\
\eta_{10} - (\eta_{01} - \xi_{10})(e^{-x}y^2 + y + e^x) + \xi_{01}(e^{-x}y^2 + y + e^x)^2 = 0.
\]

This leads to a system of 15 equations, which we solve and discover that there exist a nontrivial solution that is

\[
\begin{align*}
\xi_{00} &= \eta_{01}, \\
\xi_{10} &= \xi_{01} = \eta_{00} = \eta_{10} = 0.
\end{align*}
\]

Thus, we can choose \( \xi(x, y) = 1 \) and \( \eta(x, y) = y \). The respective derivatives inserted in (2.10) yields that \( \zeta(x, y, p) = p \). Therefore, the infinitesimal generator is

\[
X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + p \frac{\partial}{\partial p}.
\]

We construct the new coordinates \( s(x, y), r(x, y) \) and \( t(x, y, p) \). This yields the following transformations between the coordinate systems,

\[
s = x \iff x = s, \\
r = e^{-x}y \iff y = re^s, \\
t = py^{-1} \iff p = re^st.
\]

The surface equation (3.27) transforms to

\[
e^s[r^2 + r + 1 - rt] = 0.
\]

Thus, the new surface equation is \( r^2 + r + 1 - rt = 0 \). This yields that \( t(r) = r + 1 + 1/r \). Therefore, the constraint equation is

\[
\frac{ds}{dr} = \frac{s_x + ps_y}{r_x + pr_y} = \frac{e^x}{p - y} = \frac{1}{rt - r} = \frac{1}{r^2 + 1}.
\]

This yields that

\[
s(r) = \int \frac{1}{r^2 + 1} dr = \arctan(r) + c_3, \quad c_3 \in \mathbb{R}.
\]
Changing back to the original coordinates using (3.29), we obtain that

\[ y = e^x \tan(x + c_3). \]

Thus, the solution to (3.26) is \( y = e^x \tan(x + c) \), where \( c \) is an arbitrary constant.

If we determine the characteristic of (3.26) according to (2.34) and search for invariant solutions, we will not find any. That is because the equation

\[ Q(x, y) = 0 \iff e^{-x}y^2 + e^x = 0, \]

does not admit any real solutions. This means that the symmetry corresponding to the tangent vector field \((\xi(x, y), \eta(x, y)) = (1, y)\) acts nontrivially on all the solution curves of (3.26).

A family of solution curves of (3.26) is illustrated in Figure 3.5.

![Figure 3.5: Some solution curves of (3.26).](image)

**Example 3.10.** In [20], they present the following Bernoulli-type of equation,

\[ y' = y + y^{-1}e^x, \quad (y \neq 0). \quad (3.30) \]

We use (3.30) as a third example to illustrate the method described in Section 2.2. The surface equation is

\[ F(x, y, p) = y + y^{-1}e^x - p = 0. \quad (3.31) \]
Thus, the determining equation is

\[ \xi y^{-1}e^{x} + \eta(1 - y^{-2}e^{x}) - \eta_x - (\eta_y - \xi_x)(y + y^{-1}e^{x}) + \xi_y(y + y^{-1}e^{x})^2 = 0, \tag{3.32} \]

where we have used (2.10) and \( p(x, y) = y + y^{-1}e^{x} \). If we assume that \( \xi(x, y) = \xi_{00} \) and \( \eta(x, y) = \eta_{00} \), it will lead to a system of rank two and the trivial solution \( \xi_{00} = \eta_{00} = 0 \). Instead, we let \( \xi(x, y) = \xi_{00} + \xi_{10}x + \xi_{01}y \) and \( \eta(x, y) = \eta_{00} + \eta_{10}x + \eta_{01}y \). Then (3.32) becomes

\[
(\xi_{00} + \xi_{10}x + \xi_{01}y)y^{-1}e^{x} + (\eta_{00} + \eta_{10}x + \eta_{01}y)(1 - y^{-2}e^{x}) - \eta_{10} - \\
(\eta_{10} - \xi_{10})(y + y^{-1}e^{x}) + \xi_{01}(y + y^{-1}e^{x})^2 = 0.
\]

This yields a set of 10 equations with a nontrivial solution, that is

\[
\xi_{00} = 2\eta_{01}, \\
\xi_{10} = \xi_{01} = \eta_{00} = \eta_{10} = 0.
\]

Thus, we can choose \( \xi_{00} = 2 \) and \( \eta_{01} = 1 \). This yields that \( \xi(x, y) = 2 \) and \( \eta(x, y) = y \).

The respective derivatives inserted in (2.10) yields that \( \zeta(x, y, p) = p \). The infinitesimal generator becomes

\[
X = 2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + p \frac{\partial}{\partial p}.
\]

After constructing the new coordinates, we have the following transformations between the coordinate systems,

\[
s = \frac{1}{2}x \iff x = 2s, \\
r = y^{2}e^{-x} \iff y = \pm r^{1/2}e^{s}, \\
t = py^{-1} \iff p = \pm tr^{1/2}e^{s}.
\tag{3.33}
\]

The surface equation (3.31) transforms to

\[
e^{s}[r^{1/2} + r^{-1/2} - tr^{1/2}] = 0.
\]

Thus, the new surface equation is \( r^{1/2} + r^{-1/2} - tr^{1/2} = 0 \). Solving for \( t \) yields that \( t(r) = 1 + 1/r \). Therefore, the constraint equation is

\[
\frac{ds}{dr} = \frac{s_x + p s_y}{r_x + p r_y} = \frac{e^x}{4py - 2y^2} = \frac{1}{2r + 4}.
\]
Thus,

\[ s(r) = \int \frac{1}{2r + 4} dr = \frac{1}{2} \ln(r + 2) + c_3, \quad c_3 \in \mathbb{R}. \]

Changing back to the original coordinates using (3.33), this yields that

\[ y = \pm e^{x/2} \sqrt{e^{x/2}-2}, \quad c \in \mathbb{R}, \]

is the solution to (3.30). The result is unchanged even if we use the negative \( y \)-value and \( p \)-value in (3.33). Our result is in agreement with the result in [20], obtained by a different method.

Since the equation

\[ Q(x, y) = 0 \iff y + y^{-1}e^x - \frac{y}{2} = 0, \]

does not admit any real solutions, (3.30) has no invariant solutions.

A family of solution curves of (3.30) is illustrated in Figure 3.6.

![Figure 3.6: Some solution curves of (3.30).](image)
Chapter 4

Conclusions

In this thesis we have studied Lie symmetries of ordinary differential equations. We have presented some of the main mathematical basics of Lie symmetries, including a method that describes how the underlying symmetries of a first order ordinary differential equation can be identified and used to solve the equation. We have outlined how this can be extended and used in the case of higher order ordinary differential equations. We have also presented the concept of reduction of order and treated invariant solutions.

To further clarify the theoretical basics we have given several illustrative examples, where we have used figures to enlighten the ideas in the examples graphically. The majority of the examples treated in this thesis are nonlinear ordinary differential equations, where most of the standard solution methods become insufficient. This was one of the main reasons to study symmetry methods in finding solutions to such types of equations.

Before writing this thesis, the author had only studied typical first courses in differential equations and learned how to identify and solve specific types such as homogeneous, exact, separable, etc. Now we have extended the theory and made it even more coherent. We have found a connection between these seemingly unrelated methods and are pleased with the results.

During the work of this thesis, we have noticed that the hardest step in the symmetry methods seems to be to find the symmetries of the differential equations. We have presented one method of how this can be done for first order ordinary differential equations, but it is not applicable to all of them.

To sum up this part of the thesis, we give an overview of some other aspects in the study of symmetries. Further discussions can be found in the references.
Transition to second order

In Section 2.2 we presented a method of how symmetries can be used to solve ordinary differential equations of first order. However, we also mentioned that the functions $\xi(x, y)$ and $\eta(x, y)$ are polynomials of finite degree. Unfortunately, this might not always be the case. In [1], they present a method that can be used in the study of symmetries of first order ordinary differential equations. Here we outline the main features of the result.

There are several methods of how we can use symmetries to solve different first order ordinary differential equations, but it seems to be not as useful as in the higher order case. The methods that are available to solve differential equations of higher order are in general constructed in a better way. This is an advantage that can be exploited, by deriving second order ordinary differential equations from the original ordinary differential equation of first order. Consequently, Lie symmetries of first order differential equations can be investigated via second order differential equations.

Nonlocal symmetries

In this thesis we have brought attention to local symmetries. The existence and applications of nonlocal symmetries (i.e. symmetries where the functions $\xi(x, y)$ and $\eta(x, y)$ depend upon an integral), is something that is usually not taken into account in the symmetry methods. The reason for this is simply that nonlocal symmetries are difficult to find. Therefore, researchers have done a lot of work in this area [6, 14].

Sometimes, nonlocal symmetries are also referred to as hidden symmetries, and appears when the order of a differential equation is decreased or increased. They also emerge when differential equations do not have Lie symmetries, but still can be integrated [17].

Initial-value problems

In this thesis we have concluded that symmetries are powerful tools to use when solving differential equations. Although, symmetry methods have not been as successful when it comes to dealing with initial- and boundary-value problems. It is not generally true that the symmetries of an initial-value problem (or boundary-value problem) are also symmetries of the unconstrained differential equation. This is presented in [10], which also provides a method of how to construct symmetries of a particular class of initial-value problems, and describes the difficulties that arise in the process.
Bibliography


